

A fresh look at hadronic light-by-light scattering in the muon $g - 2$ with the Dyson-Schwinger approach

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Abstract. Using the Dyson-Schwinger and Bethe-Salpeter equations, we calculate the hadronic light-by-light scattering contribution to the anomalous magnetic moment of the muon a_μ , using a phenomenological model for the gluon and quark-gluon interaction. We find $a_\mu = (84 \pm 13) \times 10^{-11}$ for meson exchange, and $a_\mu = (107 \pm 2 \pm 46) \times 10^{-11}$ for the quark loop. The former is commensurate with past calculations; the latter much larger due to dressing effects. This leads to a revised estimate of $a_\mu = 116\,591\,865.0(96.6) \times 10^{-11}$, reducing the difference between theory and experiment to $\simeq 1.9 \sigma$.

1 Introduction

The anomalous magnetic moment $a_\mu = (g_\mu - 2)/2$ of the muon is one of the most precisely determined quantities in particle physics, both theoretically and experimentally. Their impressive agreement on the level of 10^{-8} serves as one of the prime examples of the fidelity of the Standard Model (SM). Experimental efforts at Brookhaven and theoretical efforts of the past ten years have pinned a_μ down to the 10^{-11} level, leading to deviations between theory [1] and experiment [2,3] of about 3σ :

$$\text{Experiment : } 116\,592\,089.0(63.0) \times 10^{-11}, \quad (1)$$

$$\text{Theory : } 116\,591\,790.0(64.6) \times 10^{-11}. \quad (2)$$

This discrepancy is extremely interesting, since it may be a signal for New Physics beyond the SM. However, to clearly distinguish between New Physics and possible shortcomings in the SM calculations the uncertainties present in both experimental and theoretical values of a_μ need to be further reduced.

The greatest uncertainties in the theoretical determination of a_μ are encountered in the hadronic contributions, *i.e.* those terms which involve QCD beyond perturbation theory. The most prominent of these is given by the vacuum polarisation tensor dressing of the QED vertex, see fig. 1(a). Fortunately, it can be related to experimental data of e^+e^- -annihilation and τ -decay via dispersion relations and the optical theorem, resulting in a

precise determination with systematically improvable errors [1]. Although currently these uncertainties dominate the theoretical error in eq. (2) it is foreseeable that future experiments reduce this error below that of another, more problematic source. This is the hadronic light-by-light (LBL) scattering diagram, shown in fig. 1(b). This contribution cannot be directly related to experiment and must hence be calculated entirely through theory.

The central object is the photon four-point function. It receives important contributions from the small momentum region below 2 GeV, where perturbative QCD breaks down and non-perturbative methods are imperative. In the past the LBL contribution has been approximated using ideas from the large- N_c expansion and chiral effective theories and the associated ordering of diagrams [4], see fig. 2. These diagrams have been evaluated within the extended Nambu-Jona-Lasinio model (ENJL) [5–7], and the hidden local symmetry model [8–10]. Recent refined calculations have used ideas of vector meson dominance (VMD) [11–14] and a non-local chiral quark model [15,16], see [17] for a summary. In all these calculations the (pseudoscalar) meson exchange contributes the most with the meson loop found to be small. An explanation of the latter is given in [12]. As a result, we quote the recent value for LBL $a_\mu^{\text{LBL}} = 105(26) \times 10^{-11}$ proposed in ref. [17], which also agrees with [14].

One problem common to all of these approaches has seen intense debate in recent years [1]: to account for spacelike momenta flowing through the meson propagator in the exchange diagram one requires a prescription

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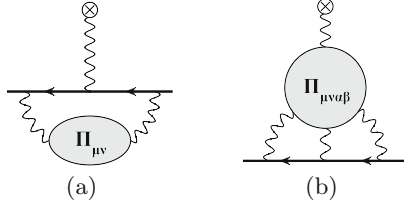


Fig. 1. Two classes of corrections to the photon-muon vertex function: (a) hadronic vacuum polarisation contribution; (b) hadronic light-by-light (LBL) scattering contribution.

