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Iterative approximation of solutions for constrained convex minimization problem

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Abstract In this paper, we propose a new iterative scheme for finding a minimizer of a constrained convex minimization problem and prove that the sequence generated by our new scheme converges strongly to a solution of the constrained convex minimization problem in a real Hilbert space.

Mathematics Subject Classification 47H09 · 47J25

الملخص

نقترح في هذه الورقة مخططاً تكرارياً جديداً لإيجاد أصغر قيمة لمسألة قيمة صغرى مقيدة محدبة ونثبت أن المتتابعة المولدة من مخططنا الجديد تتقارب بشكل قوي إلى حلّ لمسألة القيمة الصغرى المقيدة المحدبة في فضاء هلبرت.

1 Introduction

Let H be a real Hilbert space with inner product $\langle ., . \rangle$ and norm $||.||$ and let C be a nonempty, closed and convex subset of H.

Definition 1.1 A mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Construction of fixed points of nonexpansive mappings is an important subject in nonlinear mapping theory and its applications; in particular, in image recovery and signal processing (see, for example, [5,15,20]). For the past 40 years or so, the approximation of fixed points of nonexpansive mappings and fixed points of some of their generalizations and approximation of zeros of accretive-type operators have been a flourishing area of research for many mathematicians. For more details, the reader can consult [2,8,12].

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For any point $u \in H$, there exists a unique point $P_C u \in C$ such that

$$||u - P_C u|| \leq ||u - y||, \quad \forall y \in C.$$

P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping of H onto C. It is also known that P_C satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq ||P_C x - P_C y||^2, \tag{1.1}$$

for all $x, y \in H$. Furthermore, $P_C x$ is characterized by the properties $P_C x \in C$ and

$$\langle x - P_C x, P_C x - y\rangle \geq 0, \tag{1.2}$$

for all $y \in C$.

Definition 1.2 A mapping $T : H \to H$ is said to be firmly nonexpansive if $2T - I$ is nonexpansive, or equivalently

$$\langle x - y, Tx - Ty\rangle \geq ||Tx - Ty||^2, \quad \forall x, y \in H.$$

Alternatively, T is firmly nonexpansive if T can be expressed as

$$T = \frac{1}{2}(I + S),$$

where $S : H \to H$ is nonexpansive. For example, projections are firmly nonexpansive.

Definition 1.3 A mapping $T : H \to H$ is said to be an averaged mapping if it can be written as the average of the identity mapping I and a nonexpansive mapping; that is

$$T = (1 - \alpha)I + \alpha S, \tag{1.3}$$

where $\alpha \in (0, 1)$ and $S : H \to H$ is nonexpansive. More precisely, when (1.3) holds, we say that T is α-averaged. Thus, firmly nonexpansive mappings (in particular, projections) are $\frac{1}{2}$-averaged mappings.

Some properties of averaged mappings are in the following proposition below.

Proposition 1.4 ([5,9]) *For given operators $S, T, V : H \to H$:*

(a) *If $T = (1-\alpha)S + \alpha V$ for some $\alpha \in (0, 1)$ and if S is averaged and V is nonexpansive, then T is averaged.*
(b) *T is firmly nonexpansive if and only if the complement $I - T$ is firmly nonexpansive.*
(c) *If $T = (1 - \alpha)S + \alpha V$ for some $\alpha \in (0, 1)$ and if S is firmly nonexpansive and V is nonexpansive, then T is averaged.*
(d) *The composite of finitely many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^N$ is averaged, then so is the composite $T_1 \ldots T_N$. In particular, if T_1 is α_1-averaged and T_2 is α_2-averaged, where $\alpha_1, \alpha_2 \in (0, 1)$, then the composite $T_1 T_2$ is α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.*

Definition 1.5 A nonlinear operator T whose domain $D(T) \subset H$ and range $R(T) \subset H$ is said to be:

(a) monotone if

$$\langle x - y, Tx - Ty\rangle \geq 0, \quad \forall x, y \in D(T),$$

(b) β- strongly monotone if there exists $\beta > 0$ such that

$$\langle x - y, Tx - Ty\rangle \geq \beta||x - y||^2, \quad \forall x, y \in D(T),$$

(c) ν- inverse strongly monotone (for short, ν-ism) if there exists $\nu > 0$ such that

$$\langle x - y, Tx - Ty\rangle \geq \nu||Tx - Ty||^2, \quad \forall x, y \in D(T).$$

It can be easily seen that (i) if T is nonexpansive, then $I - T$ is monotone; (ii) the projection mapping P_C is a 1-ism. The inverse strongly monotone (also referred to as co-coercive) operators have been widely used to solve practical problems in various fields, for instance, in traffic assignment problems; see, for example, [3,11] and the references therein.

