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The non-minimal heterotic pure spinor string in a curved background

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ABSTRACT: We study the non-minimal pure spinor string in a curved background. We find that the minimal BRST invariance implies the existence of a non-trivial stress-energy tensor for the minimal and non-minimal variables in the heterotic curved background. We find constraint equations for the b ghost. We construct the b ghost as a solution of these constraints.

KEYWORDS: Superstrings and Heterotic Strings, Conformal Field Models in String Theory, Topological Strings

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1 Introduction

The pure spinor formalism of the superstring was constructed more than a decade ago [1]. The idea is to add a constrained ghost, which satisfies the pure spinor condition. The string sigma model is constructed in a way that it is conformal invariant. Berkovits noted that the conformal invariance of the model was not enough to get the superstring physical spectrum and invented a nilpotent charge with the help of a pure spinor variable. It turns out that the cohomology of this nilpotent charge gives the physical superstring spectrum and nothing else [2]. Unlike RNS, the pure spinor formalism does not need to make a projection to get the physical spectrum and space-time supersymmetry is manifest. In fact, not only massless states are described in terms of superfields, massive states can also be described in this language [3]. Many applications of the formalism, like computing manifestly supersymmetric scattering amplitudes, were developed later (see the review [4]).

Despite the success in reproducing known results in other formalisms and obtaining new results, the pure spinor formalism is not understood completely. Perhaps, the most important lacking ingredient is a symmetry of the world-sheet action that allows quantization. In other words, it is not known what is the fixed gauge symmetry that implies the existence of the pure spinor BRST charge. Instead of facing directly this problem, one could continue the program and determine some features that would lead, eventually, to solve the previous issue. One of this features is the inclusion of a b ghost. Since the string model does not require a pair of (b, c) ghosts, the BRST ghosts of the parameterization invariance of the string world-sheet, they have to be constructed as a functions of the string model variables. However, the minimal pure spinor formalism of [1] is not suitable to define

a b ghost. Berkovits introduced new variables and constructed a non-minimal pure spinor formalism [5]. The cohomology of the modified BRST charge does not change respect to the minimal version of the pure spinor formalism [5]. The nilpotency of the b ghost was verified in [6] and [7]. The conjugate c ghost was constructed in [8]. All this was done in flat space-time background. Recently, the construction of the b ghost in a super Maxwell background was done in [9]. The natural question is to determine the b ghost in a generic curved background. This is the purpose of this paper.

The idea is to determine a string world-sheet action in a heterotic curved background that it is consistent with the pure spinor BRST symmetry. We find that the stress-energy tensor has the form

$$T = T_0 + T_1, \tag{1.1}$$

where T_0 is the stress-energy tensor for the minimal variable and T_1 is an expression that it is reduced to the correct limit on a flat space-time background. The expression for T is determined in section 3. Once T is determined, the b ghost is obtained through the relation $Qb = T$. The result has the form

$$b = b_0 + f(\Omega_\alpha), \tag{1.2}$$

where b_0 has the same dependence on world-sheet than the ghost in flat background space-time and f is conformal weight two which depends linearly on the scalar part of the Lorentz connection Ω_α . Note that $\Omega_\alpha = \frac{1}{4}\nabla_\alpha\Phi$, where Φ is the dilaton superfield. Then, in the flat space-time background Φ vanishes and the b ghost has the correct limit. Note also that there are backgrounds where the dilaton superfield is constant, then it would be possible that in such cases, the b ghost has the same dependence on world-sheet fields than the b ghost in flat space-time background.

The plan of the paper is as it follows. In section 2, the non-minimal pure spinor formalism in flat background is reviewed. In section 3, the minimal pure spinor formalism in the heterotic curved background is reviewed. This system was studied in [10], where it was shown that nilpotency of the BRST charge implies that the background satisfies the ten-dimensional supergravity equations of motion and the $N = 1$ super Yang-Mills equations of motion in a curved background. Note that this background was shown to be conformal invariant at one-loop [11] and the one-loop BRST anomaly was studied in [12]. The BRST transformations for the world-sheet fields in a curved heterotic background were determined in [13].

The BRST transformation of the minimal and non-minimal variables are obtained in the section 4. They are obtained as consequence of the trivial cohomology of the non-minimal contribution to the BRST charge. This fact was noted in [5] in flat space-time background. We generalize this fact to the heterotic curved background. In section 5, we determine the world-sheet action of the non-minimal pure spinor string in the heterotic curved background. Here, the stress-energy tensor receives a non-trivial contribution from the non-minimal sector. It is important to determine the stress-energy tensor because it will allow to find constraint equations for the b ghost. This is done in section 6. The constraints equations come from the definition for the b ghost. It satisfies $Qb = T$, where

Q is the BRST charge and T is the stress-energy tensor. Finally, in section 7, we solve the constraint equations for the b ghost and determine that the resulting construction has the correct flat space-time background limit.

2 The non-minimal pure spinor string on a flat background

In this section we review the non-minimal pure spinor formalism [5] on a flat background. The action is given by

$$S = S_0 + \int d^2z \widehat{\omega}^\alpha \bar{\partial} \widehat{\lambda}_\alpha + s^\alpha \bar{\partial} r_\alpha, \tag{2.1}$$

where S_0 is the minimal action which is given by

$$S_0 = \int d^2z \frac{1}{2} \partial X^m \bar{\partial} X_m + p_\alpha \bar{\partial} \theta^\alpha + \omega^\alpha \bar{\partial} \lambda^\alpha, \tag{2.2}$$

where (X^m, θ^α) are the coordinates of $N = 1$ ten-dimensional superspace, p_α is the canonical conjugate of θ^α . The minimal λ and the non-minimal $(\widehat{\lambda}, r)$ ghosts are constrained to satisfy

$$\lambda \gamma^m \lambda = \widehat{\lambda} \gamma^m \widehat{\lambda} = \widehat{\lambda} \gamma^m r = 0, \tag{2.3}$$

where $\gamma_m^{\alpha\beta}$ and $\gamma_{\alpha\beta}^m$ are the symmetric gamma matrices in ten dimensions.

In order to preserve these constraints, the canonical conjugate ghosts $\omega, \widehat{\omega}$ and s are defined up to the gauge transformations

$$\begin{aligned} \delta \omega_\alpha &= (\lambda \gamma_m)_\alpha \Lambda^m, & \delta s^\alpha &= (\gamma^m \widehat{\lambda})^\alpha \widetilde{\Lambda}_m, \\ \delta \widehat{\omega}^\alpha &= (\gamma^m \widehat{\lambda})^\alpha \widetilde{\Lambda}_m - (\gamma^m r)^\alpha \widetilde{\Lambda}_m. \end{aligned} \tag{2.4}$$

The quantization of this system is performed after the inclusion of a nilpotent charge which is identified with a BRST charge. It is given by

$$Q = Q_0 + Q_1, \tag{2.5}$$

where

$$Q_0 = \oint dz \lambda^\alpha d_\alpha, \quad Q_1 = \oint dz \widehat{\omega}^\alpha r_\alpha, \tag{2.6}$$

are the the BRST charges for the minimal and non-minimal pure spinor variables. Note that the BRST charge is nilpotent because both Q_0 and Q_1 are nilpotent and anticommute.

