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How to Remove the Boundary in CFT – An Operator Algebraic Procedure

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Dedicated to Klaus Fredenhagen on the occasion of his 60th birthday

Abstract: The relation between two-dimensional conformal quantum field theories with and without a timelike boundary is explored.

1. Introduction

In [18], the authors have formulated boundary conformal field theory (BCFT) in real time (Lorentzian signature) in the algebraic framework of quantum field theory. BCFT is a local Möbius covariant QFT B_+ on the two-dimensional Minkowski halfspace M_+ (given by $x > 0$), which contains a (given) local chiral subtheory A, e.g., the stress-energy tensor. The reward of this approach was the surprisingly simple formula (1.2) below, expressing the von Neumann algebras of local observables $B_+(O)$ in a double cone $O \subset M_+$ in terms of an (in general nonlocal) chiral conformal net B of localized algebras associated with intervals along the boundary (the time axis $x = 0$). The net B is Möbius covariant and contains the local chiral observables A:

$$A(I) \subset B(I) \tag{1.1}$$

for each interval $I \subset \mathbb{R}$.

The reduction to a single chiral net is responsible for a kinematical simplification, explaining, e.g., Cardy's observation [3] that in BCFT, bulk n-point correlation functions are linear combinations of chiral $2n$-point conformal blocks.

The algebra $B_+(O)$ is a relative commutant of $B(K)$ within $B(L)$,

$$B_+(O) = B(K)' \cap B(L), \tag{1.2}$$

where $K \subset L$ are a pair of open intervals on the boundary $\mathbb{R}$ such that the disconnected complement $L \setminus \overline{K} = I \cup J$ is the set of advanced and retarded times $t \pm x$ associated with points in $(t, x) \in O$ (see Fig. 1). Although the chiral net B is not necessarily local,

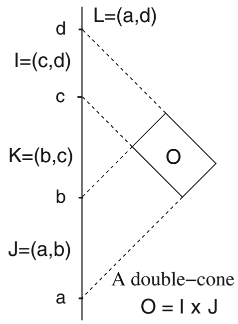

Fig. 1. Intervals on the boundary and double cones in the halfspace

the intersections (1.2) do commute with each other when two double cones are spacelike separated.

The main result in [18] is that every BCFT is contained in a maximal (Haag dual) BCFT of the form (1.2).

This leads to a somewhat paradoxical conclusion: on the one hand, each local bulk observable is defined as a (special) observable from a chiral CFT. Thus, superficially, the "degrees of freedom" of a BCFT are not more than those of a chiral CFT, containing only a single chiral component of the stress-energy tensor (Virasoro algebra). One might argue that such a "reduction of degrees of freedom" is a characteristic feature of QFT with a boundary. But this point of view cannot be maintained, because on the other hand, it was shown in [18] that the resulting BCFT B_+ is locally equivalent to another CFT B_{2D} on the full two-dimensional (2D) Minkowski spacetime, which has all the degrees of freedom of a 2D QFT, and in particular contains a full 2D stress-energy tensor (two commuting copies of the Virasoro algebra). Even in the simplest case, when the chiral net B on the boundary coincides with A (sometimes known as "the Cardy case"), the associated bulk QFT contains apart from the full 2D stress-energy tensor more ("non-chiral") local fields that factorize into chiral fields with braid group statistics. Locally, also the BCFT contains the same fields.

This paradoxical situation is not a contradiction; it rather shows that "counting degrees of freedom" of a QFT is an elusive task. Trivially, there is no obstruction against a proper inclusion of the form $\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ if $\mathcal{H}$ is an infinite-dimensional Hilbert space. But "counting degrees of freedoms", e.g. by entropy arguments, requires the specification of the Hamiltonian. The BCFT shares the Hamiltonian and ground state (vacuum) of the chiral CFT, while the associated 2D CFT has a different Hamiltonian and a different ground state. Thus, with respect to different Hamiltonians, the spacetime dimension (measured through some power law behaviour of the entropy) may assume different values (1 or 2, in the present case).

Looking at the issue from a different perspective, we may start from a vacuum representation of the Virasoro algebra. The latter integrates to a unitary projective representation of the diffeomorphism group of the circle Diff (S^1), which contains the diffeomorphism group of an interval Diff (I) as a subgroup. For two open intervals with disjoint closures, there is a canonical identification between Diff $(I \cup J)$ and Diff $(I) \times$ Diff (J). In terms of the stress-energy tensor T, this amounts to an isomorphism between $\exp iT(f+g)$ and $\exp iT(f) \otimes \exp iT(g)$, when f and g have disjoint support. It would be hard to see this local isomorphism directly in terms of the Virasoro algebra.

The mathematical theorem underlying these facts is the well-known Split Property [6], which can be derived in local QFT in any dimension under a suitable phase space assumption. In chiral local CFT, a sufficient assumption is the existence of the conformal character $\mathrm{Tr}\ \exp -\beta L_0$.

In the algebraic framework, the *chiral* observables of a BCFT (e.g., the stress-energy tensor) localized in a double cone O are operators belonging to the von Neumann algebra $A_+(O) = A(I) \vee A(J)$, where I and J are two open intervals of the time axis ("advanced and retarded times") such that $t + x \in I$, $t - x \in J$ for $(t, x) \in O$ (this justifies the notation $O = I \times J$), and $A(I)$ are the von Neumann algebras generated by the unitary exponentials of chiral fields smeared within I. In contrast, the *chiral* observables in a 2D CFT are operators in the algebra $A_{2D}(O) = A_L(I) \otimes A_R(J)$ where I and J are regarded as two open intervals of the lightcone axes, and $A_R(I)$ and $A_L(J)$ are generated by left and right chiral fields. Our present association between BCFT and 2D CFT applies to the case when $A_L(I) = A_R(I) = A(I)$, i.e., the left chiral observables $A_L(I) \otimes 1$ are isomorphic with the right chiral observables $1 \otimes A_R(I)$, and both are isomorphic with the chiral observables $A(I)$ of the BCFT.

Let $\mathcal{H}_0$ denote the vacuum Hilbert space for the chiral CFT described by the algebras $A(I)$. The split property states that if I and J are two intervals with disjoint closures, there is a canonical unitary $\mathcal{V} : \mathcal{H}_0 \to \mathcal{H}_0 \otimes \mathcal{H}_0$ implementing an isomorphism

$$\mathcal{V}\,(A(I) \vee A(J))\,\mathcal{V}^* = A(I) \otimes A(J). \tag{1.3}$$

The split isomorphism does not preserve the vacuum vector, i.e., the canonical "split vector" $\Xi = \mathcal{V}^*(\Omega \otimes \Omega)$ is an excited state in $\mathcal{H}_0$. By construction, the split state $(\Xi, \cdot\,\Xi)$ on $A(I) \vee A(J)$ has the property that its expectation values for either subalgebra $A(I)$ or $A(J)$ coincide with those in the vacuum state, but the correlations between observables $a_1 \in A(I)$ and $a_2 \in A(J)$ are suppressed:

$$(\Xi\ ,\ a_1 a_2\, \Xi) = (\Xi\ ,\ a_1\, \Xi)\ (\Xi\ ,\ a_2\, \Xi) = (\Omega\ ,\ a_1\, \Omega)\ (\Omega\ ,\ a_2\, \Omega). \tag{1.4}$$

The split isomorphism depends on the pair of intervals I and J. It trivially restricts to algebras associated with subintervals, but it does not, in general, extend to larger intervals. When the intervals touch or overlap, a split state and the split isomorphism cease to exist.

While the split isomorphism is well known, we discuss in this paper its extension to "non-chiral" local observables, which do *not* belong to $A(I) \vee A(J)$ in the BCFT, and to $A(I) \otimes A(J)$ in the 2D CFT.

As a concrete demonstration for the resolution of the above "paradox", we present two simple but nontrivial models where the algebraic relations outlined can be easily translated into the field-theoretic setting, i.e., we characterize the local algebras of the various QFTs in terms of generating local Wightman fields.

Let us translate (1.2) into the field-theoretic language. The intervals I and J shrink to the points $t \pm x$ when $O = I \times J$ shrinks to a point (t, x). Thus, we have to approximate a field $\Phi(t, x)$ of the BCFT by observables in $A(L)$ (where the interval L approximates $(t - x, t + x)$ from the outside), that commute with all fields localized in the interval K (which approximates $(t - x, t + x)$ from the inside). This will be done in Sect. 2. A crucial point here is that generating the local algebra $A(L)$ involves "non-pointwise" operations, e.g., typical observables may be exponentials of smeared field operators, so that an element of the relative commutant is not necessarily localized in the disconnected set $L \setminus \overline{K} = I \cup J$.

