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Fixed point theorems with generalized altering distance functions in partially ordered metric spaces via w-distances and applications

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Abstract

In this paper, we establish some fixed point theorems with T-contractions and w-distances in partially ordered metric spaces. The main tool used in our proof is a generalized altering distance function. Our results can be applied directly to study multidimensional fixed point which covers the concepts of coupled, tripled, quadruple fixed point etc. Moreover, a Fredholm integral equation and an initial-value problem for partial differential equation of parabolic type are also discussed.

Keywords: T-contractions; altering distance; multidimensional; w-distance; partial order

1 Introduction

Let (X,d) be a complete metric space and F be a selfmap of X. We say that F is a ϕ-contraction if

$$d(Fx,Fy) \leq \phi\bigl(d(x,y)\bigr) \quad \text{for each } x,y \in X, \tag{1.1}$$

where $\phi : [0,+\infty) \to [0,+\infty)$ with $\phi(t) < t$ for all $t > 0$. The following generalization of the Banach contraction mapping principle was proved by Rakotch [1]: if ϕ is monotone and continuous, then any ϕ-contraction is a Picard operator. Subsequently, this result was improved by Boyd and Wong [2], who showed that one need only assume that $\phi(t) < t$ for all $t > 0$, together with the right-upper semicontinuity of ϕ. Meanwhile, Meir and Keeler [3] found that the conclusion of Banach's Theorem holds more generally from the following condition of a weakly uniformly strict contraction: Given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\epsilon \leq d(x,y) < \epsilon + \delta \quad \text{implies} \quad d(Fx,Fy) < \epsilon. \tag{1.2}$$

In fact, Rakotch's corollary and Boyd and Wong's Theorem 1 easily followed from Meir and Keeler's theorem. We note that Lim [4] characterized condition (1.2) in terms of the function ϕ in (1.1). This is obviously desirable since then one can easily see how much more general is Meir and Keeler's result [3] than Boyd and Wong's Theorem 1. Moreover, Proinov [5] established two general theorems for equivalence between the Meir-Keeler type contractive conditions (1.2) and the contractive conditions involving gauge functions (1.1). Indeed, Theorem 3.5 in Proinov [5] was an extension of Theorem 1 in Lim [4]. Next, the condition (1.2) was extended by Chi *et al.* [6] from the view point of T-contractions. Let $T : X \to X$ be an injective, continuous and sequentially convergent mapping. For every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\epsilon \le d(Tx, Ty) < \epsilon + \delta \quad \text{implies} \quad d(TFx, TFy) < \epsilon. \tag{1.3}$$

On the other hand, Delbosco [7] initiated a study of the following contractive condition with the so-called altering distance function:

$$\psi\big(d(Fx, Fy)\big) \le k\psi\big(d(x, y)\big) \quad \text{for each } x, y \in X \text{ and some } k \in [0, 1), \tag{1.4}$$

where $\psi \in \widetilde{\Psi}$ and $\widetilde{\Psi}$ is defined by

$$\begin{aligned} \widetilde{\Psi} = \big\{&\psi : [0, \infty) \to [0, +\infty), \psi \text{ is continuous, nondecreasing and} \\ &\psi^{-1}\big(\{0\}\big) = \{0\}\big\}. \end{aligned} \tag{1.5}$$

In fact, Delbosco [7] only considered the particular case when ψ is a power function. Until 1984, Khan *et al.* [8] formally introduced the definition of the above family $\widetilde{\Psi}$, and proved that any mapping F satisfying (1.4) with $\psi \in \widetilde{\Psi}$ is a Picard operator.

In the recent past, the idea of altering function has been utilized by many authors. We would like to mention the work of Dutta and Choudhury [9], they presented a generalization of (1.4) to subsume the results of Rhoades [10] and Khan *et al.* [8].

Theorem 1.1 ([9]) *Let F be a selfmapping defined on a complete metric space (X, d) satisfying the following condition*:

$$\psi\big(d(Fx, Fy)\big) \le \psi\big(d(x, y)\big) - \phi\big(d(x, y)\big) \quad \textit{for each } x, y \in X, \tag{1.6}$$

where $\psi, \phi \in \widetilde{\Psi}$. Then F is a Picard operator.

Furthermore, Jachymski [11] showed that Theorem 1.1 and Theorem 3 of Khan *et al.* [8] are equivalent by establishing a geometric lemma giving a list of equivalent conditions for some subsets of the plane.

Another recent direction of such generalizations, see [12–14], has been studied by weakening the contractive conditions and, in compensation, by simultaneously enriching the metric space structure with a partial order. Very recently, Su [15] presented the definition of generalized altering distance function to prove the following new fixed point theorem of generalized contraction mappings in a complete metric space endowed with a partial order.

Theorem 1.2 ([15]) *Let X be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $F : X \to X$ be a continuous and* *nondecreasing mapping such that*

$$\psi\bigl(d(Fx,Fy)\bigr) \le \phi\bigl(d(x,y)\bigr), \quad \forall x \ge y, \tag{1.7}$$

where $\psi \in \Psi$, Ψ *is defined by*

$$\Psi = \bigl\{\psi : [0,\infty) \to [0,+\infty), \psi \text{ is nondecreasing and } \psi^{-1}\bigl(\{0\}\bigr) = \{0\}\bigr\}, \tag{1.8}$$

and $\phi : [0,+\infty) \to [0,+\infty)$ *is a right-upper semicontinuous function with the condition* $\psi(t) > \phi(t)$ *for all* $t > 0$. *If there exists* $x_0 \in X$ *such that* $x_0 \le Fx_0$, *then* F *has a fixed point.*

The purpose of this paper is to prove some fixed point theorems with respect to w-distances in partially ordered metric spaces employing altering functions and the notation of a function involving Meir-Keeler type. Recall that the concept of a w-distance was initiated by Kada *et al.* [16], and was primarily used to extend Ekeland's variational principle, Caristi's fixed point theorem and the non-convex minimization theorems whose details are available in Takahashi [17]. Very recently, He [18] established a fixed point theorem with w-distance for (1.1) in complete metric spaces. Note that Lakzian *et al.* [19] utilized the concept of a w-distance on a metric space to generalize Theorem 1.1. Furthermore, Rouzkard *et al.* [20] proved the following fixed point theorem in a complete metric space equipped with a partial order using w-distances together with altering functions. For other new results on w-distances, please see [21–23].

Theorem 1.3 ([20]) (X,d) *is a complete partially ordered metric space equipped with a* w*-distance. Let* $F : X \to X$ *be a continuous and nondecreasing mapping such that*

$$\psi\bigl(p(Fx,Fy)\bigr) \le \phi\psi\bigl(p(x,y)\bigr), \quad \forall x \ge y, \tag{1.9}$$

where $\psi \in \overline{\Psi}$, $\phi \in \overline{\Phi}$, *and* $\overline{\Psi}$ *and* $\overline{\Phi}$ *are defined by*

$$\begin{aligned}\overline{\Psi} = \bigl\{&\psi : \mathbb{R}_+ \to \mathbb{R}_+, \psi \text{ is nondecreasing, continuous, and} \\ &\psi(t) > 0 \text{ for each } t > 0\bigr\}\end{aligned} \tag{1.10}$$

and

$$\begin{aligned}\overline{\Phi} = \bigl\{&\phi : [0,+\infty) \to [0,+\infty), \phi \text{ is nondecreasing, right continuous, and} \\ &\phi(t) < t \text{ for all } t > 0\bigr\},\end{aligned} \tag{1.11}$$

respectively. If there exists $x_0 \in X$ *such that* $x_0 \le Fx_0$, *then* F *has a fixed point.*

Finally, we point out that Samet *et al.* [24] and Roldán *et al.* [25] have proved that coupled and multidimensional fixed point results can be obtained as easy consequences of fixed point results in dimension one in the setup of metric spaces. Therefore, our results can be applied directly to the coupled fixed points of mixed monotone operators and multidimensional fixed points theorems [26–31].

2 Preliminaries

Before presenting our results, we collect relevant definitions and results which will be needed in the proofs of our main results.

Definition 2.1 ([16, 17]) Let (X, d) be a metric space. Then a function $p : X \times X \to [0, +\infty)$ is called a w-distance on X if the following conditions are satisfied:

(a) $p(x, z) \leq p(x, y) + p(y, z)$ for any $x, y, z \in X$;
(b) for any $x \in X$, $p(x, \cdot) \to [0, +\infty)$ is lower semicontinuous (*i.e.*, if $x \in X$ and $y_n \to y$ in X, then $p(x, y) \leq \liminf_{n\to\infty} p(x, y_n)$);
(c) for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(x, z) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \epsilon$.

Lemma 2.1 ([16]) *Let p be a w-distance on a metric space (X, d) and $\{x_n\}$ be a sequence in X such that for each $\epsilon > 0$, there exists $N(\epsilon) \in \mathbb{N}$ such that $m > n > N(\epsilon)$ implies $p(x_n, x_m) < \epsilon$. Then $\{x_n\}$ is a Cauchy sequence.*

Lemma 2.2 ([16]) *Let (X, d) be a metric space equipped with a w-distance p. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, whereas $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, +\infty)$ converging to zero. Then the following conclusions hold (for $x, y, z \in X$):*

(i) *if $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for $n \in \mathbb{N}$, then $y = z$; in particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$;*
(ii) *if $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for $n \in \mathbb{N}$, then $\lim_{n\to\infty} d(y_n, z) = 0$;*
(iii) *if $p(x_n, x_m) \leq \alpha_n$ for $n, m \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence;*
(iv) *if $p(y, x_n) \leq \alpha_n$ for $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.*

Definition 2.2 ([4]) A function $\phi : [0, +\infty) \to [0, +\infty)$ is said to be an L-function if it satisfies the following conditions:

(a) $\phi(0) = 0$;
(b) $\phi(t) > 0$ for all $t > 0$;
(c) for every $\epsilon > 0$, there exists $\delta > 0$ such that $\phi(t) \leq \epsilon$ for all $t \in [\epsilon, \epsilon + \delta]$.

