Nonlinearly charged Lifshitz black holes for any exponent $z > 1$

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ABSTRACT: Charged Lifshitz black holes for the Einstein-Proca-Maxwell system with a negative cosmological constant in arbitrary dimension D are known only if the dynamical critical exponent is fixed as $z = 2(D-2)$. In the present work, we show that these configurations can be extended to much more general charged black holes which in addition exist for any value of the dynamical exponent $z > 1$ by considering a nonlinear electrodynamics instead of the Maxwell theory. More precisely, we introduce a two-parametric nonlinear electrodynamics defined in the more general, but less known, so-called $(\mathcal{H}, P)$-formalism and obtain a family of charged black hole solutions depending on two parameters. We also remark that the value of the dynamical exponent $z = D - 2$ turns out to be critical in the sense that it yields asymptotically Lifshitz black holes with logarithmic decay supported by a particular logarithmic electrodynamics. All these configurations include extremal Lifshitz black holes. Charged topological Lifshitz black holes are also shown to emerge by slightly generalizing the proposed electrodynamics.

KEYWORDS: Gauge-gravity correspondence, Field Theories in Higher Dimensions, Black Holes, Holography and condensed matter physics (AdS/CMT)

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Contents

1 Introduction

In the last five years, there has been an intensive activity in order to extend the standard Anti de Sitter/Conformal Field Theory (AdS/CFT) correspondence [1] to the non-relativistic physics. In this context, the AdS group is replaced by its non-relativistic cousin the Schrödinger group [2, 3] or by the Lifshitz group [4]. These two groups share in common an anisotropic scaling symmetry that allows the time and space coordinates to rescale with different weight

$$t \mapsto \lambda^z t, \qquad \vec{x} \mapsto \lambda \vec{x}, \tag{1.1}$$

where the constant z is called the dynamical critical exponent. The main difference between the Schrödinger and the Lifshitz groups lies in the fact that the former enjoys in addition a Galilean boost symmetry whose gravity dual metric requires the introduction of a closed null direction in addition to the holographic direction. This extra coordinate is hard to understand holographically which makes its physical interpretation a subject of debate. In the present work, within the spirit of the AdS/CFT correspondence, we are interested in looking for black hole solutions whose asymptotic metrics enjoy the anisotropic symmetry (1.1) without the Galilean boost symmetry. In this case, the zero temperature gravity dual is the so-called Lifshitz spacetime defined by the metric [4]

$$ds^2 = -\frac{r^{2z}}{l^{2z}} dt^2 + \frac{l^2}{r^2} dr^2 + r^2 d\vec{x}^2, \tag{1.2}$$

where $\vec{x}$ is a $(D-2)$-dimensional vector. Lifshitz black holes refer to black hole configurations whose metrics reproduce asymptotically the Lifshitz spacetime and their role holographically is to capture the non-relativistic behavior at finite temperature. This is the lesson learned from the AdS/CFT correspondence, which is recovered for $z = 1$, and that has been proved to be a promising tool for studying strongly correlated quantum systems. However, other values of z are present experimentally in several problems of condensed matter physics, and it is then natural to explore the existence of Lifshitz black holes with the aim of widening the scope of holographic applications. From the very beginning of the Lifshitz advent it becomes clear that pure Einstein gravity with eventually a cosmological constant resists to support such configurations, since it is clear from Birkhoff's theorem and its generalizations that in highly symmetric vacuum situations there is no freedom to choose the boundary conditions and they become fixed by the dynamics. Consequently, Lifshitz asymptotic requires the introduction of matter sources [5–14] or/and to consider higher-curvature gravity theories [15–19]. One of the first and simplest examples that has been used for supporting the Lifshitz spacetimes at any dimension D is a Proca field coupled to Einstein gravity with a negative cosmological constant [5]. However, as we know from the classical days of the "no-hair" conjecture Proca fields are not enough to produce black holes by itself due to its impossibility to make non-trivial contributions to global charges due to its massive behavior. This is not the case for the massless Maxwell field and electrically charged black holes are known at any dimension, however, this gauge field along is incompatible with the Lifshitz asymptotic. Therefore, the superposition of both spin-1 fields looks a priori as a promising strategy. In fact, charged black hole solutions which are asymptotically Lifshitz can be constructed adding the Maxwell action to the described Proca system [20]. Unfortunately, up to now, the only configurations that have been obtained are those with the single value $z = 2(D-2)$ for the dynamical exponent at any dimension [20].

Here, we prove that the charged Lifshitz black holes found for $z = 2(D-2)$ in [20] can be extended to much more general black hole configurations which are in addition characterized by any value of the dynamical exponent $z > 1$. This task is achieved considering a nonlinear version of the electrodynamics instead of the Maxwell theory. The interest on nonlinear electrodynamics started with the pioneer work of Born and Infeld whose main motivation was to cure the problem of infinite energy of the electron [21]. However, since the Born-Infeld model was not able to fulfill all the hope, nonlinear electrodynamics became less popular until recent years, although outstanding work was done in the middle [22]. The renewal interest on nonlinear electrodynamics matches with their emergence and importance in the low energy limit of heterotic string theory. Also, nonlinear electrodynamics have been proved to be a powerful and useful tool in order to construct black hole solutions with interesting features and properties as for example regular black holes [23–27] or black holes with nonstandard asymptotic behaviors in Einstein gravity or some of its generalizations [28–38]. Charged black hole solutions emerging from nonlinear theories also have nice thermodynamics properties which make them attractive to be studied. Usually, the presence of nonlinearities render the equations much more difficult to handle, and the hope to obtain exact analytic solutions interesting from a physical point of view may be

considerably restricted. However, as shown below the derivation of these charged black hole solutions is strongly inherent to the presence of the nonlinearities. To be more precise, we introduce a two-parametric nonlinear electrodynamics defined in the more general, but less known, so-called $(\mathcal{H}, P)$-formalism and obtain a family of charged black hole solutions depending on two parameters and valid for any exponent $z > 1$.