The following proposition gathers some results on the relationship between averaged mappings and inverse strongly monotone operators.

Proposition 1.6 ([5]) *Let $T : H \to H$ be an operator.*

(a) *T is nonexpansive if and only if the complement $I - T$ is $\frac{1}{2}$-ism.*
(b) *If T is ν-ism, then for $\gamma > 0$, γT is $\frac{\nu}{\gamma}$-ism.*
(c) *T is averaged if and only if the complement $I - T$ is ν-ism for some $\nu > 1/2$. Indeed, for $\alpha \in (0, 1)$, T is α-averaged if and only if $I - T$ is $\frac{1}{2\alpha}$-ism.*

Consider the following constrained convex minimization problem:

$$\text{minimize}\{f(x) : x \in C\}, \tag{1.4}$$

where $f : C \to \mathbb{R}$ is a real-valued convex function. We say that the minimization problem (1.4) is consistent if the minimization problem (1.4) has a solution. In the sequel, we shall denote the solution set of problem (1.4) by S. If f is (Fréchet) differentiable, then the gradient-projection method (for short, GPM) generates a sequence $\{x_n\}$ using the following recursive formula:

$$x_{n+1} = P_C(x_n - \lambda \nabla f(x_n)), \quad \forall n \geq 1, \tag{1.5}$$

or more generally,

$$x_{n+1} = P_C(x_n - \lambda_n \nabla f(x_n)), \quad \forall n \geq 1, \tag{1.6}$$

where in both (1.5) and (1.6) the initial guess x_0 is taken from C arbitrarily, and the parameters, λ or λ_n, are positive real numbers. The convergence of the algorithms (1.5) and (1.6) depends on the behaviour of the gradient ∇f. As a matter of fact, it is known that if ∇f is α-strongly monotone and L-Lipschitzian with constants $\alpha,\ L > 0$, then the operator

$$T := P_C(I - \lambda \nabla f) \tag{1.7}$$

is a contraction; hence, the sequence $\{x_n\}$ defined by the algorithm (1.5) converges in norm to the unique solution of the minimization problem (1.4). More generally, if the sequence $\{\lambda_n\}$ is chosen to satisfy the property

$$0 < \liminf \lambda_n \leq \limsup \lambda_n < \frac{2\alpha}{L^2}, \tag{1.8}$$

then the sequence $\{x_n\}$ defined by the algorithm (1.6) converges in norm to the unique minimizer of (1.4). However, if the gradient ∇f fails to be strongly monotone, the operator T defined by (1.7) would fail to be contractive; consequently, the sequence $\{x_n\}$ generated by the algorithm (1.6) may fail to converge strongly (see [18, Sect. 4]). If ∇f is Lipschitzian, then the algorithms (1.5) and (1.6) can still converge in the weak topology under certain conditions.

The GPM for finding the approximate solutions of the constrained convex minimization problem is well known; see, for example, [16] and the references therein. The convergence of the sequence generated by the this method depends on the behaviour of the gradient of the objective function. If the gradient fails to be strongly monotone, then the strong convergence of the sequence generated by GPM may fail. Recently, Xu [18] gave an alternative operator-oriented approach to algorithm (1.6); namely, an averaged mapping approach. He gave his averaged mapping approach to the gradient-projection algorithm (1.6) and the relaxed gradient-projection algorithm. Moreover, he constructed a counterexample which shows that algorithm (1.5) does not converge in norm in an infinite-dimensional space, and also presented two modifications of gradient-projection algorithms which are shown to have strong convergence. Further, he regularized the minimization problem (1.4) to devise an iterative scheme that generates a sequence converging in norm to the minimum-norm solution of (1.4) in the consistent case.