Definition 2.2′ ([32, 33]) A function $\phi : [0, +\infty) \to [0, +\infty)$ is said to be a Jachymski function if it satisfies the following conditions:

(a) $\phi(0) = 0$;
(b) for every $\epsilon > 0$ there exists $\delta > 0$ such that for any $t \in [0, +\infty)$,

$$\epsilon < t < \epsilon + \delta \quad \text{implies} \quad \phi(t) \leq \epsilon.$$

Remark 2.1 It is easy to see that each L-function is a Jachymski function. In [33], the authors gave a concrete example to illustrate that the converse does not follow even in the case that $\phi(t) < t$ for all $t > 0$.

Preliminaries and notation about coincidence points can also be found in [25, 29]. Let n be a positive integer. Henceforth, X will denote a nonempty set and X^n will denote the product space $\underbrace{X \times X \times \cdots \times X}_{n}$. In the sequel, let $F : X^n \to X$ and $g : X \to X$ be two mappings.

Definition 2.3 ([25, 34]) An ordered metric space $(X, d, \preceq)$ is said to have the sequential g-monotone property if it verifies:

(i) if $\{x_m\}$ is a nondecreasing sequence and $\{x_m\} \to x$, then $gx_m \preceq gx$ for all m;
(ii) if $\{y_m\}$ is a non-increasing sequence and $\{y_m\} \to y$, then $gy_m \succeq gy$ for all m.

If g is the identity mapping, then X is said to have the sequential monotone property.

Henceforth, fix a partition $\{A, B\}$ of $\Gamma_n = \{1, 2, \ldots, n\}$, that is, $A \cup B = \Gamma_n$ and $A \cap B = \emptyset$ such that A and B are nonempty sets. We will denote

$$\Omega_{A,B} = \{\gamma : \Gamma_n \to \Gamma_n : \gamma(A) \subseteq A \text{ and } \gamma(B) \subseteq B\}$$

and

$$\Omega'_{A,B} = \{\gamma : \Gamma_n \to \Gamma_n : \gamma(A) \subseteq B \text{ and } \gamma(B) \subseteq A\}.$$

If $(X, \preceq)$ is a partially ordered space, $x, y \in X$ and $i \in \Gamma_n$, we will use the following notation:

$$x \preceq_i y \quad \Longleftrightarrow \quad \begin{cases} x \preceq y, & \text{if } i \in A, \\ x \succeq y, & \text{if } i \in B. \end{cases}$$

Consider on the product space X^n the following partial order: for $X = (x_1, x_2, \ldots, x_n)$, $Y = (y_1, y_2, \ldots, y_n) \in X^n$,

$$X \sqsubseteq Y \quad \Longleftrightarrow \quad x_i \preceq_i y_i \quad \text{for all } i \in \Gamma_n. \tag{2.1}$$

Definition 2.4 ([29]) Let $(X, \preceq)$ be a partially ordered space. We say F has the mixed g-monotone property with respect to the partition $\{A, B\}$ if F is g-monotone nondecreasing in arguments of A and g-monotone non-increasing in arguments of B, *i.e.*, for all $x_1, x_2, \ldots, x_n, y, z \in X$, and all $i \in \Gamma_n$,

$$gy \preceq gz \quad \Longrightarrow \quad F(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) \preceq_i F(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n).$$

If g is the identity mapping, then we say that F has the mixed monotone property.

Henceforth, let $\gamma_1, \gamma_2, \ldots, \gamma_n : \Gamma_n \to \Gamma_n$ be n mapping from Γ_n into itself and let Υ be the n-tuple $(\gamma_1, \gamma_2, \ldots, \gamma_n)$.

Definition 2.5 ([25]) A point $(x_1, x_2, \ldots, x_n) \in X^n$ is called a Υ-fixed point of the mapping F if

$$F(x_{\gamma_i(1)}, x_{\gamma_i(2)}, \ldots, x_{\gamma_i(n)}) = x_i \quad \text{for all } i \in \Gamma_n.$$

Definition 2.6 ([6]) Let (X, d) be a metric space. A mapping $T : X \to X$ is called sequentially convergent if $\{Ty_n\}$ is convergent implies that $\{y_n\}$ is a convergent sequence for every sequence $\{y_n\}$.

Proposition 2.1 ([25]) *If* $(x_1, x_2, \ldots, x_n) \sqsubseteq (y_1, y_2, \ldots, y_n)$, *then*

$$(x_{\gamma(1)}, x_{\gamma(2)}, \ldots, x_{\gamma(n)}) \sqsubseteq (y_{\gamma(1)}, y_{\gamma(2)}, \ldots, y_{\gamma(n)}), \quad \text{if } \gamma \in \Omega_{A,B},$$
$$(x_{\gamma(1)}, x_{\gamma(2)}, \ldots, x_{\gamma(n)}) \sqsupseteq (y_{\gamma(1)}, y_{\gamma(2)}, \ldots, y_{\gamma(n)}), \quad \text{if } \gamma \in \Omega'_{A,B}.$$

Lemma 2.3 ([25]) *Let (X,d) be a metric space and define $D_n, \triangle_n : X^n \times X^n \to [0,+\infty)$, for all $A = (a_1, a_2, \ldots, a_n), B = (b_1, b_2, \ldots, b_n) \in X^n$, by*

$$D_n(A,B) = \max_{1\le i\le n} d(a_i,b_i) \quad \text{and} \quad \triangle_n(A,B) = \sum_{i=1}^{n} d(a_i,b_i).$$

Then D_n and $\triangle_n$ are metrics on X^n.

Lemma 2.4 ([25]) *Let $(X,d,\preceq)$ be a partially ordered metric space and let $F : X^n \to X$ be a mapping. Let $\Upsilon = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ be a n-tuple of mappings from Γ_n into itself verifying $\gamma_i \in \Omega_{A,B}$ if $i \in A$ and $\gamma_i \in \Omega'_{A,B}$ if $i \in B$. Define $F_\Upsilon = X^n \to X^n$, for all $x_1, x_2, \ldots, x_n \in X$, by*

$$\begin{aligned} F_\Upsilon(x_1,x_2,\ldots,x_n) = \bigl(&F(x_{\gamma_1(1)}, x_{\gamma_1(2)}, \ldots, x_{\gamma_1(n)}), F(x_{\gamma_2(1)}, x_{\gamma_2(2)}, \ldots, x_{\gamma_2(n)}),\\ &\ldots, F(x_{\gamma_n(1)}, x_{\gamma_n(2)}, \ldots, x_{\gamma_n(n)})\bigr). \end{aligned} \tag{2.2}$$

(1) *if F has the mixed monotone property, then F_Υ is monotone nondecreasing with respect to $\sqsubseteq$ on X^n given by* (2.1);
(2) *if F is continuous, then F_Υ is also continuous;*
(3) *a point $(x_1, x_2, \ldots, x_n) \in X^n$ is a Υ-fixed point of F if, and only if, $(x_1, x_2, \ldots, x_n)$ is a fixed point of F_Υ.*

Lemma 2.5 ([35, 36]) *Let X be a nonempty set and $T : X \to X$ be a function. Then there exists a subset $E \subseteq X$ such that $T(E) = T(X)$ and $T : E \to X$ is one-to-one.*

3 Main results

Theorem 3.1 *Let (X,d) be a complete partially ordered metric space equipped with a w-distance p and $T : X \to X$ be an injective, continuous and sequentially convergent mapping. Suppose that $F : X \to X$ is a nondecreasing and continuous mapping such that*

$$\psi\bigl(p(TFx, TFy)\bigr) < \psi\bigl(p(Tx, Ty)\bigr), \quad x \neq y \tag{3.1}$$

and

$$\psi\bigl(p(TFx, TFy)\bigr) \le \phi\bigl(\psi\bigl(p(Tx, Ty)\bigr)\bigr), \quad \forall x \ge y, \tag{3.2}$$

where $\psi \in \Psi$, and $\phi : [0,+\infty) \to [0,+\infty)$ is a Jachymski function. If there exists $x_0 \in X$ such that $x_0 \le F(x_0)$, then F has a fixed point.

Proof Let $x_0 \in X$ be an arbitrary point. We construct two iterative sequences $\{x_n\}$ and $\{y_n\}$ in the following way: $y_n = Tx_n$, $x_n = Fx_{n-1}$, $n = 1, 2, \ldots$. Since F is a nondecreasing operator, we obtain by induction that

$$x_0 \le Fx_0 \le F^2x_0 \le \cdots \le F^n x_0 \le F^{n+1}x_0 \le \cdots. \tag{3.3}$$

By (3.3) and, as the elements x_{n+1} and x_n are comparable, we get

$$\psi\bigl(p(y_{n+1}, y_n)\bigr) = \psi\bigl(p(TFx_n, TFx_{n-1})\bigr) \le \phi\bigl(\psi\bigl(p(Tx_n, Tx_{n-1})\bigr)\bigr) = \phi\bigl(\psi\bigl(p(y_n, y_{n-1})\bigr)\bigr).$$

It is easy to show that $y_n \neq y_{n-1}$ for $n \in \mathbb{N}$. In fact, if there exists some $n_0 \in \mathbb{N}$ such that $y_{N_0} = y_{N_0-1}$. Then $Tx_{N_0} = Tx_{N_0-1}$. Notice that T is an injective mapping, $x_{N_0} = x_{N_0-1}$ if and only if $Fx_{N_0-1} = x_{N_0-1}$. Thus x_{N_0-1} is a fixed point of F. Therefore, the condition (3.1) of Theorem 3.1 tells us that

$$\psi\bigl(p(y_{n+1}, y_n)\bigr) < \psi\bigl(p(y_n, y_{n-1})\bigr).$$

Since ψ is nondecreasing, then the sequence $K = \{p(y_{n+1}, y_n)\}$ of real numbers is decreasing and is bounded below by 0. Hence, K converges to $r \geq 0$, the greatest lower bound of K. We assert that $r = 0$. Assume on the contrary that $r > 0$. Then there exists $\delta = \delta(r)$ and some $m \in \mathbb{N}$ such that $r < \psi(p(y_{m+1}, y_m)) < r + \delta$, we have $\psi(p(y_{m+2}, y_{m+1})) \leq \phi(\psi(p(y_{m+1}, y_m))) \leq r$, which contradicts the fact that r is the greatest lower bound of K. Thus,

$$0 \leq \psi\Bigl(\lim_{n\to\infty} p(y_{n+1}, y_n)\Bigr) \leq \lim_{n\to\infty} \psi\bigl(p(y_{n+1}, y_n)\bigr) = 0.$$

Notice that $\psi(t) = 0$ if and only if $t = 0$. Therefore, we have $r = 0$. Our idea comes from the proof of Theorem 2.7 in Suzuki [37].