The plan of the paper is organized as follows. In the next section, we present the $(\mathcal{H}, P)$-formalism together with the new proposed electrodynamics, the action including the Proca system and derive the corresponding field equations. In section 3, we consider the simplest Lifshitz black hole ansatz, as well as electric ansatze for the spin-1 fields and solve the field equations. We exhibit a family of two-parametric charged black hole solutions for a generic value of the dynamical exponent $z > 1$ and $z \neq D - 2$. We prove that this last value of the dynamical exponent $z = D - 2$ turns out to be critical in the sense that it yields asymptotically Lifshitz black holes with logarithmic decay supported by a particular logarithmic electrodynamics which can be obtained as a nontrivial limit of the initially proposed one. We also construct charged topological Lifshitz black holes by slightly generalizing the proposed electrodynamics. In section 4, we analyze in detail a particular case of our $(\mathcal{H}, P)$-electrodynamics with invertible constitutive relations, for which the nonlinear electrodynamics Lagrangian can be written as a single function of the Maxwell invariant of the strength $F_{\mu\nu}$, concretely becoming a power law intensively studied in the recent literature, see e.g. [28–38]. In section 5, we exhaustively study the existence of event horizons for all the values of the parameters. We conclude that there are eight kinds of different black hole families depending on the structural coupling constant of the electrodynamics and the dynamical critical exponent z. We present graphs for each of these families in order to make transparent the involved arguments. Finally, the last section is devoted to our conclusions and further works.

2 Action and field equations

We are interested in extending the charged Lifshitz black holes obtained in [20] for a particular value of the dynamical exponent to much more general black hole configurations valid for any value $z > 1$. We show that this task can be achieved by allowing the electrodynamics to behave nonlinearly. More precisely, in addition to the Proca field that supports the anisotropic asymptotic in the standard gravity we incorporate an appropriate nonlinear electrodynamics, see [22] for a clever exposition of the formalism. The dynamics of this theory is governed by the following action

$$S\left[g^{\mu\nu}, B_\mu, A_\mu, P^{\mu\nu}\right] = \int d^D x \sqrt{-g} \left[\frac{1}{2\kappa}(R - 2\lambda) - \frac{1}{4}H_{\mu\nu}H^{\mu\nu} - \frac{1}{2}m^2 B_\mu B^\mu \right.$$
$$\left. -\frac{1}{2}P^{\mu\nu}F_{\mu\nu} + \mathcal{H}(P) \right]. \tag{2.1}$$

Here, R stands for the scalar curvature of the metric $g_{\mu\nu}$ and λ is the cosmological constant. The tensor $H_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ is the field strength of the Proca field B_μ having mass m

and whose field equation is given by

$$\nabla_\mu H^{\mu\nu} = m^2 B^\nu \,. \tag{2.2a}$$

The last two terms in the action describe the nonlinear behavior of the electromagnetic field A_μ with field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The origin of these terms is the following. The naive way to introduce nonlinear effects in electrodynamics is to consider a Lagrangian given by a general function of the quadratic invariants build out from the strength field $F_{\mu\nu}$. However, a different formulation is to rewrite this Lagrangian in terms of a Legendre transform $\mathcal{H}(P)$, function of the invariants formed with the conjugate antisymmetric tensor $P^{\mu\nu}$, namely $P \equiv \frac{1}{4}P_{\mu\nu}P^{\mu\nu}$. The resulting action turns out to be also a functional of the conjugate antisymmetric tensor $P^{\mu\nu}$ in addition to the electromagnetic field A_μ, and is given by the last two terms of (2.1). The advantage of this formulation lies in the fact that the variation with respect to the electromagnetic field A_μ gives the nonlinear version of the Maxwell equations

$$\nabla_\mu P^{\mu\nu} = 0 \,, \tag{2.2b}$$

and these equations are now simply expressed in terms of the antisymmetric tensor $P^{\mu\nu}$. These latter can also be easily integrated and from the expression, the strength field of the original electromagnetic field can be calculated using now the variation of the action with respect to the conjugate antisymmetric tensor $P^{\mu\nu}$, which gives the so-called constitutive relations

$$F_{\mu\nu} = \mathcal{H}_P P_{\mu\nu} \,, \tag{2.2c}$$

where $\mathcal{H}_P \equiv \partial\mathcal{H}/\partial P$ and the structural function $\mathcal{H}(P)$ defines the concrete electrodynamics. Note that the standard linear Maxwell theory is recovered for $\mathcal{H}(P) = P$. Obviously, this formulation resembles the standard electrodynamics in a media in four dimensions where the components of the antisymmetric tensors $P_{\mu\nu}$ and $F_{\mu\nu}$ are just the components of the vector fields $\left(\vec{D}, \vec{H}\right)$ and $\left(\vec{E}, \vec{B}\right)$, respectively. In this sense a given nonlinear electrodynamics is equivalent to a given media, and then from this analogy it is clear that the components of the tensor $P_{\mu\nu}$ are the appropriate field strengths to describe nonlinear electrodynamics. It is important to emphasize that in many cases both formalisms are not equivalent since as it occurs in standard Legendre transforms, the equivalence depends on the invertibility of the conjugate relations, which for electrodynamics are embodied in the constitutive relations (2.2c). In fact, there are well behaved nonlinear electrodynamics where the Lagrangian cannot be written as a single function of the invariants of $F_{\mu\nu}$; relevant examples are those giving rise to regular black holes [23–27]. This is also the case we consider here. Hence, the use of the described formalism is compulsory since the precise nonlinear electrodynamics we propose as source for the Lifshitz black holes with generic dynamical critical exponent z is determined by the following structural function

$$\mathcal{H}(P) = -\frac{\left[2z^2 - Dz + 2(D-2)\right]}{2\kappa l^2}\beta_1\sqrt{-2l^2 P} - \frac{(z-1)(z-D+2)^2}{\kappa z}\beta_1{}^2 P$$
$$+ \frac{(D-2)^2}{2\kappa l^2}\beta_2\left(-2l^2 P\right)^{\frac{z}{2(D-2)}} \,, \tag{2.2d}$$

where β_1 and β_2 are dimensionless coupling constants for any dimensionless election of the Einstein constant κ. The invariant P is negative definite since we will only consider purely electric configurations.