Very recently, motivated by the work of Xu [18], Ceng et al. [6] proposed the following implicit iterative scheme

$$x_\lambda = P_C(s\gamma V x_\lambda + (I - s\mu F)T_\lambda x_\lambda)$$

and the following explicit iterative scheme

$$x_{n+1} = P_C(s_n \gamma V x_n + (I - s\mu F)T_n x_n)$$

for finding the approximate minimizer of a constrained convex minimization problem and prove that the sequences generated by their schemes converge strongly to a solution of the constrained convex minimization problem (see [6] for more details). Such a solution is also a solution of a variational inequality defined over the set of fixed points of a nonexpansive mapping. Also, based on Yamada hybrid steepest descent method, Tian and Huang [17] proposed respectively the following implicit and explicit iterative scheme:

$$x_s = P_C(I - s_n \mu F)T_{\lambda_s}(x_s)$$

and

$$x_{n+1} = P_C(I - s_n \mu F)T_{\lambda_n}(x_n).$$

They proved that the sequences generated by their implicit and explicit schemes converge strongly to a solution of the constrained convex minimization problem, which also solves a certain variational inequality (see [17] for more details).

Motivated by the work of Xu [18], Ceng et al. [6] and Tian and Huang [17], we introduce a new iterative scheme for finding the approximate minimizer of a constrained convex minimization problem and prove that the sequence generated by our scheme converge strongly to a solution of the constrained convex minimization problem.

2 Preliminaries

In the sequel, we shall also make use of the following lemmas.

Lemma 2.1 *Let H be a real Hilbert space. Then, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle \quad \forall\, x, y \in H.$$

Lemma 2.2 *Let H be a real Hilbert space. The following inequality holds:*

$$||\lambda x + (1 - \lambda)y||^2 = \lambda||x||^2 + (1 - \lambda)||y||^2 - \lambda(1 - \lambda)||x - y||^2, \quad \forall x, y \in H, \quad \lambda \in [0, 1].$$

Lemma 2.3 (Browder [4], Goebel and Kirk [10]) *Let H be a real Hilbert space, C a closed convex subset of H, and $T : C \to C$ a nonexpansive mapping with a fixed point. Assume that a sequence $\{x_n\}$ in C is such that $x_n \rightharpoonup x$ and $x_n - Tx_n \to y$. Then $x - Tx = y$.*

Lemma 2.4 (Xu [19]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \sigma_n + \gamma_n, \quad n \geq 0,$$

where, (i)$\{\alpha_n\} \subset [0, 1]$, $\sum \alpha_n = \infty$;(ii) $\lim sup\, \sigma_n \leq 0$;(iii) $\gamma_n \geq 0$; $(n \geq 0)$, $\sum \gamma_n < \infty$. *Then, $a_n \to 0$ as $n \to \infty$.*

We adopt the following notations:

- $x_n \to x$ means that $x_n \to x$ strongly;
- $x_n \rightharpoonup x$ means that $x_n \to x$ weakly;
- $w_w(x_n) := \{x : \exists x_{n_j} \rightharpoonup x\}$ is the weak w-limit set of the sequence $\{x_n\}_{n=1}^{\infty}$.

3 Main results

In this section, we modify the gradient projection method so as to have strong convergence. Below we include such modification. Our result in this section complements the results of Xu [18]. Furthermore, using the technique in [18,14], we obtain the following theorem.

Theorem 3.1 *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Suppose that the minimization problem (1.4) is consistent and let S denote its solution set. Assume that the gradient ∇f is L-Lipschitzian with constant $L > 0$. For any given $u \in C$, let the sequences $\{x_n\}$ and $\{y_n\}$ be generated iteratively by $x_1 \in C$,*

$$\begin{cases} y_n = \alpha_n u + (1-\alpha_n)x_n, \\ x_{n+1} = P_C(y_n - \lambda_n \nabla f(y_n)), \quad n \geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ in [0,1] and $\{\lambda_n\}$ in $\left(0, \frac{2}{L}\right)$ satisfy the following conditions:

(C1) $\lim\limits_{n\to\infty} \alpha_n = 0$;

(C2) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $0 < \liminf \lambda_n \leq \limsup \lambda_n < \frac{2}{L}$.

Then the sequence $\{x_n\}$ converges strongly to a minimizer $\hat{x}$ of (1.4) which is the closest to u from the solution set S. In other words, $\hat{x} = P_S u$.