Now, we will prove that for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\epsilon < p(Tx, Ty) < \epsilon + \delta \quad \text{implies} \quad p(TFx, TFy) \leq \epsilon, \quad x, y \in X. \tag{3.4}$$

Fix $\epsilon > 0$ and put $\alpha = \lim_{t\to\epsilon^+} \psi(t)$. We consider the following two cases:

(i) $\alpha < \psi(\epsilon + \beta)$ holds for every $\beta > 0$;

(ii) there exists $\delta_1 > 0$ such that $\alpha = \psi(\epsilon + \delta_1)$.

In the first case, it follows from (3.2) that there exists $\gamma > 0$ such that

$$\psi\bigl(p(Tx, Ty)\bigr) < \alpha + \gamma \quad \text{implies} \quad \psi\bigl(p(TFx, TFy)\bigr) \leq \phi\bigl(\psi\bigl(p(Tx, Ty)\bigr)\bigr) \leq \alpha.$$

We can choose $\delta_2 > 0$ satisfying $\psi(\epsilon + \delta_2) < \alpha + \gamma$. Fix $x, y \in X$ with $p(Tx, Ty) < \epsilon + \delta_2$. Then we have $\psi(p(Tx, Ty)) \leq \psi(\epsilon + \delta_2)$, and hence $\psi(p(TFx, TFy)) \leq \alpha < \psi(\epsilon + \beta)$. The monotonicity property of ψ tells us that $p(TFx, TFy) \leq \epsilon$. In the second case, we also fix $x, y \in X$ with $p(Tx, Ty) < \epsilon + \delta_1$. If $p(TFx, TFy) > \epsilon$, then we obtain

$$\alpha \leq \psi\bigl(p(TFx, TFy)\bigr) < \psi\bigl(p(Tx, Ty)\bigr) \leq \alpha.$$

This is a contradiction. Therefore we get $p(TFx, TFy) \leq \epsilon$.

In the following, we will show that $\{y_n\}$ is a Cauchy sequence. Take $\epsilon > 0$ and choose $\tilde{\delta} = \tilde{\delta}(\epsilon)$ with $\tilde{\delta} \leq \epsilon$. Since $r = 0$, there exists some positive integer $\tilde{m}$ such that

$$p(y_n, y_{n-1}) < \tilde{\delta} \quad \text{for all } n > \tilde{m}. \tag{3.5}$$

Now, let us fix $n > \tilde{m}$. To conclude that $\{y_n\}$ is a Cauchy sequence, it suffices to show that

$$p(y_{n+k}, y_n) \leq \epsilon \quad \text{for } k = 1, 2, \ldots. \tag{3.6}$$

We prove (3.6) by induction. Since $\tilde{\delta} \leq \epsilon$, the inequality (3.6) for the case $k = 1$ follows from (3.5). Suppose that (3.6) holds for some fixed $k \in \mathbb{N}$. Then by (3.5) and the assumption we have

$$p(y_{n+k}, y_{n-1}) \leq p(y_{n+k}, y_n) + p(y_n, y_{n-1}) < \tilde{\delta} + \epsilon.$$

Thus, by (3.2), we obtain $p(y_{n+k+1}, y_n) \leq \epsilon$. Hence, we deduce that $\{y_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, there exists $w \in X$ such that $d(y_n, w) \to 0$ $(n \to \infty)$. Since T is sequentially convergent, $\{x_n\}$ converges to some point in X say z. By the continuity of T, we have $Tz = w$. Hence, $d(y_n, Tz) \to 0$ $(n \to \infty)$.

Next, we show that z is a fixed point of F. For m sufficiently large, it follows from the lower semicontinuity of $p(y_m, \cdot)$ that

$$p(y_m, Tz) \leq \liminf_{n\to\infty} p(y_m, y_n) = \alpha_m \quad \text{(say)},$$

$$p(y_m, TFz) \leq \liminf_{n\to\infty} p(y_m, y_{n+1}) = \beta_m \quad \text{(say)}.$$

Note $\lim_{m\to\infty} \alpha_m = \beta_m = 0$. Therefore, we have $Tz = TFz$. Since T is injective, we get $Fz = z$. □

Remark 3.1 As shown in [13], the continuity assumption of F in Theorem 3.1 can be replaced by the following alternative condition imposed on the ambient space X:

if a nondecreasing sequence $\{x_n\}_{n=0}^{\infty} \subset X$ converges to x, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \leq x$ for all $k \in \mathbb{N}$.

Remark 3.2 In Theorem 3.1, the monotonicity of F is not essential for the existence of a fixed point. In fact, we can replace the nondecreasing property of F with the non-increasing property of F. In this case, the condition that $x_0 \leq F(x_0)$ should replaced by $x_0 \geq F(x_0)$.

Remark 3.3 From the process of the proof of Theorem 3.1, the monotonicity of ψ in (1.8) can be replaced by the continuity of its. In fact, Sastry and Babu [38] have addressed a similar problem (see Theorem 2.1 of [38]).

Theorem 3.2 *Let (X, d) be a complete partially ordered metric space equipped with a w-distance p and $T : X \to X$ be an injective, continuous, and sequentially convergent mapping. Suppose that $F : X \to X$ is a nondecreasing and continuous mapping such that*

$$\psi\bigl(p(TFx, TFy)\bigr) < \psi\bigl(M_T(x, y)\bigr), \quad x \neq y \tag{3.7}$$

and

$$\psi\bigl(p(TFx, TFy)\bigr) \leq \phi\bigl(\psi\bigl(M_T(x, y)\bigr)\bigr), \quad \forall x \geq y, \tag{3.8}$$

where

$$M_T(x, y) = \left\{p(Tx, Ty), p(Tx, TFx), p(Ty, TFy), \frac{p(Tx, TFy) + p(Ty, TFx)}{2}\right\},$$

$\psi \in \Psi$, and $\phi : [0, +\infty) \to [0, +\infty)$ is a Jachymski function. If there exists $x_0 \in X$ such that $x_0 \leq F(x_0)$, then F has a fixed point.

Proof Similar to proof of Theorem 3.1, we only need to show that $\{y_n\}$ is a Cauchy sequence. Denote $r_n = p(y_{n+1}, y_n)$ for all $n \in \mathbb{N}$. If $r_{n-1} \le r_n$ for some $n \in \mathbb{N}$, then we have

$$\begin{aligned} &M_T(x_n, x_{n-1}) \\ &\quad = \max\biggl\{p(Tx_n, Tx_{n-1}), p(Tx_n, TFx_n), p(Tx_{n-1}, TFx_{n-1}), \\ &\qquad \frac{p(Tx_n, TFx_{n-1}) + p(Tx_{n-1}, TFx_n)}{2}\biggr\} \\ &\quad = \max\biggl\{p(y_n, y_{n-1}), p(y_n, y_{n+1}), p(y_{n-1}, y_n), \frac{p(y_n, y_n) + p(y_{n-1}, y_{n+1})}{2}\biggr\} \\ &\quad \le r_n, \end{aligned}$$

where we have used that

$$\frac{p(y_{n-1}, y_{n+1}) + p(y_n, y_n)}{2} \le \max\bigl\{p(y_{n-1}, y_n), p(y_n, y_{n+1})\bigr\}.$$

On the other hand,

$$M_T(x_n, x_{n-1}) \ge r_n.$$

Hence

$$M_T(x_n, x_{n-1}) = r_n.$$

It follows from (3.8) that

$$\psi(r_n) = \psi\bigl(p(y_{n+1}, y_n)\bigr) = \psi\bigl(p(TFx_n, TFx_{n-1})\bigr) \le \phi\bigl(\psi\bigl(M_T(x_n, x_{n-1})\bigr)\bigr) < \psi(r_n),$$

a contradiction. Therefore $r_{n-1} > r_n$ for all $n \in \mathbb{N}$. Thus $\{r_n\}$ is a monotone decreasing sequence of positive real numbers and there exists $r \ge 0$ such that $\lim_{n\to\infty} r_n = r$. Similarly to Theorem 3.1, we obtain $r = 0$. By (3.4), we see that, for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$M_T(x, y) \le \epsilon + \delta \quad \text{implies} \quad p(TFx, TFy) \le \epsilon.$$

Since $r = 0$, there exist $\delta \ge \delta_1 > 0$ and $N \in \mathbb{N}$ such that $r_n < \delta_1/4$ for $n \ge N$.

Now, we show that $\{y_n\}$ is a Cauchy sequence. If otherwise, there exist $\tilde{\epsilon} > 0$ and $\delta \ge \delta_2 \ge \delta_1$ for which we can find two sequences of positive integers $\{n(k)\}$ and $\{m(k)\}$ such that $n(k) > m(k) \ge N + 1$, $p(y_{m(k)}, y_{n(k)}) \ge \tilde{\epsilon} + \delta_2/2$, and $p(y_{m(k)}, y_{n(k)-1}) < \tilde{\epsilon} + \delta_2/2$. Thus

$$\tilde{\epsilon} + \delta_2/2 \le p(y_{m(k)}, y_{n(k)}) \le p(y_{m(k)}, y_{n(k)-1}) + p(y_{n(k)-1}, y_{n(k)}) < \tilde{\epsilon} + \delta_2/2 + \delta_1/4 < \tilde{\epsilon} + \delta. \quad (3.9)$$

Again,

$$p(y_{m(k)}, y_{n(k)+1}) \le p(y_{m(k)}, y_{n(k)}) + p(y_{n(k)}, y_{n(k)+1}) < \tilde{\epsilon} + \delta_2/2 + \delta_1/4 + \delta_1/4 \le \tilde{\epsilon} + \delta. \quad (3.10)$$

Similarly,

$$p(y_{n(k)}, y_{m(k)+1}) < \tilde{\epsilon} + \delta. \tag{3.11}$$

By (3.9)-(3.11), we have

$$M_T(x_{n(k)}, x_{m(k)}) < \tilde{\epsilon} + \delta.$$

Hence

$$p(y_{n(k)+1}, y_{m(k)+1}) \leq \tilde{\epsilon}.$$

On the other hand, it follows from (3.9)-(3.11) that

$$\begin{aligned}
&p(y_{n(k)+1}, y_{m(k)+1}) \\
&\quad \geq p(y_{m(k)}, y_{n(k)}) - p(y_{m(k)}, y_{m(k)+1}) - p(y_{n(k)+1}, y_{n(k)}) \\
&\quad > \tilde{\epsilon} + \delta_2/2 - \delta_1/4 - \delta_1/4 \geq \tilde{\epsilon}.
\end{aligned}$$

This is a contradiction. Therefore $\{y_n\}$ is a Cauchy sequence. □

It is therefore our interest now to provide additional conditions to ensure that the fixed point in Theorems 3.1 and 3.2 is in fact unique. Such a condition has been used in many results [12, 13] and says:

$$\text{For } x, y \in X, \text{there exists } z \in X \text{ which is comparable to } x \text{ and } y. \tag{3.12}$$

Theorem 3.3 *In addition to the hypotheses of Theorems* 3.1 *and* 3.2, *suppose that condition* (3.12) *holds. Then F has a unique fixed point.*

The proof is trivial, here we omit the details. The readers are referred to the proof of Theorem 2.7 in [13].