The cohomology of the minimal BRST operator, the first term in (2.5), describes the physical superstring states. For massless states, the (unintegrated) vertex operator is $U = \lambda^\alpha A_\alpha(X, \theta)$. This state is in the cohomology of Q_0 if $Q_0 U = 0$ and $U \sim U + Q_0 U$. These conditions imply that U contains the photon and the photino as physical degrees of freedom and nothing else [1]. Similarly, vertex operators for higher mass states can be defined and cohomology conditions put the superfields on-shell. This is the case for the first massive state where the only physical states are the massive spin-3/2 multiplet [3]. The cohomology of the non-minimal pure spinor BRST operator (2.5) is equivalent to the cohomology of the minimal pure spinor BRST operator in (2.5) because the non-minimal

contribution has trivial cohomology [5]. Consequently, the action of the non-minimal model is BRST equivalent to the action of the minimal model. In fact

$$S = S_0 + Q \int d^2z s^\alpha \widehat{\partial} \lambda_\alpha. \tag{2.7}$$

Below, this relation will be used to determine the superstring world-sheet action on a curved background.

3 The minimal heterotic pure spinor string on a curved background

In this section we review the construction of the action for the heterotic string in a curved background (see the appendix for a short review of our conventions). The action can be obtained by adding to the flat action of (2.1) the integrated vertex operator and then covariantize respect to background invariance. The action becomes [10]

$$S_0 = \int d^2z \left[\frac{1}{2} \Pi^a \bar{\Pi}^b \eta_{ab} + \frac{1}{2} \Pi^A \bar{\Pi}^B B_{BA} + d_\alpha (\bar{\Pi}^\alpha + \bar{J}^I W_I^\alpha) + \lambda^\alpha \omega_\beta \bar{J}^I U_{I\alpha}{}^\beta + \omega_\alpha \bar{\nabla} \lambda^\alpha + (\rho_A \nabla \rho_A) \right] + S_{FT}, \tag{3.1}$$

where Π^A and $\bar{\Pi}^A$ are defined from the background supevielbein E_M^A and the superfield coordinates Z^M as

$$\Pi^A = \partial Z^M E_M^A, \quad \bar{\Pi}^A = \bar{\partial} Z^M E_M^A. \tag{3.2}$$

The variable d_α is interpreted as the world-sheet generator for translations in superspace. The world-sheet covariant derivative on the pure spinor variable is defined by

$$\bar{\nabla} \lambda^\alpha = \bar{\partial} \lambda^\alpha + \lambda^\beta \bar{\Omega}_\beta{}^\alpha, \tag{3.3}$$

where $\Omega_\beta{}^\alpha = \bar{\partial} Z^M \Omega_{M\beta}{}^\alpha$ with Ω being the Lorentz connection. Note that the connection $\Omega_{A\alpha}{}^\beta = E_M^A \Omega_{M\alpha}{}^\beta$ has the index structure

$$\Omega_{A\alpha}{}^\beta = \Omega_A \delta_\alpha^\beta + \frac{1}{4} (\gamma_{ab})_\alpha{}^\beta \Omega_A{}^{ab}, \tag{3.4}$$

where Ω_A is the scalar connection and $\Omega_A{}^{ab}$ is the usual Lorentz connection. The right-moving heterotic fermions ρ_A transform in the fundamental representation of $E_8 \times E_8$ or $SO(32)$ and its covariant derivative is defined such that

$$(\rho_A \nabla \rho_A) = (\rho_A \partial \rho_A) + \Pi^A \bar{J}^I A_{IA},$$

where $\bar{J}^I = \frac{1}{2} K_{AB}^I \rho_A \rho_B$ with K^I represents the generators of the Lie algebra of $E_8 \times E_8$ or $SO(32)$ in the fundamental representation, and A_{IA} is the corresponding gauge field. Finally, S_{FT} is the Fradkin-Tseytlin term given by the world-sheet integral of the dilaton superfield Φ .

In [10] it was shown that the charge Q_0 in (2.6) is nilpotent and conserved if the background is constrained to satisfy the supergravity and SYM equations of motion in ten

dimensions. Alternatively, in [13] it was found how the world-sheet fields of the action (3.1) transform under Q_0 and it was verified that the action S_0 is BRST invariant. We will assume that the minimal variables are unaffected by the non-minimal BRST charge. Below, we will need the BRST transformation of the connection $\bar{\Omega}_\alpha{}^\beta$ which is equal to

$$Q_0 \bar{\Omega}_\alpha{}^\beta = \bar{\nabla}(\lambda^\gamma \Omega_{\gamma\alpha}{}^\beta) - \lambda^\gamma \bar{\Pi}^A R_{A\gamma\alpha}{}^\beta, \quad (3.5)$$

where the covariant derivative was defined in (3.3) and R is the curvature superfield. Note that the first term in this transformation is a Lorentz transformation with the field-dependent parameter $\lambda^\gamma \Omega_{\gamma\alpha}{}^\beta$. In [13] it was shown that this property is true for all the world-sheet fields of the action (3.1), that is, the BRST transformation of the fields always contains a gauge and Lorentz transformation. We will denote as \tilde{Q}_0 on the world-sheet fields as the minimal pure spinor BRST transformation without including the Lorentz rotation. Below we will need the action of \tilde{Q}_0 on the other world-sheet fields. These transformations were derived in [13] and the non-vanishing variations are

$$\begin{aligned} \tilde{Q}_0 \Pi^A &= \delta_\alpha^A \nabla \lambda^\alpha - \lambda^\alpha \Pi^B T_{B\alpha}{}^A, & \tilde{Q}_0 \omega_\alpha &= d_\alpha, \\ \tilde{Q}_0 d_\alpha &= \lambda^\beta \Pi^a (\gamma_a)_{\beta\alpha} + \lambda^\beta \lambda^\gamma \omega_\delta R_{\alpha\beta\gamma}{}^\delta, \end{aligned} \quad (3.6)$$

where we have used some local Lorentz symmetry to gauge fix the torsion component $T_{\alpha\beta}{}^\gamma$ to zero [10].

4 BRST transformations of the non-minimal variables

One could think that the Q_0 BRST transformations of the non-minimal variables are trivial. However, all the fields in the minimal model transform, at least, with a Lorentz rotation term. Since the non-minimal variables transform under Lorentz rotations, then we expect a non trivial action of Q_0 on the non-minimal pure spinor variables. One could argue that the Lorentz index in the non-minimal variables just counts number of fields and it is not a vector index. This is not the case. The b ghost in flat space is constructed from contractions between minimal and non-minimal variables such that it is Lorentz invariant. Using cohomology arguments, we will find the form in which Q_0 acts on the non-minimal variables.