The Einstein equations resulting from varying the action with respect to the metric yield

$$
G_{\mu\nu} + \lambda g_{\mu\nu} = \kappa \left[H_{\mu\alpha} H_\nu{}^\alpha - \frac{1}{4} g_{\mu\nu} H_{\alpha\beta} H^{\alpha\beta} + m^2 \left(B_\mu B_\nu - \frac{1}{2} g_{\mu\nu} B^\alpha B_\alpha \right) \right.
$$
$$
\left. + \mathcal{H}_P P_{\mu\alpha} P_\nu{}^\alpha - g_{\mu\nu} (2P\mathcal{H}_P - \mathcal{H}) \right]. \tag{2.2e}
$$

The Einstein-Proca-Nonlinear Electrodynamics system (2.2) is the one we pretend to solve in the context of a Lifshitz asymptotic with generic anisotropy. A last remark on this subject, usually a nonlinear electrodynamics is considered plausible if it satisfies the correspondence principle of approaching the Maxwell theory ($\mathcal{H} \approx P$) in the weak field limit ($|l^2 P| \ll 1$), which is obviously not satisfied for our proposal (2.2d). However, this expectation is realized for self-gravitating charged configurations if the simultaneous weak field limit of the gravitational background is flat spacetime, where the Maxwell theory is phenomenologically viable, or another spacetime where at least there are no theoretical obstructions for the behavior of standard electrodynamics. We emphasize that Lifshitz asymptotic with generic anisotropy is apparently out of this category, since Lifshitz black holes charged in standard way [20] are known only if the dynamical critical exponent is fixed to $z = 2(D-2)$. Notice this is precisely the exponent for which Maxwell theory is recovered in our electrodynamics (2.2d) if additionally the structural constants are chosen as $\beta_1 = 0$ and β_2 is appropriately fixed. Summarizing, the proposed electrodynamics has no correspondence to Maxwell theory for weak fields, but apparently this is a desirable feature if one intents to obtain charged Lifshitz black holes with generic anisotropy. An objective that we will be able to achieve in the next section.

3 Solving field equations

We look for an asymptotically Lifshitz ansatz of the form

$$
ds^2 = -\frac{r^{2z}}{l^{2z}} f(r) dt^2 + \frac{l^2}{r^2} \frac{dr^2}{f(r)} + r^2 d\vec{x}^2 , \tag{3.1}
$$

where the gravitational potential satisfies $f(\infty) = 1$. For spin-1 fields we assume electric ansatze $B_\mu = B_t(r) \delta_\mu^t$ and $P_{\mu\nu} = 2\delta_{[\mu}^t \delta_{\nu]}^r D(r)$. We start by integrating the nonlinear Maxwell equations (2.2b) which give the generalization of the Coulomb law

$$
P_{\mu\nu} = 2\delta_{[\mu}^t \delta_{\nu]}^r \frac{Q}{l^{D-2}} \left(\frac{l}{r} \right)^{D-z-1} \quad \longrightarrow \quad P = -\frac{Q^2}{2r^{2(D-2)}} , \tag{3.2}
$$

where the integration constant Q is related to the electric charge, and for spacetime dimension D it has dimension of length^{D-3}. The corresponding electric field can be evaluated from the constitutive relations (2.2c), $E \equiv F_{tr} = \mathcal{H}_P D$, after substituting the Coulomb

law above in the structural function (2.2d). The Proca field is obtained algebraically from the difference between the diagonal temporal and radial components of the mixed version of Einstein equations (2.2e)

$$B_t(r) = \sqrt{\frac{(D-2)(z-1)}{\kappa}} \left(\frac{r}{l}\right)^z \frac{f(r)}{lm} \,, \tag{3.3}$$

which is only possible for $z > 1$. Substituting now this expression in the Proca equation (2.2a) and taking the gravitational potential as $f = 1 - h$ we get

$$r^2 h'' + (z + D - 1)rh' + (D-2)zh = -l^2 m^2 + (D-2)z \,, \tag{3.4}$$

which is an Euler differential equation with a constant inhomogeneity for the function h. The function h must vanish at infinity in order to satisfy the Lifshitz asymptotic; this implies the vanishing of the constant inhomogeneity which fixes the value of the Proca field mass to

$$m = \frac{\sqrt{(D-2)z}}{l} \,. \tag{3.5}$$

The general solution of the Euler equation (3.4) compatible with the Lifshitz asymptotic is a superposition of two negative powers of r which are roots of the following characteristic polynomial, obtained from substituting $h \propto 1/r^\alpha$ in the differential equation,

$$(\alpha - z)(\alpha - D + 2) = 0 \,. \tag{3.6}$$

Hence, the solution of the gravitational potential for a generic value $z > 1$ of the dynamical critical exponent is

$$f(r) = 1 - M_1 \left(\frac{l}{r}\right)^{D-2} + M_2 \left(\frac{l}{r}\right)^z \,. \tag{3.7}$$

This is true except for $z = D-2$ where there is a double root of the characteristic polynomial and the single negative power must be supplemented with a logarithmic behavior; we analyze this case in the next subsection.

Returning to Einstein equations we choose the parameterization for the cosmological constant supporting the Lifshitz spacetime (1.2) in the absence of an electromagnetic field

$$\lambda = -\frac{z^2 + (D-3)z + (D-2)^2}{2l^2} \,. \tag{3.8}$$

It is straightforward to show that using all the above ingredients, e.g. substituting the generalized Coulomb law (3.2) in the structural function (2.2d) and the result in the energy-momentum tensor, all the remaining Einstein equations are satisfied if the previous integration constants are fixed in terms of the charge via the structural coupling constants as

$$M_1 = \beta_1 \frac{|Q|}{l^{D-3}}, \quad M_2 = \beta_2 \left(\frac{|Q|}{l^{D-3}}\right)^{\frac{z}{D-2}} \,. \tag{3.9}$$

Finally, a comment is needed regarding the AdS generalization $z = 1$ of these solutions. It cannot be obtained taking trivially the limit $z \to 1$. The reason is that as it can be

anticipated from the expression (3.3), the Proca field is not needed in this case, and this implies that the equation (3.4) no longer determines the gravitational potential. In other words, there is no constraint fixing the electrodynamics, and solutions for any of such theories are possible. In particular, for the case considered in the paper we recover the terms of the above gravitational potential in addition to a new term accompanied by a genuine integration constant related to the standard AdS mass.

