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Research Article

Global Exponential Stability of Delayed Cohen-Grossberg BAM Neural Networks with Impulses on Time Scales

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Based on the theory of calculus on time scales, the homeomorphism theory, Lyapunov functional method, and some analysis techniques, sufficient conditions are obtained for the existence, uniqueness, and global exponential stability of the equilibrium point of Cohen-Grossberg bidirectional associative memory (BAM) neural networks with distributed delays and impulses on time scales. This is the first time applying the time-scale calculus theory to unify the discrete-time and continuous-time Cohen-Grossberg BAM neural network with impulses under the same framework.

1. Introduction

In the recent years, bidirectional associative memory (BAM) neural networks and Cohen-Grossberg neural networks (CGNNs) with their various generalizations have attracted the attention of many mathematicians, physicists, and computer scientists (see [1–17]) due to their wide range of applications in, for example, pattern recognition, associative memory, and combinatorial optimization. Particularly, as discussed in [18–20], in the hardware implementation of the neural networks, when communication and response of neurons happens time delays may occur. Actually, time delays are known to be a possible source of instability in many real-world systems in engineering, biology, and so forth. (see, e.g., [21] and references therein). However, besides delay effect, impulsive effect likewise exists in a wide variety of evolutionary processes in which states are changed abruptly at certain moments of time, involving fields such as medicine and biology, economics, mechanics, electronics, and telecommunications. As artificial electronic systems, neural networks such as Hopfield neural networks, bidirectional neural networks, and recurrent neural networks often are subject to impulsive perturbations which can affect dynamical behaviors of the systems just as time delays. Therefore, it is necessary to consider both impulsive effect and delay effect on the stability of neural networks.

As is well known, both continuous and discrete systems are very important in implementation and applications. However, it is troublesome to study the stability for continuous and discrete systems, respectively. Therefore, it is worth studying a new method, such as the time-scale theory, which can unify the continuous and discrete situations.

Motivated by the above discussions, the objective of this paper is to study the global exponential stability of the following Cohen-Grossberg bidirectional associative memory networks with impulses and time delays on time scales:

$$\begin{aligned}
x_i^{\Delta}(t) &= -a_i(x_i(t))\Bigg[b_i(x_i(t)) - \sum_{j=1}^{m} p_{ji}^0 f_j\left(y_j(t-\tau_{ji})\right) \\
&\qquad - \sum_{j=1}^{m} p_{ji}^1 \int_0^{\infty} h_{ij}(s) f_j\left(y_j(t-s)\right)\Delta s + r_i\Bigg], \quad t \geq 0,\ t \neq t_k,\ t \in \mathbb{T}, \\
\Delta x_i(t_k) &= I_k(x_i(t_k)), \quad i = 1,2,\ldots,n,\ k = 1,2,\ldots, \\
y_j^{\Delta}(t) &= -c_j\left(y_j(t)\right)\Bigg[d_j\left(y_j(t)\right) - \sum_{i=1}^{n} q_{ij}^0 g_i\left(x_i(t-\sigma_{ij})\right) \\
&\qquad - \sum_{i=1}^{n} q_{ij}^1 \int_0^{\infty} k_{ij}(s) g_i(x_i(t-s))\Delta s + s_j\Bigg], \quad t \geq 0,\ t \neq t_k,\ t \in \mathbb{T}, \\
\Delta y_j(t_k) &= J_k\left(y_j(t_k)\right), \quad j = 1,2,\ldots,m,\ k = 1,2,\ldots,
\end{aligned} \tag{1.1}$$

where $\mathbb{T}$ is a time scale; $I_k, J_k : \mathbb{R} \to \mathbb{R}$ are continuous, $x_i(t)$, $y_j(t)$ are the states of the ith neuron from the neural field F_X and the jth neuron from the neural field F_Y at time t, respectively; f_j, g_i denote the activation functions of the jth neuron from F_Y and the ith neuron from F_X, respectively; r_i and s_j are constants, which denote the external inputs on the ith neuron from F_X and the jth neuron from F_Y, respectively; τ_{ji} and σ_{ij} correspond to the transmission delays; $a_i(x_i(t))$ and $c_j(y_j(t))$ represent amplification functions; $b_i(x_i(t))$ and $d_j(y_j(t))$ are appropriately behaved functions such that the solutions of system (1.1) remain bounded; $p_{ji}^0, p_{ji}^1, q_{ij}^0$, and q_{ij}^1 denote the connection strengths which correspond to the neuronal gains associated with the neuronal activations; I_i and J_j denote the external inputs. For each interval I of $\mathbb{R}$, we denote that by $\mathrm{I}_{\mathbb{T}} = I \bigcap T$, $\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-)$, $\Delta y_j(t_k) = y_j(t_k^+) - y_j(t_k^-)$ are the impulses at moments t_k, and $x_i(t_k^+), x_i(t_k^-), y_j(t_k^+), y_j(t_k^-)$ $(i = 1,2,\ldots,n, j = 1,2,\ldots,m)$ represent the right and left limits of $x_i(t_k^+)$ and $y_j(t_k^+)$ in the sense of time scales; $0 < t_1 < t_2 < \cdots < t_k \to \infty$ is a strictly increasing sequence.

The system (1.1) is supplement with initial values given by

$$\begin{aligned}
x_i(s) &= \varphi_i(s), \quad s \in [-\infty, 0]_{\mathbb{T}},\ i = 1,2,\ldots,n, \\
y_j(s) &= \psi_j(s), \quad s \in [-\infty, 0]_{\mathbb{T}},\ j = 1,2,\ldots,m,
\end{aligned} \tag{1.2}$$

where φ_i, ψ_j are continuous real-valued functions defined on their respective domains.

As usual in the theory of impulsive differential equations, at the points of discontinuity t_k of the solution $t \to (x_1(t), x_2(t), \ldots, x_n(t), y_1(t), y_2(t), \ldots, y_m(t))^T$ we assume that

$$x_i(t_k) = x_i(t_k^-), \qquad y_j(t_k) = y_j(t_k^-), \qquad x_i^{\Delta}(t_k) = x_i^{\Delta}(t_k^-), \qquad y_j^{\Delta}(t_k) = y_j^{\Delta}(t_k^-), \tag{1.3}$$

for $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$.

The organization of the rest of this paper is as follows. In Section 2, we introduce some notations and definitions, and state some preliminary results which are needed in later sections. In Section 3, by means of homeomorphism theory, we study the existence and uniqueness of the equilibrium point of system (1.1). In Section 4, by constructing a suitable Lyapunov function, we establish the exponential stability of the equilibrium of (1.1). In Section 5, we present an example to illustrate the feasibility and effectiveness of our results obtained in previous sections.

2. Preliminaries

In this section, we will cite some definitions and lemmas which will be used in the proofs of our main results.