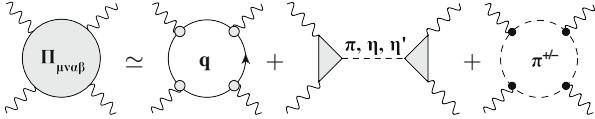


Fig. 2. The hadronic light-by-light (LBL) contribution to a_μ and its leading expansion in a quark loop (left), pseudoscalar meson exchange (middle) and a meson loop (right).

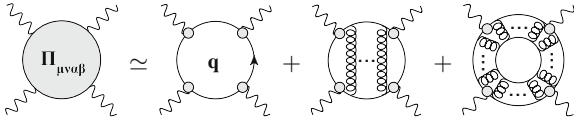


Fig. 3. The LBL contribution to a_μ and its expansion in a quark loop part (left), a ladder exchange part (middle) and a ladder ring part (right). All propagators are fully dressed.

to take the meson “off-shell”. Such a procedure is not free of ambiguities thus generating systematic uncertainties which are hard to quantify.

A possibility to avoid such problems is to abandon effective degrees of freedom by working on the level of (dressed) quarks and gluons. Such a (non-perturbative) expansion is given in fig. 3. Here, all propagators are fully dressed, with the quark-gluon interaction given in terms of a rainbow-ladder (RL) approximation. No double-counting is involved. On the meson mass shells, the ladder sum in the exchange and ring diagrams of fig. 3 reduce to the corresponding diagrams of fig. 2. However, fig. 3 accounts systematically and unambiguously for off-shell effects which may be more accurate than the effective description of fig. 2. Clearly, the diagrams in fig. 3 are not the full story and must be supplemented by non-RL-like diagrams. Nevertheless, they provide a systematically improvable starting point for a complementary evaluation of hadronic LBL scattering.

In this letter we briefly report on the first steps in the evaluation of fig. 3; a detailed account on our calculation can be found in ref. [18], a very brief summary is given in the proceedings contribution ref. [19]. Our framework for the QCD part of the calculation are the Dyson-Schwinger (DSE) and Bethe-Salpeter equations (BSE) of Landau gauge QCD [20–22]. We present results for the quark loop diagram and for the resonant part of the middle diagram of fig. 3 in the form of the pseudoscalar meson exchange. To our knowledge our calculation is the first to employ

fully dressed, momentum-dependent quark propagators and vertices in the quark loop; for the meson exchange diagrams we calculate the $\pi\gamma\gamma$ form factors from the underlying theory as opposed to the frequently employed strategy of using ansätze [11, 12, 14]. As we will see, our results for the pseudoscalar meson exchange agree nicely with those of previous approaches, whereas the contribution from the dressed quark loop turns out to be considerably larger. We discuss these findings at the end of this letter.

2 Computational scheme: quarks and mesons

Here we summarize our calculational scheme for LBL; the details are given in ref. [18]. We determine the fully dressed inverse quark propagator,

$$S^{-1}(p) = i\not{p}A(p^2) + B(p^2) \quad (3)$$

with vector and scalar dressing functions $A(p^2)$ and $B(p^2)$ from the DSE shown in fig. 4. The Landau gauge gluon propagator $D_{\mu\nu}(k^2)$ is given by

$$D_{\mu\nu}(k) = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \frac{Z(k^2)}{k^2} \quad (4)$$

with the non-perturbative dressing function $Z(k^2)$. The quark-gluon vertex $\Gamma_\mu(p, q)$ with quark momenta p, q in principle consists of contributions from twelve different tensor structures. Here we take the phenomenologically important γ_μ -part into account leading to $\Gamma_\mu(p, q) = \Gamma^{\text{YM}}(k^2)\gamma_\mu$ with gluon momentum k . Both functions $Z(k^2)$ and $\Gamma^{\text{YM}}(k^2)$ are combined into a single model function. For the Maris-Tandy (MT) model [23] this is