3.1 Nonlinearly charged logarithmic Lifshitz black hole for $z = D - 2$

As mentioned previously, when the dynamic critical exponent takes the value $z = D - 2$ we find a double multiplicity for the decay power $D - 2$, which forces the incorporation of a logarithmic behavior for the gravitational potential according to

$$f(r) = 1 - \left(\frac{l}{r}\right)^{D-2}\left[M_1 \ln\left(\frac{r}{l}\right) - M_2\right].\tag{3.10}$$

It is interesting and useful to realize the above solution as a non-trivial limit of the generic case (3.7). In order to make clear this limit is possible, we redefine the integration constants of the generic case by

$$\left(M_1, M_2\right) \mapsto \left((z - D + 2)M_1, M_2 - M_1\right),\tag{3.11}$$

which allows to rewrite (3.7) in the following way

$$f(r) = 1 - \left(\frac{l}{r}\right)^z\left\{\frac{M_1}{z - D + 2}\left[\left(\frac{r}{l}\right)^{z-D+2} - 1\right] - M_2\right\}.\tag{3.12}$$

Taking now the limit $z \to D - 2$ and using the following definition for the logarithmic function

$$\ln(x) \equiv \lim_{\alpha \to 0}\frac{x^\alpha - 1}{\alpha},\tag{3.13}$$

we obtain the logarithmic Lifshitz potential (3.10). The fact that the gravitational potential of the critical case $z = D - 2$ can be achieved through a limiting procedure allows to guess its supporting electrodynamics by a similar reasoning. Using the linear dependence between the integration and structural coupling constants of the generic case (3.9), the redefinitions (3.11) are equivalent to redefine the structural coupling constants along the same way

$$\left(\beta_1, \beta_2\right) \mapsto \left((z - D + 2)\beta_1, \beta_2 - \beta_1\right),\tag{3.14}$$

which allows to rewrite the generic structural function (2.2d) as

$$\mathcal{H}(P) = \frac{1}{2\kappa l^2}\left\{\frac{(D-2)^2}{z - D + 2}\left[\left(-2l^2P\right)^{\frac{z-D+2}{2(D-2)}} - 1\right] - 2z - D + 4\right\}\beta_1\sqrt{-2l^2P}$$
$$- \frac{(z-1)}{\kappa z}\beta_1{}^2 P + \frac{(D-2)^2}{2\kappa l^2}\beta_2\left(-2l^2P\right)^{\frac{z}{2(D-2)}}.\tag{3.15}$$

Taking again the limit $z \to D - 2$ and using the above definition for the logarithmic function, we obtain the following structural function

$$\mathcal{H}(P) = \frac{1}{2\kappa l^2} \left\{ \left[(D-2) \ln \sqrt{-2l^2 P} - 3D + 8 \right] \beta_1 + (D-2)^2 \beta_2 \right\} \sqrt{-2l^2 P}$$
$$- \frac{(D-3)}{(D-2)\kappa} \beta_1{}^2 P \,, \tag{3.16}$$

which is in fact the logarithmic electrodynamics supporting the solution for $z = D - 2$ as it can be checked directly from the remaining Einstein equations. The fixing of the integration constants in terms of the charge via the structural coupling constants can be understood straightforwardly in this process. However, it is more illustrative to deduce them from the limiting procedure. Writing the redefined integration constants (3.11) in terms of the old ones, using the relation between the last and their structural coupling constants as corresponds to the generic case (3.9), writing these old coupling constants in terms of the redefined ones (3.14), and finally taking the limit $z \to D - 2$ we obtain the following relations

$$M_1 = \beta_1 \frac{|Q|}{l^{D-3}}, \quad M_2 = \left(\beta_2 + \frac{\beta_1}{D-2} \ln \frac{|Q|}{l^{D-3}} \right) \frac{|Q|}{l^{D-3}} \,. \tag{3.17}$$

These fixings are confirmed by the Einstein equations when $z = D-2$. Finally, we comment that logarithmic Lifshitz black holes are rare, the other example we know is produced using higher curvature gravity [15, 16].

3.2 Switching on topology

Here, we show that the previous solutions can be extended to topological Lifshitz charged black holes that are solutions with more general constant curvature horizon topologies. This can be done by changing the $(D-2)$-dimensional flat base metric $d\vec{x}^2$ in (3.1) with the metric $d\Omega_k^2 = \gamma_{ij}(\vec{x})dx^i dx^j$ of a space of constant curvature $k = \pm 1, 0$, i.e.

$$ds^2 = -\frac{r^{2z}}{l^{2z}} f(r) dt^2 + \frac{l^2}{r^2} \frac{dr^2}{f(r)} + r^2 d\Omega_k^2 \,. \tag{3.18}$$

The resulting potentials for the metric and matter fields will be all the same that the previously studied flat ones ($k = 0$), which at least in the case of the metric is surprising since the gravitational potential is usually topology dependent. The only change is that in order to support nontrivial topologies, $k \neq 0$, we need to consider a different nonlinear electrodynamics by generalizing the structural functions (2.2d) and (3.16) to

$$\mathcal{H}_k(P) = \mathcal{H}(P) - \frac{(D-2)(D-3)}{2\kappa l^2} \beta_k \left(-2l^2 P \right)^{1/(D-2)} \,. \tag{3.19}$$

Unfortunately, for nontrivial topologies the charge is no longer an independent integration constant, as in the flat case, since it becomes related to the structural coupling constant of the topological contribution according to

$$\beta_k \left(\frac{|Q|}{l^{D-3}} \right)^{\frac{2}{D-2}} = k \,. \tag{3.20}$$

4 Invertible power-law electrodynamics for $\beta_1 = 0$

An interesting special case is when the structural coupling constant β_1 vanishes, the structural function becomes a power-law

$$\mathcal{H}(P) = \frac{(D-2)^2}{2\kappa l^2} \beta_2 \left(-2l^2 P\right)^{\frac{z}{2(D-2)}}, \tag{4.1}$$

and the constitutive relations (2.2c) are invertible. This case can be described by a Lagrangian, $\mathcal{L} \equiv \frac{1}{2} P^{\mu\nu} F_{\mu\nu} - \mathcal{H} = 2P\mathcal{H}_P - \mathcal{H}$, which can be written as a standard function of the invariant $F \equiv \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \mathcal{H}_P^2 P$ according to

$$\mathcal{L}(F) = \frac{(z-D+2)}{l^2 z} \bar{\beta}_2 \left(-\frac{2l^2 F}{\bar{\beta}_2^2}\right)^{\frac{z}{2(z-D+2)}}, \tag{4.2}$$

where we have redefined appropriately the structural coupling constant as $\bar{\beta}_2 = (D-2)z\beta_2/(2\kappa)$. Notice that for $z = 2(D-2)$ and fixing $\bar{\beta}_2 = -1$ we recover Maxwell theory. This kind of power-law Lagrangian has been exhaustively studied in refs. [28–38]. Note that in ref. [39], the authors consider such nonlinear electrodynamics and study the implications concerning the thermodynamics of Lifshitz black holes charged in this way.