Consider the following conditions:

- (i) (X, d) is complete and p is a w-distance on X;
- (ii) the mapping $F : X^n \to X$ has the mixed monotone property;
- (ii)′ let $F : X^n \to X$ and $g : X \to X$ be two mappings such that F has the mixed g-monotone property;
- (iii) F is continuous or $(X, d, \preceq)$ has the sequential monotone property;
- (iii)′ F is continuous or $(X, d, \preceq)$ has the sequential g-monotone property;
- (iv) there exist $x_0^1, x_0^2, \ldots, x_0^n \in X$ verifying

$$x_0^i \preceq_i F\big(x_0^{\gamma_i(1)}, x_0^{\gamma_i(2)}, \ldots, x_0^{\gamma_i(n)}\big) \quad \text{for all } i \in \Gamma_n;$$

- (iv)′ there exist $x_0^1, x_0^2, \ldots, x_0^n \in X$ verifying

$$gx_0^i \preceq_i F\big(x_0^{\gamma_i(1)}, x_0^{\gamma_i(2)}, \ldots, x_0^{\gamma_i(n)}\big) \quad \text{for all } i \in \Gamma_n;$$

- (v) let $T : X \to X$ be an injective, continuous and sequentially convergent mapping.

Corollary 3.1 *Under hypotheses* (i)-(v). *Assume that the following contraction condition is satisfied*:

$$\psi\left(\frac{\sum_{i=1}^{n} p(TF(x_{\gamma_i(1)}, x_{\gamma_i(2)}, \ldots, x_{\gamma_i(n)}), TF(y_{\gamma_i(1)}, y_{\gamma_i(2)}, \ldots, y_{\gamma_i(n)}))}{n}\right) \\ \leq \phi\left(\frac{\sum_{i=1}^{n} p(Tx_i, Ty_i)}{n}\right) \tag{3.13}$$

for which $x_i \preceq_i y_i$ for all $i \in \Gamma_i$, where $\psi \in \Psi$, and $\phi : [0, +\infty) \to [0, +\infty)$ is a Jachymski function with the condition: $\phi(t) < \psi(t)$ for all $t > 0$. Moreover, if for all $A, B \in X^n$ there exists $U \in X^n$ such that $A \sqsubseteq U$ and $B \sqsubseteq U$. Then F has a unique Υ-fixed point.

Proof Consider the functional $\triangle_n : X^n \times X^n \to [0, +\infty)$ defined by

$$\triangle_n\big((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)\big) = \frac{1}{n}\sum_{i=1}^{n} d(Tx_i, Ty_i)$$

and

$$\widetilde{\triangle}_n\big((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)\big) = \frac{1}{n}\sum_{i=1}^{n} p(Tx_i, Ty_i)$$

for all $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X^n$. Combining Lemmas 2.3 and 2.4, we see that $\triangle_n$ is a metric on X^n and $\widetilde{\triangle}_n$ is a w-distance on X^n. Moreover, if (X, d) is complete, then $(X^n, \triangle_n)$ is a complete metric space, too. Now, consider the operator $F_\Upsilon : X^n \to X^n$ defined by (2.2). Clearly, for $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X^n$, in view of Proposition 2.1, we have

$$\widetilde{\triangle}_n\big(F_\Upsilon(x_1, x_2, \ldots, x_n), F_\Upsilon(y_1, y_2, \ldots, y_n)\big) \\ = \frac{\sum_{i=1}^{n} p(TF(x_{\gamma_i(1)}, x_{\gamma_i(2)}, \ldots, x_{\gamma_i(n)}), TF(y_{\gamma_i(1)}, y_{\gamma_i(2)}, \ldots, y_{\gamma_i(n)}))}{n}.$$

Thus, (3.13) implies

$$\psi\big(\widetilde{\triangle}_n\big(F_\Upsilon(x_1, x_2, \ldots, x_n), F_\Upsilon(y_1, y_2, \ldots, y_n)\big)\big) \leq \phi\big(\widetilde{\triangle}_n\big((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)\big)\big)$$

with $(x_1, x_2, \ldots, x_n) \sqsubseteq (y_1, y_2, \ldots, y_n)$.

Since F has the mixed monotone property, then F_Υ is a nondecreasing mapping with respect to $\sqsubseteq$. From the condition (iv), we have

$$\big(x_0^1, x_0^2, \ldots, x_0^n\big) \sqsubseteq F_\Upsilon\big(x_0^1, x_0^2, \ldots, x_0^n\big).$$

Now, if F is continuous, then F_Υ is continuous. Applying Theorem 3.1, we see that F_Υ has a fixed point, which implies from Lemma 2.4 that F has a Υ-fixed point. In addition, we obtain the uniqueness of the fixed point of F_Υ from Theorem 3.3, which implies the uniqueness of fixed point of F. □

Corollary 3.2 *Under hypotheses* (i), (ii)′-(iv)′, (v). *Assume that the following contraction condition is satisfied*:

$$\psi\big(p\big(TF(x_1, x_2, \ldots, x_n), TF(y_1, y_2, \ldots, y_n)\big)\big) \leq \phi\Big(\max_{1 \leq i \leq n} p(Tgx_i, Tgy_i)\Big) \tag{3.14}$$

for which $gx_i \preceq_i gy_i$ for all $i \in \Gamma_i$, where $\psi \in \Psi$, and $\phi : [0,+\infty) \to [0,+\infty)$ is a Jachymski function with the condition: $\phi(t) < \psi(t)$ for all $t > 0$. In addition, $F(X^n) \subseteq g(X)$, $g(X)$ is complete and g is continuous. Then F and g have at least one Υ-coincidence point.

Proof By Lemma 2.5, there exists $E \subseteq X$ such that $g(E) = g(X)$ and $g : E \to X$ is one-to-one. Define a map $G : g^n(E) \to g(E)$ by $G(gx_1, gx_2, \ldots, gx_n) = F(x_1, x_2, \ldots, x_n)$ and $G(gy_1, gy_2, \ldots, gy_n) = F(y_1, y_2, \ldots, y_n)$. Since g is one-to-one on $g(E)$, G is well defined. Consider the product space $Y = X^n$ provided with the metric D_n (as in Lemma 2.3) and the partial order $\sqsubseteq$ on Y given by (2.1). Then $(Y, D_n, \sqsubseteq)$ is a complete ordered metric space.

Since G has the mixed monotone property, it follows from Lemma 2.4(1) that $G_\Upsilon : Y \to Y$ is nondecreasing with respect to $\sqsubseteq$. The continuity of G tells us that G_Υ is also continuous. If $\mathbb{x}_0 = (gx_0^1, gx_0^2, \ldots, gx_0^n) \in Y$, then condition (iv)$'$ is equivalent to $\mathbb{x}_0 \sqsubseteq G_\Upsilon \mathbb{x}_0$. Define $\widetilde{D}_n(A, B) = \max_{1\le i\le n} p(a_i, b_i)$. Then $\widetilde{D}_n$ is a w-distance on X^n. For given $\mathbb{X} = (gx_1, gx_2, \ldots, gx_n)$, and $\mathbb{Y} = (gy_1, gy_2, \ldots, gy_n) \in Y$ such that $\mathbb{X} \sqsubseteq \mathbb{Y}$, we can derive that the points $(x_{\gamma_i(1)}, x_{\gamma_i(2)}, \ldots, x_{\gamma_i(n)})$ and $(y_{\gamma_i(1)}, y_{\gamma_i(2)}, \ldots, y_{\gamma_i(n)})$ are comparable by $\sqsubseteq$ from Proposition 2.1, and

$$\widetilde{D}_n(TG_\Upsilon \mathbb{X}, TG_\Upsilon \mathbb{Y}) = \max_{1\le i\le n} p\bigl(TF(x_{\gamma_i(1)}, x_{\gamma_i(2)}, \ldots, x_{\gamma_i(n)}), TF(y_{\gamma_i(1)}, y_{\gamma_i(2)}, \ldots, y_{\gamma_i(n)})\bigr).$$

Thus

$$\psi\bigl(\widetilde{D}_n(TG_\Upsilon \mathbb{X}, TG_\Upsilon \mathbb{Y})\bigr) \le \phi\bigl(\widetilde{D}_n(T\mathbb{X}, T\mathbb{Y})\bigr)$$

for all $gx_i, gy_j \in g(E)$ $(i, j = 1, 2, \ldots, n)$. Since $g(E) = g(X)$ is complete, it follows from Theorem 3.1 that F and g have at least one Υ-coincidence point. □

Remark 3.4 Compared with Theorem 14 in [27], the condition that O-compatibility between F and g is removed and replaced by the completeness of $g(X)$. Moreover, we adopt generalized altering functions and a Jachymski function.

4 Some examples

Firstly, we show how to take appropriate operator T by Examples 4.1 and 4.2. Our partial idea comes from [20, 39].