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \qquad \mu(t) = \sigma(t) - t. \tag{2.1}$$

A point $t \in \mathbb{T}$ is called left dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left scattered if $\rho(t) < t$, right dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is right dense continuous provided that it is continuous at right dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. If f is continuous at each right dense point and each left-dense point, then f is said to be a continuous function on $\mathbb{T}$. The set of continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C(\mathbb{T})$.

For $y : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, we define the delta derivative of $y(t), y^{\Delta}(t)$ to be the number (if it exists) with the property that for a given $\varepsilon > 0$, there exists a neighborhood U of t such that

$$\left| [y(\sigma(t)) - y(s)] - y^{\Delta}(t)[\sigma(t) - s] \right| < \varepsilon |\sigma(t) - s| \tag{2.2}$$

for all $s \in U$.

If y is continuous, then y is right dense continuous, and y is delta differentiable at t, then y is continuous at t.

Let y be right dense continuous. If $y^{\Delta}(t) = y(t)$, then we define the delta integral by

$$\int_a^t y(s)\Delta s = Y(t) - Y(a). \tag{2.3}$$

Definition 2.1 (see [22]). For each $t \in \mathbb{T}$, let N be a neighborhood of t, then, for $V \in C_{\mathrm{rd}}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^+]$, define $D^+V^{\Delta}(t, x(t))$ to mean that, given $\varepsilon > 0$, there exists a right neighborhood $N_{\varepsilon} \subset N$ of t such that

$$\frac{\left[V(\sigma(t), x(\sigma(t))) - V(s, x(\sigma(t))) - \mu(t,s) f(t, x(t))\right]}{\mu(t,s)} < D^+V^{\Delta}(t, x(t)) + \varepsilon \tag{2.4}$$

for each $s \in N_{\varepsilon}$, $s > t$, where $\mu(t,s) \equiv \sigma(t) - s$. If t is rd and $V(t, x(t))$ is continuous at t, this reduce to

$$D^+V^{\Delta}(t, x(t)) = \frac{V(\sigma(t), x(\sigma(t))) - V(t, x(\sigma(t)))}{\sigma(t) - t}. \tag{2.5}$$

Definition 2.2 (see [23]). If $a \in \mathbb{T}, \sup \mathbb{T} = \infty$, and f is right dense continuous on $[a, \infty)$, then we define the improper integral by

$$\int_a^{\infty} f(t) \Delta t = \lim_{b \to \infty} \int_a^b f(t) \Delta t \tag{2.6}$$

provided that this limit exists, and we say that the improper integral converges in this case. If this limit does not exist, then we say that the improper integral diverges.

A function $r : \mathbb{T} \to R$ is called regressive if

$$1 + \mu(t) r(t) \neq 0 \tag{2.7}$$

for all $t \in \mathbb{T}^k$.

If r is regressive function, then the generalized exponential function e_r is defined by

$$e_r(t,s) = \exp\left\{\int_s^t \xi_{\mu(\tau)}(r(\tau)) \Delta\tau\right\} \quad \text{for } s, t \in \mathbb{T}, \tag{2.8}$$

with the cylinder transformation

$$\xi_h(z) = \begin{cases} \dfrac{\mathrm{Log}(1 + hz)}{h}, & \text{if } h \neq 0, \\ z, & \text{if } h = 0. \end{cases} \tag{2.9}$$

Let $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, then we define

$$p \oplus q := p + q + \mu p q, \qquad p \ominus q := p \oplus (\ominus q), \qquad \ominus p := \frac{p}{1 + \mu p}. \tag{2.10}$$

Then the generalized exponential function has the following properties.

Lemma 2.3 (see [24]). *Assume that* $p,\ q : \mathbb{T} \to \mathbb{R}$ *are two regressive functions, then*

(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$

(ii) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$

(iii) $e_p(t,\sigma(s)) = e_p(t,s)/(1+\mu(s)p(s))$

(iv) $1/e_p(t,s) = e_{\ominus p}(t,s)$

(v) $e_p(t,s) = 1/(e_p(s,t)) = e_{\ominus p}(s,t)$

(vi) $e_p(t,s)e_p(s,r) = e_p(t,r)$

(vii) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$

(viii) $e_p(t,s)/e_q(t,s) = e_{p\ominus q}(t,s)$.

Definition 2.4. The equilibrium point $u^* = (x_1^*, x_2^*, \dots, x_n^*, y_1^*, y_2^*, \dots, y_m^*)^T$ of system (1.1) is said to be exponentially stable if there exists a positive constant α such that for every $\delta \in \mathbb{T}$, there exists $N = N(\delta) \geq 1$ such that the solution $u(t) = (x_1(t), x_2(t), \dots, x_n(t), y_1(t), y_2(t), \dots, y_m(t))^T$ of (1.1) with initial value $(\varphi_1(s), \varphi_2(s), \dots, \varphi_n(s), \psi_1(s), \psi_2(s), \dots, \psi_m(s))^T$ satisfies

$$\|u-u^*\| \leq N e_{-\alpha}(t,\delta)\left[\sum_{i=1}^{n}\max_{\delta\in(-\infty,0]_{\mathbb{T}}}\left|\varphi_i(\delta)-x_i^*\right| + \sum_{j=1}^{m}\max_{\delta\in(-\infty,0]_{\mathbb{T}}}\left|\psi_j(\delta)-y_j^*\right|\right]. \tag{2.11}$$

Lemma 2.5 (see [25]). *If* $H(x) \in C(\mathbb{R}^{n+m}, \mathbb{R}^{n+m})$ *satisfies the following conditions:*

(i) $H(x)$ *is injective on* $\mathbb{R}^{n+m}$,

(ii) $\|H\| \to +\infty$ *as* $\|x\| \to +\infty$,

then $H(x)$ *is a homeomorphism of* $\mathbb{R}^{n+m}$ *onto itself.*

For $z = (x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m)^T \in \mathbb{R}^{n+m}$, we define the norm as

$$\|z\| = \sum_{i=1}^{n}|x_i| + \sum_{j=1}^{m}|y_j|. \tag{2.12}$$

Throughout this paper, we assume that

(H_1) $a_i, c_j \in C(\mathbb{T}, \mathbb{R}^+)$, and satisfy $0 < \underline{a}_i \leq a_i(x) \leq \overline{a}_i, 0 < \underline{c}_j \leq c_j(x) \leq \overline{c}_j$, $\forall x \in \mathbb{R}$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$;

(H_2) the activation functions $f_j, g_i \in C(\mathbb{R}, \mathbb{R})$ and there exist positive constants M_j, N_i such that

$$\left|f_j(x)-f_j(y)\right| \leq M_j|x-y|, \qquad \left|g_i(x)-g_i(y)\right| \leq N_i|x-y|, \tag{2.13}$$

for all $x, y \in \mathbb{R}$, $i = 1, \dots, n$, $j = 1, \dots, m$;

(H_3) $b_i, d_j \in C(\mathbb{R}, \mathbb{R}), b_i(0) = d_j(0) = 0$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, and there exist positive constants η_i, ω_j such that

$$\frac{b_i(x) - b_i(y)}{x - y} \geq \eta_i, \quad \frac{d_j(x) - d_j(y)}{x - y} \geq \omega_j, \quad \forall x \neq y; \tag{2.14}$$

(H_4) the kernels h_{ji} and k_{ij} defined on $[0, \infty)_{\mathbb{T}}$ are nonnegative continuous integral functions such that $\int_0^\infty h_{ji}(s)\Delta s = 1, \int_0^\infty s h_{ji}(s)\Delta s < +\infty$, $\int_0^\infty k_{ij}(s)\Delta s = 1, \int_0^\infty s k_{ij}(s)\Delta s < +\infty$.

