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The solution for a class of sum operator equation and its application to fractional differential equation boundary value problems

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Abstract
In this paper we study a class of sum operator equation $Ax + Bx + C(x, x) = x$ on ordered Banach spaces, where A is an increasing operator, B is a decreasing operator, and C is a mixed monotone operator. The existence and uniqueness of its positive solution are obtained by using the properties of cone and fixed point theorems for mixed monotone operators. As an application, we utilize the obtained results to study the existence and uniqueness of positive solutions for nonlinear fractional differential equation boundary value problems.

Keywords: operator equation; fixed point theorem; fractional differential equation; positive solution

1 Introduction

Over the past several decades, nonlinear functional analysis has been an active area of research in mechanics, elasticity, fluid dynamics, and so on. As an important branch of nonlinear functional analysis, the nonlinear operator theorem and its application in nonlinear differential equations have attracted much attention (see [1–7]). It is well known that the existence and uniqueness of positive solutions to nonlinear operator equations are very important in theory and applications. Many authors have studied this problem; for a small sample of such work, we refer the reader to [8–16].

Reference [14] has successively considered the sum operator equation $Mx + Qx + Nx = x$ on ordered Banach spaces, where M is an increasing, α-concave operator, Q is an increasing sub-homogeneous operator, and N is a homogeneous operator. The existence and uniqueness of its positive solutions are obtained by using the properties of cones and a fixed point theorem for increasing general β-concave operators.

In [15], the sum operator equation $A(x, x) + Bx = x$ has been considered. A is a mixed monotone operator and B is an increasing α-concave (or sub-homogeneous) operator. By using the properties of cones and a fixed point theorem for mixed monotone operators, respectively, the author established the existence and uniqueness of positive solutions for the operator equation.

In most of the literature, people pay more attention to the study of the increasing and mixed-monotone operators. However, studying the decreasing operator is equality important. So inspired by [14] and [15], we study the following sum operator equations on ordered Banach spaces in this paper:

$$Ax + Bx + C(x,x) = x, \tag{1.1}$$

where A is an increasing α-concave (or sub-homogeneous) operator, B is a decreasing operator, C is a mixed monotone operator. By using the properties of cones and the fixed point theorem for mixed monotone operator, the existence and uniqueness of the positive solution are obtained. Our research methods are different from those in the related literature. As an application, we utilize the obtained results to study the existence and uniqueness of positive solutions for nonlinear fractional differential equation boundary value problems. Our results extend and improve the related conclusions in the literature. Besides, it provides a new way to study the differential equations. To the best of our knowledge, the fixed point results on the operator equation (1.1) with α*-concave* (or sub-homogeneous) increasing, decreasing and mixed monotone operators are still very few. So it is worthwhile to investigate the operator equation (1.1).

The content of this paper is organized as follows. In Section 2, we present some definitions, lemmas and basic results that will be used in the proofs of our theorems. In Section 3, we consider the existence and uniqueness of positive solutions for the operator equation (1.1). In Section 4, we utilize the results obtained in Section 3 to study the existence and uniqueness of positive solutions for nonlinear fractional differential equation boundary value problems.

2 Preliminaries

For convenience of the reader, we present here some definitions, lemmas, and basic results that will be used in the proofs of our theorems.

Suppose that $(E, \|\cdot\|)$ is a real Banach space which is partially ordered by a cone $P \subset E$, *i.e.*, $x \leq y$ if and only if $y - x \in P$. If $x \leq y$ and $x \neq y$, then we denote $x < y$ or $y > x$. By θ we denote the zero element of E. Recall that a non-empty closed convex set $P \subset E$ is a cone if it satisfies (i) $x \in P$, $\lambda \geq 0 \Rightarrow \lambda x \in P$; (ii) $x \in P$, $-x \in P \Rightarrow x = \theta$.

Putting $\mathring{P} = \{x \in P \mid x \text{ is an interior point of } P\}$, a cone P is said to be solid if $\mathring{P}$ is nonempty. Moreover, P is called normal if there exists a constant $N > 0$ such that, for all $x, y \in E$, $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$. In this case, N is called the normality constant of P. If $x_1, x_2 \in E$, the set $[x_1, x_2] = \{x \in E \mid x_1 \leq x \leq x_2\}$ is called the order interval between x_1 and x_2. We say that an operator $A : E \to E$ is increasing (decreasing) if $x \leq y$ implies $Ax \leq Ay$ ($Ax \geq Ay$).

For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leq y \leq \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > \theta$ (i.d., $h \geq \theta$ and $h \neq \theta$), we denote by P_h the set $P_h = \{x \in E \mid x \sim h\}$. It is easy to see that $P_h \subset P$.

Definition 2.1 ([17]) An operator $A : P \to P$ is said to be sub-homogeneous if it is satisfies

$$A(tx) \geq tAx, \quad \forall t \in (0,1), x \in P. \tag{2.1}$$

Definition 2.2 ([17]) Let $D = P$ or $D = \mathring{P}$ and α be a real number with $0 \le \alpha < 1$. An operator $A : D \to D$ is said to be α-concave if it satisfies

$$A(tx) \ge t^{\alpha} Ax, \quad \forall t \in (0,1), x \in D. \tag{2.2}$$

Definition 2.3 ([17]) $A : P \times P \to P$ is said to be a mixed monotone operator if $A(x,y)$ is increasing in x and decreasing in y, *i.e.*, u_i, v_i $(i = 1,2) \in P$, $u_1 \le u_2$, $v_1 \ge v_2$ imply $A(u_1, v_1) \le A(u_2, v_2)$. An element $x \in P$ is called a fixed point of A if $A(x,x) = x$.

Lemma 2.4 (See Lemma 2.1 and Theorem 2.1 in [12]) *Let P be a normal cone in E. Assume that $T : P \times P \to P$ is a mixed monotone operator and satisfies*

(A_1) *there exists $h \in P$ with $h \ne \theta$ such that $T(h,h) \in P_h$;*
(A_2) *for any $u, v \in P$ and $t \in (0,1)$, there exists $\varphi(t) \in (t,1]$ such that $T(tu, t^{-1}v) \ge \varphi(t)T(u,v)$.*

Then
(1) *$T : P_h \times P_h \to P_h$;*
(2) *there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that $rv_0 \le u_0 < v_0$,*
$u_0 \le T(u_0, v_0) \le T(v_0, u_0) \le v_0$;
(3) *T has a unique fixed point x^* in P_h;*
(4) *for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences*

$$x_n = T(x_{n-1}, y_{n-1}), \qquad y_n = T(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots,$$

we have $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$.