$$Z(k^2)\Gamma^{\text{YM}}(k^2) = \frac{4\pi}{g^2} \left(\frac{\pi}{\omega^6} Dq^4 \exp(-q^2/\omega^2) + \frac{2\pi\gamma_m}{\log(\tau + (1+q^2/\Lambda_{\text{QCD}}^2)^2)} \left[1 - e^{-q^2/(4m_t^2)} \right] \right), \quad (5)$$

with $m_t = 0.5 \text{ GeV}$, $\tau = e^2 - 1$, $\gamma_m = 12/(33 - 2N_f)$, $\Lambda_{\text{QCD}} = 0.234 \text{ GeV}$, $\omega = 0.4 \text{ GeV}$ and $D = 0.93 \text{ GeV}^2$. The Gaussian factor gives the interaction strength necessary for dynamical chiral symmetry breaking, whereas the logarithm represents the one-loop behavior of the running coupling at large perturbative momenta. The latter is mandatory for the correct short-distance behavior of the quark propagator and all derived quantities.

The rainbow approximation eq. (5) leads to the corresponding ladder approximation in the kernel of the BSE, fig. 5, describing mesons as bound states of quarks and antiquarks. The Bethe-Salpeter vertex function for a pseudoscalar meson is

$$\Gamma^{q\bar{q}}(p; P) = \gamma_5 [F_1 - i\not{P}F_2 - i\not{p}F_3 - [\not{P}, \not{p}]F_4], \quad (6)$$

with P, p the total and the relative momenta of the two constituents and $F_i := F_i(p, P)$.

The approximation (figs. 4 and 5), with the MT model eq. (5) is very successful from a phenomenological perspective [23–25, 21]. This is especially true for the pseudoscalar



Fig. 4. Dyson-Schwinger equation for the quark propagator.

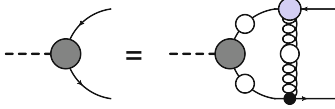


Fig. 5. Bethe-Salpeter equation for mesonic bound states.

meson sector, wherein the Goldstone nature of the pion is realized in the chiral limit. While tuned to reproduce experimental values for the pion mass and decay constant, it also reproduces the pion charge radius and $\pi\gamma\gamma$ transition form factors on the percent level. In the vector channel, agreement with experimental masses and decay constants is on the five and ten percent level. Thus, we view the MT model as a promising starting point for our calculation of the LBL amplitude.

3 The pseudoscalar meson exchange contribution to LBL

We begin by calculating the pseudoscalar meson exchange diagram of fig. 2, which first requires that we numerically determine the full quark-photon vertex, given by its inhomogeneous BSE shown in fig. 6. This vertex is non-perturbative and necessary for the calculation of the $\pi\gamma\gamma$ form-factor. With one Lorentz and two spinor indices, it is decomposed into twelve Dirac structures; a suitable basis is specified in ref. [26]. A quark-photon vertex determined thus also contains time-like poles corresponding to the vector meson exchange [24]. Thus, the main ideas of VMD are naturally included here.

Next, we determine the $\pi \rightarrow \gamma\gamma$ form-factor $F^{\pi\gamma\gamma}(k_1, k_2)$, with the two photon momenta k_1 and k_2 . In the impulse approximation, consistent with the RL truncation introduced above, this is given by the diagram in fig. 7. The π^0 electromagnetic form-factor has been explored in detail in ref. [25], wherein it has been confirmed that the correct limit at vanishing photon momenta, given by the Abelian anomaly, is obtained:

$$\Lambda_{\mu\nu}^{\pi\gamma\gamma}(k_1^2, k_2^2) = i \frac{\alpha_{\text{em}}}{\pi f_\pi} \varepsilon_{\mu\nu\alpha\beta} k_1^\alpha k_2^\beta F^{\pi\gamma\gamma}(k_1^2, k_2^2), \quad (7)$$

where α_{em} is the fine structure constant and f_π the pion decay constant. The prefactors are such that $F^{\pi\gamma\gamma}(0, 0) = 1$. It has been shown analytically and numerically that the form factor has the correct asymptotic behaviour [25]:

$$\begin{aligned} \lim_{Q^2 \rightarrow \infty} F^{\pi\gamma\gamma*}(0, Q^2) &\propto \frac{1}{Q^2}, \\ \lim_{Q^2 \rightarrow \infty} F^{\pi\gamma\gamma*}(Q^2, Q^2) &\propto \frac{1}{Q^2}. \end{aligned} \quad (8)$$