A second, somewhat puzzling feature of the algebraic treatment of BCFT is the fact that the description of the local algebras $B_+(O)$ in terms of the chiral boundary net (Eq. (1.2)) is much simpler than that of the local algebras $B_{2D}(O)$ of the associated $2D$ conformal QFT without a boundary. The latter are (rather clumsily) defined as Jones extensions of the tensor products $A(I) \otimes A(J)$ in terms of a Q-system constructed from the chiral extension $A \subset B$ with the help of α-induction [20].

One purpose of this work is to present a more direct construction of the 2D CFT without boundary from the BCFT. The obvious idea is to take a limit as the boundary is "shifted to infinity". But we shall do more, and establish the *covariant* local isomorphism between the subnets $O \mapsto B_+(O)$ and $O \mapsto B_{2D}(O)$ as $O \subset O_0$, i.e., the restriction of the AQFTs to any fixed double cone O_0 within the halfspace $x > 0$, at finite distance from the boundary.

The main problem here is, of course, the enhancement of the conformal symmetry, i.e., the reconstruction of the unitary positive-energy representation of the two-dimensional conformal group Möb × Möb from that of the chiral conformal group Möb. This is done by a "lift" of the chiral Möbius covariance of the local chiral net A, using the split property which allows to "embed" the 2D chiral algebra $A(I) \otimes A(J)$ into a local BCFT algebra $B_+(O)$. This will be done in Sect. 3. The point is that only a single local algebra of the BCFT is needed for this reconstruction of the 2D conformal group and the full 2D CFT.

In Sect. 4, we show that the 2D CFT can also be obtained through a limit where the boundary is "shifted to the left", or equivalently, the BCFT observables are "shifted to the right". The translations in the spatial direction "away from the boundary" do not belong to the chiral Möbius group of the BCFT. But they are at our disposal by the previous lifting of the 2D Möbius group into the BCFT. Therefore, we can study the behavior of correlation functions in the limit of "removing the boundary". As we shift the boundary, the retarded and advanced times are shifted apart from each other. The convergence of the vacuum correlations of the BCFT to the vacuum correlations of the 2D CFT is therefore a consequence of the cluster behavior of vacuum correlations of the chiral CFT A.

We add three appendices containing some related observations.

2. Models

The purpose of this section is to illustrate the construction (1.2) in a field-theoretic setting. It is convenient to assume the trivial chiral extension $B = A$ since even in this case the construction (1.2) is nontrivial, i.e., non-chiral local BCFT fields that factorize into nonlocal chiral fields can be constructed from local chiral fields only. We exhibit local BCFT fields in a region $O = I \times J \subset M_+$ as "neutral" chiral operators, that behave like products of "charged" chiral operators localized in I and J in the limit of large distance from the boundary. The limit of pointlike localization is also discussed, and reproduces familiar vertex operators.

Consider the free $U(1)$ current j with commutator $[j(x), j(y)] = 2\pi i\delta'(x - y)$ and charge operator $Q = (2\pi)^{-1} \int j(x)dx$. The unitary Weyl operators $W(f) = e^{ij(f)}$ for real test functions f satisfy the Weyl relation

$$W(f)\, W(g) = e^{-i\pi\sigma(f,g)} \cdot W(f + g) = e^{-2\pi i\sigma(f,g)} \cdot W(g)\, W(f) \qquad (2.1)$$

and have the vacuum expectation value

$$\omega(W(f)) = e^{-i\pi\sigma(f_-,f_+)} = e^{-\frac{1}{2}\int_{\mathbb{R}_+} k\,dk|\hat{f}(k)|^2}, \qquad (2.2)$$

where the symplectic form is

$$\sigma(f,g) = \frac{1}{2}\int_{\mathbb{R}} dx\ \big(f(x)\,g'(x) - f'(x)\,g(x)\big) = \frac{1}{2\pi i}\int_{\mathbb{R}} k\,dk\ \hat{f}(-k)\,\hat{g}(k), \tag{2.3}$$

and f_+ (f_-) correspond to the restrictions to positive (negative) values of k of the Fourier transform $\hat{f}(k) = \int_{\mathbb{R}} dx\, e^{ikx}\, f(x)$. With these conventions, $W(f)\Omega$ is a state with charge density $-f'(x)$.

The vacuum correlations of Weyl operators are

$$\omega(W(f_1)\cdots W(f_n)) = e^{-i\pi\left(\sum_i \sigma(f_{i-},f_{i+}) + 2\sum_{i<j}\sigma(f_{i-},f_{j+})\right)}. \tag{2.4}$$

The Weyl operators $W(f)$ with supp $f \subset I$ generate the local von Neumann algebras of the chiral net $I \mapsto A(I)$. We fix a double cone $O = I \times J \in M_+$. Let $K \subset L$ be the open intervals such that $L\backslash\bar{K} = I \cup J$, as before. If f is a test function that vanishes outside L and is constant in K, then $W(f)$ belongs to $A(L)$ and commutes with $A(K)$ by (2.1) and (2.3), hence

$$W(f) \subset B_+(O) = A(K)' \cap A(L). \tag{2.5}$$

These are examples of operators that belong to $B_+(O)$ but (if $f|_K \neq 0$) not to $A_+(O) = A(I) \vee A(J)$.

Weyl operators can also be defined for smooth functions f such that f' has compact support, and the relation (2.1) holds. Then $q = f(-\infty) - f(\infty)$ is called the charge. However, $i\sigma(f_-, f_+)$ diverges, and the vacuum expectation value (2.2) vanishes unless $q = 0$ (see below). This implies that correlation functions (2.4) of charged Weyl operators vanish whenever the total charge is non-zero (charge conservation), while the IR divergences in each term in the exponent of (2.4) cancel for neutral correlations. The neutral Weyl operators (2.5) in $B_+(O)$ are (up to a phase factor) products of charged Weyl operators with charge densities localized in J and in I.

In the limit of sharp step functions $G_u(x) = q\cdot\theta(x-u)$ (requiring a regularization [4]), the regularized Weyl operators $W(G_u)$ become the well-known vertex operators of charge $-q$ and scaling dimension $\frac{1}{2}q^2$ [21], which are formally written as

$$V_{-q}(u) = \,:\!\exp\left(iq\int_u^\infty j(y)dy\right)\!:. \tag{2.6}$$

Thus, as O shrinks to a point $(t,x) \in M_+$, and I and J shrink to the points $t+x$ and $t-x$, the (regularised) Weyl operators $W(G_{t-x} - G_{t+x})$ behave as

$$\Phi_q(t,x) = V_q(t+x)V_{-q}(t-x). \tag{2.7}$$

The correlation functions of vertex operators are computed from (2.4), giving

$$\langle \ldots \cdot V_{q_i}(u_i)\cdot \ldots\rangle = \lim_{\varepsilon\searrow 0}\prod_{i<j}\left(\frac{-i}{u_i - u_j - i\varepsilon}\right)^{-q_iq_j} \tag{2.8}$$

if $\sum_i q_i = 0$, and $= 0$ otherwise, from which the well-known anyonic commutation relations can be read off. It is then easily seen that $\Phi_{q_1}(t_1,x_1)$ commutes with $\Phi_{q_2}(t_2,x_2)$ when either $t_1+x_1 > t_2+x_2 > t_2-x_2 > t_1-x_1$ or when $t_2+x_2 > t_1+x_1 > t_1-x_1 > t_2-x_2$, because in these cases the anyonic phase factors cancel. It also commutes with

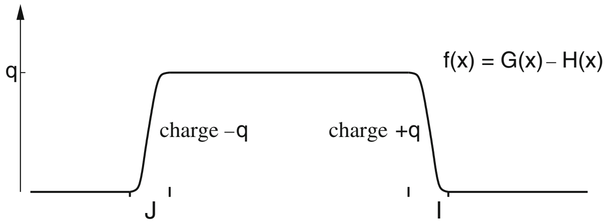

Fig. 2. A test function f such that $W(f)$ belongs to $B_+(O)$, but not to $A_+(O)$. G and H are smooth step functions, supp $G' \subset J$, supp $H' \subset I$

$j(t_2 \pm x_2)$ if $t_2 \pm x_2 \neq t_1 \pm x_1$. These are precisely the requirements for locality of the fields $\Phi_q(t,x)$ among each other, and relative to the conserved current

$$j_0(t,x) = j(t+x) + j(t-x), \quad j_1(t,x) = j(t+x) - j(t-x) \tag{2.9}$$

defined for $x > 0$, i.e., Φ_q and j^μ are local fields on the halfspace M_+. The correlation functions of n fields $\Phi_{q_i}(t_i, x_i)$ are correlations of $2n$ vertex operators ($2n$-point conformal blocks).