3. Existence and Uniqueness of the Equilibrium

In this section, using homeomorphism theory, we will study the existence and uniqueness of the equilibrium point of system (1.1).

An equilibrium point of (1.1) is a constant vector $(x_1^*, x_2^*, \ldots, x_n^*, y_1^*, y_2^*, \ldots, y_m^*)^T \in \mathbb{R}^{n+m}$ which satisfies the system

$$\begin{gathered} a_i(x_i^*)\left[b_i(x_i^*) - \sum_{j=1}^m \left(p_{ji}^0 + p_{ji}^1\right) f_j\left(y_j^*\right) + r_i\right] = 0, \quad i = 1, 2, \ldots, n, \\ c_j\left(y_j^*\right)\left[d_j\left(y_j^*\right) - \sum_{i=1}^n \left(q_{ij}^0 + q_{ij}^1\right) g_i(x_i^*) + s_j\right] = 0, \quad j = 1, 2, \ldots, m, \end{gathered} \tag{3.1}$$

where the impulsive jumps $I_k(\cdot), J_k(\cdot)$ satisfy

$$I_k(x_i^*) = 0, \quad i = 1, 2, \ldots, n, \quad J_k\left(y_j^*\right) = 0, \quad j = 1, 2, \ldots, m. \tag{3.2}$$

From the assumptions (H_1) and (H_4), it follows that

$$\begin{gathered} b_i(x_i^*) = \sum_{j=1}^m \left(p_{ji}^0 + p_{ji}^1\right) f_j\left(y_j^*\right) + r_i, \quad i = 1, 2, \ldots, n, \\ d_j\left(y_j^*\right) = \sum_{i=1}^n \left(q_{ij}^0 + q_{ij}^1\right) g_i(x_i^*) + s_j, \quad j = 1, 2, \ldots, m. \end{gathered} \tag{3.3}$$

Noting that if $b_i^{-1}(\cdot), d_j^{-1}(\cdot)$ exist and activation functions $f_j(\cdot)$ and $g_j(\cdot)$ are bounded, then the existence of an equilibrium point of system (1.1) is easily obtained from Brouwer's fixed point theorem. We can refer to [2–8].

Theorem 3.1. *Assume that* (H_2) *and* (H_3) *hold. Suppose further that for each* $i = 1,2,\ldots,n, j = 1,2,\ldots,m$*, the following inequalities are satisfied:*

$$\eta_i > \sum_{j=1}^{m}\left|q_{ij}^0 + q_{ij}^1\right|N_i, \qquad \omega_j > \sum_{i=1}^{n}\left|p_{ji}^0 + p_{ji}^1\right|M_j. \tag{3.4}$$

Then there exists a unique equilibrium point of system (1.1).

Proof. Consider a mapping $\Phi : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ defined by

$$\begin{aligned}
\Phi_i(z) &= b_i(x_i) - \sum_{j=1}^{m}\left(p_{ji}^0 + p_{ji}^1\right)f_j(y_j) + r_i, \quad i = 1,2,\ldots,n, \\
\Phi_i(z) &= d_j(y_j) - \sum_{i=1}^{n}\left(q_{ij}^0 + q_{ij}^1\right)g_i(x_i) + s_j, \quad j = 1,2,\ldots,m,
\end{aligned} \tag{3.5}$$

where $z = (x_1,x_2,\ldots,x_n,y_1,y_2,\ldots,y_m)^T \in \mathbb{R}^{n+m}, \Phi(z) = (\Phi_1(z),\ldots,\Phi_n(z),\ldots,\Phi_{n+m}(z))^T \in \mathbb{R}^{n+m}$. First, we want to show that Φ is an injective mapping on $\mathbb{R}^{n+m}$. By contradiction, suppose that there exists a distinct z, $\overline{z} \in \mathbb{R}^{n+m}$ such that $\Phi(z) = \Phi(\overline{z})$, where $z = (x_1,x_2,\ldots,x_n,y_1,y_2,\ldots,y_m)^T \in \mathbb{R}^{n+m}$ and $\overline{z} = (\overline{x}_1,\overline{x}_2\ldots,\overline{x}_n,\overline{y}_1,\overline{y}_2,\ldots,\overline{y}_m)^T \in \mathbb{R}^{n+m}$. Then it follows from (3.5) that

$$\begin{aligned}
b_i(x_i) - b_i(\overline{x}_i) &= \sum_{j=1}^{m}\left(p_{ji}^0 + p_{ji}^1\right)\left[f_j(y_j) - f_j\left(\overline{y}_j\right)\right], \quad i = 1,2,\ldots,n, \\
d_j(y_j) - d_j\left(\overline{y}_j\right) &= \sum_{i=1}^{n}\left(q_{ij}^0 + q_{ij}^1\right)\left[g_i(x_i) - g_j(\overline{x}_i)\right], \quad j = 1,2,\ldots,m.
\end{aligned} \tag{3.6}$$

In view of (H_2)-(H_3) and (3.6), we have

$$\begin{aligned}
\sum_{i=1}^{n}\eta_i|x_i - \overline{x}_i| &\le \sum_{i=1}^{n}\sum_{j=1}^{m}\left|p_{ji}^0 + p_{ji}^1\right|M_j\left|y_j - \overline{y}_j\right|, \\
\sum_{j=1}^{m}\omega_j\left|y_j - \overline{y}_j\right| &\le \sum_{j=1}^{m}\sum_{i=1}^{n}\left|q_{ij}^0 + q_{ij}^1\right|N_i|x_i - \overline{x}_i|.
\end{aligned} \tag{3.7}$$

Thus, we can obtain

$$\sum_{i=1}^{n}\left[\eta_i - \sum_{j=1}^{m}\left|q_{ij}^0 + q_{ij}^1\right|N_i\right]|x_i - \overline{x}_i| + \sum_{j=1}^{m}\left[\omega_j - \sum_{i=1}^{n}\left|p_{ji}^0 + p_{ji}^1\right|M_j\right]\left|y_j - \overline{y}_j\right| \le 0. \tag{3.8}$$

It follows from (3.4) and (3.8) that $|x_i - \overline{x}_i| = 0$ and $|y_j - \overline{y}_j| = 0$, $i = 1,2,\ldots, n$, $j = 1,2,\ldots,m$. That is $z = \overline{z}$, which leads to a contradiction. Therefore, Φ is an injective on $\mathbb{R}^{n+m}$.