3 Main results

In this section we consider the existence and uniqueness of positive solutions for the operator equation $Ax + Bx + C(x,x) = x$. We assume that E is a real Banach space with a partial order introduced by a normal cone P of E. Take $h \in E$, $h > \theta$, P_h is given as in the preliminaries.

Theorem 3.1 *Let $\alpha \in (0,1)$. Suppose that $A : P \to P$ is an increasing sub-homogeneous operator, $B : P \to P$ is a decreasing operator, $C : P \times P \to P$ is a mixed monotone operator, and that they satisfy the following conditions*:

$$B\bigl(t^{-1}y\bigr) \ge tBy, \qquad C\bigl(tx, t^{-1}y\bigr) \ge t^{\alpha} C(x,y), \quad \forall t \in (0,1), x, y \in P. \tag{3.1}$$

Assume that

(H_1) *there is $h_0 \in P_h$ such that $Ah_0 \in P_h$, $Bh_0 \in P_h$, $C(h_0, h_0) \in P_h$;*
(H_2) *there exists a constant $\delta > 0$ such that $C(x,y) \ge \delta(Ax + By)$, $\forall x, y \in P$.*

Then
(1) *$A : P_h \to P_h$, $B : P_h \to P_h$, $C : P_h \times P_h \to P_h$;*
(2) *there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that*

$$rv_0 \le u_0 < v_0, \qquad u_0 \le Au_0 + Bv_0 + C(u_0, v_0) \le Av_0 + Bu_0 + C(v_0, u_0) \le v_0;$$

(3) *the operator equation* $Ax + Bx + C(x,x) = x$ *has a unique solution* x^* *in* P_h;
(4) *for any initial values* $x_0, y_0 \in P_h$, *constructing successively the sequences*

$$x_n = Ax_{n-1} + By_{n-1} + C(x_{n-1}, y_{n-1}),$$
$$y_n = Ay_{n-1} + Bx_{n-1} + C(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots,$$

we have $x_n \to x^*$ *and* $y_n \to y^*$ *as* $n \to \infty$.

Proof From Definition 2.1 and (3.1), we have

$$\begin{aligned} &A(t^{-1}x) \le \frac{1}{t}Ax, \qquad B(ty) \le \frac{1}{t}By, \\ &C(t^{-1}x, ty) \le \frac{1}{t^\alpha}C(x,y) \quad \text{for } t \in (0,1), x, y \in P. \end{aligned} \tag{3.2}$$

First step: we will demonstrate $A : P_h \to P_h$, $B : P_h \to P_h$, $C : P_h \times P_h \to P_h$.

Since $Ah_0 \in P_h, Bh_0 \in P_h, C(h_0, h_0) \in P_h$ there exist constants $\lambda_1, \lambda_2, \lambda_3, \nu_1, \nu_2, \nu_3 > 0$, such that

$$\lambda_1 h \le Ah_0 \le \nu_1 h, \qquad \lambda_2 h \le Bh_0 \le \nu_2 h, \qquad \lambda_3 h \le C(h_0, h_0) \le \nu_3 h.$$

From $h_0 \in P_h$, there exists a constant $t_0 \in (0,1)$ such that

$$t_0 h \le h_0 \le \frac{1}{t_0}h.$$

Combine (2.1), (3.2) with the increasing property of operator A and the decreasing property of operator B, we have

$$Ah \le A\left(\frac{1}{t_0}h_0\right) \le \frac{1}{t_0}Ah_0 \le \frac{1}{t_0}\nu_1 h, \qquad Ah \ge A(t_0 h_0) \ge t_0 Ah_0 \ge t_0\lambda_1 h,$$
$$Bh \le B(t_0 h_0) \le \frac{1}{t_0}Bh_0 \le \frac{1}{t_0}\nu_2 h, \qquad Bh \ge B\left(\frac{1}{t_0}h_0\right) \ge t_0 Bh_0 \ge t_0\lambda_2 h.$$

For any $x \in P_h$, we can choose a sufficiently small number $\mu \in (0,1)$ such that

$$\mu h \le x \le \frac{1}{\mu}h.$$

Then

$$Ax \le A\left(\frac{1}{\mu}h\right) \le \frac{1}{\mu}\cdot\frac{1}{t_0}\nu_1 h, \qquad Ax \ge A(\mu h) \ge \mu t_0\lambda_1 h,$$
$$Bx \le B(\mu h) \le \frac{1}{\mu}\cdot\frac{1}{t_0}\nu_2 h, \qquad Bx \ge B\left(\frac{1}{\mu}h\right) \ge \mu t_0\lambda_2 h.$$

Evidently, $\frac{1}{\mu t_0}\nu_1, \frac{1}{\mu t_0}\nu_2, \mu t_0\lambda_1, \mu t_0\lambda_2 > 0$. Thus $Ax \in P_h$, $Bx \in P_h$; that is, $A : P_h \to P_h$, $B : P_h \to P_h$. Also from (3.1), (3.2), and the properties of mixed monotone operator C, we have

$$C(h,h) \leq C\left(\frac{1}{t_0}h_0, t_0 h_0\right) \leq \frac{1}{t_0^\alpha}C(h_0,h_0) \leq \frac{\nu_3}{t_0^\alpha}h,$$
$$C(h,h) \geq C\left(t_0 h_0, \frac{1}{t_0}h_0\right) \geq t_0^\alpha C(h_0,h_0) \geq t_0^\alpha \lambda_3 h.$$

Noting that $\frac{\nu_3}{t_0^\alpha}, t_0^\alpha \lambda_3 > 0$, we can get $C(h,h) \in P_h$. An application of Lemma 2.4 implies that $C : P_h \times P_h \to P_h$. So the conclusion (1) is true.

The second step is to demonstrate the conclusions (2)-(4) are correct.