Following the procedure outlined above we obtain the $\pi \rightarrow \gamma\gamma$ form-factor $F^{\pi\gamma\gamma}(k_1, k_2)$ numerically for all momenta k_1 and k_2 necessary to evaluate the meson exchange

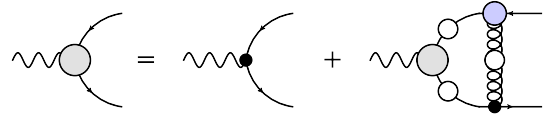


Fig. 6. Equation for the quark-photon vertex.

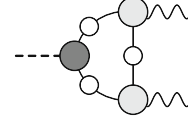


Fig. 7. The $\pi^0\gamma\gamma$ form-factor in impulse approximation.

diagram in LBL, with a total numerical error of the order of one percent. Similarly, we evaluate the corresponding form-factors for the η and η' mesons.

Our solutions for $F^{\pi\gamma\gamma}(k_1, k_2)$ have the same overall structure as the VMD inspired ansätze used in refs. [11, 12, 14, 17]. There are non-negligible differences in the mid-momentum region of the order of five percent, whereas the asymptotic behavior is the same [24].

Our results for the form factors are used to evaluate the pseudoscalar meson exchange contribution to LBL. This requires an off-shell prescription for the exchanged mesons. We use the chiral limit axial-vector Ward-Takahashi identity

$$2P_\mu I_\mu^5(k, P) = iS^{-1}(k_+) \gamma_5 + i\gamma_5 S^{-1}(k_-) \quad (9)$$

that establishes the following form for the dominant π^0 -amplitude:

$$F_1(k, P) = \lambda_3 (B(k_+) + B(k_-)) / (2f_\pi), \quad (10)$$

with $k_\pm = k \pm P/2$ and λ_3 a Gell-Mann matrix. On mass-shell in the chiral limit it reduces to the exact result $F_1(k, 0) := \lambda_3 B(k^2) / f_\pi$. The pseudoscalar amplitude of eq. (6) is generalized via eq. (10) for all Dirac structures.

The systematic error of our calculation can be attributed entirely to the validity of the RL approximation within the MT model, eq. (5), and the off-shell prescription eq. (10). No other approximations have been used. While in the Goldstone-Boson sector the MT model works well, there is certainly a larger error in the flavor singlet sector. We therefore guesstimate a total systematic error: ten percent for the pion contribution, and twenty percent for the η and η' contributions. With a numerical error of two percent we obtain: $a_\mu^{\text{LBL}; \pi^0} = (57.5 \pm 6.9) \times 10^{-11}$, $a_\mu^{\text{LBL}; \eta} = (15.8 \pm 3.5) \times 10^{-11}$ and $a_\mu^{\text{LBL}; \eta'} = (11.0 \pm 2.4) \times 10^{-11}$ leading to

$$a_\mu^{\text{LBL}; \text{PS}} = (84.3 \pm 12.8) \times 10^{-11}. \quad (11)$$

This number is compatible with previous results [13, 14, 17, 16].