Then we will prove Φ is a homeomorphism on $\mathbb{R}^{n+m}$. For convenience, we let $\widetilde{\Phi}(z) = \Phi(z) - \Phi(0)$, where

$$\begin{aligned}\widetilde{\Phi}_i(z) &= b_i(x_i) - \sum_{j=1}^{m}\left(p_{ji}^0 + p_{ji}^1\right)\left[f_j(y_j) - f_j(0)\right], \quad i = 1,2,\ldots,n,\\ \widetilde{\Phi}_{n+j}(z) &= d_j(y_j) - \sum_{i=1}^{n}\left(q_{ij}^0 + q_{ij}^1\right)\left[g_i(x_i) - g_i(0)\right], \quad j = 1,2,\ldots,m.\end{aligned} \tag{3.9}$$

We assert that $\|\widetilde{\Phi}\| \to \infty$ as $\|z\| \to \infty$. Otherwise there is a sequence $\{z^v\}$ such that $\|z^v\| \to \infty$ and $\|\widetilde{\Phi}(z^v)\|$ is bounded as $v \to \infty$, where $z^v = (x_1^v, x_2^v, \ldots, x_n^v, y_1^v, y_2^v, \ldots, y_m^v)^T \in \mathbb{R}^{n+m}$. Noting that

$$\begin{aligned}\sum_{i=1}^{n}\mathrm{sgn}(x_i^v)\left\{b_i(x_i^v) - \widetilde{\Phi}_i(z^v)\right\} &= \sum_{i=1}^{n}\mathrm{sgn}(x_i^v)\sum_{j=1}^{m}\left(p_{ji}^0 + p_{ji}^1\right)\left[f_j\left(y_j^v\right) - f_j(0)\right]\\ &\le \sum_{i=1}^{n}\sum_{j=1}^{m}\left|p_{ji}^0 + p_{ji}^1\right|M_j\left|y_j^v\right|,\\ \sum_{j=1}^{m}\mathrm{sgn}\left(y_j^v\right)\left\{d_j\left(y_j^v\right) - \widetilde{\Phi}_{n+j}(z^v)\right\} &= \sum_{j=1}^{m}\mathrm{sgn}\left(y_j^v\right)\sum_{i=1}^{n}\left(q_{ij}^0 + q_{ij}^1\right)\left[g_i(x_i^v) - g_i(0)\right]\\ &\le \sum_{j=1}^{m}\sum_{i=1}^{n}\left|q_{ij}^0 + q_{ij}^1\right|N_i\left|x_i^v\right|,\end{aligned} \tag{3.10}$$

we have

$$\begin{aligned}&\sum_{i=1}^{n}\mathrm{sgn}(x_i^v)\left\{b_i(x_i^v) - \widetilde{\Phi}_i(z^v)\right\} + \sum_{j=1}^{m}\mathrm{sgn}\left(y_j^v\right)\left\{d_j\left(y_j^v\right) - \widetilde{\Phi}_{n+j}(z^v)\right\}\\ &\quad\le \sum_{i=1}^{n}\sum_{j=1}^{m}\left|p_{ji}^0 + p_{ji}^1\right|M_j\left|y_j^v\right| + \sum_{j=1}^{m}\sum_{i=1}^{n}\left|q_{ij}^0 + q_{ij}^1\right|N_i\left|x_i^v\right|.\end{aligned} \tag{3.11}$$

On the other hand, we have

$$\begin{aligned}&\sum_{i=1}^{n}\mathrm{sgn}(x_i^v)\left\{b_i(x_i^v) - \widetilde{\Phi}_i(z^v)\right\} + \sum_{j=1}^{m}\mathrm{sgn}\left(y_j^v\right)\left\{d_j\left(y_j^v\right) - \widetilde{\Phi}_{n+j}(z^v)\right\}\\ &\quad\ge \sum_{i=1}^{n}\eta_i\left|x_i^v\right| - \sum_{i=1}^{n}\left|\widetilde{\Phi}_i(z^v)\right| + \sum_{j=1}^{m}\omega_j\left|y_j^v\right| - \sum_{j=1}^{m}\left|\widetilde{\Phi}_{n+j}(z^v)\right|.\end{aligned} \tag{3.12}$$

It follows from (3.11) and (3.12) that

$$\Theta\left(\sum_{i=1}^{n}\left|x_i^v\right|+\sum_{j=1}^{m}\left|y_j^v\right|\right)\leq\sum_{i=1}^{n}\left|\widetilde{\Phi}_i(z^v)\right|+\sum_{j=1}^{m}\left|\widetilde{\Phi}_{n+j}(z^v)\right|, \tag{3.13}$$

where

$$\Theta=\min\left\{\min_{1\leq i\leq n}\left\{\eta_i-\sum_{j=1}^{m}\left|q_{ij}^0+q_{ij}^1\right|N_i\right\},\min_{1\leq j\leq m}\left\{\omega_j-\sum_{i=1}^{n}\left|p_{ji}^0+p_{ji}^1\right|M_j\right\}\right\}>0. \tag{3.14}$$

That is

$$\|z^v\|\leq\frac{1}{\Theta}\left\|\widetilde{\Phi}(z^v)\right\|, \tag{3.15}$$

which contradicts our choice of $\{z^v\}$. Hence, Φ satisfies $\|\Phi\|\to\infty$ as $\|z\|\to\infty$. By Lemma 2.5, Φ is a homeomorphism on $\mathbb{R}^{n+m}$ and there exists a unique point $z^*=(x_1^*,x_2^*,\ldots,x_n^*,y_1^*,y_2^*,\ldots,y_m^*)^T$ such that $\Phi(z^*)=0$. From the definition of Φ, we know that $z^*=(x_1^*,x_2^*,\ldots,x_n^*,y_1^*,y_2^*,\ldots,y_m^*)^T$ is the unique equilibrium point of (1.1). □

4. Global Exponential Stability of the Equilibrium

In this section, we will construct some suitable Lyapunov functions to derive the sufficient conditions which ensure that the equilibrium of (1.1) is globally exponentially stable.