Now we define an operator $T = A + B + C$ by $T(x,y) = Ax + By + C(x,y)$ for $A(x,y) = Ax$, $B(x,y) = By$. Then $T : P \times P \to P$ is a mixed monotone operator and $T(h,h) \in P_h$. In the following, we show that there exists $\varphi(t) \in (t,1]$ with respect to $t \in (0,1)$ such that

$$T(tx, t^{-1}y) \geq \varphi(t)T(x,y), \quad \forall x,y \in P.$$

From (H_2), we have

$$C(x,y) \geq \frac{Ax + By + C(x,y)}{1+\delta^{-1}} = \frac{T(x,y)}{1+\delta^{-1}}, \quad \forall x,y \in P.$$

Also from (2.1), (3.1), we can obtain

$$\begin{aligned} T(tx, t^{-1}y) - tT(x,y) &= A(tx) + B(t^{-1}y) + C(tx, t^{-1}y) - tAx - tBy - tC(x,y) \\ &\geq (t^\alpha - t)C(x,y) \geq \frac{t^\alpha - t}{1+\delta^{-1}}T(x,y), \quad \forall x,y \in P, t \in (0,1). \end{aligned}$$

Consequently, for any $x,y \in P$, $t \in (0,1)$,

$$T(tx, t^{-1}y) \geq tT(x,y) + \frac{t^\alpha - t}{1+\delta^{-1}}T(x,y) = \left(t + \frac{t^\alpha - t}{1+\delta^{-1}}\right)T(x,y).$$

Let

$$\varphi(t) = t + \frac{t^\alpha - t}{1+\delta^{-1}}, \quad t \in (0,1).$$

Then $\varphi(t) \in (t,1]$ and $T(tx, t^{-1}y) \geq \varphi(t)T(x,y)$ for any $t \in (0,1)$ and $x,y \in P$. Hence the condition (A_2) in Lemma 2.4 is satisfied. An application of Lemma 2.4 implies: (c_1) there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that $rv_0 \leq u_0 < v_0$, $u_0 \leq T(u_0,v_0) \leq T(v_0,u_0) \leq v_0$; ($c_2$) the operator T has a unique fixed point x^* in P_h; (c_3) for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences

$$x_n = T(x_{n-1}, y_{n-1}), \qquad y_n = T(y_{n-1}, x_{n-1}), \quad n = 1,2,\ldots,$$

we have $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$. That is, the conclusions (2)-(4) hold. □

From the proof of Theorem 3.1, we can easily prove the following conclusion.

Corollary 3.2 *Let $\alpha \in (0,1)$. Suppose that $A : P_h \to P_h$ is an increasing sub-homogeneous operator, $B : P_h \to P_h$ is a decreasing operator, $C : P_h \times P_h \to P_h$ is a mixed monotone operator, assume that* (3.1) *and* (H_2) *hold. Then the conclusions* (2)-(4) *of Theorem* 3.1 *hold.*

Corollary 3.3 *Let $\alpha \in (0,1)$. Suppose that $A : P \to P$ is an increasing sub-homogeneous operator, $C : P \times P \to P$ is a mixed monotone operator and satisfies $C(tx, t^{-1}y) \geq t^{\alpha} C(x,y)$, $\forall t \in (0,1)$, $x, y \in P$. Assume that*

(H_3) *there is $h_0 \in P_h$ such that $Ah_0 \in P_h$, $C(h_0, h_0) \in P_h$;*
(H_4) *there exists a constant $\delta > 0$ such that $C(x,y) \geq \delta Ax$, $\forall x, y \in P$.*

Then

(1) $A : P_h \to P_h$, $C : P_h \times P_h \to P_h$;
(2) *there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that*

$$rv_0 \leq u_0 < v_0, \qquad u_0 \leq Au_0 + C(u_0, v_0) \leq Av_0 + C(v_0, u_0) \leq v_0;$$

(3) *the operator equation $Ax + C(x,x) = x$ has a unique solution x^* in P_h;*
(4) *for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences*

$$x_n = Ax_{n-1} + C(x_{n-1}, y_{n-1}), \qquad y_n = Ay_{n-1} + C(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots,$$

we have $x_n \to x^*$ and $y_n \to y^*$ as $n \to \infty$.

Corollary 3.4 *Let $\alpha \in (0,1)$. Suppose that $B : P \to P$ is a decreasing operator, $C : P \times P \to P$ is a mixed monotone operator and satisfy* (3.1)*. Assume that*

(H_5) *there is $h_0 \in P_h$ such that $Bh_0 \in P_h$, $C(h_0, h_0) \in P_h$;*
(H_6) *there exists a constant $\delta > 0$ such that $C(x,y) \geq \delta By$, $\forall x, y \in P$.*

Then

(1) $B : P_h \to P_h$, $C : P_h \times P_h \to P_h$;
(2) *there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that*

$$rv_0 \leq u_0 < v_0, \qquad u_0 \leq Bv_0 + C(u_0, v_0) \leq Bu_0 + C(v_0, u_0) \leq v_0;$$

(3) *the operator equation $Bx + C(x,x) = x$ has a unique solution x^* in P_h;*
(4) *for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences*

$$x_n = By_{n-1} + C(x_{n-1}, y_{n-1}), \qquad y_n = Bx_{n-1} + C(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots,$$

we have $x_n \to x^*$ and $y_n \to y^*$ as $n \to \infty$.

Corollary 3.5 *Let $\alpha \in (0,1)$. Suppose that $C : P \times P \to P$ is a mixed monotone operator, and satisfies $C(tx, t^{-1}y) \geq t^{\alpha} C(x,y)$, $\forall t \in (0,1)$, $x, y \in P$. Assume that there is $h_1 > \theta$, such that $C(h_1, h_1) \in P_h$ hold. Then*

(1) *there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that*

$$rv_0 \leq u_0 < v_0, \qquad u_0 \leq C(u_0, v_0) \leq C(v_0, u_0) \leq v_0;$$

(2) *the operator equation* $C(x,x) = x$ *has a unique solution* x^* *in* P_h;
(3) *for any initial values* $x_0, y_0 \in P_h$, *constructing successively the sequences*

$$x_n = C(x_{n-1}, y_{n-1}), \qquad y_n = C(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots,$$

we have $x_n \to x^*$ *and* $y_n \to y^*$ *as* $n \to \infty$.

Remark 3.6 Corollaries 3.3, 3.4, 3.5 which have been studied in [12, 15, 16] are special cases of Theorem 3.1. In this sense, our results extend and supplement the results in [12, 15, 16].

Theorem 3.7 *Let* $\alpha \in (0,1)$. *Suppose that* $A : P \to P$ *is an increasing sub-homogeneous operator,* $B : P \to P$ *is a decreasing operator,* $C : P \times P \to P$ *is a mixed monotone operator, and satisfy*

$$B(t^{-1}y) \geq t^\alpha By, \qquad C(tx, t^{-1}y) \geq tC(x,y), \quad \forall t \in (0,1), x, y \in P. \tag{3.3}$$

Assume that

(H_1') *there is* $h_0 \in P_h$ *such that* $Ah_0 \in P_h$, $Bh_0 \in P_h$, $C(h_0, h_0) \in P_h$;
(H_2') *there exists a constant* $\delta > 0$ *such that* $Ax + C(x,y) \leq \delta By$, $\forall x, y \in P$.

