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RESEARCH ARTICLE

Dual gravity and matter

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Abstract We consider the problem of finding a dual formulation of gravity in the presence of non-trivial matter couplings. In the absence of matter a dual graviton can be introduced only for linearised gravitational interactions. We show that the coupling of linearised gravity to matter poses obstructions to the usual construction and comment on possible resolutions of this difficulty.

Keywords Gravity · Duality · Matter

1 Introduction

One of the remarkable features of $D = 4$ electrodynamics is that it allows for both an electric formulation, using the vector potential A_μ, and for a magnetic formulation, using a dual vector potential $\tilde{A}_\mu$, in any background described by a metric $g_{\mu\nu}$. The duality relation between these two fields can be written as

$$F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\tau}\tilde{F}^{\rho\tau}, \quad F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}, \quad \tilde{F}_{\mu\nu} = 2\partial_{[\mu}\tilde{A}_{\nu]}. \tag{1}$$

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The integrability condition for the local existence of $\tilde{A}_\mu$ is the original field equation: $-\frac{1}{2}\epsilon^{\nu\sigma_1\sigma_2\sigma_3}\nabla_{\sigma_1}\tilde{F}_{\sigma_2\sigma_3} = \nabla_\mu F^{\mu\nu} = 0$. Generally, the duality exchanges field equations and Bianchi identities. The duality property can be preserved when the Maxwell field is coupled to other matter, e.g., axion/dilaton scalar fields, but breaks down when generalised to non-abelian gauge groups. The construction (1) in $D = 4$ generalises to any p-form A_p (that is any field with p antisymmetric spacetime indices) in D dimensions, the dual of the field A_p being a $(D-p-2)$-form $\tilde{A}_{D-p-2}$.

It is natural to ask whether a similar dual formulation exists for the gravitational field. For *linearised* gravity in vacuo[1] such a formulation is known to exist [3–13] but a BRST analysis reveals, under rather general assumptions [14], obstructions to extend this to a theory with covariant and local interactions (see also [15]).

Expanded around a flat background the metric takes the form

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} + O\left(\kappa^2 h^2\right) \tag{2}$$

and the curvature tensors simplify in linear order to

$$R_{\mu\nu\,\rho\sigma} = 2\partial_{[\mu}\omega_{\nu]\,\rho\sigma}, \quad R_{\mu\nu} = -\partial_\mu\omega_{\rho\,\rho\nu} - \partial_\rho\omega_{\mu\,\nu\rho}, \quad R = -2\partial_\rho\omega_{\sigma\,\sigma\rho}, \tag{3}$$

where now all derivatives are partial and indices are raised and lowered with the flat Minkowski metric and we disregard higher order terms in the graviton $h_{\mu\nu}$ from now on. Evidently, the curvature tensors are of order $O(\kappa h)$.[2] The spin connection is $\omega_{\mu\,\nu\rho} = 2\kappa\partial_{[\nu}h_{\rho]\mu}$ in terms of the graviton and satisfies $\omega_{[\mu\,\nu\rho]} = 0$. The linearised vacuum Einstein equations in D-dimensional space–time can be written as [5]

$$0 = R^\nu{}_\rho - \frac{1}{2}\delta^\nu_\rho R = -\frac{1}{(D-2)!}\epsilon^{\nu\sigma_1\ldots\sigma_{D-1}}\partial_{\sigma_1}Y_{\sigma_2\ldots\sigma_{D-1},\rho}, \tag{4}$$

where

$$Y_{\mu_1\ldots\mu_{D-2},\rho} = \frac{1}{2}\epsilon_{\mu_1\ldots\mu_{D-2}}{}^{\sigma_1\sigma_2}\left(\omega_{\rho\,\sigma_1\sigma_2} - 2\eta_{\rho\sigma_1}\omega^\nu{}_{\nu\sigma_2}\right) \tag{5}$$

is obtained from dualising the spin connection and its trace. Y is contained in the tensor product of a vector with a $(D-2)$-form, we use a comma to seperate the antisymmetric indices from the single vector index. Equation (4) suggests the introduction of a dual graviton $D_{\mu_1\ldots\mu_{D-3},\rho}$ via

$$(D-2)\partial_{[\mu_1}D_{\mu_2\ldots\mu_{D-2}],\rho} = Y_{\mu_1\ldots\mu_{D-2},\rho} \tag{6}$$

as solution to the Y-Bianchi identity (4), which is equivalent to the graviton equation of motion. The consequence of linearisation $\omega_{[\mu\,\nu\rho]} = 0$ is equivalent to $Y_{\mu_1\ldots\mu_{D-3}\nu}{}^\nu = 0$, which is a differential condition on the dual graviton. It was argued in [7] that the