Example 4.1 Let $X = \{(1,0), (0,1), (1,3), (1,2{,}015)\}$ with the metric $d((x_1,x_2),(y_1,y_2)) = |x_1 - y_1| + |x_2 - y_2|$. Define $F(x,y) : X \to X$ by

$$F(x,y) = \begin{cases} (1,0), & \text{if } (x,y) = (1,0), \\ (1,0), & \text{if } (x,y) = (0,1), \\ (1,2015), & \text{if } (x,y) = (1,3), \\ (1,0), & \text{if } (x,y) = (1,2{,}015) \end{cases}$$

and

$$p\bigl((x_1,x_2),(y_1,y_2)\bigr) = \max\bigl\{d\bigl(F(x_1,x_2),(y_1,y_2)\bigr), d\bigl(F(x_1,x_2),F(y_1,y_2)\bigr)\bigr\}.$$

On the set X, we consider the following relation:

$$\begin{aligned} &\text{for } (x_1,x_2),(y_1,y_2) \in X, \quad (x_1,x_2) \preceq (y_1,y_2) \\ &\quad \Longleftrightarrow \quad (x_1,x_2) = (y_1,y_2) \quad \text{or} \quad \bigl((x_1,x_2),(y_1,y_2) \in \bigl\{(1,0),(0,1),(1,3)\bigr\} \quad \text{and} \\ &\qquad\qquad (x_1,x_2) \leq_2 (y_1,y_2), \end{aligned}$$

where $(x_1,x_2) \leq_2 (y_1,y_2) \Longleftrightarrow x_1 \leq y_1$, $x_2 \leq y_2$, '$\leq$' is the usual order).

It is clear that $(X,\preceq)$ is a complete partially ordered metric space and p is a w-distance on (X,d). Obviously, F is a nondecreasing map on the partial order $\preceq$. We claim that the condition (1.9) is not true for every $\psi \in \overline{\Psi}$, $\phi \in \overline{\Phi}$. Indeed, for $(x_1,x_2) = (y_1,y_2) = (1,3)$, we have

$$\begin{aligned} \psi(2{,}015) &= \psi\bigl(p\bigl(F(1,3),F(1,3)\bigr)\bigr) \leq \phi\psi\bigl(p\bigl((1,3),(1,3)\bigr)\bigr) \\ &< \psi\bigl(p\bigl((1,3),(1,3)\bigr)\bigr) = \psi(2{,}012), \end{aligned}$$

which is a contradiction.

If we define the mapping $T : X \to X$ as follows: $T(1,0) = (1,0)$, $T(1,2{,}015) = (0,1)$, $T(0,1) = (1,2{,}015)$, $T(1,3) = (1,3)$, then $T : X \to X$ be an injective, continuous and sequentially convergent mapping.

Now, we show that F and T satisfy the condition (3.2). In fact, choose $\psi(x) = x$, $\phi(x) = \ln(1+x)$. We have the following cases:

Case 1. $(x_1,x_2) = (y_1,y_2) = (1,0)$, we have

$$0 = p\bigl(TF(1,0),TF(1,0)\bigr) \leq \phi\bigl(p\bigl(T(1,0),T(1,0)\bigr)\bigr) = 0.$$

Case 2. $(x_1,x_2) = (y_1,y_2) = (0,1)$, we have

$$0 = p\bigl(TF(0,1),TF(0,1)\bigr) \leq \phi\bigl(p\bigl(T(0,1),T(0,1)\bigr)\bigr) = \ln(1+2{,}015).$$

Case 3. $(x_1,x_2) = (y_1,y_2) = (1,3)$, we have

$$2 = p\bigl(TF(1,3),TF(1,3)\bigr) \leq \phi\bigl(p\bigl(T(1,3),T(1,3)\bigr)\bigr) = \phi(2{,}012) = \ln(1+2{,}012).$$

Case 4. $(x_1,x_2) = (y_1,y_2) = (1,2{,}015)$, we have

$$0 = p\bigl(TF(1,2{,}015),TF(1,2{,}015)\bigr) \leq \phi\bigl(p\bigl(T(1,2{,}015),T(1,2{,}015)\bigr)\bigr) = \phi(2) = \ln(1+2).$$

Case 5. $(x_1,x_2) = (1,3)$, $(y_1,y_2) = (1,0)$, we have

$$2 = p\bigl(TF(1,3),TF(1,0)\bigr) \leq \phi\bigl(p\bigl(T(1,3),T(1,0)\bigr)\bigr) = \phi(2{,}015) = \ln(1+2{,}015).$$

Case 6. $(x_1,x_2) = (1,3)$, $(y_1,y_2) = (0,1)$, we have

$$2 = p\bigl(TF(1,3),TF(0,1)\bigr) \leq \phi\bigl(p\bigl(T(1,3),T(0,1)\bigr)\bigr) = \phi(2{,}015) = \ln(1+2{,}015).$$

Consequently, for all $(x_1,x_2),(y_1,y_2) \in X$ with $(x_1,x_2) \succeq (y_1,y_2)$, we obtain

$$p\bigl(TF(x_1,x_2),TF(y_1,y_2)\bigr) \le \phi\bigl(p\bigl(T(x_1,x_2),T(y_1,y_2)\bigr)\bigr).$$

Example 4.2 Let $X = [1,+\infty)$, $p = d(x,y) = |x-y|$ and $Fx = \frac{2\sqrt{6}}{\sqrt{3x}}$, $x \in X$. Then F satisfies condition (3.2) but does not satisfy (1.7).

In fact, set

$$\frac{2\sqrt{6}}{\sqrt{3}(\sqrt{y}+\sqrt{x})\sqrt{xy}} > 1, \quad x,y \in X. \tag{4.1}$$

Assume that there exist $\psi \in \Psi$ and a right-upper semicontinuous function ϕ such that (1.7) holds. This means that

$$\psi\bigl(d(Fx,Fy)\bigr) = \psi\left(\frac{2\sqrt{6}|y-x|}{\sqrt{3}(\sqrt{y}+\sqrt{x})\sqrt{xy}}\right) \le \phi\bigl(|x-y|\bigr) < \psi\bigl(|x-y|\bigr).$$

Since ψ is monotone nondecreasing, we have $\frac{2\sqrt{6}}{\sqrt{3}(\sqrt{y}+\sqrt{x})\sqrt{xy}} \le 1$, which contradicts (4.1).

Now, we prove that (3.2) holds. Indeed, consider the map $T : [1,+\infty) \to [1,+\infty)$ defined by $Tx = \ln x + 1$. For every $\epsilon > 0$, if we choose $\delta = \epsilon/5$ and $\epsilon < d(Tx,Ty) = |\ln x - \ln y| < \epsilon + \delta = 6\epsilon/5$. Then

$$d(TFx,TFy) = \left|\ln\frac{2\sqrt{6}}{\sqrt{3x}} - \ln\frac{2\sqrt{6}}{\sqrt{3y}}\right| = \frac{1}{2}|\ln x - \ln y| < \frac{3\epsilon}{5} < \epsilon.$$

Secondly, Example 4.3 illustrates that the partial order on the underlying metric space how to play necessary role in Theorem 3.1.

Example 4.3 Let $X = [0,1]$ with Euclidean distance $d(x,y) = |x-y|$ for all $x,y \in X$ and $p(x,y) = 3d(x,y)$ be a w-distance on (X,d). We consider the order $\prec$ in X given by

$$x = y \quad \text{or} \quad \left[x,y \in \left\{0,1,\frac{1}{2},\frac{1}{3},\dots\right\} \text{ with } x \le y\right],$$

where $\le$ is usual order.

Consider the operator $F : X \to X$ defined by

$$F(x) = \begin{cases} 0, & x = 0, \\ \frac{1}{\sqrt{2}}x^2, & x = \frac{1}{n}, n = 1,2,\dots, \\ \frac{1}{3\sqrt{2}}, & \text{otherwise.} \end{cases}$$

Then F satisfies condition (3.2) but does not satisfy condition (1.3).

Indeed, for every $\epsilon > 0$, assume that there exists $\delta > 0$ such that $\epsilon \le p(T0, T\frac{1}{3\sqrt{2}}) < \epsilon + \delta$ implies that $p(TF0, TF\frac{1}{3\sqrt{2}}) = p(T0, T\frac{1}{3\sqrt{2}}) < \epsilon$, a contradiction. Hence F does not satisfy (1.3).

Furthermore, F does not also satisfy condition (1.9). In fact, take $x = 1$, $y = \frac{1}{2}$, we have

$$\psi\left(\frac{3}{\sqrt{2}}\left|1 - \frac{1}{2^2}\right|\right) \le \phi\left(\psi\left(3\left|1-\frac{1}{2}\right|\right)\right) < \psi\left(\frac{3}{2}\right),$$

a contradiction.

On the other hand, we choose $Tx = x^2$, $x \in X$, $\psi(x) = \frac{x}{2}$, $\phi(x) = kx$, $k \in (0,1)$. It follows that

$$\begin{aligned}\psi\bigl(p(TFx,TFy)\bigr) &= \frac{3}{4}\bigl|x^4 - y^4\bigr| \le \frac{3k}{2}\bigl|x^2 - y^2\bigr| \\ &= \phi\bigl(\psi\bigl(p(Tx,Ty)\bigr)\bigr), \quad x,y \in X \text{ and } x \ge y.\end{aligned}$$

Theorem 3.1 gives us the existence of a fixed point of F.

Next, superiority of L-function is embodied in Example 4.4.