Then the conclusions (1)-(4) *of Theorem* 3.1 *hold.*

Proof According to Definition 2.1 and (3.3), we obtain

$$\begin{aligned} &A(t^{-1}x) \leq \frac{1}{t}Ax, \qquad B(ty) \leq \frac{1}{t^\alpha}By, \\ &C(t^{-1}x, ty) \leq \frac{1}{t}C(x,y) \quad \text{for } t \in (0,1), x, y \in P. \end{aligned} \tag{3.4}$$

Similarly to the proof of Theorem 3.1, we have $A : P_h \to P_h$, $B : P_h \to P_h$, $C : P_h \times P_h \to P_h$.

In the following, we will prove the conclusions (2)-(4) are true. Define an operator $T = A + B + C$ by $T(x,y) = Ax + By + C(x,y)$. Then $T : P \times P \to P$ is a mixed monotone operator and $T(h,h) \in P_h$. Next, we show that there exists $\varphi(t) \in (t, 1]$ with respect to $t \in (0,1)$ such that

$$T(tx, t^{-1}y) \geq \varphi(t) T(x,y), \quad \forall x, y \in P.$$

From (H_2'), we have

$$By \geq \frac{Ax + By + C(x,y)}{1 + \delta} = \frac{T(x,y)}{1 + \delta}, \quad \forall x, y \in P.$$

Also from (2.1), (3.3), we have

$$\begin{aligned} T(tx, t^{-1}y) - tT(x,y) &= A(tx) + B(t^{-1}y) + C(tx, t^{-1}y) - tAx - tBy - tC(x,y) \\ &\geq (t^\alpha - t)By \geq \frac{t^\alpha - t}{1 + \delta} T(x,y), \quad \forall x, y \in P, t \in (0,1). \end{aligned}$$

Consequently, for any $x, y \in P$, $t \in (0,1)$,

$$T\left(tx, t^{-1}y\right) \geq tT(x,y) + \frac{t^\alpha - t}{1+\delta} T(x,y) = \left(t + \frac{t^\alpha - t}{1+\delta}\right) T(x,y).$$

Let

$$\varphi(t) = t + \frac{t^\alpha - t}{1+\delta}, \quad t \in (0,1).$$

Then $\varphi(t) \in (t,1]$ and $T(tx, t^{-1}y) \geq \varphi(t)T(x,y)$ for any $t \in (0,1)$ and $x, y \in P$. Hence the condition (A_2) in Lemma 2.4 is satisfied. As an application of Lemma 2.4, we can get the conclusions (2)-(4). □

Theorem 3.8 *Let* $\alpha \in (0,1)$. *Suppose that* $A : P \to P$ *is an increasing* α*-concave operator,* $B : P \to P$ *is a decreasing operator,* $C : P \times P \to P$ *is a mixed monotone operator, and they satisfy*

$$B\left(t^{-1}y\right) \geq tBy, \qquad C\left(tx, t^{-1}y\right) \geq tC(x,y), \quad \forall t \in (0,1), x, y \in P. \tag{3.5}$$

Assume that

(H_1'') *there is* $h_0 \in P_h$ *such that* $Ah_0 \in P_h$, $Bh_0 \in P_h$, $C(h_0,h_0) \in P_h$;
(H_2'') *there exists a constant* $\delta > 0$ *such that* $By + C(x,y) \leq \delta Ax$, $\forall x, y \in P$.

Then the conclusions (1)-(4) *of Theorem* 3.1 *hold.*

Proof By Definition 2.2 and (3.5), we have

$$\begin{aligned} &A\left(t^{-1}x\right) \leq \frac{1}{t^\alpha} Ax, \qquad B(ty) \leq \frac{1}{t} By, \\ &C\left(t^{-1}x, ty\right) \leq \frac{1}{t} C(x,y) \quad \text{for } t \in (0,1), x, y \in P. \end{aligned} \tag{3.6}$$

Similarly to the proof of Theorem 3.1, we have $A : P_h \to P_h$, $B : P_h \to P_h$, $C : P_h \times P_h \to P_h$.

Now we define an operator $T = A + B + C$ by $T(x,y) = Ax + By + C(x,y)$ for $A(x,y) = Ax$, $B(x,y) = By$. Then $T : P \times P \to P$ is a mixed monotone operator and $T(h,h) \in P_h$. In the following, we show that there exists $\varphi(t) \in (t,1]$ with respect to $t \in (0,1)$ such that

$$T\left(tx, t^{-1}y\right) \geq \varphi(t) T(x,y), \quad \forall x, y \in P.$$

By (H_2''), we can obtain

$$Ax \geq \frac{Ax + By + C(x,y)}{1+\delta} = \frac{T(x,y)}{1+\delta}, \quad \forall x, y \in P.$$

Also from (2.2), (3.5), we have

$$\begin{aligned} T\left(tx, t^{-1}y\right) - tT(x,y) &= A(tx) + B\left(t^{-1}y\right) + C\left(tx, t^{-1}y\right) - tAx - tBy - tC(x,y) \\ &\geq \left(t^\alpha - t\right) Ax \geq \frac{t^\alpha - t}{1+\delta} T(x,y), \quad \forall x, y \in P, t \in (0,1). \end{aligned}$$

So, for any $x,y \in P$, $t \in (0,1)$,

$$T\big(tx,t^{-1}y\big) \geq tT(x,y) + \frac{t^\alpha - t}{1+\delta}T(x,y) = \left(t + \frac{t^\alpha - t}{1+\delta}\right)T(x,y).$$

Let

$$\varphi(t) = t + \frac{t^\alpha - t}{1+\delta}, \quad t \in (0,1).$$

Then $\varphi(t) \in (t,1]$ and $T(tx,t^{-1}y) \geq \varphi(t)T(x,y)$ for any $t \in (0,1)$ and $x,y \in P$. Hence the condition (A_2) in Lemma 2.4 is satisfied. An application of Lemma 2.4, we see the conclusions (2)-(4) hold. □

4 Applications

Fractional differential equations arise in many field, such as physics, mechanics, chemistry, engineering and biological sciences, *etc.* In recent years, many authors have investigated the existence of positive solutions for nonlinear fractional differential equation boundary value problems (see [18–21]). However, there are few papers concerned with the uniqueness of positive solutions. In this section, we only apply the results in Section 3 to study nonlinear fractional differential equation boundary value problems. We study the existence and uniqueness of positive solutions for the following nonlinear fractional differential equation boundary value problem:

$$\begin{cases} -D^{\nu}_{0^+}u(t) = f(t,u(t),u(t)) + g(t,u(t)) + q(t,u(t)), & 0 < t < 1, 3 < \nu \leq 4, \\ u(0) = u'(0) = u''(0) = u''(1) = 0. \end{cases} \tag{4.1}$$