Proof Inspired by the method of proof of [18], it is well known that

(a) $x^* \in C$ solves the minimization problem (1.4) if and only if x^* solves the fixed-point equation

$$x^* = P_C(I - \lambda\nabla f)x^*,$$

where $\lambda > 0$ is any fixed positive number. For the sake of simplicity, we may assume that (due to condition (C3))

$$0 < a \leq \lambda_n \leq b < \frac{2}{L}, \quad n \geq 1$$

where a and b are constants;

(b) the gradient ∇f is $\frac{1}{L}$-ism [1];

(c) $P_C(I - \lambda\nabla f)$ is $\frac{2+\lambda L}{4}$- averaged for $0 < \lambda < \frac{2}{L}$. Hence we have that, for each n, $P_C(I - \lambda_n\nabla f)$ is $\frac{2+\lambda_n L}{4}$-averaged. Therefore, we can write

$$P_C(I - \lambda_n\nabla f) = \frac{2-\lambda_n L}{4} I + \frac{2+\lambda_n L}{4} T_n = (1-\beta_n)I + \beta_n T_n, \tag{3.2}$$

where T_n is nonexpansive and $\beta_n = \frac{2+\lambda_n L}{4} \in [a_1, b_1] \subset (0, 1)$, where $a_1 = \frac{2+aL}{4}$ and $b_1 = \frac{2+bL}{4} < 1$. Then we can rewrite (3.1) as

$$\begin{cases} y_n = \alpha_n u + (1-\alpha_n)x_n, \\ x_{n+1} = (1-\beta_n)y_n + \beta_n T_n y_n, \quad n \geq 1, \end{cases} \tag{3.3}$$

For any $x^* \in S$, noticing that $T_n x^* = x^*$, we have

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq (1-\beta_n)\|y_n - x^*\| + \beta_n\|T_n y_n - x^*\| \\ &\leq \|y_n - x^*\| \\ &= \|(1-\alpha_n)(x_n - x^*) + \alpha_n(u - x^*)\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|u - x^*\| \\ &\leq \max\{\|x_n - x^*\|, \|u - x^*\|\}. \end{aligned}$$

By induction, it is easy to see that

$$\|x_n - x^*\| \leq \max\{\|x_1 - x^*\|, \|u - x^*\|\}, \quad \forall n \geq 1.$$

Hence $\{x_n\}$ is bounded and so are $\{y_n\}$ and $\{T_n y_n\}$.

Using Lemma 2.2 and (3.3), we have

$$\begin{aligned}||x_{n+1}-x^*||^2 &= ||(1-\beta_n)(y_n-x^*)+\beta_n(T_n y_n-x^*)||^2\\ &= (1-\beta_n)||y_n-x^*||^2+\beta_n||T_n y_n-x^*||^2-\beta_n(1-\beta_n)||T_n y_n-y_n||^2\\ &\le (1-\beta_n)||y_n-x^*||^2+\beta_n||y_n-x^*||^2-\beta_n(1-\beta_n)||T_n y_n-y_n||^2\\ &= ||y_n-x^*||^2-\beta_n(1-\beta_n)||T_n y_n-y_n||^2.\end{aligned}$$

Therefore, by Lemma 2.1, we have

$$\begin{aligned}\beta_n(1-\beta_n)||T_n y_n-y_n||^2 &\le ||y_n-x^*||^2-||x_{n+1}-x^*||^2\\ &\le ||x_n-x^*||^2-||x_{n+1}-x^*||^2+2\alpha_n\langle u-x^*, y_n-x^*\rangle. \end{aligned}\tag{3.4}$$

Since $\{y_n\}$ is bounded, then there exists a constant $M \ge 0$ such that

$$\langle u-x^*, y_n-x^*\rangle \le M \quad \text{for all} \quad n \ge 1.$$

So, from (3.4) we have

$$\beta_n(1-\beta_n)||T_n y_n-y_n||^2 \le ||x_n-x^*||^2-||x_{n+1}-x^*||^2+2\alpha_n M. \tag{3.5}$$

Now, we divide the rest of the proof into two cases.