Example 4.4 Let $X = \{1 - \frac{1}{n}, n = 4,5,\ldots\} \cup \{0,1\} \cup \{2n, n = 1,2,\ldots\}$, where $(X,d,\preceq)$ is a complete partially ordered metric space with a metric d and usual order $\preceq$. We define $p: X \times X \to [0,+\infty)$ by

$$p(x,y) = \begin{cases} \max\{x,y\}, & \text{if } x,y \in \{1-\frac{1}{n}, n = 4,5,\ldots\} \cup \{0,1\}, x \neq y, \\ x + y, & \text{if at least one of } x \text{ or } y \notin \{1-\frac{1}{n}, n = 4,5,\ldots\} \cup \{0,1\}, x \neq y, \\ 1, & \text{if } x = y, \end{cases}$$

and set

$$F(x) = \begin{cases} 0, & x = 0, \\ 1 - \frac{1}{n}, & x = 1 - \frac{1}{2n}, n = 2,3,\ldots, \\ \frac{3}{4}, & x = 1, \\ 1 - \frac{1}{n}, & x = 1 - \frac{1}{2n+1}, n = 2,3,\ldots, \\ 1, & x = 2n, n = 1,2,\ldots. \end{cases}$$

It is easy to see that conditions (1.6) and (1.9) are not satisfied. In fact, assume that there exist $\varphi, \psi \in \widetilde{\Psi}$ such that

$$\varphi p(Fx,Fy) \le \varphi p(x,y) - \psi p(x,y).$$

Take $x = 1 - \frac{1}{2n}$, $y = 1$. We have

$$\varphi\left(1 - \frac{1}{n}\right) \le \varphi(1) - \psi(1), \quad n = 2,3,\ldots.$$

Thus $\psi(1) = 0$, which contradicts the definition of $\widetilde{\Psi}$.

Furthermore, if there exist $\phi \in \overline{\Psi}$ and $\psi \in \overline{\Phi}$ such that

$$\phi p(Fx,Fy) \le \psi\phi\bigl(p(x,y)\bigr).$$

Again take $x = 1 - \frac{1}{2n}$, $y = 1$. We obtain

$$\phi\left(1 - \frac{1}{n}\right) \le \psi\phi(1).$$

Thus $0 < \phi(1) \le \psi\phi(1)$, which contradicts the fact that $\psi(t) < t$ for all $t > 0$.

Now, to verify (3.2), choose $\psi(x) = x$ and

$$\phi(x) = \begin{cases} x^2, & x \in [0,1] \cap X, \\ \frac{2}{3}x + \frac{1}{3}, & x \in [1,+\infty) \cap X. \end{cases}$$

Without loss of generality, we assume that $x > y$ and discuss the following cases:

Case 1. Let $x, y \in \{1 - \frac{1}{2n}, n = 2,3,\ldots\}$ ($\{1 - \frac{1}{2n+1}, n = 2,3,\ldots\}$). We take $x = 1 - \frac{1}{2(n+1)}$ ($1 - \frac{1}{2(n+1)+1}$), $y = 1 - \frac{1}{2n}$ ($1 - \frac{1}{2n+1}$), $n = 2,3,\ldots$. Then

$$\begin{aligned} &\psi p\left(F\left(1 - \frac{1}{2(n+1)}\right), F\left(1 - \frac{1}{2n}\right)\right) \\ &\quad = 1 - \frac{1}{n+1} \\ &\quad \leq \left(1 - \frac{1}{2(n+1)}\right)^2 \\ &\quad = \phi\psi p\left(1 - \frac{1}{2(n+1)}, 1 - \frac{1}{2n}\right), \quad n = 2,3,\ldots \end{aligned}$$

or

$$\begin{aligned} &p\left(F\left(1 - \frac{1}{2(n+1)+1}\right), F\left(1 - \frac{1}{2n+1}\right)\right) \\ &\quad = 1 - \frac{1}{n+1} \\ &\quad \leq \left(1 - \frac{1}{2(n+1)+1}\right)^2 \\ &\quad = \phi p\left(1 - \frac{1}{2(n+1)+1}, 1 - \frac{1}{2n+1}\right), \quad n = 2,3,\ldots. \end{aligned}$$

Case 2. Let $x = 1$ and $y \in \{1 - \frac{1}{2n}, n = 2,3,\ldots\}$ ($\{1 - \frac{1}{2n+1}, n = 2,3,\ldots\}$). It follows that

$$\psi p\left(F1, F\left(1 - \frac{1}{2n}\right)\right) = 1 - \frac{1}{n} < 1 = \phi\psi p\left(1, 1 - \frac{1}{2n}\right)$$

or

$$p\left(F1, F\left(1 - \frac{1}{2n+1}\right)\right) = 1 - \frac{1}{n} < 1 = \phi p\left(1, 1 - \frac{1}{2n+1}\right).$$

Case 3. Let $y = 0$. It follows that

$$\begin{aligned} &p\left(F\left(1 - \frac{1}{2n}\right), F0\right) = 1 - \frac{1}{n} < \left(1 - \frac{1}{2n}\right)^2 = \phi p\left(1 - \frac{1}{2n}, 0\right) \quad \text{or} \\ &p(F1, F0) = \frac{3}{4} < 1 = \phi p(1,0), \quad \text{or} \\ &p\left(F\left(1 - \frac{1}{2n+1}\right), F0\right) = 1 - \frac{1}{n} < \left(1 - \frac{1}{2n+1}\right)^2 = p\left(1 - \frac{1}{2n+1}, 0\right). \end{aligned}$$

Case 4. Let $x \in \{1 - \frac{1}{2n+1}, n = 2,3,\ldots\}$ ($\{1 - \frac{1}{2n}, n = 2,3,\ldots\}$) and $y \in \{1 - \frac{1}{2n}, n = 2,3,\ldots\}$ ($\{1 - \frac{1}{2n+1}, n = 2,3,\ldots\}$). It follows that

$$\begin{aligned} &p\left(F\left(1 - \frac{1}{2n+1}\right), F\left(1 - \frac{1}{2n}\right)\right) \\ &\quad = 1 - \frac{1}{n} < \left(1 - \frac{1}{2n+1}\right)^2 = \phi p\left(1 - \frac{1}{2n+1}, 1 - \frac{1}{2n}\right) \end{aligned}$$

or

$$\begin{aligned} &p\left(F\left(1 - \frac{1}{2(n+1)}\right), F\left(1 - \frac{1}{2n+1}\right)\right) \\ &\quad = 1 - \frac{1}{n+1} < \left(1 - \frac{1}{2(n+1)}\right)^2 = \phi p\left(1 - \frac{1}{2(n+1)}, 1 - \frac{1}{2n+1}\right). \end{aligned}$$

Case 5. Let $x \in \{2,4,6,\ldots\}$. We choose $x = 2n$, $n = 1,2,\ldots$. It follows that

$$\begin{aligned} &p(Fx, F0) = 1 < \frac{4n}{3} + \frac{1}{3} = \phi(2n) = \phi\bigl(p(x,0)\bigr), \\ &p\left(Fx, F\left(1 - \frac{1}{2n}\right)\right) \\ &\quad = 2 - \frac{1}{n} < \frac{2}{3}\left(1 + 2n - \frac{1}{2n}\right) + \frac{1}{3} = \phi\left(1 + 2n - \frac{1}{2n}\right) = \phi\left(p\left(x, 1 - \frac{1}{2n}\right)\right), \\ &p(Fx, F1) = \frac{7}{4} < \frac{2}{3}(1 + 2n) + \frac{1}{3} = \phi(1 + 2n) = \phi\bigl(p(x,1)\bigr), \\ &p\left(Fx, F\left(1 - \frac{1}{2n+1}\right)\right) = 2 - \frac{1}{n} < \frac{2}{3}\left(1 + 2n - \frac{1}{2n+1}\right) + \frac{1}{3} = \phi\left(p\left(x, 1 - \frac{1}{2n+1}\right)\right), \\ &2 = p\bigl(Fx, F2(n-1)\bigr) < \frac{2}{3}(4n - 2) + \frac{1}{3} = \phi\bigl(2n + 2(n-1)\bigr), \quad n = 2,3,\ldots. \end{aligned}$$

Consequently,

$$p(Fx, Fy) \le \phi\bigl(p(x,y)\bigr), \quad x \in \{2,4,6,\ldots\}, x \ge y.$$

Considering all the above cases, F has a unique fixed point by an application of Theorem 3.1.

In Example 4.5, $\psi(x) = x$ does not meet the requirement and hence we take a different function to alter the distances.

Example 4.5 Let $X = [0,1] \cup \{2,3,\ldots\}$, where $(X, d, \le)$ is a complete partially ordered metric space with a metric d and the usual order $\le$. We define $p : X \times X \to [0, +\infty)$ by $p(x,y) = \max\{x,y\}$. Set

$$F(x) = \begin{cases} \frac{x}{3}, & x \in [0,1], \\ x^2, & x \in \{2,3,4,\ldots\}. \end{cases}$$

We note that (3.2) fails to hold when $\psi(t) = t$. In fact, for $x \in \{2,3,4,\ldots\}$, we see that

$$p(Fx, F1) = Fx \le \phi p(x,1) = \phi(x) \le x$$

does not hold. Choose

$$\psi(x) = \begin{cases} x, & x \in [0,1], \\ \frac{1}{x}, & x > 1 \end{cases}$$

and

$$\phi(x) = \frac{2}{3}x, \quad x \in X.$$

Without loss of generality, we put $x > y$, and consider the following cases:

Case 1. $x \in [0,1]$. Then

$$\psi\bigl(p(Fx,Fy)\bigr) = Fx = \frac{x}{3} < \frac{2x}{3} = \phi\psi\bigl(p(x,y)\bigr).$$

Case 2. $x \in \{2,3,4,\ldots\}$. It follows that

$$\psi p(Fx,Fy) = \psi Fx = \frac{1}{x^2} < \frac{2}{3x} = \phi\psi\bigl(p(x,y)\bigr).$$

That is (3.2) holds. Thus, Theorem 3.1 implies that F has a unique fixed point $0 \in X$.

In the following, we apply the results of Section 3 to study the existence and uniqueness of positive solution for a nonlinear integral equation.

In order to compare our results to the ones in [26, 40], we shall consider the same integral equation, that is,

$$x(t) = \int_a^b \bigl(K_1(t,s) + K_2(t,s)\bigr)\bigl(f\bigl(s,x(s)\bigr) + g\bigl(s,x(s)\bigr)\bigr)\,ds + h(t), \quad t \in I = [a,b]. \tag{4.2}$$

Let Ψ denote the set of all functions $\psi : [0,+\infty) \to [0,+\infty)$ satisfying

(i) ψ is nondecreasing;

(ii) there exists a Jachymski function $\phi : [0,+\infty) \to [0,+\infty)$ with $\phi(r) < r$ for all $r > 0$ such that $\psi(r) = \phi(r/2)$.