Here $D^{\alpha}_{0^+}$ is the Riemann-Liouville fractional derivative of order $\nu > 0$, defined by

$$D^{\nu}_{0^+}u(t) = \frac{1}{\Gamma(n-\nu)}\left(\frac{d}{dt}\right)^n \int_0^t (t-\tau)^{n-\nu-1}u(\tau)\,d\tau,$$

where $n = [\nu] + 1$. $[\nu]$ denotes the integer part of the number ν; see [22]. $f(t,u,v) : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is continuous, and $g(t,u), q(t,v) : [0,1] \times [0,+\infty) \to [0,+\infty)$ are continuous.

In our considerations we will work in the Banach apace $E = C[0,1] = \{x : [0,1] \to \mathbb{R}$ is continuous$\}$ with the standard norm $\|x\| = \sup\{|x(t)| : t \in [0,1]\}$. Notice that this space can be equipped with a partial order given by

$$x,y \in C[0,1], \quad x \leq y \quad \Leftrightarrow \quad x(t) \leq y(t) \quad \text{for all } t \in [0,1].$$

Set $P = \{x \in C[0,1] \mid x(t) \geq 0, t \in [0,1]\}$, the standard cone. It is clear that P is a normal cone in $C[0,1]$ and the normality constant is 1.

Definition 4.1 ([23]) The integral

$$I^{\nu}_{0^+}f(x) = \frac{1}{\Gamma(\nu)}\int_0^x \frac{f(t)}{(x-t)^{1-\nu}}\,dt, \quad x > 0,$$

where $\nu > 0$, is called the Riemann-Liouville fractional integral of order ν and $\Gamma(\nu)$ is the Euler gamma function defined by

$$\Gamma(\nu) = \int_0^{+\infty} t^{\nu-1}e^{-t}\,dt, \quad \nu > 0.$$

Lemma 4.2 ([19, 24]) *Let $\nu > 0$ and $u \in C(0,1) \cap L(0,1)$. The fractional differential equation*

$$D_{0^+}^{\nu} u(t) = 0$$

has

$$u(t) = c_1 t^{\nu-1} + c_2 t^{\nu-2} + \cdots + c_n t^{\nu-n}, \quad c_i \in R, i = 0,1,\ldots,n, n = [\nu] + 1,$$

as unique solution.

Lemma 4.3 ([19, 24]) *Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\nu > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then*

$$I_{0^+}^{\nu} D_{0^+}^{\nu} u(t) = u(t) + c_1 t^{\nu-1} + c_2 t^{\nu-2} + \cdots + c_n t^{\nu-n}$$

for some $c_i \in R$, $i = 0,1,\ldots,n$, $n = [\nu] + 1$.

Lemma 4.4 *If $f(t,u(t),u(t)) + g(t,u(t)) + q(t,u(t)) \geq 0$, then the fractional boundary value problem* (4.1) *has a unique positive solution*

$$u(t) = \int_0^1 G(t,s)\bigl[f\bigl(s,u(s),u(s)\bigr) + g\bigl(s,u(s)\bigr) + q\bigl(s,u(s)\bigr)\bigr]\,ds,$$

where

$$G(t,s) = \frac{1}{\Gamma(\nu)} \begin{cases} t^{\nu-1}(1-s)^{\nu-3} - (t-s)^{\nu-1}, & 0 \leq s \leq t \leq 1, \\ t^{\nu-1}(1-s)^{\nu-3}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{4.2}$$

Proof Lemma 4.3 and Definition 4.1 imply that

$$\begin{aligned} u(t) &= c_1 t^{\nu-1} + c_2 t^{\nu-2} + c_3 t^{\nu-3} + c_4 t^{\nu-4} \\ &\quad - \int_0^t \frac{(t-s)^{\nu-1}}{\Gamma(\nu)}\bigl[f\bigl(s,u(s),u(s)\bigr) + g\bigl(s,u(s)\bigr) + q\bigl(s,u(s)\bigr)\bigr]\,ds. \end{aligned}$$

From (4.1), we know that $c_2 = c_3 = c_4 = 0$ and

$$c_1 = \frac{1}{\Gamma(\nu)} \int_0^t (1-s)^{\nu-3}\bigl[f\bigl(s,u(s),u(s)\bigr) + g\bigl(s,u(s)\bigr) + q\bigl(s,u(s)\bigr)\bigr]\,ds.$$

Then the unique solution of (4.1) is given by

$$\begin{aligned} u(t) &= \int_0^1 \frac{t^{\nu-1}(1-s)^{\nu-3}}{\Gamma(\nu)} \bigl[f\bigl(s,u(s),u(s)\bigr) + g\bigl(s,u(s)\bigr) + q\bigl(s,u(s)\bigr)\bigr]\,ds \\ &\quad - \int_0^t \frac{(t-s)^{\nu-1}}{\Gamma(\nu)} \bigl[f\bigl(s,u(s),u(s)\bigr) + g\bigl(s,u(s)\bigr) + q\bigl(s,u(s)\bigr)\bigr]\,ds \\ &= \int_0^1 G(t,s)\bigl[f\bigl(s,u(s),u(s)\bigr) + g\bigl(s,u(s)\bigr) + q\bigl(s,u(s)\bigr)\bigr]\,ds. \end{aligned}$$

This completes the proof of Lemma 4.4. □

Lemma 4.5 *Let* $3 < \nu \leq 4$. *Then the function* $G(t,s)$ *defined by* (4.2) *satisfies the following conditions*:

(1) $G(t,s) \geq 0$, $(t,s) \in [0,1] \times [0,1]$;

(2) $\frac{1}{\Gamma(\nu)} s(2-s)(1-s)^{\nu-3} t^{\nu-1} \leq G(t,s) \leq \frac{1}{\Gamma(\nu)}(1-s)^{\nu-3}t^{\nu-1}$ *for* $t,s \in [0,1]$.