Theorem 4.1. *Assume that* (H_1)*–*(H_4) *hold, suppose further that*

(H_5) *for each* $i=1,2,\ldots,n,\ j=1,2,\ldots,m$*, the following inequalities are satisfied:*

$$\underline{a}_i\eta_i>\sum_{j=1}^{m}\overline{c_j}\left(\left|q_{ij}^0\right|+\left|q_{ij}^1\right|\right)N_i,\qquad \underline{c}_j\omega_j>\sum_{i=1}^{n}\overline{a}_i\left(\left|p_{ji}^0\right|+\left|p_{ji}^1\right|\right)M_j \tag{4.1}$$

(H_6) *the impulsive operators* $I_{ik}(x_i(t))$ *and* $J_{jk}(y_j(t))$ *satisfy*

$$\begin{gathered}I_{ik}(x_i(t_k))=-\gamma_{ik}\left(x_i(t_k)-x_i^*\right),\quad 0<\gamma_{ik}<2,\ i=1,\ldots,n,\ k\in\mathbb{Z}^+,\\ J_{jk}\left(y_j(t_k)\right)=-\widetilde{\gamma}_{jk}\left(y_j(t_k)-y_j^*\right),\quad 0<\widetilde{\gamma}_{jk}<2,\ j=1,\ldots,m,\ k\in\mathbb{Z}^+.\end{gathered} \tag{4.2}$$

Then the unique equilibrium point of system (1.1) *is globally exponentially stable.*

Proof. According to Theorem 3.1, we know that (1.1) has a unique equilibrium point $z^*=(x_1^*,x_2^*,\ldots,x_n^*,y_1^*,\ldots,y_m^*)^T$. In view of (H_6), it is easy to see that $I_i(x_i^*)=0$ and $J_j(y_j^*)=0$.

Suppose that $z(t) = (x_1(t), x_2(t), \ldots, x_n(t), y_1(t), y_2(t), \ldots, y_m(t))^T$ is an arbitrary solution of (1.1). Let $u_i(t) = x_i(t) - x_i^*,\ v_j(t) = y_j(t) - y_j^*,\ t \geq 0$, then system (1.1) can be rewritten as

$$\begin{aligned}
u_i^{\Delta}(t) = -\tilde{a}_i(u_i(t)) \Bigg[\tilde{b}_i(u_i(t)) &- \sum_{j=1}^{m} p_{ji}^{0} \tilde{f}_j(v_j(t-\tau_{ji})) \\
&- \sum_{j=1}^{m} p_{ji}^{1} \int_0^{\infty} h_{ji}(s) \tilde{f}_j(v_j(t-s)) \Delta s - r_i \Bigg], \\
&i = 1,2,\ldots,n,\ t > 0,\quad t \neq t_k,\ t \in \mathbb{T}, \\
v_j^{\Delta}(t) = -\tilde{c}_j(v_j(t)) \Bigg[\tilde{d}_j(v_j(t)) &- \sum_{i=1}^{n} q_{ij}^{0} \tilde{g}_i(u_i(t-\sigma_{ij})) \\
&- \sum_{i=1}^{n} q_{ij}^{1} \int_0^{\infty} k_{ij}(s) \tilde{g}_i(u_i(t-s)) \Delta s - s_j \Bigg], \\
&j = 1,2,\ldots,m,\ t > 0,\ t \neq t_k,\ t \in \mathbb{T},
\end{aligned} \tag{4.3}$$

where, for $i = 1,2,\ldots,n,\ j = 1,2,\ldots,m$,

$$\begin{aligned}
\tilde{a}_i(u_i(t)) &= a_i(u_i(t) + x_i^*), & \tilde{b}_i(u_i(t)) &= b_i(u_i(t) + x_i^*) - b_i(x_i^*), \\
\tilde{c}_j(v_j(t)) &= c_j\left(v_j(t) + y_j^*\right), & \tilde{f}_i(v_i(t)) &= f_j\left(v_j(t) + y_j^*\right) - f_j\left(y_j^*\right), \\
\tilde{d}_j(v_j(t)) &= d_j\left(v_j(t) + y_j^*\right) - b_i\left(y_j^*\right), & \tilde{g}_i(u_i(t)) &= g_i(u_i(t) + x_i^*) - b_i(x_i^*).
\end{aligned} \tag{4.4}$$

Also, for all $t = t_k,\ k \in \mathbb{Z}^+,\ i = 1,2,\ldots,n,\ j = 1,2,\ldots,m$,

$$\begin{aligned}
\left|u_i(t_k^+)\right| &= \left|x_i(t_k^+) - x_i^*\right| = \left|x_i(t_k) + I_k(x_i(t_k)) - x_i^*\right| \\
&= \left|(1-\gamma_{ik})(x_i(t_k) - x_i^*)\right| \leq \left|x_i(t_k) - x_i^*\right| \leq |u_i(t_k)|, \\
\left|v_j(t_k^+)\right| &= \left|y_j(t_k^+) - y_j^*\right| = \left|y_j(t_k) + I_k(y_j(t_k)) - y_j^*\right| \\
&= \left|(1-\tilde{\gamma}_{jk})\left(y_j(t_k) - y_j^*\right)\right| \leq \left|y_j(t_k) - y_j^*\right| \leq \left|v_j(t_k)\right|.
\end{aligned} \tag{4.5}$$

Hence by (H_1) and (H_2), we have

$$\begin{aligned} D^+|u_i(t)|^\Delta \le & -\underline{a}_i\eta_i|u_i(t)| + \overline{a}_i\sum_{j=1}^{m}\left|p_{ji}^0\right|M_j\left|v_j(t-\tau_{ji})\right| \\ & + \overline{a}_i\sum_{j=1}^{m}\left|p_{ji}^1\right|M_j\int_0^\infty h_{ji}(s)\left|v_j(t-s)\right|\Delta s, \quad i=1,2,\ldots,n, \end{aligned} \tag{4.6}$$

$$\begin{aligned} D^+\left|v_j(t)\right|^\Delta \le & -\underline{c}_j\left|v_j(t)\right| + \overline{c}_j\sum_{i=1}^{n}\left|q_{ij}^0\right|N_i\left|u_i(t-\sigma_{ij})\right| \\ & + \overline{c}_j\sum_{i=1}^{n}\left|q_{ij}^1\right|N_i\int_0^\infty k_{ij}(s)|u_i(t-s)|\Delta s, \quad j=1,2,\ldots,m. \end{aligned} \tag{4.7}$$

Also, for $i=1,2,\ldots,n$,

$$\begin{aligned} x_i(t_k+0) - x_i^*(t_k+0) &= x_i(t_k) + I_{ik}(x_i(t_k)) - x_i^*(t_k) - I_{ik}(x_i^*(t_k)) \\ &= (1-\gamma_{ik})(x_i(t_k) - x_i^*(t_k)), \quad k\in\mathbb{Z}^+, \end{aligned} \tag{4.8}$$

thus

$$\begin{aligned} \left|x_i(t_k^+) - x_i^*(t_k^+)\right| &= |1-\gamma_{ik}|\left|x_i(t_k) - x_i^*(t_k)\right| \\ &\le \left|x_i(t_k) - x_i^*(t_k)\right|, \quad i=1,\ldots,n,\ k\in\mathbb{Z}^+. \end{aligned} \tag{4.9}$$

Similarly, we have

$$\begin{aligned} \left|y_j(t_k^+) - y_j^*(t_k^+)\right| &= \left|1-\gamma_{jk}^*\right|\left|y_j(t_k) - y_j^*(t_k)\right| \\ &\le \left|y_j(t_k) - y_j^*(t_k)\right|, \quad j=1,2,\ldots,m,\ k\in\mathbb{Z}^+. \end{aligned} \tag{4.10}$$