After this digression to pointlike fields, let us resume the study of the correlation functions (2.4) of the smooth Weyl operators $W(f_i) \in B_+(O)$, and their behavior as O is shifted away from the boundary. We choose n test functions of the form

$$f_i = G_i - H_i, \tag{2.10}$$

where G_i, H_i are smooth step functions with values 0 at $-\infty$ and q_i at $+\infty$, such that $G_i' = g_i$ is supported in J and $H_i' = h_i$ is supported in I (see Fig. 2).
The neutral states $W(f_i)\Omega$ carry the charge q_i in I and the charge $-q_i$ in J.

The neutrality condition for each Weyl operator $W(f_i)$ can be written

$$\int_{\mathbb{R}} dx\; g_i(x) - \int_{\mathbb{R}} dx\; h_i(x) = 0 \quad \Leftrightarrow \quad \hat{g}_i(0) - \hat{h}_i(0) = 0. \tag{2.11}$$

The exponent in (2.4) is a linear combination of terms of the form (using $\hat{f}_i = i(\hat{g}_i - \hat{h}_i)/k$)

$$2\pi i\, \sigma(f_{i-}, f_{j+}) = \int_{\mathbb{R}_+} \frac{dk}{k} \int dx\; (g_i(x) - h_i(x)) \int dy\; \big(g_j(y) - h_j(y)\big)\; e^{-ik(x-y)} \tag{2.12}$$

which are IR finite because of (2.11). The separate contributions from g_i and h_i, however, are IR divergent. Therefore, we first regularize at $k = 0$ by the subtraction $e^{-ik(x-y)} \to e^{-ik(x-y)} - e^{-k/\mu}$ ($\mu > 0$ arbitrary), which does not change the result because of (2.11), and then compute the contributions from g and h separately.

We are interested in the behavior of the correlation function (2.4) as O is shifted away from the boundary. This means that the functions g_i are shifted by a distance a to the left, and h_i are shifted by the same distance to the right. The g-g contributions and the h-h contributions to $\sigma(f_{i-}, f_{j+})$ are obviously invariant under this shift, while in the mixed h-g contributions $x - y$ is replaced by $x - y + 2a$:

$$2\pi i\, \sigma_{h_i,g_j}(a) := -\int_I dx\, h_i(x) \int_J dy\, g_j(y) \int_{\mathbb{R}_+} \frac{dk}{k} \left[e^{-ik(x-y+2a)} - e^{-k/\mu}\right], \tag{2.13}$$

and similarly for the g-h contributions. The last integrand can be split into two parts:

$$\left(e^{-ik(x-y+2a)} - 1\right) e^{-k/\mu} \ + \ e^{-ik(x-y+2a)} \left(1 - e^{-k/\mu}\right) \tag{2.14}$$

so that the first contribution to the momentum integral equals

$$-\log\left(1 + i\mu(x - y + 2a)\right) \tag{2.15}$$

while the second (distributional) contribution is of order $O(a^{-1})$ in the limit of large a. Because the remaining integrals have compact support, we obtain

$$\lim_{a\to\infty} \sigma_{h_i,g_j}(a) = q_i q_j \cdot \log(2i\, a\mu) + O(a^{-1}). \tag{2.16}$$

Together with the g-h contributions $q_i q_j \cdot \log(-2i\, a\mu)$, these terms in the exponent of (2.4) cumulate up to the factor

$$\prod_i (2a\mu)^{-q_i^2} \prod_{i<j} (2a\mu)^{-2q_iq_j} = (2a\mu)^{-q^2}, \tag{2.17}$$

where $q = \sum_i q_i$ is the total charge within I. Thus (2.4) vanishes in the limit $a \to \infty$ if $q \neq 0$, enforcing "chiral charge conservation" in the limit. If $q = 0$, the mixed contributions give 1, and the remaining g-g and h-h contributions yield

$$\begin{aligned} &\lim_{a\to\infty} \omega\left(W(f_1)\cdots W(f_n)\right) \\ &\quad = \omega\left(W(G_1)\cdots W(G_n)\right) \cdot \omega\left(W(-H_1)\cdots W(-H_n)\right) \end{aligned} \tag{2.18}$$

involving charged Weyl operators. These expressions are well-defined (and independent of μ) because $\sum_i G_i$ and $\sum_i H_i$ are neutral precisely due to $q = 0$.

The factorization of the vacuum correlations in the limit $a \to \infty$ is the desired feature we wanted to illustrate by this example. In the limit, $W(f_i)$ have the same correlations as $W(-H_i) \otimes W(G_i)$, which are charged observables of the associated 2D CFT. Notice that in the limit of sharp test functions (see above), one obtains

$$V_q(t + x) \otimes V_{-q}(t - x), \tag{2.19}$$

which are local fields in the entire two-dimensional Minkowski spacetime M^2.

Remark. The above construction can be generalized to the $SU(2)$ current algebra. The Frenkel-Kac representation of $SU(2)$ currents at level 1 is given by $j^3 \equiv j$ and $j^{\pm}(x) = j^1(x) \pm i j^2(x) = V_{\pm\sqrt{2}}(x)$. Then $V_q(x) \cdot V_{-q}(y)$ commutes with $V_{q'}(w)$ at $w \neq x, y$ provided $qq' \in \mathbb{Z}$. Hence the field

$$\Phi_{\frac{1}{2}\sqrt{2}}(t, x) = V_{\frac{1}{2}\sqrt{2}}(t + x) \cdot V_{-\frac{1}{2}\sqrt{2}}(t - x) \tag{2.20}$$

is local (as before) and relatively local w.r.t. the conserved currents j^a $(a = 1, 2, 3)$

$$j_0^a(t, x) = j^a(t + x) + j^a(t - x), \quad j_1^a(t, x) = j^a(t + x) - j^a(t - x). \tag{2.21}$$

$\Phi_{\frac{1}{2}\sqrt{2}}(t, x)$ is a neutral combination of charged primary fields of dimension $\frac{1}{4}$, transforming in the spin-$\frac{1}{2}$ representation of $SU(2)$, localized at $t+x$ and $t-x$. The description of this model in terms of smooth Weyl operators is rather straightforward, see e.g., [2]: Weyl operators with integer multiples of the charge $\sqrt{2}$ belong to $A(I)$, while operators with half-integer multiples of the charge $\sqrt{2}$ in I and in J belong to $A(K)' \cap A(L)$.

The mechanism of "charge separation" described here for obtaining elements of $B_+(O)$ that do not belong to $A_+(O)$ is very general [18], although in general it cannot be formulated in terms of Weyl operators. In Sect. 4 we shall show that also the factorization behavior far away from the boundary is a general feature, which allows to recover the 2D CFT from the BCFT.

3. Reconstruction of the 2D Symmetry

We work in this section with a fixed "chiral extension" $A \subset B$. Here, A is a Haag dual Möbius covariant local net $\mathbb{R} \supset I \mapsto A(I)$ of von Neumann algebras on its vacuum Hilbert space $\mathcal{H}_0$, satisfying the split property and having finitely many irreducible DHR sectors of finite dimension (these properties together are called "complete rationality" [14]; in the case of diffeomorphism covariant nets, Haag duality = strong additivity is a consequence of the other properties [19]. The fact that the $U(1)$ Weyl algebra in Sect. 2 is *not* completely rational, indicates that the results to be reported in this section hold also in more general situations).

B is a Möbius covariant net $\mathbb{R} \supset I \mapsto B(I)$ on its vacuum Hilbert space $\mathcal{H}_0^B$ such that for each I the inclusion $A(I) \subset B(I)$ holds and is an irreducible subfactor, which has automatically finite Jones index [13] equal to the statistical dimension of the (reducible) representation of A on $\mathcal{H}_0^B$ [17]. The net B may be non-local, but is required to be relatively local w.r.t. A.

If only A is specified, the irreducible chiral extensions B of A can be classified in terms of Q-systems of A [17]. The complete classification has been computed for A the Virasoro nets with central charge $c < 1$ (and implicitly also for the $SU(2)$ current algebras) in [15].

With $A \subset B$ one can associate a boundary CFT B_+ on the halfspace M_+ and a two-dimensional CFT B_{2D} on Minkowski spacetime M^2. To describe the former, we introduce a convenient notation (see Fig. 1). For any quadruples of four real numbers such $a < b < c < d$ we define $I = (c, d)$, $J = (a, b)$, $K = (b, c)$, $L = (a, d)$, and $O = \{(t, x) : t + x \in I,\ t - x \in J\} \subset M_+$. Every double cone $O \subset M_+$ is of this form and determines I, J, K, L, and similarly every pair of open intervals $J < I$ ("I is to the right = future of J") determines K, L, and $O = I \times J$.

Then the BCFT associated with $A \subset B$ is the net (1.2), i.e., $O \mapsto B_+(O) = B(K)' \cap B(L)$. We have shown in [18] that $B_+(O)$ contains $A_+(O) = A(I) \vee A(J)$ as a subfactor with finite index, B_+ is local and Haag dual on M_+, every Haag dual BCFT with chiral observables A arises in this way (namely the chiral extension B can be recovered from the BCFT), and every non-Haag-dual local BCFT net is intermediate between A_+ and B_+. If $B = A$, $B_+(O)$ equals the four-interval subfactor $A(E) \subset A(E')'$ on the circle [14] ($E = I \cup J$).