Proof For the condition (1), when $0 \leq t \leq s \leq 1$ it is obvious that

$$G(t,s) = \frac{t^{\nu-1}(1-s)^{\nu-3}}{\Gamma(\nu)} \geq 0.$$

In the case $0 \leq s \leq t \leq 1$ $(s \neq 1)$, we have

$$\begin{aligned} G(t,s) &= \frac{1}{\Gamma(\nu)} \biggl[\frac{t^{\nu-1}(1-s)^{\nu-1}}{(1-s)^2} - (t-s)^{\nu-1} \biggr] \\ &\geq \frac{1}{\Gamma(\nu)} \bigl[t^{\nu-1}(1-s)^{\nu-1} - (t-s)^{\nu-1} \bigr] \\ &= \frac{1}{\Gamma(\nu)} \bigl[(t-ts)^{\nu-1} - (t-s)^{\nu-1} \bigr] \geq 0. \end{aligned}$$

Moreover, as $G(t,1) = 0$, then we conclude that $G(t,s) \geq 0$ for all $(t,s) \in [0,1] \times [0,1]$. So the condition (1) is true.

For the condition (2), first we prove the left inequality. If $0 \leq s \leq t \leq 1$, then we have $0 \leq t-s \leq t-ts = (1-s)t$, and thus

$$(t-s)^{\nu-1} \leq (1-s)^{\nu-1} t^{\nu-1}.$$

Hence,

$$\begin{aligned} G(t,s) &= \frac{1}{\Gamma(\nu)} \bigl[(1-s)^{\nu-3} t^{\nu-1} - (t-s)^{\nu-1} \bigr] \geq \frac{1}{\Gamma(\nu)} \bigl[(1-s)^{\nu-3} t^{\nu-1} - (1-s)^{\nu-1} t^{\nu-1} \bigr] \\ &= \frac{1}{\Gamma(\nu)} \bigl[(1-s)^{\nu-3} - (1-s)^{\nu-1} \bigr] t^{\nu-1} = \frac{1}{\Gamma(\nu)} s(2-s)(1-s)^{\nu-3} t^{\nu-1}. \end{aligned}$$

If $0 \leq t \leq s \leq 1$, then we have

$$G(t,s) = \frac{1}{\Gamma(\nu)} (1-s)^{\nu-3} t^{\nu-1} \geq \frac{1}{\Gamma(\nu)} s(2-s)(1-s)^{\nu-3} t^{\nu-1}.$$

So the left inequality holds. Evidently, the right inequality also holds. The proof is completed. □

Theorem 4.6 *Let* $3 < \nu \le 4$. *Assume that*

(L_1) $f : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ *is continuous, and* $g, q : [0,1] \times [0,+\infty) \to [0,+\infty)$ *are continuous with* $g(t,0) \not\equiv 0$, $q(t,1) \not\equiv 0$, *and* $f(t,0,1) \not\equiv 0$;

(L_2) $f(t,u,v)$ *is increasing in* $u \in [0,+\infty)$ *for fixed* $t \in [0,1]$ *and* $v \in [0,+\infty)$, *decreasing in* $v \in [0,+\infty)$ *for fixed* $t \in [0,1]$ *and* $u \in [0,+\infty)$, *and* $g(t,u)$ *is increasing in* $u \in [0,+\infty)$ *for fixed* $t \in [0,1]$, *and* $q(t,v)$ *is decreasing in* $v \in [0,+\infty)$ *for fixed* $t \in [0,1]$;

(L_3) $g(t,\lambda u) \ge \lambda g(t,u)$ *for* $\lambda \in (0,1)$, $t \in [0,1]$, $u \in [0,+\infty)$, *and* $q(t,\lambda^{-1}v) \ge \lambda q(t,v)$ *for* $\lambda \in (0,1)$, $t \in [0,1]$, $v \in [0,+\infty)$, *and there exists a constant* $\alpha \in (0,1)$ *such that* $f(t,\lambda u,\lambda^{-1}v) \ge \lambda^{\alpha} f(t,u,v)$, $\forall t \in [0,1]$, $\lambda \in (0,1)$, $u,v \in [0,+\infty)$;

(L_4) *there exists a constant* $\delta > 0$ *such that* $f(t,u,v) \ge \delta(g(t,u) + q(t,v))$, $t \in [0,1]$, $u,v \ge 0$.

Then

(1) *there exists* $u_0, v_0 \in P_h$ *and* $r \in (0,1)$ *such that* $rv_0 \le u_0 < v_0$ *and*

$$\begin{cases} u_0(t) \le \int_0^1 G(t,s)[f(s,u_0(s),v_0(s)) + g(s,u_0(s)) + q(s,v_0(s))]\,ds, & t \in [0,1], \\ v_0(t) \ge \int_0^1 G(t,s)[f(s,v_0(s),u_0(s)) + g(s,v_0(s)) + q(s,u_0(s))]\,ds, & t \in [0,1], \end{cases}$$

where $h(t) = t^{\nu-1}$, $t \in [0,1]$;

(2) *the problem* (4.1) *has a unique positive solution* u^* *in* P_h;

(3) *for any* $x_0, y_0 \in P_h$, *constructing successively the sequences*

$$\begin{cases} x_{n+1}(t) = \int_0^1 G(t,s)[f(s,x_n(s),y_n(s)) + g(s,x_n(s)) + q(s,y_n(s))]\,ds, & n = 0,1,2,\ldots, \\ y_{n+1}(t) = \int_0^1 G(t,s)[f(s,y_n(s),x_n(s)) + g(s,y_n(s)) + q(s,x_n(s))]\,ds, & n = 0,1,2,\ldots, \end{cases}$$

we have $\|x_n - u^*\| \to 0$ *and* $\|y_n - u^*\| \to 0$ *as* $n \to \infty$.

Proof To begin with, from Lemma 4.4, problem (4.1) has an integral formation given by

$$u(t) = \int_0^1 G(t,s)\bigl[f\bigl(s,u(s),u(s)\bigr) + g\bigl(s,u(s)\bigr) + q\bigl(s,u(s)\bigr)\bigr]\,ds.$$

Define three operators $A : P \to E$; $B : P \to E$; $C : P \to E$ by

$$(Au)(t) = \int_0^1 G(t,s)g\bigl(s,u(s)\bigr)\,ds, \qquad (Bv)(t) = \int_0^1 G(t,s)q\bigl(s,v(s)\bigr)\,ds,$$
$$C(u,v)(t) = \int_0^1 G(t,s)f\bigl(s,u(s),v(s)\bigr)\,ds.$$

It is easy to prove that u is the solution of problem (4.1) if and only if $u = Au + Bu + C(u,u)$. From (L_1), we know that $A : P \to P$, $B : P \to P$, $C : P \times P \to P$. In the sequel, we check that A, B, C satisfy all the assumptions of Theorem 3.1.