Case 1 Assume that the sequence $\{||x_n-x^*||\}$ is a monotonically decreasing sequence. Then $\{||x_n-x^*||\}$ is convergent. Clearly, we have

$$||x_{n+1}-x^*||^2-||x_n-x^*||^2 \to 0.$$

It then implies from (3.5) that

$$\lim_{n\to\infty}\beta_n(1-\beta_n)||T_n y_n-y_n|| = 0.$$

Using the condition $\beta_n \in [a_1, b_1] \subset (0, 1)$, we have

$$\lim_{n\to\infty}||T_n y_n-y_n|| = 0. \tag{3.6}$$

Now from (3.3), we obtain

$$||y_n-x_{n+1}|| = \beta_n||T_n y_n-y_n|| \to 0 \quad \text{as} \quad n\to\infty. \tag{3.7}$$

From (3.3), we know that

$$\begin{aligned}||y_n-x_n|| &= ||\alpha_n u+(1-\alpha_n)x_n-x_n||\\ &= \alpha_n||u-x_n|| \le \alpha_n M_1 \to 0,\end{aligned}\tag{3.8}$$

where $M_1 \ge ||u-x_n||$, $\forall n \ge 1$. Therefore, from (3.8) and (3.7), we have

$$||x_{n+1}-x_n|| \to 0. \tag{3.9}$$

Also, from (3.6) and (3.8), we have

$$\begin{aligned}||T_n x_n-x_n|| &\le ||T_n x_n-T_n y_n||+||T_n y_n-y_n||+||y_n-x_n||\\ &\le 2||x_n-y_n||+||T_n y_n-y_n|| \to 0 \quad \text{as} \quad n\to\infty.\end{aligned}\tag{3.10}$$

Next, we prove that

$$w_w(x_n) \subset S.$$

Suppose that $p \in w_w(x_n)$ and $\{x_{n_j}\}$ is a subsequence of $\{x_n\}$ such that $x_{n_j} \rightharpoonup p$; thus $x_{n_j+1} \rightharpoonup p$ by (3.9) and $y_{n_j} \rightharpoonup p$ by (3.8). We may assume that $\lambda_{n_j} \to \lambda$; then we have $0 < \lambda < \frac{2}{L}$. Set $T := P_C(I - \lambda\nabla f)$; then T is nonexpansive. Since $x_{n_j+1} = P_C(y_{n_j} - \lambda_{n_j}\nabla f(y_{n_j}))$ and $x_{n_j+1} - x_{n_j} \to 0$, we get

$$\begin{aligned} ||x_{n_j} - Ty_{n_j}|| &\leq ||x_{n_j+1} - x_{n_j}|| + ||P_C(y_{n_j} - \lambda_{n_j}\nabla f(y_{n_j})) - P_C(y_{n_j} - \lambda\nabla f(y_{n_j}))|| \\ &\leq ||x_{n_j+1} - x_{n_j}|| + |\lambda_{n_j} - \lambda|||\nabla f(y_{n_j})|| \\ &\leq ||x_{n_j+1} - x_{n_j}|| + M_2|\lambda_{n_j} - \lambda| \to 0. \end{aligned}$$

Furthermore, by (3.8) we obtain

$$\begin{aligned} ||x_{n_j} - Tx_{n_j}|| &\leq ||x_{n_j} - Ty_{n_j}|| + ||Ty_{n_j} - Tx_{n_j}|| \\ &\leq ||x_{n_j} - Ty_{n_j}|| + ||y_{n_j} - x_{n_j}|| \to 0. \end{aligned}$$

Also, we have

$$\begin{aligned} ||y_{n_j} - Ty_{n_j}|| &\leq ||y_{n_j} - x_{n_j}|| + ||x_{n_j} - Tx_{n_j}|| + ||Ty_{n_j} - Tx_{n_j}|| \\ &\leq 2||y_{n_j} - x_{n_j}|| + ||x_{n_j} - Tx_{n_j}|| \to 0. \end{aligned}$$

Lemma 2.3 guarantees that $w_w(x_n) \subset F(T) = S$ and $w_w(y_n) \subset F(T) = S$.

Next, we prove that $\{x_n\}$ converges strongly to $\hat{x} \in S$, where $\hat{x}$ is the solution of (1.4) which is closest to u from the solution set S. First, we show that $\limsup_{n\to\infty}\langle y_n - \hat{x}, u - \hat{x}\rangle \leq 0$. Observe that there exists a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ satisfying

$$\limsup_{n\to\infty}\langle y_n - \hat{x}, u - \hat{x}\rangle = \lim_{j\to\infty}\langle y_{n_j} - \hat{x}, u - \hat{x}\rangle.$$