The 2D CFT B_{2D} associated with $A \subset B$ has been constructed in [20]. Its local algebras are extensions (with finite Jones index) of the tensor products $A(I) \otimes A(J)$, specified in terms of a Q-system constructed from the chiral extension $A \subset B$ with the help of α-induction.

We know from [18] that B_+ and B_{2D} are locally isomorphic, i.e., for each $O \subset M_+$ there is an isomorphism $\varphi^O : B_+(O) \to B_{2D}(O)$ such that

$$\varphi^O\,(B_+(O_1)) = B_{2D}(O_1) \quad \text{for all } O_1 \subset O. \tag{3.1}$$

However, the Hilbert space and the vacuum state for the two theories are very different.

In this section, we wish to understand the relation between these two nets, by giving an alternative construction of the 2D CFT directly from the BCFT. The crucial point is the construction of the enhanced Möbius symmetry of the 2D CFT, and its ground state (the 2D vacuum) which is different from the BCFT vacuum.

We first construct the Hilbert space $\mathcal{H}_{2D}$ for the 2D CFT. We choose a fixed reference double cone $O_0 = I_0 \times J_0 \subset M_+$. The subfactor $A_+(O_0) = A(I_0) \vee A(J_0) \subset B_+(O_0) = B(K_0)' \vee B(L_0)$ is irreducible with finite index [18], and hence has a unique conditional expectation $\mu : B_+(O_0) \to A_+(O_0)$, which is automatically normal and faithful. Let $\Xi \in \mathcal{H}_0$ be the canonical split vector for $A(I_0) \vee A(J_0)$ as in (1.4). The split state $\xi = (\Xi, \cdot \Xi)$ on $A_+(O_0)$ extends to the state $\hat{\xi} = \xi \circ \mu$ on $B_+(O_0)$. Let $\hat{\mathcal{H}}$, $\hat{\Xi}$ and $\hat{\pi}$ denote the GNS Hilbert space, GNS vector and GNS representation for $(B_+(O_0), \hat{\xi})$. We also write $|b\rangle$ for $\hat{\pi}(b)\hat{\Xi}$. Let us analyze the structure of $\hat{\mathcal{H}}$.

The structure of $B_+(O_0)$ has been described in [18]. By complete rationality, A has finitely many irreducible superselection sectors [14]. Choose for each irreducible sector of A a representative DHR endomorphism [7] σ localized in I_0, and a representative τ localized in J_0. (For the vacuum sector, $\sigma = \tau = \mathrm{id}$. $\bar{\sigma}$ and $\bar{\tau}$ are the representatives of the conjugate sector.) Then the elements of $B_+(O_0)$ are (weak limits of) sums of operators of the form $\iota(a_1a_2) \cdot \psi$ where ι is the injection $A \to B$, $a_1 \in A(I_0)$, $a_2 \in A(J_0)$, and $\psi \in B(L_0)$ generalize the Weyl operators $W(f)$ (2.10) of Sect. 2: they are (for each pair σ, τ) "charged" intertwiners in $\mathrm{Hom}(\iota, \iota\sigma\bar{\tau}) \cap B(K_0)'$. We may express these intersections in a different way: Let $\alpha^{\pm}_{\rho}$ denote the endomorphisms of B extending the DHR endomorphisms ρ of A by "α-induction" [17], where α^+_{ρ} (α^-_{ρ}) acts trivially on $b \in B$ localized to the right = future (left = past) of the interval where ρ is localized. Thus $\alpha^-_{\sigma}\alpha^+_{\bar{\tau}}$ acts trivially on $B(K_0)$, because $J_0 < K_0 < I_0$. Hence

$$\mathrm{Hom}\,(\iota, \iota\sigma\bar{\tau}) \cap B(K_0)' = \mathrm{Hom}\left(\mathrm{id}_B, \alpha^-_{\sigma}\alpha^+_{\bar{\tau}}\right). \tag{3.2}$$

(For an alternative characterization of the charged intertwiners by means of an eigenvalue condition, see App. B.) If $O_1 \subset M_+$ is another double cone in the halfspace, the algebra $B_+(O_1)$ is generated by $A(I_1) \vee A(J_1)$ and charged intertwiners

$$\psi_1 = \iota\,(u \times \bar{u}) \cdot \psi \in \mathrm{Hom}\left(\mathrm{id}_B, \alpha^-_{\sigma_1}\alpha^+_{\bar{\tau}_1}\right) \tag{3.3}$$

with unitary charge transporters $u \in \mathrm{Hom}(\sigma, \sigma_1)$ and $\bar{u} \in \mathrm{Hom}(\bar{\tau}, \bar{\tau}_1)$, where σ_1 is localized in I_1 and $\bar{\tau}_1$ is localized in J_1.

E.g., if $B = A$ (the "Cardy case"), the charged intertwiners (generalizing the Weyl operators $W(f)$ in (2.10) of Sect. 2) are of the form $\psi \in \mathrm{Hom}(\mathrm{id}, \sigma\bar{\tau})$. This implies that τ and σ are representatives of the same sector. Thus, the charges of BCFT fields are in 1:1 correspondence with the DHR sectors of A.

In the general case, when ψ and ψ' are two charged intertwiners, $\mu(\psi'\psi^*)$ is an intertwiner $\in \mathrm{Hom}(\sigma'\bar{\tau}', \sigma\bar{\tau}) \cap (A(I_0) \vee A(J_0))$. This space is zero unless $\sigma' = \sigma$ and $\tau' = \tau$, and $\mathrm{Hom}(\sigma\bar{\tau}, \sigma\bar{\tau}) \cap (A(I_0) \vee A(J_0)) = \mathbb{C} \cdot 1$ [16]. Therefore, we may choose (for each pair σ, τ) a basis of charged intertwiners ψ which is orthonormal w.r.t. the inner product $\mu(\psi'\psi^*)$.

Lemma 1. *The subspaces $\hat{\mathcal{H}}_{\psi}$ of $\hat{\mathcal{H}}$ spanned by $|\psi^* \cdot \iota(A(I_0) \vee A(J_0))\rangle$ are mutually orthogonal. Each subspace $\hat{\mathcal{H}}_{\psi}$ factorizes as a representation of $A_+(O_0)$ according to*

$$\hat{\mathcal{H}}_{\psi} \cong \mathcal{H}_{\sigma} \otimes \mathcal{H}_{\bar{\tau}}, \tag{3.4}$$

where $\mathcal{H}_{\sigma}$ and $\mathcal{H}_{\bar{\tau}}$ carry the representations σ and $\bar{\tau}$ of $A(I_0)$ and $A(J_0)$, respectively.

Proof. The computation of matrix elements in a dense set of vectors

$$\begin{aligned}
&\langle \psi^* \cdot \iota(a_1''a_2'')|\,\hat\pi\,(\iota(a_1a_2))\,|\psi^* \cdot \iota(a_1'a_2')\rangle \\
&= \big(\varXi\,,\,a_1''^*a_2''^*\,\mu\left(\psi\,\iota(a_1a_2)\,\psi^*\right)\,a_1'a_2'\,\varXi\big) \\
&= \big(\varXi\,,\,a_1''^*a_2''^*\,\sigma\bar\tau(a_1a_2)\,a_1'a_2'\,\varXi\big) \\
&= \big(a_1''\,\Omega\,,\,\sigma(a_1)\,a_1'\,\Omega\big)\cdot\big(a_2''\,\Omega\,,\,\bar\tau(a_2)\,a_2'\,\Omega\big)
\end{aligned} \tag{3.5}$$

proves the claim. □

We may therefore identify the vectors $|\psi^*\iota(a_1'a_2')\rangle$ with $a_1'\Omega \otimes a_2'\Omega \in \mathcal{H}_\sigma \otimes \mathcal{H}_{\bar\tau}$ in the representation $\sigma \otimes \bar\tau$ under the split isomorphism, such that in particular, the GNS vector $\hat\varXi = |1\rangle \in \hat{\mathcal{H}}$ corresponds to the 2D vacuum vector $\Omega \otimes \Omega \subset \mathcal{H}_0 \otimes \mathcal{H}_0$. We write the extended Hilbert space $\hat{\mathcal{H}}$ in the form

$$\hat{\mathcal{H}} \equiv \mathcal{H}_{2D} \cong \bigoplus\nolimits_{\sigma,\tau} Z_{\sigma,\tau}\,\mathcal{H}_\sigma \otimes \mathcal{H}_{\bar\tau} \tag{3.6}$$

(the "2D Hilbert space"). The nonnegative integer multiplicities are

$$Z_{\sigma,\tau} = \dim \mathrm{Hom}(\alpha^+_\tau, \alpha^-_\sigma) \tag{3.7}$$

by the above characterization (3.2) of the spaces of charged intertwiners. The chiral factorization (3.6) of the GNS construction from the extended state $\xi \circ \mu$ may be viewed as the remnant of the original "splitting behavior" of the split vector $\varXi$.