First, we prove that C is a mixed monotone operator, A is increasing and B is decreasing.

In fact, for $u_i, v_i \in P$, $i = 1,2$ with $u_1 \geq u_2$, $v_1 \leq v_2$, we know that $u_1(t) \geq u_2(t)$, $v_1(t) \leq v_2(t)$, $t \in [0,1]$, and by (L_2) and Lemma 4.5

$$\begin{aligned} C(u_1,v_1)(t) &= \int_0^1 G(t,s)f\bigl(s,u_1(s),v_1(s)\bigr)\,ds \\ &\geq \int_0^1 G(t,s)f\bigl(s,u_2(s),v_2(s)\bigr)\,ds = C(u_2,v_2)(t). \end{aligned}$$

That is, $C(u_1,v_1) \geq C(u_2,v_2)$. Similarly, it follows from (L_2) and Lemma 4.5 that A is increasing and B is decreasing.

Second, we show that B, C satisfies the condition (3.1) and A is sub-homogeneous operator.

For any $\lambda \in (0,1)$ and $u, v \in P$, by (L_3) we have

$$\begin{aligned} B\bigl(\lambda^{-1}v\bigr)(t) &= \int_0^1 G(t,s)q\bigl(s,\lambda^{-1}v(s)\bigr)\,ds \\ &\geq \lambda \int_0^1 G(t,s)q\bigl(t,v(s)\bigr)\,ds = \lambda Bv(t), \\ C\bigl(\lambda u,\lambda^{-1}v\bigr)(t) &= \int_0^1 G(t,s)f\bigl(s,\lambda u(s),\lambda^{-1}v(s)\bigr)\,ds \\ &\geq \lambda^{\alpha} \int_0^1 G(t,s)f\bigl(s,u(s),v(s)\bigr)\,ds = \lambda^{\alpha} C(u,v)(t), \end{aligned}$$

that is, $B(\lambda^{-1}v) \geq \lambda Bv$ for $\lambda \in (0,1)$, $u \in P$, $C(\lambda u,\lambda^{-1}v) \geq \lambda^{\alpha} C(u,v)$ for $\lambda \in (0,1)$, $u, v \in P$.

So, the operators B, C satisfy (3.1). Also, for any $\lambda \in (0,1)$, $u \in P$, from (L_3) we know that

$$A(\lambda u)(t) = \int_0^1 G(t,s)g\bigl(s,\lambda u(s)\bigr)\,ds \geq \lambda \int_0^1 G(t,s)g\bigl(s,u(s)\bigr)\,ds = \lambda Au(t),$$

that is, $A(\lambda u) \geq \lambda Au$ for $\lambda \in (0,1)$, $u \in P$. So, the operator A is sub-homogeneous.

Third, we show that $Ah \in P_h$, $Bh \in P_h$, and $C(h,h) \in P_h$.

In fact, from (L_1), (L_2) and Lemma 4.5, for any $t \in [0,1]$, we have

$$\begin{aligned} C(h,h)(t) &= \int_0^1 G(t,s)f\bigl(s,h(s),h(s)\bigr)\,ds = \int_0^1 G(t,s)f\bigl(s,s^{\nu-1},s^{\nu-1}\bigr)\,ds \\ &\geq \frac{1}{\Gamma(\nu)}h(t)\int_0^1 s(2-s)(1-s)^{\nu-3}f(s,0,1)\,ds, \\ C(h,h)(t) &= \int_0^1 G(t,s)f\bigl(s,h(s),h(s)\bigr)\,ds = \int_0^1 G(t,s)f\bigl(s,s^{\nu-1},s^{\nu-1}\bigr)\,ds \\ &\leq \frac{1}{\Gamma(\nu)}h(t)\int_0^1 (1-s)^{\nu-3}f(s,1,0)\,ds, \end{aligned}$$

from (L_2), (L_4), we have

$$f(s,1,0) \geq f(s,0,1) \geq 0.$$

Since $f(t,0,1) \not\equiv 0$, we get

$$\int_0^1 f(s,1,0)\,ds \geq \int_0^1 f(s,0,1)\,ds > 0,$$

and in consequence

$$l_1 = \frac{1}{\Gamma(\nu)} \int_0^1 s(2-s)(1-s)^{\nu-3} f(s,0,1)\,ds > 0,$$
$$l_2 = \frac{1}{\Gamma(\nu)} \int_0^1 (1-s)^{\nu-3} f(s,1,0)\,ds > 0.$$

So, $l_1 h(t) \leq C(h,h)(t) \leq l_2 h(t)$, $t \in [0,1]$, and hence we have $C(h,h) \in P_h$. Similarly,

$$\frac{1}{\Gamma(\nu)} h(t) \int_0^1 s(2-s)(1-s)^{\nu-3} g(s,0)\,ds \leq Ah(t) \leq \frac{1}{\Gamma(\nu)} h(t) \int_0^1 (1-s)^{\nu-3} g(s,1)\,ds,$$
$$\frac{1}{\Gamma(\nu)} h(t) \int_0^1 s(2-s)(1-s)^{\nu-3} q(s,1)\,ds \leq Bh(t) \leq \frac{1}{\Gamma(\nu)} h(t) \int_0^1 (1-s)^{\nu-3} q(s,0)\,ds,$$

from $g(t,0) \not\equiv 0$, $q(t,1) \not\equiv 0$, we easily prove $Ah \in P_h$, $Bh \in P_h$. Hence the condition (H_1) of Theorem 3.1 is satisfied.

Lastly, we show the condition (H_2) of Theorem 3.1 is satisfied.

For $u, v \in P$ and any $t \in [0,1]$. From (L_4)

$$\begin{aligned} C(u,v)(t) &= \int_0^1 G(t,s) f\bigl(s,u(s),v(s)\bigr)\,ds \\ &\geq \delta \int_0^1 G(t,s)\bigl(g\bigl(s,u(s)\bigr) + q\bigl(s,v(s)\bigr)\bigr)\,ds \\ &= \delta \left[\int_0^1 G(t,s) g\bigl(s,u(s)\bigr)\,ds + \int_0^1 G(t,s) q\bigl(s,v(s)\bigr)\,ds\right] \\ &= \delta\bigl[(Au)(t) + (Bv)(t)\bigr], \end{aligned}$$

then we get $C(u,v) \geq \delta(Au + Bv)$ for $u, v \in P$.

So the condition of Theorem 4.6 follows from Theorem 3.1. □

By using Theorem 3.7, we can easily prove the following conclusion.