Let G_i and G_j^* be defined by

$$\begin{aligned} G_i(\varepsilon_i) = & \ \underline{a}_i\eta_i - \varepsilon_i - \sum_{j=1}^{m}\overline{c}_j\left|q_{ij}^0\right|N_i e_{\varepsilon_i}(\sigma(t), t-\sigma_{ij}) \\ & - \sum_{j=1}^{m}\overline{c}_j\left|q_{ij}^1\right|N_i\int_0^\infty k_{ij}(s)e_{\varepsilon_i}(\sigma(t), t-s)\Delta s, \quad i=1,2,\ldots,n, \\ G_j^*(\xi_j) = & \ \underline{c}_j\omega_j - \xi_j - \sum_{i=1}^{n}\overline{a}_i\left|p_{ji}^0\right|M_j e_{\xi_j}(\sigma(t), t-\tau_{ji}) \\ & - \sum_{i=1}^{n}\overline{a}_i\left|p_{ji}^1\right|M_j\int_0^\infty h_{ji}(s)e_{\xi_j}(\sigma(t), t-s)\Delta s, \quad j=1,2,\ldots,m, \end{aligned} \tag{4.11}$$

respectively, where $\varepsilon_i, \xi_j \in [0,\infty)$. By (H_5), we have

$$G_i(0) = \underline{a}_i\eta_i - \sum_{j=1}^{m}\overline{c}_j\left(\left|q_{ij}^0\right| + \left|q_{ij}^1\right|\right)N_i > 0, \quad i = 1,2,\ldots,n,$$
$$G_j^*(0) = \underline{c}_j\omega_j - \sum_{i=1}^{n}\overline{a}_i\left(\left|p_{ji}^0\right| + \left|p_{ji}^1\right|\right)M_j > 0, \quad j = 1,2,\ldots,m. \tag{4.12}$$

Since G_i, G_j^* are continuous on $[0,\infty)$ and $G_i(\varepsilon_i) \to -\infty, G_j^*(\xi_j) \to -\infty$, as $\varepsilon_i \to +\infty, \xi_j \to +\infty$, there exist $\varepsilon_i^*, \xi_j^* > 0$ such that $G_i(\varepsilon_i^*) = 0, G_j^*(\xi_j^*) = 0$ and $G_i(\varepsilon_i) > 0$, for $\varepsilon_i \in (0,\varepsilon_i^*), G_j^*(\xi_j) > 0$, for $\xi_j \in (0,\xi_j^*)$. By choosing $\alpha = \min_{1\le i\le n, 1\le j\le m}\{\varepsilon_i^*, \xi_j^*\}$, we obtain

$$\begin{aligned}
G_i(\alpha) &= \underline{a}_i\eta_i - \alpha - \sum_{j=1}^{m}\overline{c}_j\left|q_{ij}^0\right|N_i e_\alpha(\sigma(t), t-\sigma_{ij}) \\
&\quad - \sum_{j=1}^{m}\overline{c}_j\left|q_{ij}^1\right|N_i\int_0^\infty k_{ij}(s)e_\alpha(\sigma(t), t-s)\Delta s \\
&\ge 0, \quad i = 1,2,\ldots,n, \\
G_j^*(\alpha) &= \underline{c}_j\omega_j - \alpha - \sum_{i=1}^{n}\overline{a}_i\left|p_{ji}^0\right|M_j e_\alpha(\sigma(t), t-\tau_{ji}) \\
&\quad - \sum_{i=1}^{n}\overline{a}_i\left|p_{ji}^1\right|M_j\int_0^\infty h_{ji}(s)e_\alpha(\sigma(t), t-s)\Delta s \\
&\ge 0, \quad j = 1,2,\ldots,m,
\end{aligned} \tag{4.13}$$

Denote

$$\mu_i(t) = e_\alpha(t,\delta)|u_i(t)|, \quad t \in \mathbb{R},\ i = 1,2,\ldots,n, \tag{4.14}$$
$$\nu_j(t) = e_\alpha(t,\delta)\left|v_j(t)\right|, \quad t \in \mathbb{R},\ j = 1,2,\ldots,m, \tag{4.15}$$

where $\delta \in [-\infty, 0]_{\mathbb{T}}$. For $t > 0,\ t \neq t_k,\ k \in \mathbb{Z}^+,\ i = 1,2,\ldots,n,\ j = 1,2,\ldots,m,$ it follows from (4.6)–(4.15), we can obtain

$$\begin{aligned}
D^+\mu_i^\Delta(t) &= \alpha e_\alpha(t,\delta)|u_i(t)| + e_\alpha(\sigma(t),\delta)D^+|u_i(t)|^\Delta \\
&\le \alpha e_\alpha(t,\delta)|u_i(t)| + e_\alpha(\sigma(t),\delta) \\
&\quad \times \Bigg[-\underline{a}_i\eta_i|u_i(t)| + \overline{a}_i\sum_{j=1}^{m}\left|p_{ji}^0\right|M_j\left|v_j(t-\tau_{ji})\right| \\
&\qquad + \overline{a}_i\sum_{j=1}^{m}\left|p_{ji}^1\right|M_j\int_0^\infty h_{ji}(s)\left|v_j(t-s)\right|\Delta s\Bigg]
\end{aligned}$$

$$\begin{aligned}
&\leq -(\underline{a}_i\eta_i - \alpha)\mu_i(t) + \overline{a}_i\sum_{j=1}^{m}\left|p_{ji}^0\right|M_j e_\alpha(\sigma(t), t-\tau_{ji})\nu_j(t-\tau_{ji}) \\
&\quad + \overline{a}_i\sum_{j=1}^{m}\left|p_{ji}^1\right|M_j\int_0^\infty h_{ji}(s)e_\alpha(\sigma(t), t-s)\nu_j(t-s)\Delta s, \\
D^+\nu_j^\Delta(t) &\leq -\left(\underline{c}_j\omega_j - \alpha\right)\nu_j(t) + \overline{c}_j\sum_{i=1}^{n}\left|q_{ij}^0\right|N_i e_\alpha(\sigma(t), t-\sigma_{ij})\mu_i(t-\sigma_{ij}) \\
&\quad + \overline{c}_i\sum_{i=1}^{n}\left|q_{ij}^1\right|N_i\int_0^\infty k_{ij}(s)e_\alpha(\sigma(t), t-s)\mu_i(t-s)\Delta s.
\end{aligned} \tag{4.16}$$

Also,

$$\mu_i(t_k^+) \leq \mu_i(t_k),\ \nu_j(t_k^+) \leq \nu_j(t_k), \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, m,\ k \in \mathbb{Z}^+. \tag{4.17}$$