The BRST charge in a curved background has the same form as in flat space, that is,

$$Q = Q_0 + Q_1, \quad (4.1)$$

where both Q_0 and Q_1 are nilpotent. Because Q is nilpotent, we obtain the Q_0 and Q_1 anticommute. The action of Q_1 on the non-minimal variables is

$$Q_1 \hat{\lambda}_\alpha = -r_\alpha, \quad Q_1 s^\alpha = \hat{\omega}^\alpha, \quad Q_1 \hat{\omega}^\alpha = Q_1 r_\alpha = 0. \quad (4.2)$$

Consider the last two equations here. Acting with Q_0 and using anticommutation with Q_1 , we obtain

$$Q_1(Q_0 \hat{\omega}^\alpha) = 0, \quad Q_1(Q_0 r_\alpha) = 0. \quad (4.3)$$

Because the cohomology of Q_1 is trivial [5], we determine the form of $Q_0\widehat{\omega}^\alpha$ and Q_0r_α to be

$$Q_0\widehat{\omega}^\alpha = \widehat{\omega}^\beta \mathcal{A}_\beta{}^\alpha + r_\beta \mathcal{G}^{\beta\alpha}, \quad Q_0r_\alpha = \mathcal{B}_\alpha{}^\beta r_\beta + \widehat{\omega}^\beta \mathcal{H}_{\beta\alpha}, \quad (4.4)$$

where $\mathcal{A}, \mathcal{B}, \mathcal{G}, \mathcal{H}$ depend on the minimal variables. Note that the terms involving \mathcal{A} and \mathcal{B} contains Lorentz rotations, so we identify them with the field-dependent Lorentz parameter of the minimal sector. Recall that Q_0 maps a λ -ghost number n field to a λ -ghost number $n + 1$ field, so we need to add a factor of λ to the above transformations. Then, we have

$$Q_0\widehat{\omega}^\alpha = -\widehat{\omega}^\beta \lambda^\gamma (\Omega_{\gamma\beta}{}^\alpha + \mathcal{A}_{\gamma\beta}{}^\alpha) + \lambda^\beta r_\gamma \mathcal{G}_\beta{}^{\gamma\alpha}, \quad Q_0r_\alpha = \lambda^\gamma (\Omega_{\gamma\alpha}{}^\beta + \mathcal{B}_{\gamma\alpha}{}^\beta) r_\beta + \lambda^\beta \widehat{\omega}^\gamma \mathcal{H}_{\beta\gamma\alpha}, \quad (4.5)$$

where $\mathcal{A}, \mathcal{B}, \mathcal{G}$ and \mathcal{H} depend on the minimal variables only. Note that \mathcal{H} has conformal weight -1 , then it has to vanish because it is not possible to write a quantity of such conformal weight in the minimal formalism, at least if one requires Lorentz covariance. Let us try the possibility that both \mathcal{G} and \mathcal{H} vanish. Below, we will determine \mathcal{A} and \mathcal{B} by requiring that Q_0 is nilpotent. Before that, we obtain the action of Q_0 on the other non-minimal variable $\widehat{\lambda}$ and s . Analogous to (4.3) we have

$$Q_1(Q_0\widehat{\lambda}_\alpha) = Q_0r_\alpha, \quad Q_1(Q_0s^\alpha) = -Q_0\omega^\alpha, \quad (4.6)$$

from which we obtain

$$Q_0\widehat{\lambda}_\alpha = \lambda^\gamma (\Omega_{\gamma\alpha}{}^\beta + \mathcal{B}_{\gamma\alpha}{}^\beta) \widehat{\lambda}_\beta, \quad Q_0s^\alpha = s^\beta \lambda^\gamma (\Omega_{\gamma\beta}{}^\alpha + \mathcal{A}_{\gamma\beta}{}^\alpha). \quad (4.7)$$

Because Q_0 is nilpotent, \mathcal{A} and \mathcal{B} in (4.5) and (4.7) satisfy certain constraints. Applying Q_0 to first equation in (4.5), we obtain

$$Q_0^2\widehat{\omega}^\alpha = -\frac{1}{2}\widehat{\omega}^\beta \lambda^\gamma \lambda^\delta (R_{\gamma\delta\beta}{}^\alpha + \nabla_{(\gamma}\mathcal{A}_{\delta)\beta}{}^\alpha - \mathcal{A}_{\gamma\beta}{}^\rho \mathcal{A}_{\delta\rho}{}^\alpha - \mathcal{A}_{\delta\beta}{}^\rho \mathcal{A}_{\gamma\rho}{}^\alpha), \quad (4.8)$$

where we used the definition of the curvature R in terms of the connection Ω and the Berkovits-Howe constraint $\lambda^\gamma \lambda^\delta T_{\gamma\delta}{}^A = 0$, where T is the torsion in superspace [10]. The solution for \mathcal{A} is

$$\mathcal{A}_{\gamma\beta}{}^\alpha = -\frac{1}{4}(\gamma^{ab})_\beta{}^\alpha T_{\gamma ab}. \quad (4.9)$$

To verify that (4.9) is the solution of (4.8), we use

$$R_{\gamma\delta\beta}{}^\alpha = R_{\gamma\delta}{}^\alpha{}_\beta + \frac{1}{4}(\gamma^{ab})_\beta{}^\alpha R_{\gamma\delta ab},$$

where $R_{\gamma\delta} = \nabla_{(\gamma}\Omega_{\delta)}$. Note that $\lambda^\gamma \lambda^\delta R_{\gamma\delta}$ vanishes after using that Ω_α is proportional to $\nabla_\alpha\Phi$, where Φ is the dilaton superfield [10]. The last step is to use the Bianchi identity involving $R_{\gamma\delta ab}$ and, again, the constraint $\lambda^\gamma \lambda^\delta T_{\gamma\delta}{}^A = 0$.

We proceed similarly to determine \mathcal{B} by demanding that $Q_0^2 r_\alpha$ vanishes. It turns out that $\mathcal{B} = \mathcal{A}$. In summary, the action of Q_0 on the non-minimal variables is given by

$$\begin{aligned}
 Q_0 \widehat{\omega}^\alpha &= -\widehat{\omega}^\beta \lambda^\gamma \left(\Omega_{\gamma\beta}{}^\alpha - \frac{1}{4} (\gamma^{ab})_\beta{}^\alpha T_{\gamma ab} \right), \\
 Q_0 r_\alpha &= -\lambda^\gamma r_\beta \left(\Omega_{\gamma\alpha}{}^\beta - \frac{1}{4} (\gamma^{ab})_\alpha{}^\beta T_{\gamma ab} \right), \\
 Q_0 \widehat{\lambda}_\alpha &= \lambda^\gamma \widehat{\lambda}_\beta \left(\Omega_{\gamma\alpha}{}^\beta - \frac{1}{4} (\gamma^{ab})_\alpha{}^\beta T_{\gamma ab} \right), \\
 Q_0 s^\alpha &= s^\beta \lambda^\gamma \left(\Omega_{\gamma\beta}{}^\alpha - \frac{1}{4} (\gamma^{ab})_\beta{}^\alpha T_{\gamma ab} \right).
 \end{aligned} \tag{4.10}$$

5 The non-minimal heterotic pure spinor string on a curved background

We now define a world-sheet action for the heterotic pure spinor string in a curved background. We start with the generalization of (2.7) in this case,

$$S = S_0 + Q \int d^2 z s^\alpha \overline{\nabla} \widehat{\lambda}_\alpha, \tag{5.1}$$

where S_0 is given in (3.1). Using the above BRST transformations on the minimal and non-minimal variables we obtain

$$\begin{aligned}
 S &= S_0 + \int d^2 z \widehat{\omega}^\alpha \overline{\nabla} \widehat{\lambda}_\alpha + s^\alpha \overline{\nabla} r_\alpha \\
 &\quad + \frac{1}{4} s^\beta \widehat{\lambda}_\gamma \overline{\nabla} \lambda^\alpha (\gamma^{ab})_\beta{}^\gamma T_{\alpha ab} + \lambda^\alpha s^\beta \widehat{\lambda}_\gamma \overline{\Pi}^A \left(\frac{1}{4} (\gamma^{ab})_\beta{}^\gamma \nabla_A T_{\alpha ab} - R_{A\alpha\beta}{}^\gamma \right).
 \end{aligned} \tag{5.2}$$

Because the pure spinor BRST operator is nilpotent on gauge invariant operator, the action (5.1) is BRST invariant.