4 Quark loop contribution to LBL

We now focus on the quark loop contribution to LBL. Here we follow the strategy employed in refs. [27, 28], and differentiate the four-point function with respect to the external

photon momentum [5]. We verified that we reproduce the well-known perturbative result for the electron loop with an accuracy of better than one per mille. For the quark loop we use the fully dressed quark propagators for the up, down, strange and charm quarks, extracted from their DSE, fig. 4. For the quark-photon vertex one would like to use the full numerical solution of fig. 6, which satisfies the Ward identity of the vertex and contains transverse parts including vector meson contributions. However, this is a formidable numerical task which is beyond the scope of the present work. Instead, we concentrate on that part of the vertex which is constrained by gauge invariance and compare the result for bare vertices with one where we use the first term of the Ball-Chiu representation (1BC) [26]

$$\Gamma_\mu(p, q) = \gamma_\mu (A(p^2) + A(q^2)) / 2, \quad (12)$$

where p, q are the quark and antiquark momenta. This expression is known to be a reasonable approximation of the full Ball-Chiu vertex, which is constructed to satisfy the Ward-Takahashi identity (WTI) of the complete quark-photon vertex. Thus one may hope that eq. (12) gives a good first estimate for those parts of the vertex relevant for gauge invariance. Nevertheless, one has to keep in mind that potentially important transverse contributions due to vector meson poles [5,6] are neglected. Possible quantitative implications of this omission are discussed in [18].

Here, comparing between the bare result and that with eq. (12) may serve as a first guide for the systematic error due to the relevance of vertex effects. We find

$$\begin{aligned} a_\mu^{\text{LBL; quark loop (bare vertex)}} &= (61 \pm 2) \times 10^{-11}, \\ a_\mu^{\text{LBL; quark loop (1BC)}} &= (107 \pm 2) \times 10^{-11} \end{aligned} \quad (13)$$

for the quark loop contribution. Clearly, this is a sizeable contribution. Whereas the bare vertex result roughly agrees with the number 60×10^{-11} given in [12], the dressing effects of the vertex lead to a drastic increase. Unfortunately this makes it very hard if not impossible to guess the effect of the total vertex dressing without an explicit calculation. Certainly given these findings, all previously given estimates for the systematic error in the quark loop contributions seem to be an order of magnitude too small.

5 Conclusions

In this letter we have presented a new approach towards the anomalous magnetic moment of the muon. We have used a combination of Dyson-Schwinger and Bethe-Salpeter equations to evaluate the pseudoscalar meson exchange contribution and the quark loop contribution to LBL. Our only input is the Maris-Tandy model, a phenomenologically successful ansatz for the combined strength of the gluon propagator and the quark-gluon vertex. We believe this is a systematic improvement as compared to previous approaches. When combining our two results, eq. (11) and eq. (13), we also have to add the effects from the right-hand diagrams of fig. 2 or fig. 3. These are contributions involving an additional quark loop which

are typically negative and of the order of ten to twenty percent of the leading- N_c contributions [29–31]. Since, on the other hand, we also expect positive contributions of a similar size from non-pseudoscalar exchange diagrams [1] we choose to subsume all these contributions to another $a_\mu^{\text{LBL; other}} = (0 \pm 20) \times 10^{-11}$, where the error is clearly subjective. This amounts to a total error so far of 35×10^{-11} . Because of the importance of dressing effects, we use the quark loop in the 1BC approximation for the central value and gauge an additional systematic error of 46×10^{-11} by comparing with the bare vertex result. This gives us the following total hadronic LBL contribution:

$$a_\mu^{\text{LBL; (1BC)}} = (191 \pm 35 \pm 46) \times 10^{-11}, \quad (14)$$

in our approach. Taken at face value these numbers together with the other contributions quoted in [1] clearly reduce the discrepancy between theory and experiment. Combining our light-by-light scattering results with the other SM contributions gives

$$a_\mu^{\text{theor.}} = 116\,591\,865.0(96.6) \times 10^{-11}. \quad (15)$$

With the positive shift and larger error, the deviation between theory and experiment is reduced to $\simeq 1.9 \sigma$.

Whether the neglected parts of the fermion-photon vertex have the potential to drive this theoretical result further towards the experiment or back to previous results [7, 1] remains at present a subject of mere speculation. Certainly, our results do not provide final answers in any sense but may serve as a starting point towards a more fundamental determination of a_μ along the lines discussed around fig. 3.

What we have shown, however, is that dressing effects in the quark-photon vertex due to gauge invariance constraints are important and cannot be ignored in determining reliable estimates of hadronic LBL.

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