As shown in [18] by comparison of the Q-system, the local subfactor $\hat\pi(A_+(O_0)) \subset \hat\pi(B_+(O_0))$ on $\hat{\mathcal{H}}$ is isomorphic to $A(I_0) \otimes A(J_0) \subset B_{2D}(O_0)$ constructed in [20]. We may therefore consistently denote also the former by $A_{2D}(O_0) \subset B_{2D}(O_0)$.

Next, we construct the action of the 2D Möbius group on $\mathcal{H}_{2D}$, by a "lift" of the Möbius transformations of the chiral net A, using the split isomorphism and the conditional expectation μ. The action of Möb × Möb on $\mathcal{H}_{2D}$ will then be used to define $B_{2D}(O)$ as the images of the reference algebra $B_{2D}(O_0)$ under a 2D Möbius transformation $g = (g_1, g_2)$ taking O_0 to O.

The 2D Möbius group Möb × Möb is unitarily represented in the vacuum Hilbert space $\mathcal{H}_0$ of the chiral net A by U_+U_-, the preimage of $U_0 \otimes U_0$ on $\mathcal{H}_0 \otimes \mathcal{H}_0$ under the split isomorphism. (See App. A, how U_+ and U_- can be obtained by modular theory directly on the boundary Hilbert space.) We need to lift U_+U_- to $\mathcal{H}_{2D}$.

Let $\Sigma_I \subset$ Möb denote the connected semigroup taking the interval I into itself, generated by the one-parameter subgroup preserving I and two one-parameter semigroups fixing either of its endpoints. Then $\Sigma = \Sigma_{I_0} \times \Sigma_{J_0} \subset$ Möb × Möb is the connected semigroup taking the reference double cone O_0 into itself.

For $g = (g_1, g_2) \in \Sigma$, the adjoint action of $U_+(g_1)U_-(g_2)$ on $a_1 \in A(I_0)$, $a_2 \in A(J_0)$ is given by the *independent* (= product) action of the chiral Möbius transformations given by geometric automorphisms α_g of the chiral net A:

$$\alpha^+_{g_1}\alpha^-_{g_2}(a_1 \cdot a_2) = \alpha_{g_1}(a_1) \cdot \alpha_{g_2}(a_2). \tag{3.8}$$

We extend these endomorphisms of $A_+(O_0)$ to endomorphisms of $B_+(O_0)$ by

$$\beta^+_{g_1}\beta^-_{g_2}\,(\iota(a_1a_2)\cdot\psi) := \iota\left(\alpha_{g_1}(a_1)\alpha_{g_2}(a_2)\right)\cdot\iota\left(z^\sigma(g_1)z^{\bar\tau}(g_2)\right)\cdot\psi. \tag{3.9}$$

Here $z^\rho(g) \in \mathrm{Hom}(\rho, \alpha_g\rho\alpha_g^{-1})$ are the unitary cocycles $z^\rho(g) = U_0(g)U_\rho(g)^* \in A$ [10,16], where U_0 and U_ρ are the representations of the Möbius group in the vacuum representation and in the DHR representation ρ.

Proposition 1. (i) *The maps $\beta^+_{g_1}\beta^-_{g_2}$ defined by (3.9) for $g \in \Sigma$ are homomorphisms from $B_+(O_0)$ onto $B_+(g\,O_0) \subset B_+(O_0)$.*
(ii) *For $O_1 \subset O_0$ we have $\beta^+_{g_1}\beta^-_{g_2}(B_+(O_1)) = B_+(g\,O_1)$, i.e., $\beta^+_{g_1}\beta^-_{g_2}$ "act geometrically inside $B_+(O_0)$".*
(iii) *$\beta^+_{g_1}\beta^-_{g_2}$ respect the group composition law within the semigroup Σ.*
(iv) *The conditional expectation μ intertwines $\beta^+_{g_1}\beta^-_{g_2}$ with $\alpha^+_{g_1}\alpha^-_{g_2}$.*

Proof. (i) The homomorphism property follows from the composition and conjugation laws of charged intertwiners [18] and the intertwining and localization properties of the operators and endomorphisms involved. The statement about the range is just a special case of (ii).
(ii) It is sufficient to show that a charged intertwiner $\psi_1 \in B_+(O_1)$ is mapped to a charged intertwiner in $B_+(g\,O_1)$. By virtue of (3.3), we compute

$$\beta^+_{g_1}\beta^-_{g_2}(\psi_1) = \iota\left(\alpha_{g_1}(u)z^{\sigma}(g_1)\alpha_{g_2}(\bar{u})z^{\bar{\tau}}(g_2)\right) \cdot \psi. \tag{3.10}$$

Then the claim follows, because $\alpha_{g_1}(u)\,z^{\sigma}(g) \in \mathrm{Hom}(\sigma, \alpha_{g_1}\sigma_1\alpha^{-1}_{g_1})$, and $\alpha_{g_1}\sigma_1\alpha^{-1}_{g_1}$ is localized in $g_1\,I_1$, and similarly $\alpha_{g_2}\bar{\tau}_1\alpha^{-1}_{g_2}$ is localized in $g_2\,J_1$.
(iii) The group composition law follows from the cocycle properties [10,16] of z^{ρ}.
(iv) The intertwining property of μ is due to the fact that μ annihilates all charged intertwiners except the neutral one ($\sigma = \bar{\tau} = \mathrm{id}$).

□

Next, we adapt a well-known lemma about the implementation of (groups of) automorphisms to the case of (semigroups of) endomorphisms.

Lemma 2. *Let M be a von Neumann algebra on a Hilbert space $\mathcal{H}$ with a cyclic and separating vector Ψ. Let β be an endomorphism of M, preserving the state $(\Psi, \cdot\Psi)$. Then the closure of the map $m\Psi \mapsto \beta(m)\Psi$ is an isometry U_β. If Ψ is cyclic also for $\beta(M)$, then U_β is unitary. For two endomorphisms β, β' with the same properties, such that Ψ is cyclic for $\beta(M)$, one has $U_{\beta'\beta} = U_{\beta'}U_\beta$.*

Proof. That U_β is an isometry is an obvious consequence of the invariance of the state. Since $\beta(M)\Psi$ is a dense subset, U_β is surjective, hence unitary. For the last statement it is sufficient to notice that $U_{\beta'}$ is densely defined on $\beta(M)\Psi$. □

We apply the lemma to the endomorphisms $\beta^+_{g_1}\beta^-_{g_2}$ of $B_+(O_0)$. Using (iv) of Prop. 1, we see that $\beta^+_{g_1}\beta^-_{g_2}$ leave the GNS state $(\hat{\Xi}, \cdot\hat{\Xi})$ invariant because the split state $(\Xi, \cdot\Xi)$ on $A_+(O_0)$ is invariant under $\alpha^+_{g_1}\alpha^-_{g_2}$. The vector $\hat{\Xi}$ is cyclic and separating for each $\hat{\pi}\,(B_+(O_1))$ $(O_1 \subset O_0)$ because μ is faithful and Ξ is cyclic and separating for each $A_+(O)$, which in turn follows by the split isomorphism because Ω is cyclic and separating for $A(I_1)$ and for $A(J_1)$. Thus, lemma 2 applies:

Corollary 1. *The homomorphisms $\beta^+_{g_1}\beta^-_{g_2}$ induce unitary operators on $\hat{\mathcal{H}} = \mathcal{H}_{2D}$, which satisfy the group composition law within the semigroup Σ. Together with the inverse unitary operators, they generate a covering representation $\hat{U}(g_1, g_2) = \hat{U}_+(g_1)\hat{U}_-(g_2)$ of Möb × Möb on $\mathcal{H}_{2D}$.*

The last statement is due to the fact that Σ and its inverse generate Möb × Möb, and the group law within Σ secures the commutation relations of the Lie algebra.

By construction, for $g = (g_1, g_2) \in \Sigma$, $\hat{U}(g_1, g_2)$ on the subspace $\mathcal{H}_\psi$ is equivalent to $U_\sigma(g_1) \otimes U_{\bar{\tau}}(g_2)$ on $\mathcal{H}_\sigma \otimes \mathcal{H}_{\bar{\tau}}$ under the isomorphism (3.4). By (ii) of Prop. 1, the adjoint action of $\hat{U}(g_1, g_2)$ takes $B_+(O_1)$ to $B_+(g\,O_1)$ for $O_1 \subset O_0$.