We will analyze (4.2) under the following assumptions:

(i) $K_1, K_2 \in C(I \times I, \mathbb{R})$, $K_1(t,s) \ge 0$, and $K_2(t,s) \le 0$;

(ii) $h(t) \in C(I, \mathbb{R})$;

(iii) there exist positive numbers μ, ν such that for all $x, y \in \mathbb{R}$, with $x \ge y$, the following Lipschitzian type conditions hold:

$$0 \le f(t,x) - f(t,y) \le \mu\psi(x-y)$$

and

$$-\nu\psi(x-y) \le g(t,x) - g(t,y) \le 0;$$

(iv) there exist $p > 1$ and $q > 0$ with $1/p + 1/q = 1$ such that

$$(\mu + \nu)\sup_{t\in I}\left(\int_a^b \bigl(K_1(t,s) - K_2(t,s)\bigr)^p\,ds\right)^{1/p}(b-a)^{1/q} \le 1;$$

(v) a pair $(\alpha, \beta) \in X^2$ with $X = C(I, \mathbb{R})$ is called a coupled lower-upper solution of (4.2) if, for all $t \in I$,

$$\begin{aligned} \alpha(t) \leq & \int_a^b K_1(t,s)\bigl[f\bigl(s,\alpha(s)\bigr) + g\bigl(s,\beta(s)\bigr)\bigr]\,ds \\ & + \int_a^b K_2(t,s)\bigl[f\bigl(s,\beta(s)\bigr) + g\bigl(s,\alpha(s)\bigr)\bigr]\,ds + h(t) \end{aligned}$$

and

$$\begin{aligned} \beta(t) \geq & \int_a^b K_1(t,s)\bigl[f\bigl(s,\beta(s)\bigr) + g\bigl(s,\alpha(s)\bigr)\bigr]\,ds \\ & + \int_a^b K_2(t,s)\bigl[f\bigl(s,\alpha(s)\bigr) + g\bigl(s,\beta(s)\bigr)\bigr]\,ds + h(t). \end{aligned}$$

Theorem 4.1 *Under assumptions* (i)-(v), (4.2) *has a unique solution in* $C(I, \mathbb{R})$.

Proof Consider on $X = C(I, \mathbb{R})$ the natural partial order relation, that is, for $x, y \in X$,

$$x \leq y \quad \Longleftrightarrow \quad x(t) \leq y(t), \quad \forall t \in I.$$

It is well known that X is a complete metric space with respect to the sup metric

$$d(x,y) = \sup_{t \in I} \bigl|x(t) - y(t)\bigr|, \quad x, y \in C(I, \mathbb{R}).$$

It is obviously that for any $(x, y) \in X^2$, the functions $\max\{x, y\}$, $\min\{x, y\}$ are the upper and lower bounded of x, y, respectively. Therefore, for every $(x, y), (u, v) \in X^2$, there exists the element $(\max\{x, y\}, \min\{x, y\})$ which is comparable to (x, y) and (u, v).

Now, define the mapping $F : X \times X \to X$ by

$$\begin{aligned} F(x,y)(t) = & \int_a^b K_1(t,s)\bigl[f\bigl(s,x(s)\bigr) + g\bigl(s,y(s)\bigr)\bigr]\,ds \\ & + \int_a^b K_2(t,s)\bigl[f\bigl(s,y(s)\bigr) + g\bigl(s,x(s)\bigr)\bigr]\,ds + h(t) \quad \text{for all } t \in I. \end{aligned}$$

It is not difficult to prove that F has the mixed monotone property. Now, for $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$, we have

$$\begin{aligned} d\bigl(F(x,y), F(u,v)\bigr) &= \sup_{t \in I} \bigl|F(x,y)(t) - F(u,v)(t)\bigr| \\ &= \sup_{t \in I} \biggl| \int_a^b K_1(t,s)\bigl[f\bigl(s,x(s)\bigr) - f\bigl(s,u(s)\bigr) + g\bigl(s,y(s)\bigr) - g\bigl(s,v(s)\bigr)\bigr]\,ds \\ &\quad - \int_a^b K_2(t,s)\bigl[f\bigl(s,v(s)\bigr) - f\bigl(s,y(s)\bigr) + g\bigl(s,u(s)\bigr) - g\bigl(s,x(s)\bigr)\bigr]\,ds \biggr| \\ &\leq \sup_{t \in I} \biggl| \int_a^b K_1(t,s)\bigl[\mu\psi\bigl(x(s) - u(s)\bigr) + \nu\psi\bigl(v(s) - y(s)\bigr)\bigr]\,ds \\ &\quad - \int_a^b K_2(t,s)\bigl[\mu\psi\bigl(v(s) - y(s)\bigr) + \nu\psi\bigl(x(s) - u(s)\bigr)\bigr]\,ds \biggr|. \end{aligned}$$

It follows from the monotonicity of ψ and $K_2(t,s) \leq 0$ that

$$d\big(F(x,y),F(u,v)\big) \leq \sup_{t\in I}\left|\int_a^b K_1(t,s)\big[\mu\psi\big(d(x,u)\big)+\nu\psi\big(d(y,v)\big)\big]\,ds - \int_a^b K_2(t,s)\big[\mu\psi\big(d(v,y)\big)+\nu\psi\big(d(x,u)\big)\big]\,ds\right|. \tag{4.3}$$

Similarly, we have

$$d\big(F(y,x),F(v,u)\big) \leq \sup_{t\in I}\left|\int_a^b K_1(t,s)\big[\nu\psi\big(d(x,u)\big)+\mu\psi\big(d(y,v)\big)\big]\,ds - \int_a^b K_2(t,s)\big[\nu\psi\big(d(v,y)\big)+\mu\psi\big(d(x,u)\big)\big]\,ds\right|. \tag{4.4}$$

By summing (4.3) and (4.4), and using the condition (iv), we obtain

$$\begin{aligned}&\frac{d(F(x,y),F(u,v))+d(F(y,x),F(v,u))}{2}\\ &\quad\leq \sup_{t\in I}\left(\int_a^b \big(K_1(t,s)-K_2(t,s)\big)\,ds\right)^{1/p}(b-a)^{1/q}(\mu+\nu)\frac{\psi(d(x,u))+\psi(d(v,y))}{2}\\ &\quad\leq \frac{\psi(d(x,u))+\psi(d(v,y))}{2}.\end{aligned}$$

Since ψ is non-increasing, we have

$$\frac{\psi(d(x,u))+\psi(d(y,v))}{2} \leq \psi\big(d(x,u)+d(y,v)\big).$$

Combining the definition ψ, we finally obtain

$$\begin{aligned}&\frac{d(F(x,y),F(u,v))+d(F(y,x),F(v,u))}{2}\\ &\quad\leq \psi\big(d(x,u)+d(y,v)\big)=\phi\left(\frac{d(x,u)+d(y,v)}{2}\right),\end{aligned}$$

which is just the contractive condition (3.13) in Corollary 3.1.

Now, the condition (v) implies that

$$\alpha(t)\leq F\big(\alpha(t),\beta(t)\big) \quad\text{and}\quad \beta(t)\geq F\big(\beta(t),\alpha(t)\big) \quad \text{for all } t\in I,$$

which show that all hypotheses of Corollary 3.1 hold. This proves that F has a unique coupled fixed point $(\tilde{x},\tilde{y})$ in X^2. Since $\alpha\leq\beta$, it follows that $\tilde{x}=\tilde{y}$. Thus $\tilde{x}\in C(I,\mathbb{R})$ is the unique solution of the integral equation (4.2). □

Finally, we show the existence of solution for the following initial-value problem:

$$\begin{cases} u_t(x,t)=u_{xx}(x,t)+f(x,t,u,u_x)+g(x,t,u,u_x), & -\infty<x<\infty, 0<t\leq T,\\ u(x,0)=\varphi(x), & -\infty<x<\infty,\end{cases} \tag{4.5}$$

where φ is continuously differentiable and that φ and φ' are bounded and f, g are continuous functions.

Now, we consider the space

$$\Omega = \bigl\{v(x,t) : v, v_x \in C\bigl(\mathbb{R} \times [0,T]\bigr) \text{ and } \|v\| < \infty\bigr\},$$

where

$$\|v\| = \sup_{x\in\mathbb{R},t\in[0,T]} \bigl|v(x,t)\bigr| + \sup_{x\in\mathbb{R},t\in[0,T]} \bigl|v_x(x,t)\bigr|.$$

The set Ω with the norm $\|\cdot\|$ is a Banach space. Obviously, the space with metric given by

$$d(u,v) = \sup_{x\in\mathbb{R},t\in[0,T]} \bigl|u(x,t) - v(x,t)\bigr| + \sup_{x\in\mathbb{R},t\in[0,T]} \bigl|u_x(x,t) - v_x(x,t)\bigr|$$

is a complete metric space. The set Ω can also equipped with a partial order given by

$$u,v \in \Omega, \quad u \le v \quad \Longleftrightarrow \quad u(x,t) \le v(x,t), \qquad u_x(x,t) \le v_x(x,t), \quad x \in \mathbb{R}, t \in [0,T].$$

Definition 4.1 ([41]) A pair $u, v \in \Omega \times \Omega$ is called a coupled lower-upper solution of (4.5) if

$$\begin{cases} u_t(x,t) \le u_{xx}(x,t) + f(x,t,u,u_x) + g(x,t,v,v_x), & -\infty < x < \infty, 0 < t \le T, \\ u(x,0) \le \varphi(x), & -\infty < x < \infty \end{cases}$$

and

$$\begin{cases} v_t(x,t) \ge v_{xx}(x,t) + f(x,t,v,v_x) + g(x,t,u,u_x), & -\infty < x < \infty, 0 < t \le T, \\ v(x,0) \ge \varphi(x), & -\infty < x < \infty. \end{cases}$$

Theorem 4.2 *Consider the problem* (4.5) *with $f, g : \mathbb{R} \times [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ continuous and assume that the following conditions are satisfied*:

(i) *for any $c > 0$ with $|\xi| < c$ and $|\eta| < c$, the functions $f(x,t,\xi,\eta)$, $g(x,t,\xi,\eta)$ are uniformly Hölder continuous in x and t for each compact subset of $\Omega \times [0,T]$*;

(ii) *for all (ξ_1,η_1) and (ξ_2,η_2) in $\mathbb{R} \times \mathbb{R}$ with $\xi_1 \le \xi_2$ and $\eta_1 \le \eta_2$, there exist two positive constants c_f and c_g such that*

$$0 \le f(x,t,\xi_2,\eta_2) - f(x,t,\xi_1,\eta_1) \le c_f \phi\left(\frac{\xi_2 - \xi_1 + \eta_2 - \eta_1}{2}\right),$$

$$-c_g \phi\left(\frac{\xi_2 - \xi_1 + \eta_2 - \eta_1}{2}\right) \le g(x,t,\xi_2,\eta_2) - g(x,t,\xi_1,\eta_1) \le 0,$$

where $\phi(t) : [0,+\infty) \to [0,+\infty)$ is a nondecreasing Jachymski function with $\phi(t) < t$ for all $t > 0$;

(iii) *f, g are bounded for bounded ξ and η*;

(iv) $c_f + c_g \le (T + 2\pi^{-1/2}T^{1/2})^{-1}$.