Since $\{y_{n_j}\}$ is bounded, there exists a subsequence $\{y_{n_{j_i}}\}$ of $\{y_{n_j}\}$ such that $y_{n_{j_i}} \rightharpoonup p \in F(T) = S$. Without loss of generality, we assume that $y_{n_j} \rightharpoonup p \in F(T) = S$. Then, we obtain

$$\limsup_{n\to\infty}\langle y_n - \hat{x}, u - \hat{x}\rangle = \lim_{j\to\infty}\langle y_{n_j} - \hat{x}, u - \hat{x}\rangle = \langle p - \hat{x}, u - \hat{x}\rangle \leq 0.$$

Using Lemma 2.1, we get from (3.3) that

$$\begin{aligned} ||x_{n+1} - \hat{x}||^2 &\leq ||y_n - \hat{x}||^2 \\ &= ||(1-\alpha_n)(x_n - \hat{x}) + \alpha_n(u - \hat{x})||^2 \\ &\leq (1-\alpha_n)||x_n - \hat{x}||^2 + 2\alpha_n\langle y_n - \hat{x}, u - \hat{x}\rangle. \end{aligned} \tag{3.11}$$

By Lemma 2.4, we obtain $\lim_{n\to\infty}||x_n - \hat{x}|| = 0$.

Case 2 Assume that $\{||x_n - x^*||\}$ is not a monotonically decreasing sequence. Set $\Gamma_n = ||x_n - x^*||^2$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough) by

$$\tau(n) = \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}.$$

Clearly, τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ for $n \geq n_0$. From (3.5), we see that

$$\beta_{\tau(n)}(1 - \beta_{\tau(n)})||T_{\tau(n)}y_{\tau(n)} - y_{\tau(n)}||^2 \leq 2\alpha_{\tau(n)}M \to 0 \quad \text{as} \quad n \to \infty.$$

Furthermore, we have

$$||T_{\tau(n)}y_{\tau(n)} - y_{\tau(n)}|| \to 0 \quad \text{as} \quad n \to \infty.$$

By the same argument as in Case 1, we can show that $x_{\tau(n)}$ converges weakly to $p \in w_w(x_{\tau(n)})$ as $\tau(n) \to \infty$ and $\limsup_{\tau(n)\to\infty}\langle u - \hat{x}, y_{\tau(n)} - \hat{x}\rangle \leq 0$. We know that for all $n \geq n_0$,

$$0 \leq ||x_{\tau(n)+1} - \hat{x}||^2 - ||x_{\tau(n)} - \hat{x}||^2 \leq \alpha_n[2\langle u - \hat{x}, y_{\tau(n)} - \hat{x}\rangle - ||x_{\tau(n)} - \hat{x}||^2],$$

which implies that

$$||x_{\tau(n)} - \hat{x}||^2 \leq 2\langle u - \hat{x}, y_{\tau(n)} - \hat{x}\rangle.$$

Then we conclude that

$$\lim_{n\to\infty} ||x_{\tau(n)} - \hat{x}|| = 0.$$

Therefore

$$\lim_{n\to\infty} \Gamma_{\tau(n)} = \lim_{n\to\infty} \Gamma_{\tau(n)+1} = 0.$$

Furthermore, for $n \geq n_0$, it is easily observed that $\Gamma_n \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is, $\tau(n) < n$), because $\Gamma_j > \Gamma_{j+1}$ for $\tau(n) + 1 \leq j \leq n$. As a consequence, we obtain for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

Hence $\lim_{n\to\infty} \Gamma_n = 0$, that is, $\{x_n\}$ converges strongly to $\hat{x}$. This completes the proof. □

Corollary 3.2 *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Suppose that the minimization problem (1.4) is consistent and let S denote its solution set. Assume that the gradient ∇f is L-Lipschitzian with constant $L > 0$. For any given $u \in C$, let the sequences $\{x_n\}$ and $\{y_n\}$ be generated iteratively by $x_1 \in C$,*

$$\begin{cases} y_n = \alpha_n u + (1 - \alpha_n)x_n, \\ x_{n+1} = P_C(y_n - \lambda \nabla f(y_n)), \quad n \geq 1, \end{cases} \tag{3.12}$$

where $0 < \lambda < \frac{2}{L}$ and $\{\alpha_n\}$ in [0,1] satisfies the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}$ converges strongly to a minimizer $\hat{x}$ of (1.4).

4 An application

In this section, we give an application of Theorem 3.1 to the split feasibility problem (say SFP, for short), which was introduced by Censor and Elfving [7]. SFP problem has gained much attention of several authors due to its applications to image reconstruction, signal processing and intensity-modulated radiation therapy (see [5,13,17]).