Since we are interested in the construction of the b ghost, we need to know the left-moving stress-energy tensor derived from (5.2). Under an holomorphic conformal transformation, the superspace coordinate Z , the pure spinor ghosts $\lambda, \widehat{\lambda}$ and r carry conformal weight zero. While the conjugate pure spinor variables $\omega, \widehat{\omega}$ and s carry conformal weight one. Noether theorem determines the conserved charge due to this transformation to be

$$\begin{aligned}
 T &= T_0 - \widehat{\omega}^\alpha \nabla \widehat{\lambda}_\alpha - s^\alpha \nabla r_\alpha \\
 &\quad - \frac{1}{4} s^\beta \widehat{\lambda}_\gamma \nabla \lambda^\alpha (\gamma^{ab})_\beta{}^\gamma T_{\alpha ab} - \lambda^\alpha s^\beta \widehat{\lambda}_\gamma \Pi^A \left(\frac{1}{4} (\gamma^{ab})_\beta{}^\gamma \nabla_A T_{\alpha ab} - R_{A\alpha\beta}{}^\gamma \right),
 \end{aligned} \tag{5.3}$$

where T_0 is the stress-tensor from the action S_0 which is given by

$$T_0 = -\frac{1}{2} \Pi_a \Pi^a - d_\alpha \Pi^\alpha - \omega_\alpha \nabla \lambda^\alpha. \tag{5.4}$$

Note that (5.3) reduces to the correct expression in the flat space limit because

$$T_{\alpha ab} = 2(\gamma_{ab})_\alpha{}^\beta \Omega_\beta \rightarrow 0, \quad R_{A\alpha\beta}{}^\gamma \rightarrow 0.$$

In the next section we will use the stress tensor (5.3) to determine the b ghosts satisfying $Qb = T$.

6 The b ghost

The b ghost was constructed in [5] for the case in which the background is flat. It is given by

$$b = -s^\alpha \partial \widehat{\lambda}_\alpha + \frac{1}{(\widehat{\lambda\lambda})} \widehat{\lambda}_\alpha G^\alpha - \frac{1}{(\widehat{\lambda\lambda})^2} \widehat{\lambda}_\alpha r_\beta H^{\alpha\beta} - \frac{1}{(\widehat{\lambda\lambda})^3} \widehat{\lambda}_\alpha r_\beta r_\gamma K^{\alpha\beta\gamma} + \frac{1}{(\widehat{\lambda\lambda})^4} \widehat{\lambda}_\alpha r_\beta r_\gamma r_\delta L^{\alpha\beta\gamma\delta}, \quad (6.1)$$

where G, H, K and L have conformal weight two, λ -ghost number zero and depend on the minimal variables. They have to satisfy the relations

$$QG^\alpha = \lambda^\alpha T, \quad QH^{\alpha\beta} = \lambda^{[\alpha} G^{\beta]}, \quad QK^{\alpha\beta\gamma} = \lambda^{[\alpha} H^{\beta\gamma]}, \quad QL^{\alpha\beta\gamma\delta} = \lambda^{[\alpha} K^{\beta\gamma\delta]}, \quad \lambda^{[\alpha} L^{\beta\gamma\delta\rho]} = 0. \quad (6.2)$$

to satisfy $Qb = T$. The expressions for them were derived in [5], we quote the result here

$$\begin{aligned} G^\alpha &= -\frac{1}{2} \Pi^m \gamma_m^{\alpha\beta} d_\beta - \frac{1}{4} J \partial \theta^\alpha + \frac{1}{8} N^{mn} (\gamma_{mn} \partial \theta)^\alpha, \\ H^{\alpha\beta} &= \frac{1}{192} ((d\gamma^{mnp} d) + 24 \Pi^m N^{np}) \gamma_{mnp}^{\alpha\beta}, \\ K^{\alpha\beta\gamma} &= \frac{1}{16} \gamma_{mnp}^{[\alpha\beta} (\gamma^m d)^\gamma] N^{np}, \quad L^{\alpha\beta\gamma\delta} = \frac{1}{128} \gamma_{mnp}^{[\alpha\beta} (\gamma^{pqr})^\gamma] N^{mn} N_{qr}, \end{aligned} \quad (6.3)$$

where $\Pi^m = \partial X^m + \frac{1}{2} (\theta \gamma^m \partial \theta)$ is the supersymmetric world-sheet momentum, $J = -\lambda^\alpha \omega_\alpha$ is the λ -ghost number current, and $N^{mn} = \frac{1}{2} (\lambda \gamma^{mn} \omega)$ is the generator for Lorentz transformations of the pure spinor variables.

We generalize the expression (6.1) to a curved background and find the relations analogous to (6.2). We propose

$$b = -s^\alpha \nabla \widehat{\lambda}_\alpha + \frac{1}{(\widehat{\lambda\lambda})} \widehat{\lambda}_\alpha G^\alpha - \frac{1}{(\widehat{\lambda\lambda})^2} \widehat{\lambda}_\alpha r_\beta H^{\alpha\beta} - \frac{1}{(\widehat{\lambda\lambda})^3} \widehat{\lambda}_\alpha r_\beta r_\gamma K^{\alpha\beta\gamma} + \frac{1}{(\widehat{\lambda\lambda})^4} \widehat{\lambda}_\alpha r_\beta r_\gamma r_\delta L^{\alpha\beta\gamma\delta}, \quad (6.4)$$

and now we compute Qb and impose it is equal to (5.3). Because of the form in which Q acts on both minimal and no-minimal variables, we can organize Qb in an expansion in powers of r . Note that the b ghost is a Lorentz scalar, so all the terms which depend on the Lorentz connection Ω will produce a zero variation of b because they are a field-dependent Lorentz transformation.