By constructing U_+U_-, we have thus furnished the local subnet $O_0 \supset O_1 \mapsto B_+(O_1)$ of the BCFT with a covariant "two-dimensional re-interpretation". In the representation $\hat{\pi}$ on $\hat{\mathcal{H}} = \mathcal{H}_{2D}$, this is precisely the local isomorphism φ^{O_0} referred to in (3.1). The present discussion shows that φ^{O_0} intertwines the global 2D Möbius covariance with a "hidden" symmetry of the BCFT, which is induced by the extended split state $\hat{\xi}$ and acts locally geometric.

We now define for arbitrary double cones $O \subset M^2$ the associated local algebras of the 2D conformal net on $\mathcal{H}_{2D}$ by varying $g = (g_1, g_2) \in$ Möb × Möb in the connected neighborhood of unity for which $g\,O_0 \subset M^2$, and putting

$$B_{2D}(O) := \hat{U}(g_1, g_2)\ B_{2D}(O_0)\ \hat{U}(g_1, g_2)^* \quad \text{if} \quad O = g\,O_0 \subset M^2. \tag{3.11}$$

For $O \subset O_0$, this coincides with $\hat{\pi}\left(\beta^+_{g_1}\beta^-_{g_2}(B_+(O_0))\right) = \hat{\pi}\,(B_+(O))$ by virtue of (ii) of Prop. 1. Notice that $B_{2D}(g\,O_0)$ is uniquely defined as long as $O = g\,O_0 \subset M^2$ because in this case any two g with the same image $g\,O_0$ differ by an element of Σ, while it requires the passage to a covering space when M^2 is conformally completed.

Theorem 1. *The net of von Neumann algebras $O \mapsto B_{2D}(O)$ defined by (3.11) is covariant, isotonous, and local.*

Proof. The covariance is by construction. Isotony and locality of the 2D net follow from the geometric action inside O_0, (ii) of Prop. 1, and the fact that every pair of double cones in M^2 such that either $O_1 \subset O_2$ are $O_1 \subset O_2'$ can be moved inside O_0 by a Möbius transformation, where we know (from the boundary CFT) that isotony and locality hold. □

Corollary 2. *The extension $A_{2D} \subset B_{2D}$ is isomorphic to the extension constructed in* [20].

Proof. Since the local subfactor $A_{2D}(O_0) \subset B_{2D}(O_0)$ constructed in [20] is isomorphic to $A_+(O_0) \subset B_+(O_0)$, and the isomorphism intertwines the representations of the 2D Möbius group, the global isomorphism follows. □

We have associated with the BCFT a 2D local CFT, that is locally isomorphic. The association is intrinsic in the sense that it requires only the subnet $O_0 \supset O_1 \mapsto B_+(O_1)$ together with the covariance of the DHR sectors of the underlying chiral CFT A.

It should be noticed that the construction is up to unitary equivalence independent of the choice of the reference double cone $O_0 \subset M_+$. The reason is essentially that the charge structure of $B(K)' \cap B(L)$ exhibited by the multiplicities $Z_{\sigma,\tau}$ in (3.6) is independent of the pair $K \subset L$.

We conclude this section with an observation concerning diffeomorphism covariance:

Proposition 2. *If $A \subset B$ is a chiral extension of a diffeomorphism covariant chiral net A, then the (possibly non-local) chiral net B, the BCFT net B_+ defined by (1.2), and the 2D net B_{2D} associated with B_+ by Thm. 1 are also diffeomorphism covariant.*

Proof. The chiral net A is diffeomorphism covariant if for a diffeomorphism γ of S^1 there is a unitary operator w_γ on $\mathcal{H}_0$ such that $u_\gamma A(I) u_\gamma^* = A(\gamma I)$. Haag duality of A implies that if γ is localized in an interval I (i.e., acts trivially on the complement), then w_γ is an observable in $A(I)$.

For a chiral extension $A \subset B$ we claim that if γ is localized in I_0, then for $I_1 \subset I_0$ one has $\iota(w_\gamma) B(I_1) \iota(w_\gamma^*) = B(\gamma I_1)$, i.e., $\iota(w_\gamma)$ implement the local diffeomorphisms. Namely, $B(I_1)$ is generated by $\iota(A(I_1))$ and $v_1 = \iota(u) \cdot v$, where $v \in B(I_0)$ is the canonical charged intertwiner $v \in \mathrm{Hom}(\iota, \iota\theta)$ for the canonical DHR endomorphism θ localized in I_0 [17] (see also App. B), and θ_1 is an equivalent DHR endomorphism localized in I_1. We find

$$\iota(w_\gamma)\ v_1\ \iota(w_\gamma^*) = \iota(w_\gamma u \theta(w_\gamma^*)) \cdot v. \tag{3.12}$$

Now, $w_\gamma u \theta(w_\gamma^*) \in \mathrm{Hom}(\theta, \gamma\theta_1\gamma^{-1})$, and $\gamma\theta_1\gamma^{-1}$ is localized in γI_1. This proves the claim. The diffeomorphism covariance of the chiral net B follows because the diffeomorphisms localized in I_0 together with the Möbius group generate the diffeomorphism group of S^1.

The argument for the boundary CFT and for the 2D CFT are very similar: we first show that for diffeomorphisms $\gamma = \gamma_1\gamma_2$ where γ_1 is localized in I_0 and γ_2 localized in J_0, the adjoint action with $\iota(w_{\gamma_1} w_{\gamma_2})$ takes $B_+(O_1)$ to $B_+(\gamma O_1)$ if $O_1 \subset O_0$. Again, it is sufficient to verify the action on the charged intertwiners (3.3) of $B_+(O_1)$:

$$\iota(w_{\gamma_1} w_{\gamma_2}) \cdot \psi_1 \cdot \iota(w_{\gamma_1} w_{\gamma_2})^* = \iota\Big((w_{\gamma_1} u \sigma(w_{\gamma_1}^*))(w_{\gamma_2} \bar{u} \bar{\tau}(w_{\gamma_2}^*))\Big) \cdot \psi, \tag{3.13}$$

where $w_{\gamma_1} u \sigma(w_{\gamma_1}^*) \in \mathrm{Hom}(\sigma, \gamma_1\sigma_1\gamma_1^{-1})$ and $w_{\gamma_2} \bar{u} \bar{\tau}(w_{\gamma_2}^*) \in \mathrm{Hom}(\bar{\tau}, \gamma_2\bar{\tau}_1\gamma_2^{-1})$, and $\gamma_1\sigma_1\gamma_1^{-1}$ is localized in $\gamma_1 I_1$ and $\gamma_2\bar{\tau}_1\gamma_2^{-1}$ is localized in $\gamma_2 J_1$. Hence (3.13) is a charged intertwiner of $B_+(\gamma O_1)$. This proves the claim. Then the diffeomorphism covariance of B_+ and B_{2D} follow because the diffeomorphisms localized in O_0 together with the Möbius group generate all diffeomorphisms. □

4. Cluster Limit

Let $b_1, \dots, b_n \in B_+(O)$ be BCFT observables localized within any fixed double cone $O = I \times J \subset M_+$. We wish to consider the behavior of a vacuum correlation

$$(\Omega\ ,\ \beta_x(b_1 \cdots b_n)\ \Omega)\ , \tag{4.1}$$

where $\beta_x = \beta_x^+ \beta_{-x}^-$ is the one-parameter semigroup of "right shifts" ($x > 0$, away from the boundary), that take I to $I + x$ and J to $J - x$, represented as homomorphisms from $B_+(O)$ to $B_+(I + x \times J + x)$, see (3.9).

In Sect. 3 (with O as the fixed reference double cone) we have given the re-interpretation of b_i in the GNS representation $\hat{\pi}$ of the state $\xi \circ \mu$ as observables of the associated 2D CFT, with the 2D vacuum Ω_{2D} given by the GNS vector. We shall show

Theorem 2. *Let each $b_i \in B_+(O)$ ($i = 1, \dots, n$) be of the form $\iota(a_1^{(i)} a_2^{(i)}) \cdot \psi^{(i)}$ with charged intertwiners $\psi^{(i)}$ and $a_1^{(i)} \in A(I)$ and $a_2^{(i)} \in A(J)$. As x goes to $+\infty$, the BCFT vacuum correlations (4.1) converge to the 2D vacuum correlations*

$$\big(\Omega_{2D}\ ,\ \hat{\pi}(b_1 \cdots b_n)\ \Omega_{2D}\big) = \xi \circ \mu(b_1 \cdots b_n). \tag{4.2}$$

Proof. We compute the limit and the 2D vacuum expectation value separately.

Using the decomposition of products $\psi_1\psi_2$ into finite sums of operators of the form $\iota(T_1T_2)\cdot\psi$ [18], where T_i are intertwiners between DHR endomorphisms of A, we see that the product $b_1\cdots b_n$ is a finite sum of operators of the same form $\iota(a_1a_2)\cdot\psi$.