Then the existence of coupled lower-upper solution for the initial-value problem (4.5) *provides the existence of the unique solution of the problem* (4.5).

Proof The problem (4.5) is equivalent to the integral equation

$$\begin{aligned} u(x,t) = {} & \int_{-\infty}^{\infty} k(x-\xi,t)\varphi(\xi)\,d\xi + \int_0^t \int_{-\infty}^{\infty} k(x-\xi,t-\tau)\bigl[f\bigl(\xi,\tau,u(\xi,\tau),u_x(\xi,\tau)\bigr) \\ & + g\bigl(\xi,\tau,u(\xi,\tau),u_x(\xi,\tau)\bigr)\bigr]\,d\xi\,d\tau \end{aligned}$$

for all $x \in \mathbb{R}$ and $0 < t \leq T$, where

$$k(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\left\{\frac{-x^2}{4t}\right\} \tag{4.6}$$

for all $x \in \mathbb{R}$ and $t > 0$. The initial-value problem (4.5) possesses a unique solution if and only if the above integral-differential equation possesses a unique solution u such that u and u_x are continuous and bounded for all $x \in \mathbb{R}$ and $0 < t \leq T$.

Define a mapping $F : \Omega \times \Omega \to \Omega$ by

$$\begin{aligned} F(u,v)(x,t) = {} & \int_{-\infty}^{\infty} k(x-\xi,t)\varphi(\xi)\,d\xi \\ & + \int_0^t \int_{-\infty}^{\infty} k(x-\xi,t-\tau)\bigl[f\bigl(\xi,\tau,u(\xi,\tau),u_x(\xi,\tau)\bigr) \\ & + g\bigl(\xi,\tau,v(\xi,\tau),v_x(\xi,\tau)\bigr)\bigr]\,d\xi\,d\tau \end{aligned}$$

for all $x \in \mathbb{R}$ and $t \in [0,T]$. Note that, if $(u,v) \in \Omega \times \Omega$ is a fixed point of F, then (u,v) is a solution of the problem (4.5).

From the condition (ii) of Theorem 4.2, It is not difficult to prove that F has the mixed monotone property. Now, for $u_1, v_1, u_2, v_2 \in \Omega$ with $u_1 \geq u_2$ and $v_1 \leq v_2$, we have

$$\begin{aligned} & \bigl|F(u_1,v_1)(x,t) - F(u_2,v_2)(x,t)\bigr| \\ & \quad \leq \int_0^t \int_{-\infty}^{\infty} k(x-\xi,t-\tau)\bigl[f\bigl(\xi,\tau,u_1(\xi,\tau),(u_1)_x(\xi,\tau)\bigr) + f\bigl(\xi,\tau,u_2(\xi,\tau),(u_2)_x(\xi,\tau)\bigr) \\ & \qquad + g\bigl(\xi,\tau,v_1(\xi,\tau),(v_1)_x(\xi,\tau)\bigr) - g\bigl(\xi,\tau,v_2(\xi,\tau),(v_2)_x(\xi,\tau)\bigr)\bigr]\,d\xi\,d\tau \\ & \quad \leq \int_0^t \int_{-\infty}^{\infty} k(x-\xi,t-\tau)\biggl[c_f\phi\biggl(\frac{u_1(\xi,\tau)-u_2(\xi,\tau)+(u_1)_x(\xi,\tau)-(u_2)_x(\xi,\tau)}{2}\biggr) \\ & \qquad + c_g\phi\biggl(\frac{v_2(\xi,\tau)-v_1(\xi,\tau)+(v_2)_x(\xi,\tau)-(v_1)_x(\xi,\tau)}{2}\biggr)\biggr]\,d\xi\,d\tau. \end{aligned}$$

Since the function ϕ is nondecreasing, we have

$$\begin{aligned} & \phi\biggl(\frac{u_1(\xi,\tau)-u_2(\xi,\tau)+(u_1)_x(\xi,\tau)-(u_2)_x(\xi,\tau)}{2}\biggr) \\ & \quad \leq \phi\biggl(\frac{\sup_{\xi\in\mathbb{R},\tau\in[0,T]}|u_1(\xi,\tau)-u_2(\xi,\tau)| + \sup_{\xi\in\mathbb{R},t\in[0,T]}|(u_1)_x(\xi,\tau)-(u_2)_x(\xi,\tau)|}{2}\biggr) \\ & \quad \leq \phi\biggl(\frac{d(u_1,u_2)}{2}\biggr) \end{aligned}$$

and

$$\begin{aligned}
&\phi\left(\frac{v_2(\xi,\tau)-v_1(\xi,\tau)+(v_2)_x(\xi,\tau)-(v_1)_x(\xi,\tau)}{2}\right)\\
&\quad\le\phi\left(\frac{\sup_{\xi\in\mathbb{R},\tau\in[0,T]}|v_2(\xi,\tau)-v_1(\xi,\tau)|+\sup_{\xi\in\mathbb{R},t\in[0,T]}|(v_2)_x(\xi,\tau)-(v_1)_x(\xi,\tau)|}{2}\right)\\
&\quad\le\phi\left(\frac{d(v_2,v_1)}{2}\right).
\end{aligned}$$

Hence, by (4.6), we obtain

$$\begin{aligned}
&\bigl|F(u_1,v_1)(x,t)-F(u_2,v_2)(x,t)\bigr|\\
&\quad\le\int_0^t\int_{-\infty}^{\infty}k(x-\xi,t-\tau)\left[c_f\phi\left(\frac{d(u_1,u_2)}{2}\right)+c_g\phi\left(\frac{d(v_2,v_1)}{2}\right)\right]d\xi\,d\tau\\
&\quad\le T\left[c_f\phi\left(\frac{d(u_1,u_2)}{2}\right)+c_g\phi\left(\frac{d(v_2,v_1)}{2}\right)\right].
\end{aligned}\tag{4.7}$$

Furthermore, we have

$$\begin{aligned}
&\left|\frac{\partial F(u_1,v_1)}{\partial x}(x,t)-\frac{\partial F(u_2,v_2)}{\partial x}(x,t)\right|\\
&\quad\le\int_0^t\int_{-\infty}^{\infty}\left|\frac{\partial k}{\partial x}(x-\xi,t-\tau)\right|\left[c_f\phi\left(\frac{d(u_1,u_2)}{2}\right)+c_g\phi\left(\frac{d(v_2,v_1)}{2}\right)\right]d\xi\,d\tau\\
&\quad\le 2\pi^{-\frac{1}{2}}T^{\frac{1}{2}}\left[c_f\phi\left(\frac{d(u_1,u_2)}{2}\right)+c_g\phi\left(\frac{d(v_2,v_1)}{2}\right)\right].
\end{aligned}\tag{4.8}$$

Combining (4.7) with (4.8), we obtain

$$d\bigl(F(u_1,v_1),F(u_2,v_2)\bigr)\le\bigl(T+2\pi^{-\frac{1}{2}}T^{\frac{1}{2}}\bigr)\left[c_f\phi\left(\frac{d(u_1,u_2)}{2}\right)+c_g\phi\left(\frac{d(v_2,v_1)}{2}\right)\right].\tag{4.9}$$

Similarly, we obtain

$$d\bigl(F(v_1,u_1),F(v_2,u_2)\bigr)\le\bigl(T+2\pi^{-\frac{1}{2}}T^{\frac{1}{2}}\bigr)\left[c_g\phi\left(\frac{d(u_1,u_2)}{2}\right)+c_f\phi\left(\frac{d(v_2,v_1)}{2}\right)\right].\tag{4.10}$$

By summing (4.9) and (4.10), we get

$$\begin{aligned}
&\frac{d(F(u_1,v_1),F(u_2,v_2))+d(F(v_1,u_1),F(v_2,u_2))}{2}\\
&\quad\le\bigl(T+2\pi^{-\frac{1}{2}}T^{\frac{1}{2}}\bigr)\left(\frac{c_f+c_g}{2}\right)\left[\phi\left(\frac{d(u_1,u_2)}{2}\right)+\phi\left(\frac{d(v_2,v_1)}{2}\right)\right].
\end{aligned}$$

Now, since ϕ is nonincreasing, we have

$$\phi\left(\frac{d(u_1,u_2)}{2}\right)+\phi\left(\frac{d(v_2,v_1)}{2}\right)\le 2\phi\left(\frac{d(u_1,u_2)+d(v_2,v_1)}{2}\right).$$

Thus, by using the condition (iv) of Theorem 4.2, we finally get

$$\frac{d(F(u_1,v_1),F(u_2,v_2)) + d(F(v_1,u_1),F(v_2,u_2))}{2} \le \phi\left(\frac{d(u_1,u_2),d(v_2,v_1)}{2}\right).$$

Now, let $(u,v) \in \Omega \times \Omega$ be a coupled lower-upper solution of (4.5). Then we have

$$u(x,t) \le F\big(u(x,t),v(x,t)\big) \quad \text{and} \quad v(x,t) \ge F\big(v(x,t),u(x,t)\big) \quad \text{for all } x \in \mathbb{R}, t \in [0,T],$$

which show that all hypotheses of Corollary 3.1 are satisfied. □

Remark 4.1 Gordji *et al.* [41] considered the initial-value problem (4.5) when $g(x,t,u,u_x) \equiv 0$. Notice that if take $\phi(x) = \ln(1+x)$, then we derive Theorem 3.3 in [41].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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