The term independent from r is

$$T - T_0 + \frac{1}{(\widehat{\lambda\lambda})} \widehat{\lambda}_\alpha \left(\widetilde{Q}_0 G^\alpha - \frac{1}{4} \lambda^\beta T_{\beta ab} (\gamma^{ab})_\gamma^\alpha G^\gamma - 5 \lambda^\beta \Omega_\beta G^\alpha \right), \quad (6.5)$$

where $\widetilde{Q}_0 G^\alpha$ is the minimal BRST transformation without the Lorentz rotation term. It is required that

$$\widetilde{Q}_0 G^\alpha = \lambda^\alpha T_0 + \frac{1}{4} \lambda^\beta T_{\beta ab} (\gamma^{ab})_\gamma^\alpha G^\gamma + 5 \lambda^\beta \Omega_\beta G^\alpha. \quad (6.6)$$

In this way, (6.5) is equal to the stress-energy tensor T . The remaining terms in Qb has to vanish and we find constraints equations for H, K, L . The order 1 in r determines an equation for H ,

$$\widetilde{Q}_0 H^{\alpha\beta} = \lambda^{[\alpha} G^{\beta]} - \frac{1}{4} \lambda^\gamma T_{\gamma ab} (\gamma^{ab})_\delta^{[\alpha} H^{\beta]\delta} + 10 \lambda^\gamma \Omega_\gamma H^{\alpha\beta}. \quad (6.7)$$

The order 2 in r determines an equation for K ,

$$\tilde{Q}_0 K^{\alpha\beta\gamma} = \lambda^{[\alpha} H^{\beta\gamma]} + \frac{1}{4} \lambda^\delta T_{\delta ab} (\gamma^{ab})_\rho^{[\alpha} K^{\beta\gamma]\rho} + 15 \lambda^\delta \Omega_\delta K^{\alpha\beta\gamma}. \quad (6.8)$$

The order 3 in r determines an equation for L ,

$$\tilde{Q}_0 L^{\alpha\beta\gamma\delta} = \lambda^{[\alpha} K^{\beta\gamma\delta]} - \frac{1}{4} \lambda^\rho T_{\rho ab} (\gamma^{ab})_\sigma^{[\alpha} L^{\beta\gamma\delta]\sigma} + 20 \lambda^\rho \Omega_\rho L^{\alpha\beta\gamma\delta}. \quad (6.9)$$

Finally, the order 4 in r determines the constraint for L ,

$$\lambda^{[\alpha} L^{\beta\gamma\delta\rho]} = 0. \quad (6.10)$$

Note that, after adding the Lorentz rotation term, the nilpotency of Q_0 on G, H, K, L is verified. Note that the equations (6.6) to (6.10) have the correct flat space limit because $\Omega_\alpha = 0$ in this case.

7 Construction of the b ghost

We look for the fields (G, H, K, L) satisfying equations (6.6) to (6.10). They have conformal dimension two and minimal ghost number zero. Quite general, they all have the form

$$\begin{aligned} U^A = & \Pi^a \Pi^b (u_1)_{ab}{}^A + \Pi^a \Pi^\beta (u_2)_{a\beta}{}^A + \Pi^a d_\beta (u_3)_a{}^{\beta A} + \Pi^a \lambda^\beta \omega_\gamma (u_4)_{a\beta}{}^{\gamma A} \\ & + \Pi^\beta \Pi^\gamma (u_5)_{\beta\gamma}{}^A + \Pi^\beta d_\gamma (u_6)_\beta{}^{\gamma A} + \Pi^\beta \lambda^\gamma \omega_\delta (u_7)_{\beta\gamma}{}^{\delta A} + d_\beta d_\gamma (u_8)^{\beta\gamma A} \\ & + d_\beta \lambda^\gamma \omega_\delta (u_9)_\gamma{}^{\delta\beta A} + \lambda^\beta \omega_\gamma \lambda^\delta \omega_\rho (u_{10})_{\beta\delta}{}^{\gamma\rho A}, \end{aligned} \quad (7.1)$$

where the u 's are super fields of the background and the index A is α for G , $\alpha\beta$ for H , $\alpha\beta\gamma$ for K and $\alpha\beta\gamma\delta$ for L . There are possible terms involving $\nabla \Pi^A, \nabla d_\alpha, \nabla \lambda^\alpha, \nabla \omega_\alpha$. We will not need these terms, so we do not include them. Note that all the terms involving ω must be invariant under the gauge transformation $\delta \omega_\alpha = (\lambda^\gamma)_\alpha \Lambda_\gamma$. It constrains the Lorentz index structure of these terms above.

We need to know the action of \tilde{Q}_0 on G, H, K, L to solve the equations (6.6) to (6.10). Then, we compute \tilde{Q}_0 on the general world-sheet field U^A . Using (3.6) we obtain

$$\begin{aligned} \tilde{Q}_0 U^A = & \lambda^\beta \Pi^a \Pi^b \left(-2 T_{a\beta}{}^c (u_1)_{cb}{}^A + \nabla_\beta (u_1)_{ab}{}^A + (\gamma_a)_{\beta\gamma} (u_3)_b{}^{\gamma A} \right) \\ & + \lambda^\beta \Pi^a \Pi^\gamma \left(-2 T_{\gamma\beta}{}^b (u_1)_{ba}{}^a + T_{a\beta}{}^b (u_2)_{b\gamma}{}^A - \nabla_\beta (u_2)_{a\gamma}{}^A - (\gamma_a)_{\beta\delta} (u_6)_\gamma{}^{\delta A} \right) \\ & + \lambda^\beta \Pi^\gamma \Pi^\delta \left(-T_{\gamma\beta}{}^a (u_2)_{a\delta}{}^A + \nabla_\beta (u_5)_{\gamma\delta}{}^A \right) + \nabla \lambda^\beta \Pi^a (u_2)_{a\beta}{}^A \\ & + \lambda^\beta d_\gamma \Pi^a \left(T_{a\beta}{}^b (u_3)_b{}^{\gamma A} - \nabla_\beta (u_3)_a{}^{\gamma A} + (u_4)_{a\beta}{}^{\gamma A} + 2(\gamma_a)_{\beta\delta} (u_8)^{\delta\gamma A} \right) \\ & + \lambda^\beta d_\gamma \Pi^\delta \left(T_{\delta\beta}{}^a (u_3)_a{}^{\gamma A} - \nabla_\beta (u_6)_\delta{}^{\gamma A} + (u_7)_{\delta\beta}{}^{\gamma A} \right) \\ & + \lambda^\beta \lambda^\gamma \omega_\delta \Pi^a \left(R_{\rho\beta\gamma}{}^\delta (u_3)_a{}^{\rho A} - T_{a\beta}{}^b (u_4)_{b\gamma}{}^{\delta A} + \nabla_\beta (u_4)_{a\gamma}{}^{\delta A} + (\gamma_a)_{\beta\rho} (u_9)_\gamma{}^{\delta\rho A} \right) \\ & + \lambda^\beta \lambda^\gamma \omega_\delta \Pi^\rho \left(-T_{\rho\beta}{}^a (u_4)_{a\gamma}{}^{\delta A} - R_{\sigma\beta\gamma}{}^\delta (u_6)_\rho{}^{\sigma A} - \nabla_\beta (u_7)_{\rho\gamma}{}^{\delta A} \right) \\ & + 2 \nabla \lambda^\beta \Pi^\gamma (u_5)_{\beta\gamma}{}^A + \nabla \lambda^\beta d_\gamma (u_6)_\beta{}^{\gamma A} + \nabla \lambda^\beta \lambda^\gamma \omega_\delta (u_7)_{\beta\gamma}{}^{\delta A} \\ & + \lambda^\beta d_\gamma d_\delta \left(\nabla_\beta (u_8)^{\gamma\delta A} - (u_9)_\beta{}^{\delta\gamma A} \right) \\ & + \lambda^\beta \lambda^\gamma \omega_\delta d_\rho \left(2 R_{\sigma\beta\gamma}{}^\delta (u_8)^{\sigma\rho A} - \nabla_\beta (u_9)_\gamma{}^{\delta\rho A} + 2 (u_{10})_{\beta\gamma}{}^{\rho\delta A} \right) \\ & + \lambda^\beta \lambda^\gamma \omega_\delta \lambda^\rho \omega_\sigma \left(\nabla_\beta (u_{10})_{\gamma\rho}{}^{\delta\sigma A} + R_{\tau\beta\gamma}{}^\delta (u_9)_\rho{}^{\sigma\tau A} \right), \end{aligned} \quad (7.2)$$

where T is the torsion, R is the curvature, both in superspace. Now we solve the equations to obtain the b ghost. We will call the u 's superfields in (7.1) g for G , h for H , k for K and l for L .