For the present purpose, it is more convenient to write the charged intertwiners as $\psi = t\cdot\iota(\bar r)$, where $r\in\mathrm{Hom}(\mathrm{id},\tau\bar\tau)\subset A(J)$ and $t\in\mathrm{Hom}(\alpha^+_\tau,\alpha^-_\sigma)\subset\mathrm{Hom}(\iota\tau,\iota\sigma)$ (Frobenius reciprocity). Then, because $a_2=\sigma(a_2)$, we get $\iota(a_2)\cdot\psi = t\cdot\iota(\tau(a_2)\bar r)$. Hence, the product $b_1\ldots b_n$ is a finite sum of operators of the form

$$\iota(a_1)\cdot t\cdot\iota(a_2). \tag{4.3}$$

Thus, the above vacuum correlation function is a finite sum of expectation values

$$\begin{aligned} F(x) &= (\Omega\,,\ \beta_x\,(\iota(a_1)\cdot t\cdot\iota(a_2))\ \Omega)\\ &= \big(\Omega\,,\ \iota(\alpha_x(a_1)z^\sigma(x))\cdot t\cdot\iota(z^\tau(-x)^*\alpha_{-x}(a_2))\,\Omega\big)\\ &= \big(\Omega\,,\ \alpha_x(a_1)z^\sigma(x)\cdot\varepsilon(t)\cdot z^\tau(-x)^*\alpha_{-x}(a_2)\,\Omega\big)\,. \end{aligned} \tag{4.4}$$

Here, ε is the global conditional expectation $B\to A$, which preserves the vacuum state [17]. In particular, $\varepsilon(t)\in\mathrm{Hom}(\tau,\sigma)$. Therefore, the expression vanishes identically unless σ and τ belong to the same sector.

In the latter case, we express the cocycles as $z^\rho(g) = U_0(g)U_\rho(g)^*$, and $\alpha_g=\mathrm{Ad}_{U_0(g)}$, giving

$$F(x) = \big(\Omega\,,\ a_1U_\sigma(x)^*\cdot\varepsilon(t)\cdot U_\tau(x)^*a_2\,\Omega\big) = (\Omega\,,\ a_1\cdot U_\sigma(-2x)\cdot\varepsilon(t)a_2\,\Omega)\,, \tag{4.5}$$

because the intertwiners between DHR endomorphisms also intertwine the representations of the Möbius group [10]. By the spectrum condition, $F(x)$ has a bounded analytic continuation to the lower complex halfplane. $U_\sigma(-z)$ weakly converges in every direction $z=re^{i\varphi}$ $(-\pi<\varphi<0, r\to\infty)$ to the projection onto the zero eigenspace of the generator, and the latter projection is nonzero only if $\sigma=\mathrm{id}$ is the vacuum representation; in this case $t=\varepsilon(t)=1$. Thus, $F(z)$ converges in these directions to the vacuum expectation value

$$\delta_{\sigma,0}\delta_{\tau,0}\ (\Omega,a_1\Omega)\cdot(\Omega,a_2\Omega). \tag{4.6}$$

Next, we consider

$$\overline{F(x)} = \big(\Omega\,,\ \beta_x\,\big(\iota(a_2^*)\cdot t^*\cdot\iota(a_1^*)\big)\ \Omega\big)\,. \tag{4.7}$$

Let $r_\sigma\in\mathrm{Hom}(\mathrm{id},\bar\sigma\sigma)\subset A(I)$ and $r_\tau\in\mathrm{Hom}(\mathrm{id},\bar\tau\tau)\subset A(J)$. Then we can write $t^*=\iota(r_\sigma^*)\cdot\bar t\cdot\iota(r_\tau)$, where $\bar t\in\mathrm{Hom}(\alpha^+_{\bar\tau},\alpha^-_{\bar\sigma})\subset\mathrm{Hom}(\iota\bar\tau,\iota\bar\sigma)$. Using the locality properties of $a_1\in A(I)$, $a_2\in A(J)$, we can rewrite

$$\overline{F(x)} = \big(\Omega\,,\ \beta_x\,\big(\iota(r_\sigma^*\bar\sigma(a_1^*))\cdot\bar t\cdot\iota(\bar\tau(a_2^*)r_\tau)\big)\ \Omega\big)\,. \tag{4.8}$$

This expression can be computed in the same way as $F(x)$ before, giving

$$\overline{F(x)} = \big(\Omega\,,\ r_\sigma^*\bar\sigma(a_1^*))\cdot U_{\bar\sigma}(-2x)\cdot\varepsilon(\bar t)\bar\tau(a_2^*)r_\tau\,\Omega\big)\,. \tag{4.9}$$

Thus $F(x)$ also has a bounded analytic continuation to the upper complex halfplane, and converges to the same limit (4.6) also in the directions $z=re^{i\varphi}$ $(0<\varphi<\pi, r\to\infty)$. From this, we may conclude the cluster limit

$$\lim_{x\to\infty}\,(\Omega\,,\ \beta_x\,(\iota(a_1)\cdot t\cdot\iota(a_2))\ \Omega) = \delta_{\sigma,0}\delta_{\tau,0}\ (\Omega,a_1\Omega)\cdot(\Omega,a_2\Omega). \tag{4.10}$$

On the other hand, we now compute (4.2) and show that it coincides with the factorizing cluster limit of (4.1). For each contribution of the form (4.3), we have

$$(\Omega_{2D}\,,\ \hat{\pi}\,(\iota(a_1)\cdot t\cdot\iota(a_2))\ ,\ \Omega_{2D}) = \xi\circ\mu\,(\iota(a_1)\cdot t\cdot\iota(a_2)) = \xi\,(a_1\cdot\mu(t)\cdot a_2)\,. \tag{4.11}$$

But $\mu(t)\in A(I)\vee A(J)$ is an intertwiner in $\mathrm{Hom}(\sigma,\tau)$ which vanishes unless $\sigma=\mathrm{id}$ and $\tau=\mathrm{id}$ both belong to the vacuum sector. In the latter case, $t=\mu(t)=1$. Thus,

$$\langle\hat{\Xi}|\hat{\pi}\,(\iota(a_1)\cdot t\cdot\iota(a_2))\,|\hat{\Xi}\rangle = \delta_{\sigma,0}\delta_{\tau,0}\,\xi(a_1a_2) = \delta_{\sigma,0}\delta_{\tau,0}\,(\Omega,a_1\Omega)\cdot(\Omega,a_2\Omega). \tag{4.12}$$

This coincides with the cluster limit (4.10) "far away from the boundary". □

Recall that a_1 and a_2 in (4.3) were obtained by multiplying $b_1\cdots b_n$ and successively decomposing the products of the charged intertwiners. Thus, the vacuum expectation values $(\Omega,a_i\Omega)$ in (4.12) are precisely the chiral conformal blocks of the corresponding 2D correlation functions.

A variant of the conformal cluster theorem [8] should also give a quantitative estimate for the rate of the convergence, depending on the charges of the operators involved through the corresponding spectrum of L_0.

5. Conclusion

We have studied the passage from a local conformal quantum field theory defined on the halfspace $x>0$ of two-dimensional Minkowski spacetime (boundary CFT, BCFT) to an associated local conformal quantum field defined on the full Minkowski spacetime (2D CFT). There are essentially two ways: the first is to consider BCFT vacuum correlations of observables localized far away from the boundary. In the limit of infinite distance, these correlations factorize into chiral correlations (conformal blocks) of charged fields. We have traced this effect back to the cluster property of the underlying local chiral subtheory.

The second method exploits the split property, i.e., the existence of states of the underlying local chiral CFT in which correlations between observables in two fixed intervals at a finite distance are suppressed. With the help of the split property one can algebraically identify a fixed local algebra of the BCFT with a fixed local algebra of the 2D CFT, and one can generate a unitary representation of the 2D Möbius group in the GNS Hilbert space of a suitable "extended split state" of this algebra. Its ground state, the 2D vacuum, is different from the BCFT vacuum. Then, by acting with the 2D Möbius group, one can obtain *all* local algebras of the 2D CFT in the same Hilbert space.

The converse question: can one consistently "add" a boundary in any 2D CFT (without affecting the algebraic structure away from the boundary), is not addressed here. However, there arises a necessary condition from the discussion in App. C: the 2D partition function should be either modular invariant, or at least it should be intermediate between the vacuum partition function and some modular invariant partition function. We hope to return to this problem, and find also a sufficient condition.