Define a Lyapunov function

$$\begin{aligned}
V(t) = \sum_{i=1}^{n}&\Bigg[\mu_i(t) + \overline{a}_i\sum_{j=1}^{m}\left|p_{ji}^0\right|M_j e_\alpha(\sigma(t), t-\tau_{ji})\int_{t-\tau_{ji}}^{t}\nu_j(s)\Delta s \\
&+\overline{a}_i\sum_{j=1}^{m}\left|p_{ji}^1\right|M_j\int_0^\infty h_{ji}(s)e_\alpha(\sigma(t), t-s)\int_{t-s}^{t}\nu_j(z)\Delta z\Delta s\Bigg] \\
+ \sum_{j=1}^{m}&\Bigg[\nu_j(t) + \overline{c}_j\sum_{i=1}^{n}\left|q_{ij}^0\right|N_i e_\alpha(\sigma(t), t-\sigma_{ij})\int_{t-\sigma_{ij}}^{t}\mu_i(s)\Delta s \\
&+\overline{c}_j\sum_{i=1}^{n}\left|q_{ij}^1\right|N_i\int_0^\infty k_{ij}(s)e_\alpha(\sigma(t), t-s)\int_{t-s}^{t}\mu_i(z)\Delta z\Delta s\Bigg].
\end{aligned} \tag{4.18}$$

And we note that $V(t) > 0$ for $t > 0$ and $V(0) > 0$. Calculating the Δ-derivatives of V, we get

$$\begin{aligned}
D^+V^\Delta(t) \leq \sum_{i=1}^{n}&\Bigg[-(\underline{a}_i\eta_i - \eta)\mu_i(t) + \overline{a}_i\sum_{j=1}^{m}\left|p_{ji}^0\right|M_j e_\alpha(\sigma(t), t-\tau_{ji})\nu_j(t) \\
&+\overline{a}_i\sum_{j=1}^{m}\left|p_{ji}^1\right|M_j\int_0^\infty h_{ji}(s)e_\alpha(\sigma(t), t-s)\nu_j(t)\Delta s\Bigg] \\
+ \sum_{j=1}^{m}&\Bigg[-\left(\underline{c}_j\omega_j - \eta\right)\nu_j(t) + \overline{c}_j\sum_{i=1}^{n}\left|q_{ij}^0\right|N_i e_\alpha(\sigma(t), t-\sigma_{ij})\mu_i(t) \\
&+\overline{c}_j\sum_{i=1}^{n}\left|q_{ij}^1\right|N_i\int_0^\infty k_{ij}(s)e_\alpha(\sigma(t), t-s)\mu_i(t)\Delta s\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{i=1}^{n}\Bigg[\underline{a}_i\eta_i - \eta - \sum_{j=1}^{m}\overline{c}_j\left|q_{ij}^0\right|N_i e_\eta(\sigma(t), t-\sigma_{ij}) \\
&\qquad -\sum_{j=1}^{m}\overline{c}_j\left|q_{ij}^1\right|N_i\int_0^\infty k_{ij}(s)e_\alpha(\sigma(t), t-s)\Delta s\Bigg]\mu_i(t) \\
&\quad -\sum_{j=1}^{m}\Bigg[\underline{c}_j\omega_j - \eta - \sum_{i=1}^{n}\overline{a}_i\left|p_{ji}^0\right|M_j e_\eta(\sigma(t), t-\tau_{ji}) \\
&\qquad -\sum_{i=1}^{n}\overline{a}_i\left|p_{ji}^1\right|M_j\int_0^\infty h_{ji}(s)e_\alpha(\sigma(t), t-s)\Delta s\Bigg]\nu_j(t) \\
&= -\sum_{i=1}^{n}G_i(\eta)\mu_i(t) - \sum_{j=1}^{m}G_j^*(\eta)\nu_j(t) \\
&\le 0, \quad t>0,\ t\neq t_k,\ t\in\mathbb{T},\ k\in\mathbb{Z}^+.
\end{aligned}
\tag{4.19}
$$

Also,

$$
\begin{aligned}
V(t_k^+) &= \sum_{i=1}^{n}\Bigg[\mu_i(t_k^+) + \overline{a}_i\sum_{j=1}^{m}\left|p_{ji}^0\right|M_j e_\alpha(\sigma(t_k^+), t_k^+-\tau_{ji})\int_{t_k^+-\tau_{ji}}^{t_k^+}\nu_j(s)\Delta s \\
&\qquad +\overline{a}_i\sum_{j=1}^{m}\left|p_{ji}^1\right|M_j\int_0^\infty h_{ji}(s)e_\alpha(\sigma(t_k^+), t_k^+-s)\int_{t_k^+-s}^{t_k^+}\nu_j(z)\Delta z\Delta s\Bigg] \\
&\quad +\sum_{j=1}^{m}\Bigg[\nu_j(t_k^+) + \sum_{i=1}^{n}\left|q_{ij}^0\right|N_i e_\alpha(\sigma(t_k^+), t_k^+-\sigma_{ij})\int_{t_k^+-\sigma_{ij}}^{t_k^+}\mu_i(s)\Delta s \\
&\qquad +\overline{c}_j\sum_{i=1}^{n}\left|q_{ij}^1\right|N_i\int_0^\infty k_{ij}(s)e_\alpha(\sigma(t_k^+), t_k^+-s)\int_{t_k^+-s}^{t_k^+}\mu_i(z)\Delta z\Delta s\Bigg] \\
&\le \sum_{i=1}^{n}\Bigg[\mu_i(t_k) + \overline{a}_i\sum_{j=1}^{m}\left|p_{ji}^0\right|M_j e_\alpha(\sigma(t_k), t_k-\tau_{ji})\int_{t_k-\tau_{ji}}^{t_k}\nu_j(s)\Delta s \\
&\qquad +\overline{a}_i\sum_{j=1}^{m}\left|p_{ji}^1\right|M_j\int_0^\infty h_{ji}(s)e_\alpha(\sigma(t_k), t_k-s)\int_{t_k-s}^{t_k}\nu_j(z)\Delta z\Delta s\Bigg] \\
&\quad +\sum_{j=1}^{m}\Bigg[\nu_j(t_k) + \overline{c}_j\sum_{i=1}^{n}\left|q_{ij}^0\right|N_i e_\alpha(\sigma(t_k), t_k-\sigma_{ij})\int_{t_k-\sigma_{ij}}^{t_k}\mu_i(s)\Delta s \\
&\qquad +\overline{c}_j\sum_{i=1}^{n}\left|q_{ij}^1\right|N_i\int_0^\infty k_{ij}(s)e_\alpha(\sigma(t_k), t_k-s)\int_{t_k-s}^{t_k}\mu_i(z)\Delta z\Delta s\Bigg] \\
&= V(t_k), \quad k\in\mathbb{Z}^+.
\end{aligned}
\tag{4.20}
$$