Theorem 4.7 *Let* $3 < \nu \leq 4$. *Assume that* (L_1) *and* (L_2) *hold and satisfy the following conditions*:

(L_5) $f(t,\lambda u,\lambda^{-1}v) \geq \lambda f(t,u,v)$, $\forall t \in [0,1]$, $\lambda \in (0,1)$, $u,v \in [0,+\infty)$ *and* $g(t,\lambda u) \geq \lambda g(t,u)$ *for* $\lambda \in (0,1)$, $t \in [0,1]$, $u \in [0,+\infty)$, *and there exists a constant* $\alpha \in (0,1)$ *such that* $q(t,\lambda^{-1}v) \geq \lambda^{\alpha} q(t,v)$ *for* $\lambda \in (0,1)$, $t \in [0,1]$, $v \in [0,+\infty)$;

(L_6) *there exists a constant* $\delta > 0$ *such that* $g(t,u) + f(t,u,v) \leq \delta q(t,v)$, $t \in [0,1]$, $u,v \geq 0$.

Then the conclusions (1)-(3) *of Theorem* 4.6 *hold*.

By using Theorem 3.8, we can easily prove the following conclusion.

Theorem 4.8 *Let* $3 < \nu \le 4$. *Assume that* (L_1) *and* (L_2) *hold and satisfy the following conditions*:

(L_7) $f(t,\lambda u,\lambda^{-1}v) \ge \lambda f(t,u,v)$, $\forall t \in [0,1]$, $\lambda \in (0,1)$, $u,v \in [0,+\infty)$ *and there exists a constant* $\alpha \in (0,1)$ *such that* $g(t,\lambda u) \ge \lambda^{\alpha} g(t,u)$ *for* $\lambda \in (0,1)$, $t \in [0,1]$, $u \in [0,+\infty)$, *and* $q(t,\lambda^{-1}v) \ge \lambda q(t,v)$ *for* $\lambda \in (0,1)$, $t \in [0,1]$, $v \in [0,+\infty)$;

(L_8) *there exists a constant* $\delta > 0$ *such that* $q(t,v) + f(t,u,v) \le \delta g(t,u)$, $t \in [0,1]$, $u,v \ge 0$.

Then the conclusions (1)-(3) *of Theorem* 4.6 *hold*.

Example 4.9 Consider the following problem:

$$\begin{cases} -D_{0^+}^{\frac{10}{3}} u(t) = u^{\frac{1}{3}}(t) + u^{-\frac{1}{3}}(t) + u^{-1}(t) + \frac{u(t)}{1+u(t)} m(t) + a(t) + b, & 0 < t < 1, \\ u(0) = u'(0) = u''(0) = u''(1) = 0, \end{cases} \tag{4.3}$$

where $b > 0$ is a constant, $a, m : [0,1] \to [0,+\infty]$ are continuous with $m \not\equiv 0$.

In this example, we have $\nu = \frac{10}{3}$. Take $0 < c < b$ and let

$$f(t,x,y) = x^{\frac{1}{3}} + y^{-\frac{1}{3}} + a(t) + c, \qquad g(t,x) = \frac{x}{1+x} m(t) + b - c, \qquad q(t,y) = \frac{1}{y} + a(t) + c,$$

$$\alpha = \frac{1}{3}, \qquad m_{\max} = \max\bigl\{m(t) : t \in [0,1]\bigr\}, \qquad a_{\max} = \max\bigl\{a(t) : t \in [0,1]\bigr\}.$$

Obviously, $m_{\max} > 0$, $a_{\max} > 0$. $f : [0,1] \times [0,+\infty] \times [0,+\infty] \to [0,+\infty]$, and $g, q : [0,1] \times [0,+\infty] \to [0,+\infty]$ are continuous. $f(t,x,y)$ is increasing in $x \in [0,+\infty)$ for fixed $t \in [0,1]$ and $y \in [0,+\infty)$, decreasing in $y \in [0,+\infty)$ for fixed $t \in [0,1]$ and $x \in [0,+\infty)$, and $g(t,x)$ is increasing in $x \in [0,+\infty)$ for fixed $t \in [0,1]$, and $q(t,y)$ is decreasing in $y \in [0,+\infty)$ for fixed $t \in [0,1]$. $g(t,0) = b - c > 0$, $q(t,1) = 1 + a(t) + c > 0$, $f(t,0,1) = 1 + a(t) + c > 0$. Besides, for $\lambda \in (0,1)$, $t \in [0,1]$, $x \in [0,+\infty)$, $y \in [0,+\infty)$, we have

$$\begin{aligned} f\bigl(t,\lambda x,\lambda^{-1}y\bigr) &= (\lambda x)^{\frac{1}{3}} + \bigl(\lambda^{-1}y\bigr)^{-\frac{1}{3}} + a(t) + c \ge \lambda^{\frac{1}{3}}\bigl(x^{\frac{1}{3}} + y^{-\frac{1}{3}} + a(t) + c\bigr) = \lambda^{\alpha} f(t,x,y), \\ g(t,\lambda x) &= \frac{\lambda x}{1+\lambda x} m(t) + b - c \ge \frac{\lambda x}{1+x} m(t) + \lambda(b-c) = \lambda g(t,x), \\ q\bigl(t,\lambda^{-1}y\bigr) &= \bigl(\lambda^{-1}y\bigr)^{-1} + a(t) + c \ge \lambda\bigl(y^{-1} + a(t) + c\bigr) = \lambda q(t,y). \end{aligned}$$

Moreover, if we take $\delta \in (0, \frac{c}{m_{\max}+b+a_{\max}}]$, then we obtain

$$\begin{aligned} f(t,x,y) = x^{\frac{1}{3}} + y^{-\frac{1}{3}} + a(t) + c \ge c + y^{-1} &= \frac{c}{m_{\max} + b + a_{\max}}(m_{\max} + b + a_{\max}) + y^{-1} \\ &\ge \delta\left[\frac{x}{1+x} m(t) + b - c + a(t) + c + \frac{1}{y}\right] = \delta\bigl[g(t,x) + q(t,y)\bigr]. \end{aligned}$$

Hence all the conditions of Theorem 4.6 are satisfied. An application of Theorem 4.6 implies that problem (4.3) has a unique positive solution in P_h, where $h(t) = t^{\nu-1} = t^{\frac{7}{3}}$, $t \in [0,1]$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

HW participated in the design of the study and drafted the manuscript. LZ carried out the theoretical studies and helped to draft the manuscript. All authors read and approved the final manuscript.

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