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A. Modular Construction of Möb × Möb in the Split State

In [12] it was shown that a unitary representation of the Möbius group Möb is generated by the modular groups of a "halfsided modular triple", i.e., three von Neumann algebras A_i $(i = 0, 1, 2)$ with a joint cyclic and separating vector Ψ such that if σ_t^i is the modular group for (A_i, Ψ), then $\sigma_t^i(A_{i+1}) \subset A_{i+1}$ for $t \leqslant 0$. (Here, $i + 1$ is understood mod 3.) Specifically, when I is an open interval and I_1, I_2 are the subintervals obtained by removing an interior point from I, the three algebras $A_1 = A(I_1)$, $A_2 = A(I_2)$, $A_3 = A(I)'$ in a local chiral CFT together with the vacuum vector Ω define a halfsided modular triple. This means that the entire local net can be recovered from these data.
We want to show here how this construction can be applied to construct a unitary representation of the 2D Möbius group Möb × Möb from six suitable algebras in the split state Ξ associated with a pair of intervals I and J, see (1.4).
Let I_1, I_2 arise from I by removing a point, and similarly J_1, J_2 from J. Tensoring by 1, the two halfsided modular triples

$$\begin{aligned} &\big(A(I)' \otimes 1\,,\ A(I_1) \otimes 1\,,\ A(I_2) \otimes 1 \big)\,, \\ &\big(1 \otimes A(J)'\,,\ 1 \otimes A(J_1)\,,\ 1 \otimes A(J_2) \big) \end{aligned} \tag{A.1}$$

in the state $\Omega \otimes \Omega$ generate $U_0 \otimes U_0$. Under the split isomorphism, these triples turn into

$$\begin{aligned} &\big(A(I)' \cap N, A(I_1), A(I_2) \big)\,, \\ &\big(A(J)' \cap N', A(J_1), A(J_2) \big) \end{aligned} \tag{A.2}$$

in the split state Ξ, where N is the canonical intermediate type I factor between $A(I)$ and $A(J)'$. Ξ is cyclic and separating for these algebras in the subspaces $\overline{N\Xi}$ and $\overline{N'\Xi}$, respectively. The latter halfsided modular triples thus generate the two commuting representations U_+, U_- of Möb directly in $\mathcal{H}_0$.

B. Charged Intertwiners in BCFT

The charged intertwiners ψ for a given chiral extension $A \subset B$, that together with $A_+(O)$ generate $B_+(O)$, are elements of the finite-dimensional spaces $\mathrm{Hom}(\iota, \iota\sigma\bar\tau) \cap B(K)'$. In [18, Eq. (5.12)] a linear condition on $\varphi = \bar\iota(\psi) \in \mathrm{Hom}(\theta, \theta\sigma\bar\tau)$ was given which guarantees that φ commutes with $\bar\iota(B(K))$. Here $\bar\iota : B \to A$ is a homomorphism conjugate to the injection $\iota : A \to B$, such that $\gamma = \iota\bar\iota$ on $B(K)$ is a canonical endomorphism for $A(K) \subset B(K)$ and $\theta = \bar\iota\iota$ is the dual canonical endomorphism, which is a DHR endomorphism of A localized in K [17].
Unfortunately, the condition displayed in [18] does not take into account that φ belongs to $\bar\iota(B(L))$ (i.e., is in the range of $\bar\iota$). We want to reformulate this condition so that it is equivalent to ψ belonging to $B_+(O) = B(K)' \cap B(L)$.
We first notice that every element of $B(K)$ is of the form $\psi = \iota(y)\, v$, where $v \in \mathrm{Hom}(\mathrm{id}_B, \gamma) \subset B(K)$ is the canonical isometry intertwining γ. Then $\psi \in \mathrm{Hom}(\iota, \iota\sigma\bar\tau)$ if and only if $y \in \mathrm{Hom}(\theta, \sigma\bar\tau) \subset A(L)$. This already secures that $\psi \in B(L)$, and since θ is localized in K, ψ commutes with $\iota(A(K))$. Hence it commutes with $B(K)$ iff it also commutes with $v \in \mathrm{Hom}(\mathrm{id}_B, \gamma)$. This is equivalent to the relation

$$y\,x \stackrel{!}{=} \theta(y)\,x \equiv \sigma(\varepsilon_{\theta,\bar\tau})\,\varepsilon^*_{\sigma,\theta}\,\theta(y)\,x, \tag{B.1}$$

where $x = \bar{\iota}(v) \in \mathrm{Hom}(\theta, \theta^2)$. The statistics operators ε are trivial [9] due to the localizations of σ in I, $\bar{\tau}$ in J, and θ in K, but we have displayed them in order to make the condition covariant under unitary deformations of $\bar{\iota}$ and $v \in \mathrm{Hom}(\mathrm{id}_B, \iota\bar{\iota})$, possibly changing the localization of θ and leading to nontrivial statistics operators.
The condition (B.1) can be equivalently written as the eigenvalue equation

$$\Pi(y) := \lambda^{\frac{1}{2}} \cdot \left(1_\sigma \times r^* \times 1_{\bar{\tau}}\right) \circ \left(\varepsilon^*_{\sigma,\theta} \times \varepsilon^*_{\theta,\bar{\tau}}\right) \circ (1_\theta \times y \times 1_\theta) \circ x_2 \stackrel{!}{=} y. \quad \text{(B.2)}$$

Here $r = x \circ w \in \mathrm{Hom}(\mathrm{id}_A, \theta^2)$, where $w \in \mathrm{Hom}(\mathrm{id}_A, \theta) \subset A(K)$ is the dual canonical isometry (such that $(\gamma, v, \iota(w))$ form a Q-system); $x_2 = (1_\theta \times x) \circ x = (x \times 1_\theta) \circ x \in \mathrm{Hom}(\theta, \theta^3)$, and $\lambda \geqslant 1$ is the index $[B : A]$. $\circ$ and $\times$ are the concatenation and the monoidal product in the tensor category of DHR endomorphisms of A. The map Π defined by (B.2) is a linear map $\Pi : \mathrm{Hom}(\theta, \sigma\bar{\tau}) \to \mathrm{Hom}(\theta, \sigma\bar{\tau})$. Equation (B.2) obviously follows from (B.1) by left multiplication with $\sigma(r^*)$ and right multiplication with x. To see that (B.2) implies (B.1), one may insert (B.2) into both sides of (B.1) and repeatedly use the relations of the dual Q-system (θ, w, x) to get equality.
We thank I. Runkel who has pointed out to us that Π is in fact a projection. Hence the charged intertwiners ψ are precisely given by $\iota(y) \cdot v$, where y is in the range of Π. The multiplicities $Z_{\sigma,\tau}$ in (3.7) equal the dimension of the range of these projections (for each pair σ, τ).

C. Haag Duality and Modular Invariance

If A is completely rational, the C* tensor category defined by its DHR superselection sectors is modular [14], i.e., the unitary S and T matrices defined by the statistics [10] generate a representation of the group $SL(2, \mathbb{Z})$. By the Verlinde formula [22], these matrices also describe the modular transformation behavior of chiral partition functions ("characters").
By [1], the matrix Z given by (3.7) is a modular invariant (it commutes with S and T), hence the partition function of the 2D CFT B_{2D} on $\mathcal{H}_{2D}$ is invariant under modular transformations. We want to point out an interesting relation of this fact to Haag duality of the associated BCFT.
As mentioned before, every BCFT defined by (1.2) is automatically Haag dual, and any non Haag dual BCFT $\widetilde{B}_+$ with the same chiral observables is intermediate between A_+ and B_+ [18]. Therefore, the charged intertwiners $\psi \in \widetilde{B}_+$ constitute linear subspaces of the spaces of charged intertwiners in B_+. Let the dimensions of these spaces be $\widetilde{Z}_{\sigma,\tau} \leqslant Z_{\sigma,\tau}$, and at least one of them $< Z_{\sigma,\tau}$ (i.e., $\widetilde{B}_+$ is strictly contained in B_+). Then the matrix $\widetilde{Z}$ cannot be a modular invariant by the following simple argument: consider the 00 component of $S^*\widetilde{Z}S$. Because each S_{i0} is positive,

$$\left(S^*\widetilde{Z}S\right)_{00} = \sum_{ij} S_{0i} S_{0j} \widetilde{Z}_{ij} \quad \text{(C.1)}$$

is strictly smaller than $(S^*ZS)_{00} = Z_{00} = 1$. If $\widetilde{Z}$ were modular invariant, we would conclude $\widetilde{Z}_{00} < 1$, which is impossible.
We notice that the construction of a 2D CFT associated to a BCFT described in Sect. 3 takes an intermediate BCFT $A_+ \subset \widetilde{B}_+ \subset B_+$ to an intermediate 2D CFT $A_{2D} \subset \widetilde{B}_{2D} \subset B_{2D}$. Its Hilbert space is of the form (3.6) with Z replaced by $\widetilde{Z}$. Hence, we conclude that the partition function of the associated 2D CFT is modular invariant if and only if the BCFT is Haag dual.

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