It follows that $V(t) \leq V(0)$ for $t > 0$ and hence, for $t > 0$, we can obtain

$$\begin{aligned}
\sum_{i=1}^{n}\mu_i(t) + \sum_{j=1}^{m}\nu_j(t) \leq \sum_{i=1}^{n}\Bigg[&\mu_i(0) + \sum_{j=1}^{m}\left|p_{ji}^{0}\right|M_j e_\alpha(\sigma(0), -\tau_{ji})\int_{0-\tau_{ji}}^{0}\nu_j(s)\Delta s \\
&+\overline{a}_i\sum_{j=1}^{m}\left|p_{ji}^{1}\right|M_j\int_{0}^{\infty}h_{ji}(s)e_\alpha(\sigma(0), 0-s)\int_{0-s}^{0}\nu_j(z)\Delta z\Delta s\Bigg] \\
+\sum_{j=1}^{m}\Bigg[&\nu_j(0) + \sum_{i=1}^{n}\left|q_{ij}^{0}\right|N_i e_\alpha(\sigma(0), 0-\sigma_{ij})\int_{0-\sigma_{ij}}^{0}\mu_i(s)\Delta s \\
&+\overline{c}_j\sum_{i=1}^{n}\left|q_{ij}^{1}\right|N_i\int_{0}^{\infty}k_{ij}(s)e_\alpha(\sigma(0), 0-s)\int_{0-s}^{0}\mu_i(z)\Delta z\Delta s\Bigg].
\end{aligned} \tag{4.21}$$

In view of (4.14)-(4.15) and the previous inequality, we have

$$\begin{aligned}
&\sum_{i=1}^{n}\left|x_i(t) - x_i^*(t)\right| + \sum_{j=1}^{m}\left|y_j(t) - y_j^*(t)\right| \\
&\quad\leq e_{\ominus\alpha}(t,\delta)\Bigg[\sum_{i=1}^{n}\Bigg(1 + \sum_{j=1}^{m}\overline{c}_i\left|q_{ij}^{0}\right|N_i e_\alpha(\sigma(0), -\sigma_{ij})\sigma_{ij} \\
&\qquad\qquad + \sum_{i=1}^{m}\overline{c}_i\left|q_{ij}^{1}\right|N_i\int_{0}^{\infty}k_{ij}(s)e_\alpha(\sigma(0), -s)s\Delta s\Bigg)\max_{\delta\in(-\infty,0]_{\mathbb{T}}}\left|\varphi_i(\delta) - x_i^*(\delta)\right| \\
&\qquad + \sum_{j=1}^{m}\Bigg(1 + \sum_{i=1}^{n}\overline{a}_i\left|p_{ji}^{0}\right|M_j e_\alpha(\sigma(0), -\tau_{ji})\tau_{ji} \\
&\qquad\qquad + \sum_{i=1}^{n}\overline{a}_i\left|p_{ji}^{1}\right|M_j\int_{0}^{\infty}h_{ji}(s)e_\alpha(\sigma(0), -s)s\Delta s\Bigg)\max_{\delta\in(-\infty,0]_{\mathbb{T}}}\left|\psi_j(\delta) - y_j^*(\delta)\right|\Bigg] \\
&\quad\leq N e_{\ominus\alpha}(t,\delta)\left[\sum_{i=1}^{n}\max_{\delta\in(-\infty,0]_{\mathbb{T}}}\left|\varphi_i(\delta) - x_i^*(\delta)\right| + \sum_{j=1}^{m}\max_{\delta\in(-\infty,0]_{\mathbb{T}}}\left|\psi_j(\delta) - y_j^*(\delta)\right|\right],
\end{aligned} \tag{4.22}$$

where

$$\begin{aligned}
N = \max\Bigg\{&1 + \sum_{j=1}^{m}\overline{c}_i\left|q_{ij}^{0}\right|N_i e_\alpha(\sigma(0), -\sigma_{ij})\sigma_{ij} \\
&+ \sum_{j=1}^{m}\overline{c}_i\left|q_{ij}^{1}\right|N_i\int_{0}^{\infty}k_{ij}(s)e_\alpha(\sigma(0), -s)s\Delta s, 1 + \sum_{i=1}^{n}\overline{a}_i\left|p_{ji}^{0}\right|M_j e_\alpha(\sigma(0), -\tau_{ji})\tau_{ji} \\
&+\sum_{i=1}^{n}\overline{a}_i\left|p_{ji}^{1}\right|M_j\int_{0}^{\infty}h_{ji}(s)e_\alpha(\sigma(0), -s)s\Delta s\Bigg\} \geq 1.
\end{aligned} \tag{4.23}$$

The proof is complete. □

5. An Example

In this section, we give an example to illustrate our results.

Consider the following Cohen-Grossberg BAM neural networks system with distributed delays and impulses:

$$\begin{aligned}
x_1^{\Delta}(t) &= -\left(1+\frac{1}{3}\cos x_1(t)\right)\\
&\quad\times\left[5x_1(t)-\frac{1}{4}\sin\bigl(2y_1(t-1)\bigr)-\int_0^{\infty}\frac{1}{4}e^{-s}\sin\bigl(2y_1(t-s)\bigr)\Delta s+r_1\right],\\
&\qquad\qquad t>0,\ t\neq t_k,\ t\in\mathbb{T},\\
\Delta x_1(t_k) &= I_1(x_1(t_k)),\quad k=1,2,\ldots,\\
y^{\Delta}(t) &= -\left(1+\frac{1}{3}\sin y_1(t)\right)\\
&\quad\times\left[3y_1(t)-\frac{1}{4}\cos(2x_1(t-1))-\int_0^{\infty}\frac{1}{4}e^{-s}\cos(2x_1(t-s))\Delta s+s_1\right],\\
&\qquad\qquad t>0,\ t\neq t_k,\ t\in\mathbb{T},\\
\Delta y_1(t_k) &= J_1\bigl(y_1(t_k)\bigr),\quad k=1,2,\ldots,
\end{aligned} \tag{5.1}$$

where $\mathbb{T}=\mathbb{R}, \gamma_{1k}=1+(1/2)\sin(1+k), \widetilde{\gamma}_{1k}=1+(2/3)\cos 2k, k\in\mathbb{Z}^+$. A simple computation shows that $\underline{a}=\underline{c}=2/3, \overline{a}=\overline{c}=4/3, M_1=N_1=1, \eta_1=5, \omega_1=3, p_{11}^0=p_{11}^0=q_{11}^0=q_{11}^0=1/4, 0<\gamma_{1k}, \widetilde{\gamma}_{1k}<2$. It is easy to check that all conditions of Theorems 3.1 and 4.1 are satisfied. Hence, (5.1) has a unique equilibrium point, which is globally exponentially stable.

6. Conclusion

Using the time-scale calculus theory, the homeomorphism theory and the Lyapunov functional method, some sufficient conditions are obtained to ensure the existence and the global exponential stability of the unique equilibrium point of Cohen-Grossberg BAM neural networks with distributed delays and impulses on time scales. This is the first time applying the time-scale calculus theory to unify and improve impulsive Cohen-Grossberg BAM neural networks with distributed delays on time scales under the same framework. The sufficient conditions we obtained can easily be checked in practice by simple algebraic methods.

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