[1] For a discussion of gravitational duality in (Anti) de Sitter space see [1,2].

[2] The dimensions are: $[\kappa] = \frac{2-D}{2}$, $[h_{\mu\nu}] = \frac{D-2}{2}$, $[\omega_{\mu\,\nu\rho}] = 1$ and $[R_{\mu\nu\,\rho\sigma}] = 2$.

condition $D_{[\mu_1\ldots\mu_{D-3},\rho]} = 0$ can be imposed by a local Lorentz transformation. The Eqs. (4) and (5) can be derived from the Einstein action in first order formulation as shown in [5]. One introduces Y as an auxiliary field in the action which then depends on the vielbein and Y. Substituting the solution of the algebraic equation of motion for Y gives back the Einstein action. In this framework (5) is the algebraic equation of motion for Y whereas (4) is the equation of motion obtained by varying with respect to the vielbein and linearising (see also [7]).

A slightly different approach for the introduction of a dual graviton starts from the Riemann tensor and its symmetries [3,6,8]. Dualising the full linearised Riemann tensor $R_{\mu\nu\rho\sigma}$ on one set of antisymmetric indices gives the tensor

$$S_{\mu_1\ldots\mu_{D-2}\,\rho\sigma} = \frac{1}{2}\epsilon_{\mu_1\ldots\mu_{D-2}}{}^{\nu_1\nu_2} R_{\nu_1\nu_2\,\rho\sigma}. \tag{7}$$

The (algebraic and differential) identities for the Riemann tensor together with the linearised equations of motion then imply that on-shell [6,8]

$$\begin{aligned} S_{\mu_1\ldots\mu_{D-2}\,\rho\sigma} &= \partial_\sigma\partial_{[\mu_1}\tilde{D}_{\mu_2\ldots\mu_{D-2}],\rho} - \partial_\rho\partial_{[\mu_1}\tilde{D}_{\mu_2\ldots\mu_{D-2}],\sigma} \\ &= \partial_\sigma\tilde{Y}_{\mu_1\ldots\mu_{D-2},\rho} - \partial_\rho\tilde{Y}_{\mu_1\ldots\mu_{D-2},\sigma} \end{aligned} \tag{8}$$

in terms of a dual graviton $\tilde{D}_{\mu_1\ldots\mu_{D-3},\rho}$ which manifestly satisfies $\tilde{D}_{[\mu_1\ldots\mu_{D-3},\rho]} = 0$. The linearised Einstein equation in this case is obtained by taking antisymmetric parts of S, e.g.

$$\frac{1}{(D-3)!}\epsilon^{\mu\sigma_1\ldots\sigma_{D-1}} S_{\nu\sigma_1\ldots\sigma_{D-1}} = R^\mu{}_\nu - \frac{1}{2}\delta^\mu_\nu R. \tag{9}$$

In this approach there is no local duality relation similar to (5). Arguably the best one can hope for is

$$\tilde{Y}_{\mu_1\ldots\mu_{D-2},\rho} = \frac{1}{2}\epsilon_{\mu_1\ldots\mu_{D-2}}{}^{\sigma_1\sigma_2}\omega_{\rho\,\sigma_1\sigma_2} + \partial_\rho\tilde{\Lambda}_{\mu_1\ldots\mu_{D-2}}, \tag{10}$$

where $\tilde{\Lambda}_{\mu_1\ldots\mu_{D-2}}$ is a possibly non-local term which ensures that all symmetry properties are satisfied. The term $\tilde{\Lambda}_{\mu_1\ldots\mu_{D-2}}$ is allowed for since it drops out in S, cf. (8).