We now solve the equation for G^α (6.6). The expression for $\tilde{Q}_0 G^\alpha$ can be read from (7.2) and it has to equal to the right hand side of (6.6). The r.h.s. of (6.6) does not contain a term involving $\nabla \lambda^\beta \Pi^\gamma$, then, $(g_5)_{\beta\gamma}{}^\alpha = 0$. The term with $\nabla \lambda^\beta \Pi^\alpha$ in (6.6) implies that $(g_2)_{a\beta}{}^\alpha = 0$. The term with $\nabla \lambda^\alpha d_\gamma$ in (6.6) implies that $(g_6)_{\beta}{}^{\gamma\alpha} = 0$.

Consider the term involving $\lambda^\beta \Pi^\alpha \Pi^\gamma$ in the equation (6.6). It determines

$$\gamma_{\gamma\beta}^b (g_1)_{ba}{}^\alpha = 0. \quad (7.3)$$

If we multiply this equation by $\gamma_c^{\beta\gamma}$, we obtain that $(g_1)_{ca}{}^\alpha = 0$.

Consider now the term involving $\lambda^\beta \Pi^\alpha \Pi^b$ in the equation (6.6). Because g_1 is zero, the r.h.s. vanishes and we have

$$(\gamma_{(a)\beta\gamma} (g_3)_b)^{\gamma\alpha} = -\eta_{ab} \delta_\beta^\alpha, \quad (7.4)$$

which implies

$$(g_3)_a{}^{\beta\alpha} = -\frac{1}{2} \gamma_a^{\beta\alpha}. \quad (7.5)$$

Consider the term involving $\lambda^\beta d_\gamma \Pi^a$ in (6.6). It leads to the equation

$$(g_4)_{a\beta}{}^{\gamma\alpha} + 2(\gamma_a)_{\beta\delta} (g_8)^{\delta\gamma\alpha} = \frac{3}{2} \gamma_a^{\gamma\alpha} - \frac{1}{2} (\gamma_{ab})_{\beta\rho} (\gamma^b)^{\gamma\alpha} \Omega_\rho + \frac{1}{4} \gamma_{abc}^{\gamma\alpha} (\gamma^{bc})_{\beta\rho} \Omega_\rho. \quad (7.6)$$

To solve this equation we note that g_4 contains a 0-form and a 2-form when it is expanded in β^γ , and g_8 is antisymmetric in $\delta\gamma$. That is,

$$(g_4)_{a\beta}{}^{\gamma\alpha} = \delta_\beta^\gamma (j_a^\alpha) + (\gamma^{bc})_{\beta}{}^\gamma (j_a^\alpha)_{bc}, \quad (g_8)^{\delta\gamma\alpha} = \gamma_{abc}^{\delta\gamma} (k^\alpha)^{abc}. \quad (7.7)$$

Plugging (7.7) into (7.6) we obtain

$$\begin{aligned} & \delta_\beta^\gamma (j_a^\alpha) + (\gamma^{bc})_{\beta}{}^\gamma ((j_a^\alpha)_{bc} + 6(k^\alpha)_{abc}) + 2(\gamma_{abcd})_{\beta}{}^\gamma (k^\alpha)^{bcd} \\ &= \frac{3}{2} \gamma_a^{\gamma\alpha} - \frac{1}{2} (\gamma_{ab})_{\beta\rho} (\gamma^b)^{\gamma\alpha} \Omega_\rho + \frac{1}{4} \gamma_{abc}^{\gamma\alpha} (\gamma^{bc})_{\beta\rho} \Omega_\rho. \end{aligned} \quad (7.8)$$

Multiplying by δ_γ^β we determine the 0-form of g_4 ,

$$(j_a^\alpha) = \frac{3}{2} \gamma_a^{\alpha\beta} \Omega_\beta. \quad (7.9)$$

Multiplying by $(\gamma^{aefg})_{\gamma}{}^\beta$ we obtain that g_8 vanishes. Finally, multiplying by $(\gamma^{de})_{\gamma}{}^\beta$ we obtain

$$(j_a^\alpha)_{bc} = -\frac{1}{4} (\gamma_a \gamma_{bc})^{\alpha\beta} \Omega_\beta. \quad (7.10)$$

In summary, the solution of (7.6) is

$$(g_4)_{a\beta}{}^{\gamma\alpha} = \frac{3}{2} \delta_\beta^\gamma \gamma_a^{\alpha\delta} \Omega_\delta - \frac{1}{4} (\gamma^{bc})_{\beta}{}^\gamma (\gamma_a \gamma_{bc})^{\alpha\delta} \Omega_\delta, \quad (g_8)^{\delta\gamma\alpha} = 0. \quad (7.11)$$

Because g_8 vanishes, it seems that g_9 and g_{10} also vanish in order to simplify the equations from (6.6). In this case, it remains to determine g_7 . It is obtained from the equation involving $\lambda^\beta d\gamma\Pi^\delta$. In fact

$$(g_7)_{\delta\beta}\gamma^\alpha = -\delta_\beta^\alpha\delta_\delta^\gamma + \frac{1}{2}\gamma_{\delta\beta}^a\gamma_a^\gamma\gamma^\alpha. \quad (7.12)$$

Note that g_7 satisfies

$$(\gamma^{abcd})_\gamma{}^\beta(g_7)_{\delta\beta}\gamma^\alpha = 0, \quad (7.13)$$

as it is required from the gauge symmetry for ω in the minimal pure spinor string. Note that the term involving $\nabla\lambda^\beta\lambda^\gamma\omega_\delta$ in (6.6) is satisfied because $(\nabla\lambda\gamma^a\lambda) = 0$.