5 Characterizing the black holes

The gravitational potential corresponding to a generic value of the dynamical critical exponent $z > 1$ is given by eq. (3.7). Using the relations (3.9) for the integration constants in terms of the charge via the structural coupling constants, the potential can be written as

$$f(r) = 1 - \beta_1 \left(\frac{\hat{l}}{r}\right)^{D-2} + \beta_2 \left(\frac{\hat{l}}{r}\right)^z, \tag{5.1}$$

where the role of the charge is reduced to define a new length scale $\hat{l}^{D-2} \equiv l|Q|$. In fact, using a dynamical scaling (1.1) with $\lambda^{D-2} \equiv l^{D-3}/|Q|$, we can fix $|Q| = l^{D-3}$ (or equivalently $\hat{l} = l$) in the whole solution. However, we do not consider this option in order to preserve the anisotropic scaling symmetry asymptotically. There is no restriction a priori on the values and signs of the structural coupling constants; this in turn makes possible that different kinds of black holes are encoded in that single potential. Each black hole is associated with a specific electrodynamics. We analyze all these cases in this section, and also study the logarithmic case with $z = D - 2$.

First, we notice that the potential allows an extremum if

$$(D-2)\beta_1 \left(\frac{\hat{l}}{r}\right)^{D-2} = z\beta_2 \left(\frac{\hat{l}}{r}\right)^z, \tag{5.2}$$

which implies that the constants β_1 and β_2 must have the same sign. At such extremum, the second derivative is evaluated as

$$f'' = \frac{(D-2)(z-D+2)\beta_1}{\hat{l}^2} \left(\frac{\hat{l}}{r}\right)^D = \frac{z(z-D+2)\beta_2}{\hat{l}^2} \left(\frac{\hat{l}}{r}\right)^{z+2}, \tag{5.3}$$

i.e. the extremum is a minimum if $(z - D + 2)\beta_1 > 0$ [or equivalently $(z - D + 2)\beta_2 > 0$] and a maximum if $(z - D + 2)\beta_1 < 0$ [or equivalently $(z - D + 2)\beta_2 < 0$]. In other words, the two cases allowing a minimum are $\beta_1 > 0$, $\beta_2 > 0$, $z > D - 2$ on the one hand and on the other hand $\beta_1 = -\tilde{\beta}_1 < 0$, $\beta_2 = -\tilde{\beta}_2 < 0$, $1 < z < D - 2$; both cases give rise to black holes with inner and outer horizons. The maximum is achieved also for two cases: the first one has $\beta_1 > 0$, $\beta_2 > 0$, $1 < z < D - 2$, and the second one is for the values $\beta_1 = -\tilde{\beta}_1 < 0$, $\beta_2 = -\tilde{\beta}_2 < 0$, $z > D - 2$; they produce black holes with a single horizon. This is also the case when no extremum is possible, i.e. $\beta_1 > 0$ and $\beta_2 = -\tilde{\beta}_2 < 0$, which occurs for the two cases $z > D - 2$ and $1 < z < D - 2$. Let us analyze all those cases in details in two separate subsections.

5.1 Black holes with two horizons

We start with the minimum cases having $\beta_1 > 0$, $\beta_2 > 0$ and $z > D - 2$. In this situation, the minimum is achieved at

$$r_{\min} = \hat{l} \left(\frac{z}{\beta_1} \frac{\beta_2}{D - 2} \right)^{1/(z-D+2)} , \tag{5.4}$$

and has value

$$f(r_{\min}) = 1 - (z - D + 2) \left[\left(\frac{\beta_1}{z} \right)^z \left(\frac{D - 2}{\beta_2} \right)^{D-2} \right]^{1/(z-D+2)} . \tag{5.5}$$

On the other hand, for the analyzed values, the potential goes asymptotically to one from below as $1 - \beta_1 (\hat{l}/r)^{D-2}$ and goes to infinity at $r = 0$ as $+ \beta_2 (\hat{l}/r)^z$; we show examples of these behaviors in the left graph of figure 1. Consequently, we are in the presence of black holes if $f(r_{\min}) \le 0$, i.e. if

$$\left(\frac{\beta_2}{D - 2} \right)^{D-2} \le (z - D + 2)^{z-D+2} \left(\frac{\beta_1}{z} \right)^z . \tag{5.6}$$

For the strict inequality, the corresponding black holes have two horizons $r_{\pm}$ defined by the two zeros of the potential $f(r_{\pm}) = 0$. As usual, the event horizon is defined by the largest of the zeros r_+. When the inequality is saturated we obtain an extremal black hole with zero temperature. The horizon of the extremal black hole is at

$$r_{\mathrm{e}} = \hat{l} \left(\frac{z - D + 2}{z} \beta_1 \right)^{1/(D-2)} . \tag{5.7}$$

This allows to rewrite the gravitational potential of the extremal case in a simpler form as

$$f(r) = 1 - \frac{z}{z - D + 2} \left(\frac{r_{\mathrm{e}}}{r} \right)^{D-2} + \frac{D - 2}{z - D + 2} \left(\frac{r_{\mathrm{e}}}{r} \right)^z . \tag{5.8}$$

Let us now consider the cases allowing a minimum for $\beta_1 = -\tilde{\beta}_1 < 0$, $\beta_2 = -\tilde{\beta}_2 < 0$ and $1 < z < D - 2$, i.e.