This SFP can be mathematically formulated as the problem of finding a point x with the property

$$x \in C \quad \text{and} \quad Bx \in Q, \tag{4.1}$$

where C and Q are nonempty, closed and convex subset of Hilbert space H_1 and H_2 respectively and $B : H_1 \to H_2$ is a bounded linear operator.

Clearly, x^* is a solution to the split feasibility problem (4.1) if and only if $x^* \in C$ and $Bx^* - P_Q Bx^* = 0$. The proximity function f is defined by

$$f(x) = \frac{1}{2}\left|\left|Bx - P_Q Bx\right|\right|^2 \tag{4.2}$$

and consider the constrained convex minimization problem

$$\min_{x\in C} f(x) = \min_{x\in C} \frac{1}{2}\left|\left|Bx - P_Q Bx\right|\right|^2. \tag{4.3}$$

Then x^* solves the split feasibility problem (4.1) if and only if x^* solves the minimization problem (4.3). In [5], CQ algorithm was introduced to solve the SFP.

$$x_{n+1} = P_C(I - \lambda B^*(I - P_Q)B)x_n, \quad n \geq 0, \tag{4.4}$$

where $0 < \lambda < \frac{2}{||B||^2}$. It was proved that the sequence generated by (4.4) converges weakly to a solution of the SFP.

We propose the following algorithm to obtain a strong convergence iterative sequence to solve SFP. For any given $u \in C$, let the sequences $\{x_n\}$ and $\{y_n\}$ be generated iteratively by $x_1 \in C$,

$$\begin{cases} y_n = \alpha_n u + (1 - \alpha_n)x_n, \\ x_{n+1} = P_C(I - \lambda_n(B^*(I - P_Q)B + \gamma I))y_n, \quad n \geq 1, \end{cases} \tag{4.5}$$

where $\{\alpha_n\} \subset [0, 1]$, $\gamma > 0$ and $\{\lambda_n\}$ in $\left(0, \frac{2}{L}\right)$ satisfy the following conditions:

(C1) $\lim\limits_{n\to\infty} \alpha_n = 0$;

(C2) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(C^*3) $0 < \liminf \lambda_n \leq \limsup \lambda_n < \frac{2}{||B||^2+\gamma}$.

We obtain the following convergence result for solving split feasibility problem (4.1) by applying theorem (3.1).

Theorem 4.1 *Assume that the split feasibility problem* (4.1) *is consistent. Let the sequence* $\{x_n\}$ *be generated by* (4.5), *where the sequence* $\{\alpha_n\}$ *in* $[0, 1]$ *and* $\{\lambda_n\}$ *in* $\left(0, \frac{2}{L}\right)$ *satisfy the conditions* (C1)–(C*3). *Then the sequence* $\{x_n\}$ *converges strongly to a solution of the split feasibility problem* (4.1).

Proof Using the definition of the proximity function f, we have

$$\nabla f(x) = B^*(I - P_Q)Bx, \tag{4.6}$$

and ∇f is Lipschitz continuous, that is

$$||\nabla f(x) - \nabla f(y)|| \leq L||x - y||, \tag{4.7}$$

where $L = ||B||^2$.

Set

$$f_\gamma(x) = f(x) + \frac{\gamma}{2}||x||^2.$$

Consequently,

$$\begin{aligned} \nabla f_\gamma(x) &= \nabla f(x) + \gamma x \\ &= B^*(I - P_Q)Bx + \gamma x \end{aligned}$$

and ∇f_γ is Lipschitzian with Lipschitz constant $||B||^2 + \gamma$. Then the iterative scheme (4.5) is equivalent to

$$\begin{cases} y_n = \alpha_n u + (1 - \alpha_n)x_n, \\ x_{n+1} = P_C(y_n - \lambda_n \nabla f_\gamma(y_n)), \quad n \geq 1, \end{cases} \tag{4.8}$$

where $\{\alpha_n\} \subset [0, 1]$, $\gamma > 0$ and $\{\lambda_n\}$ in $\left(0, \frac{2}{L^*}\right)$ satisfy the following conditions:

(C1) $\lim\limits_{n\to\infty} \alpha_n = 0$;

(C2) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(C^*3) $0 < \liminf \lambda_n \leq \limsup \lambda_n < \frac{2}{L^*}$,

where $L^* = ||B||^2 + \gamma$. The conclusion follows from Theorem (3.1). □

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