It remains to verify that the terms involving $\lambda^\beta\lambda^\gamma\omega_\delta\Pi^a$ and $\lambda^\beta\lambda^\gamma\omega_\delta\Pi^\rho$. Instead of plugging the values of the g superfields that we have determined. We will show that these equations are implied by the others. Consider the first term. It implies the equation

$$\begin{aligned} & \lambda^\beta\lambda^\gamma\left(\nabla_\beta(g_4)_{a\beta}\delta^\alpha + T_{\beta a}{}^b(g_4)_{b\gamma}\delta^\alpha + R_{\rho\beta\gamma}{}^\delta(g_3)_a{}^{\rho\alpha}\right) \\ &= \lambda^\beta\lambda^\gamma\left(\frac{1}{4}T_{\beta bc}(\gamma^{bc})_\rho{}^\alpha(g_4)_{a\gamma}\delta^\rho + 5\Omega_\beta(g_4)_{a\gamma}\delta^\alpha\right). \end{aligned} \quad (7.14)$$

We will show that this equation is implied by the term involving $\lambda^\beta d_\gamma\Pi^a$ which states that

$$(g_4)_{a\gamma}\delta^\alpha = \nabla_\gamma(g_3)_a{}^{\delta\alpha} + T_{\gamma a}{}^b(g_3)_b{}^{\delta\alpha} - \frac{1}{4}T_{\gamma bc}(\gamma^{bc})_\rho{}^\alpha(g_3)_a{}^{\delta\rho} - 5\Omega_\gamma(g_3)_a{}^{\delta\alpha}. \quad (7.15)$$

We act with ∇_β on this equation and symmetrize in $(\beta\gamma)$ to obtain

$$\begin{aligned} & \nabla_{(\beta}(g_4)_{a\gamma)}\delta^\alpha + T_{(\beta a}{}^b(g_4)_{b\gamma)}\delta^\alpha - \frac{1}{4}T_{(\beta bc}(\gamma^{bc})_\rho{}^\alpha(g_4)_{a\gamma)}\delta^\rho - 5\Omega_{(\beta}(g_4)_{a\gamma)}\delta^\alpha \\ &= \{\nabla_\beta, \nabla_\gamma\}(g_3)_a{}^{\delta\alpha} + \left(\nabla_{(\beta}T_{\gamma)a}{}^b - T_{a(\beta}{}^c T_{\gamma)c}{}^b\right)(g_3)_b{}^{\delta\alpha} - 5\nabla_{(\beta}\Omega_{\gamma)}(g_3)_a{}^{\delta\alpha} \\ & \quad - \frac{1}{4}\left((\gamma^{bc})_\rho{}^\alpha\nabla_{(\beta}T_{\gamma)bc} - \frac{1}{4}(\gamma^{de}\gamma^{bc})_\rho{}^\alpha T_{(\beta bc}T_{\gamma)de}\right)(g_3)_a{}^{\delta\rho}. \end{aligned} \quad (7.16)$$

where the symmetrization is on $(\beta\gamma)$ only. Note that we will multiply this expression by $\lambda^\beta\lambda^\gamma$ to obtain (7.14). Then, the last term in the second line will vanish because Ω_α is proportional to $\nabla_\alpha\Phi$, where Φ is the dilation superfield. Recall that the anticommutator in the second line is related to the curvature. In fact, the graded commutator for covariant derivatives on g_3 is

$$[\nabla_A, \nabla_B](g_3)_a{}^{\delta\alpha} = -T_{BA}{}^C\nabla_C(g_3)_a{}^{\delta\alpha} + (g_3)_a{}^{\sigma\alpha}R_{BA\sigma}{}^\delta + (g_3)_a{}^{\delta\sigma}R_{BA\sigma}{}^\alpha - R_{BAa}{}^b(g_3)_b{}^{\delta\alpha}. \quad (7.17)$$

Using this equation in (7.16) and the Bianchi identity involving the curvature $R_{\beta\gamma a}{}^b$ we obtain

$$\begin{aligned} & \nabla_{(\beta}(g_4)_{a\gamma)}\delta^\alpha + T_{(\beta a}{}^b(g_4)_{b\gamma)}\delta^\alpha - \frac{1}{4}T_{(\beta bc}(\gamma^{bc})_\rho{}^\alpha(g_4)_{a\gamma)}\delta^\rho - 5\Omega_{(\beta}(g_4)_{a\gamma)}\delta^\alpha - R_{\beta\gamma\rho}{}^\delta(g_3)_a{}^{\rho\alpha} \\ &= -\frac{1}{4}\left((\gamma^{bc})_\rho{}^\alpha\nabla_{(\beta}T_{\gamma)bc} - \frac{1}{4}(\gamma^{de}\gamma^{bc})_\rho{}^\alpha T_{(\beta bc}T_{\gamma)de} - (\gamma^{bc})_\rho{}^\alpha R_{\beta\gamma bc}\right)(g_3)_a{}^{\delta\rho} \end{aligned} \quad (7.18)$$

up to terms proportional to $\gamma_{\beta\gamma}^b$ which will be zero after hitting with $\lambda^\beta\lambda^\gamma$. In the l.h.s. here is equal to the l.h.s. of (7.14), after multiplying with $\lambda^\beta\lambda^\gamma$, if we use the Bianchi identity $R_{(\beta\gamma\rho)}^\delta = 0$. The r.h.s. here is because the Bianchi identity for $R_{\beta\gamma bc}$. Therefore, we have proved that equation (7.14) is implied by the one of the other equations. A similar calculation determines that the term involving $\lambda^\beta\lambda^\gamma\omega_\delta\Pi^\rho$ in (6.6) is satisfied.

In summary, we determine G^α to be

$$G^\alpha = -\frac{1}{2}\Pi^a d_\beta \gamma_a^{\beta\alpha} - \frac{1}{4}J\Pi^\alpha + \frac{1}{8}N^{ab}(\gamma_{ab})^\alpha{}_\beta \Pi^\beta - \frac{3}{2}J\Pi^a \gamma_a^{\alpha\beta} \Omega_\beta - \frac{1}{2}N^{ab}\Pi^c (\gamma_c \gamma_{ab})^{\alpha\beta} \Omega_\beta, \quad (7.19)$$

where $J = -\lambda^\alpha\omega_\alpha$ and $N^{ab} = \frac{1}{2}(\lambda\gamma^{ab}\omega)$. Note that this expression has the correct flat space limit. In this case

$$\Pi^\alpha \rightarrow \partial\theta^\alpha, \quad \Omega_\alpha \rightarrow 0. \quad (7.20)$$

Consider now the equation for $H^{\alpha\beta}$ (6.7). Note that the r.h.s. here does not contain terms with derivatives of λ^γ , therefore $h_2 = h_5 = h_6 = h_7 = 0$. Consider the term with $\lambda^\gamma\Pi^a\Pi^\delta$, it implies that $h_1 = 0$. Consider now the term with $\lambda^\gamma\Pi^a\Pi^b$ that implies

$$(\gamma_{(a)\gamma\delta}(h_3)_b)^{\delta\alpha\beta} = 0. \quad (7.21)$$

Multiplying by η^{ab} we obtain

$$\gamma_{\gamma\delta}^a (h_3)_a^{\delta\alpha\beta} = 0.$$

And multiplying (7.21) by $(\gamma^b)^{\sigma\gamma}$ and we use the above restriction on h_3 we obtain the it vanishes.