$$f(r) = 1 + \tilde{\beta}_1 \left(\frac{\hat{l}}{r} \right)^{D-2} - \tilde{\beta}_2 \left(\frac{\hat{l}}{r} \right)^z . \tag{5.9}$$
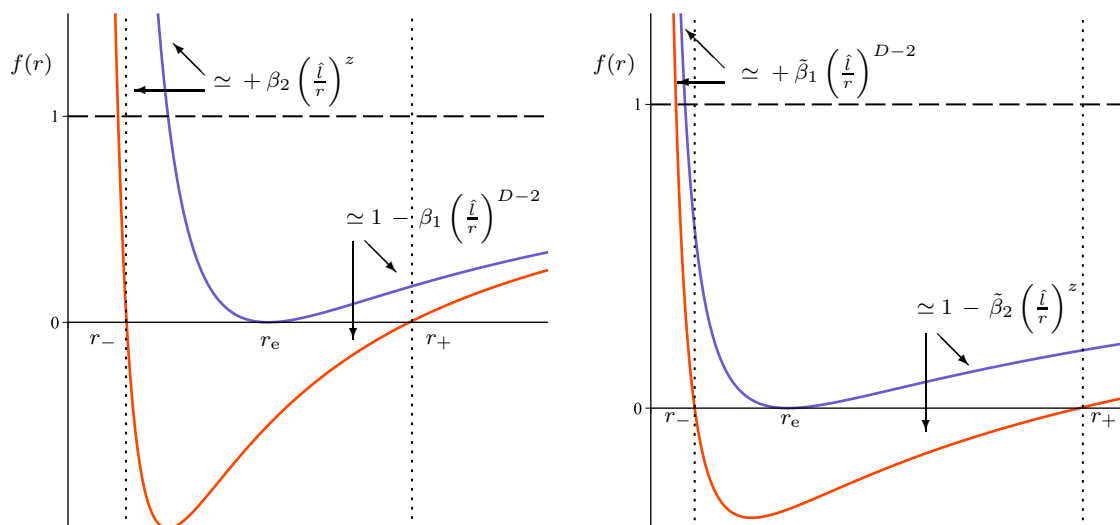
$$- 10 -$$

Figure 1. These graphs represent the generic behavior of the gravitational potential $f(r)$ when having a minimum. In the figure on the left, the red graph represent two-horizons black holes with dynamic exponent $z = 2(D-2) > D-2$ and structure coupling constants set to the values $\beta_1 > 0$ and $\beta_2 = \frac{1}{2}\beta_1^2 > 0$. As for the figure on the right it is considered $\beta_1 = -\tilde{\beta}_1 < 0$, $\beta_2 = -\tilde{\beta}_2 = -2\tilde{\beta}_1^{1/2} < 0$ and $z = (D-2)/2 < D-2$. The blue graphs represent the extremal black hole, which is the same in both cases.

These cases are similar to the previous ones, and the main difference is that now the solutions goes asymptotically to one from below as $1 - \tilde{\beta}_2(\hat{l}/r)^z$ and diverge to infinity at $r = 0$ as $+\tilde{\beta}_1(\hat{l}/r)^{D-2}$; see the right graph of figure 1. The two-horizons black holes exist now for

$$\left(\frac{\tilde{\beta}_1}{z}\right)^z \leq (D-z-2)^{D-z-2}\left(\frac{\tilde{\beta}_2}{D-2}\right)^{D-2}. \tag{5.10}$$

The extremal case saturating the inequality has exactly the same extremal horizon and functional form than the previous one, this last is just more appropriately written as

$$f(r) = 1 + \frac{z}{D-z-2}\left(\frac{r_e}{r}\right)^{D-2} - \frac{D-2}{D-z-2}\left(\frac{r_e}{r}\right)^z. \tag{5.11}$$

5.2 Black holes with a single horizon

The cases with a single horizon contain those two ones where the extremum is a maximum and other two cases where there is no extremum at all.

Let us start with the maximum options. The first option is achieved for $\beta_1 > 0$, $\beta_2 > 0$ and $1 < z < D-2$. Now, the potential decays asymptotically to one as $1 + \beta_2(\hat{l}/r)^z$ and goes to minus infinity at $r = 0$ as $-\beta_1(\hat{l}/r)^{D-2}$. In-between, increasing r the potential increases, changes the sign at the horizon and keeps increasing until it achieves a non-vanishing maximum, from which start the decay to the asymptotic Lifshitz value; see the left graph of figure 2. The second case is for $\beta_1 = -\tilde{\beta}_1 < 0$, $\beta_2 = -\tilde{\beta}_2 < 0$ [as in eq. (5.9)],
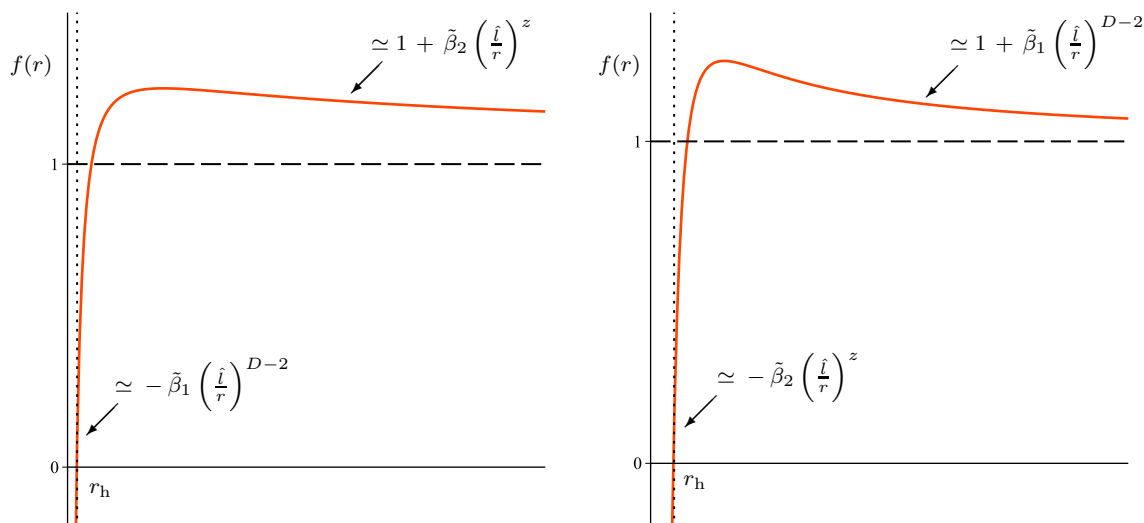
Figure 2. The graphs represent the cases with a single horizon where the extremum is a maximum. In the graph on the left the dynamic exponent takes the value $z = (D-2)/2 < D-2$ and the considered coupling constants are $\beta_1 > 0$ and $\beta_2 = \sqrt{\beta_1} > 0$. The graph on the right is for $z = 2(D-2) > D-2$, $\beta_1 = -\tilde{\beta}_1 < 0$ and $\beta_2 = -\tilde{\beta}_2 = -\tilde{\beta}_1^2 < 0$.