The term with $\lambda^\gamma d_\delta \Pi^a$ determines h_4 and h_8 . In fact, the corresponding term gives the equation

$$(h_4)_{a\gamma}{}^{\delta\alpha\beta} + 2(\gamma_a)_{\gamma\rho} (h_8)^{\rho\delta\alpha\beta} = -\frac{1}{2}\delta_\gamma^{[\alpha}\gamma_a^{\beta]\delta}. \quad (7.22)$$

Note that the four form in the expansion of h_4 in γ^δ vanishes and that h_8 is antisymmetric in $\rho\delta$. Then,

$$(h_4)_{a\gamma}{}^{\delta\alpha\beta} = \delta_\gamma^\delta (x_a^{\alpha\beta}) + (\gamma^{bc})_\gamma{}^\delta (x_a^{\alpha\beta})_{bc}, \quad (h_8)^{\rho\delta\alpha\beta} = \gamma_{bcd}^{\rho\delta} (y^{\alpha\beta})^{bcd}.$$

Plugging these expressions into (7.22) and multiplying by δ_δ^γ , then by $(\gamma^{aefg})_\delta{}^\delta$, and finally by $(\gamma^{de})_\delta{}^\gamma$ we determine h_4 and h_8 . They are

$$(h_4)_{a\gamma}{}^{\delta\alpha\beta} = -\frac{1}{16}(\gamma^{bc})_\gamma{}^\delta \gamma_{abc}^{\alpha\beta}, \quad (h_8)^{\rho\delta\alpha\beta} = \frac{1}{192}\gamma_{abc}^{\rho\delta} (\gamma^{abc})^{\alpha\beta}. \quad (7.23)$$

Consider the equation determined by the term with $\lambda^\gamma d_\delta d_\rho$ in (6.7). It determines the part of h_9 antisymmetric in $\rho\delta$ to be

$$(h_9)_\gamma{}^{[\rho\delta]\alpha\beta} = \frac{1}{16}(\gamma^{abc})^{\alpha\beta} (\gamma_{dbc})^{\rho\delta} (\gamma^d \gamma_a)_\gamma{}^\sigma \Omega_\sigma. \quad (7.24)$$

Note that the remaining equations in (6.7) are satisfied if $h_{10} = 0$. As in the case for G^α , the equations from (6.7) with curvature are implied by the equations without curvature

and the use of Bianchi identities and the pure spinor condition. It remains to determine the symmetric part in ρ^δ of h_9 . It turns out that it can be expanded as

$$(h_9)_\gamma^{(\rho\delta)\alpha\beta} = \left(H_{\gamma abcd}^1 (\gamma^d)^{\rho\delta} + H_{\gamma abcdefgh}^5 (\gamma^{defgh})^{\rho\delta} \right) (\gamma^{abc})^{\alpha\beta}. \quad (7.25)$$

The condition $(\gamma^{ijkl})_\rho^\delta (h_9)_\gamma^{\rho\delta\alpha\beta} = 0$ determines the equation,

$$(\gamma^d \gamma_{ijkl})^{\delta\gamma} H_{\gamma abcd}^1 + (\gamma^{defgh} \gamma_{ijkl})^{\delta\gamma} H_{\gamma abcdefgh}^5 + \frac{1}{16} (\gamma_{dbc} \gamma_{ijkl} \gamma^d \gamma_a)^{\delta\gamma} \Omega_\gamma = 0. \quad (7.26)$$

Note that this equation has to be completely antisymmetric in abc because we factor out the matrix $(\gamma^{abc})^{\alpha\beta}$. Because of this, we try the solution

$$\begin{aligned} H_{\gamma abcd}^1 &= (A(\gamma_{abc} \gamma_d)_\gamma^\sigma + B(\gamma_d \gamma_{abc})_\gamma^\sigma) \Omega_\sigma, \\ H_{\gamma abcd}^5 &= (C(\gamma_{abc} \gamma_{defgh})_\gamma^\sigma + D(\gamma_{defgh} \gamma_{abc})_\gamma^\sigma) \Omega_\sigma. \end{aligned} \quad (7.27)$$

The constants A, B, C, D can be determined when we plug this solution into (7.26).

Up to these constants, $H^{\alpha\beta}$ is given by

$$H^{\alpha\beta} = \frac{1}{8} \Pi^a N^{bc} \gamma_{abc}^{\alpha\beta} + \frac{1}{192} (d\gamma_{abcd}) \gamma_{abc}^{\alpha\beta} + d_\gamma \lambda^\delta \omega_\rho (h_9)_\delta^{\rho\gamma\alpha\beta}, \quad (7.28)$$

where h_9 is given above, up to some undetermined constants, and depends on Ω_α . Therefore, the flat limit of (7.28) gives the expected result because $h_9 \rightarrow 0$.

We proceed similarly to determine K and L . The calculation becomes more involving. We just can state that K has the form

$$K^{\alpha\beta\gamma} = K_0^{\alpha\beta\gamma} + \lambda^\delta \omega_\rho \lambda^\sigma \omega_\tau (k_{10})_{\delta\sigma}^{\rho\tau\alpha\beta\gamma}, \quad (7.29)$$

where K_0 is the value of K in the flat space limit and k_{10} depends linearly on Ω_α so it becomes zero in the flat space limit. Finally, the tensor L is equal to the corresponding tensor in flat space-time background.

A Review on pure spinor superspace

We review the results from [10]. The string action (3.1) is based on the superspace coordinate Z^M , where M is a target space super index and runs over ten bosonic indices and sixteen fermionic indices. We define the world-sheet fields Π^A and $\bar{\Pi}^A$ as in (3.2) by introducing the supervielbein E_M^A , where A is a local superspace index. We also need a super connection Ω_{MA}^B to write super covariant derivatives. Out of E_M^A and Ω_{MA}^B we define the super one-forms

$$E^A = dZ^M E_M^A, \quad \Omega_A^B = dZ^M \Omega_{MA}^B. \quad (A.1)$$

We can define now a covariant derivative in superspace which transform homogeneously under local Lorentz rotation. For a super p -form Ψ^A it is given by

$$\nabla \Psi_B^A = d\Psi_B^A + \Psi_B^C \Omega_C^A - (-1)^p \Omega_B^C \Psi_C^A. \quad (A.2)$$

In this formula the product between forms is a wedge product. Given the forms (A.1) and the derivative (A.2) we define the super two forms torsion T^A and curvature R_B^A as

$$T^A = \nabla E^A, \quad R_B^A = d\Omega_B^A + \Omega_B^C \Omega_C^A. \quad (\text{A.3})$$

They satisfy the Bianchi identities

$$\nabla T^A = T^B R_B^A, \quad \nabla R_B^A = 0. \quad (\text{A.4})$$

We use these identities in the torsion and curvature components,

$$T^A = \frac{1}{2} E^B E^C T_{CB}^A, \quad R_B^A = \frac{1}{2} E^C E^D R_{DCB}^A. \quad (\text{A.5})$$

In terms of the torsion and curvature components, the Bianchi identities (A.4) become

$$\nabla_{[A} T_{BC]}^D + T_{[AB}^E T_{EC]}^D - R_{[ABC]}^D = 0, \quad \nabla_{[A} R_{BC]D}^E + T_{[AB}^F R_{FC]D}^E = 0. \quad (\text{A.6})$$

In [10] and [13], the BRST invariance of the action (3.1) puts the background on-shell. In fact, the nil potency of Q implies $\lambda^\alpha \lambda^\beta T_{\alpha\beta}^A = 0$. Berkovits and Howe showed that Lorentz invariance and a symmetry involving the pure spinor variables and the connection Ω (that they call shift symmetry) allow to fix the values of the torsion component as

$$T_{\alpha\beta}^a = \gamma_{\alpha\beta}^a, \quad T_{\alpha\beta}^\gamma = 0, \quad (\text{A.7})$$

where γ^a are the symmetric gamma matrices in ten dimensions. In [10]. it was shown that (A.7) plus the Bianchi identities (A.6) put the background to satisfy the background supergravity equations of motion.

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