but now with $z > D-2$. The behavior here is similar than the one analyzed just before and the difference lies in the fact that the asymptotic decay is now given as $1 + \tilde{\beta}_1 \left(\hat{l}/r \right)^{D-2}$ and the divergence to minus infinity at $r = 0$ goes as $-\tilde{\beta}_2 \left(\hat{l}/r \right)^{z}$; see the right graph of figure 2.

As it was emphasized at the beginning of the section, the cases with no extremum at all are those with different values of the constants, the only possibility compatible with the existence of an horizon is $\beta_1 \geq 0$ and $\beta_2 = -\tilde{\beta}_2 \leq 0$, i.e.

$$f(r) = 1 - \beta_1 \left(\frac{\hat{l}}{r} \right)^{D-2} - \tilde{\beta}_2 \left(\frac{\hat{l}}{r} \right)^{z}. \tag{5.12}$$

The two cases correspond to $z > D-2$ and $1 < z < D-2$, since the behaviors at $r = 0$ and at infinity become interchanged, except when one of the structural couplings constants is zero and there is no distinction at all. However, in both cases the solutions are monotonous, diverges to minus infinity at $r = 0$, changes sign at the horizon and keep increasing to achieve the asymptotically Lifshitz value $f = 1$. Both graphs are shown in figure 3.

5.3 Logarithmic black holes

For $z = D-2$ we obtain the logarithmic potential (3.10), using the fixing of the integration constants (3.17) it can be written as

$$f(r) = 1 - \left(\frac{\hat{l}}{r} \right)^{D-2} \left[\beta_1 \ln \left(\frac{r}{\hat{l}} \right) - \beta_2 \right]. \tag{5.13}$$
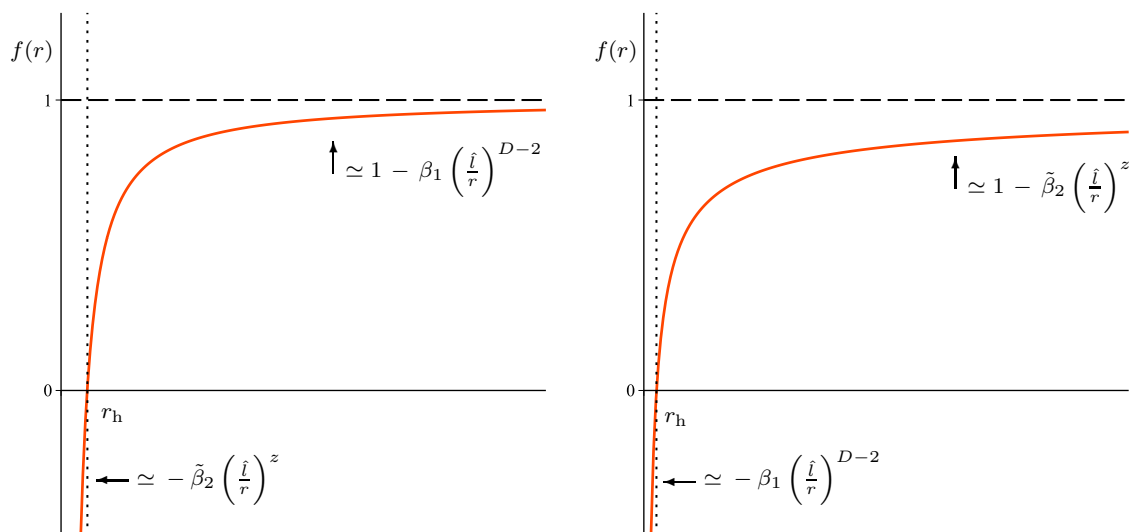
– 12 –

Figure 3. These graphics represent black holes with no extremum for the gravitational potential $f(r)$. In the left graph $z = 2(D - 2) > D - 2$ with coupling constants $\beta_1 > 0$ and $\beta_2 = -\tilde{\beta}_2 = -\beta_1^2 < 0$. The right graph is for $z = (D - 2)/2 < D - 2$, $\beta_1 > 0$ and $\beta_2 = -\tilde{\beta}_2 = -\beta_1^{1/2} < 0$.

Such potential has an extremum at

$$(D - 2)\beta_1 \ln \left(\frac{r}{\hat{l}} \right) = \beta_1 + (D - 2)\beta_2 , \tag{5.14}$$

where the second derivative is evaluated as

$$f'' = \frac{(D - 2)\beta_1}{\hat{l}^2} \left(\frac{\hat{l}}{r} \right)^D , \tag{5.15}$$

which means that the extremum is a minimum for $\beta_1 > 0$, giving two-horizons black holes, and a maximum for $\beta_1 = -\tilde{\beta}_1 < 0$ giving single horizons black holes.

For $\beta_1 > 0$ the minimum achieved at

$$r_{\min} = \hat{l} \exp \left[\frac{\beta_1 + (D - 2)\beta_2}{(D - 2)\beta_1} \right], \tag{5.16}$$

has value

$$f(r_{\min}) = 1 - \frac{\beta_1}{D - 2} \exp \left[-1 - (D - 2)\frac{\beta_2}{\beta_1} \right]. \tag{5.17}$$

From this minimum the potential grows to the right to achieve the Lifshitz asymptotic value at infinity as $1 - \beta_1 (\hat{l}/r)^{D-2} \ln (r/\hat{l})$ and to the left increase indefinitely at $r = 0$ obeying $-\beta_1 (\hat{l}/r)^{D-2} \ln (r/\hat{l})$; see the left graph of figure 4. The solution represents black holes if $f(r_{\min}) \leq 0$, which occurs for

$$\beta_2 \leq \frac{\beta_1}{D - 2} \left[\ln \left(\frac{\beta_1}{D - 2} \right) - 1 \right]. \tag{5.18}$$
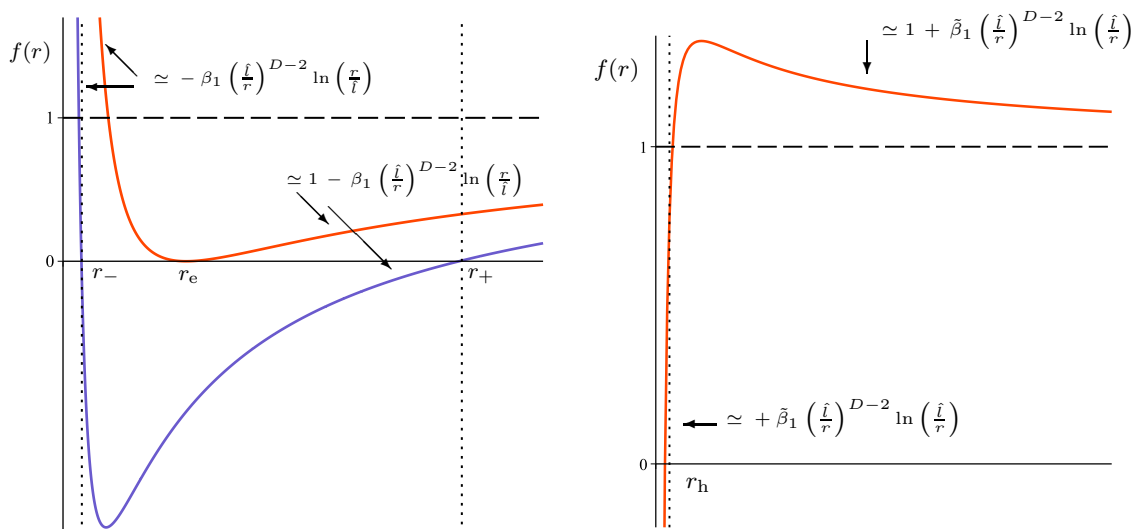
– 13 –

Figure 4. The graphics represent logarithmic black holes for $D = 5$. The left graph represents the minimum case, for which $\beta_1 > 0$. The right graph represents the maximum, where $\beta_1 = -\tilde{\beta}_1 < 0$ and $\beta_2 = -\frac{1}{D-2}\beta_1$.

Notice that using the limit procedure of subsection 3.1 we obtain exactly this inequality from the one of (5.6) in the limit $z \to (D-2)^+$. When the inequality is saturated we have an extremal black hole with horizon

$$r_{\rm e} = \hat{l}\left(\frac{\beta_1}{D-2}\right)^{1/(D-2)}, \tag{5.19}$$

and the gravitational potential is rewritten as

$$f(r) = 1 - \left(\frac{r_{\rm e}}{r}\right)^{D-2}\left[(D-2)\ln\left(\frac{r}{r_{\rm e}}\right) + 1\right]. \tag{5.20}$$

For $\beta_1 = -\tilde{\beta}_1 < 0$ we deal with a potential having a non-vanishing maximum

$$f(r) = 1 + \left(\frac{\hat{l}}{r}\right)^{D-2}\left[\tilde{\beta}_1\ln\left(\frac{r}{\hat{l}}\right) + \beta_2\right]. \tag{5.21}$$

From this maximum the potential fall to the asymptotic Lifshitz value to the right according to $1 + \tilde{\beta}_1(\hat{l}/r)^{D-2}\ln(r/\hat{l})$ and to the left it changes sign at the horizon and continues decreasing infinitely at $r = 0$ as $+\tilde{\beta}_1(\hat{l}/r)^{D-2}\ln(r/\hat{l})$; see the right graph of figure 4.

6 Conclusions

Here, using the $(\mathcal{H}, P)$-formalism, we have proposed a new nonlinear electrodynamics that together with the Proca field has allowed us to obtain charged Lifshitz black hole configurations characterized by the fact that the dynamical critical exponent can take any value $z > 1$. The solutions have as single integration constant the electric charge, but they are parameterized by the two arbitrary structural coupling constant characterizing the family of nonlinear electrodynamics, in addition to the critical exponent. After exhaustively

studying all the configurations, we conclude that the family of solutions gives rise to eight different kinds of black holes; three cases represent two-horizons black holes and the other five have a single horizon. For a generic exponent $z \neq D - 2$ the solutions decay to Lifshitz spacetime by a negative power, but the value $z = D - 2$ turns out to be critical in the sense that it yields a logarithmic decay supported by a particular logarithmic electrodynamics. This is the second example of a logarithmic Lifshitz black hole known in the literature after the higher-curvature one shown in refs. [15, 16]. We provide also a useful limiting procedure to obtain log configurations and its properties as a nontrivial limit of the generic ones. For the two asymptotically Lifshitz behaviors we exhibit examples of extremal black holes, which were known previously only in the presence of higher-order theories [15, 16]. It is interesting to stress that the derivation of these solutions has been possible because of the nonlinear character of the electrodynamics source. It is also appealing that charged topological Lifshitz black holes can also be obtained by slightly generalizing the proposed electrodynamics. Indeed, in this case, the constant curvature of the horizon is encoded in the nonlinear Lagrangian and does not appear in the gravitational potential as it usually occurs. As consequence, the electric charge is no longer an integration constant if we pursue to incorporate nontrivial horizon topologies.

It is now accepted that nonlinear electrodynamics enjoy nice thermodynamical properties since it usually satisfy the zeroth and first law. It will be nice to study the thermodynamics issue of these charged solutions in order to give a physical interpretation of the constants appearing in the solutions. In ref. [39], the authors explore the possible thermodynamics behavior of charged Lifshitz black hole solutions with an electrodynamics given by a power-law of the Maxwell invariant. However, they do not have explicit solutions in order to check if their assumptions fit or not. In section 4, we have shown that our electrodynamics contains these kind of theories as particular case, which provide an explicit working example.

We are also convinced that nonlinear electrodynamics will still be useful to explore Lifshitz black hole solutions and its generalizations in contexts beyond standard gravity, as for example for the Lovelock gravity [17] or even in higher-order gravity theories [15, 16, 18, 19].

Acknowledgments

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