This paper is organised as follows. In Sect. 2, we will show that the dual graviton can also be introduced in the context of linearized supergravity in $D = 4$. Our approach uses the duality relation (10). In Sect. 3, we discuss dual gravity in the presence of gravity and matter, in an arbitrary number of dimensions, and determine the conditions on the energy–momentum tensor that this matter coupling requires. The analysis of these conditions shows that linearised gravity and dual gravity cannot be combined with matter. In Sect. 4, we discuss these results and possible escape routes.

2 Supersymmetry in $D = 4$

In this section, we show that the supersymmetry algebra of minimal supergravity in four dimensions closes on the dual graviton $\tilde{D}_{\mu\nu}$ at the linearised level. At lowest order in the fermions, the supersymmetry transformations of the vielbein and the gravitino are

$$\begin{aligned} \delta e_\mu{}^a &= \frac{1}{2}\bar{\epsilon}\,\gamma^a\psi_\mu\,, \\ \delta\psi_\mu &= \left(\partial_\mu - \frac{1}{4}\omega_{\mu\alpha\beta}\gamma^{\alpha\beta}\right)\epsilon\,, \end{aligned} \tag{11}$$

where the spinor ϵ and the gravitino are Majorana. We want to linearise gravity around a flat background, and this corresponds to considering linearised global supersymmetry transformations

$$\begin{aligned} \delta h_{\mu\nu} &= \bar{\epsilon}\gamma_{(\mu}\psi_{\nu)}\,, \\ \delta\psi_\mu &= -\frac{1}{4}\gamma^{\alpha\beta}\omega_{\mu\,\alpha\beta}\epsilon\,, \end{aligned} \tag{12}$$

where $\omega_{\mu\,\alpha\beta}$ is the linearised spin connection, and $h_{\mu\nu}$ is the first order fluctuation of the metric.

In four dimensions the dual graviton has the same spacetime index structure as the graviton, and thus we denote it with $\tilde{D}_{\mu\nu}$, where the spacetime indices are meant to be symmetrised. This field varies with respect to general coordinate transformations, that at the linearised level are translations, but it also possesses its own gauge transformations, that have the form

$$\delta\tilde{D}_{\mu\nu} = \partial_{(\mu}\Lambda_{\nu)}\,, \tag{13}$$

where Λ_μ is an arbitrary gauge parameter. This gauge transformation has precisely the same structure as the general coordinate transformation of the linearised graviton. This would not be true in dimensions other than four.

We require the supersymmetry transformation of the dual graviton to be

$$\delta\tilde{D}_{\mu\nu} = \frac{i}{2}\bar{\epsilon}\gamma_{(\mu}\gamma_5\psi_{\nu)}\,, \tag{14}$$

where in our conventions $\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3$, and we are using mostly + signature. Using Eq. (12), the commutator of two supersymmetry transformations on $\tilde{D}_{\mu\nu}$ gives

$$[\delta_1, \delta_2]\tilde{D}_{\mu\nu} = -\frac{i}{4}\omega_{(\mu}{}^{\alpha\beta}\bar{\epsilon}_2\gamma_{\nu)\alpha\beta}\gamma_5\epsilon_1 = -\frac{1}{2}\omega_{(\mu}{}^{\alpha\beta}\epsilon_{\nu)\alpha\beta\gamma}\xi^\gamma\,, \tag{15}$$

where

$$\xi^\mu = \frac{1}{2}\bar{\epsilon}_2\gamma^\mu\epsilon_2 \tag{16}$$

is the general coordinate transformation parameter that occurs in the commutator of two supersymmetry transformations on the graviton and on the gravitino.

In this four dimensional case, the duality relation (10) becomes

$$\tilde{Y}_{\mu\nu,\rho} + \partial_\rho \tilde{\Lambda}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\omega_\rho{}^{\alpha\beta}, \tag{17}$$

where

$$\tilde{Y}_{\mu\nu,\rho} = \partial_\mu \tilde{D}_{\nu\rho} - \partial_\nu \tilde{D}_{\mu\rho}. \tag{18}$$

Using these equations, Eq. (15) becomes

$$[\delta_1, \delta_2]\tilde{D}_{\mu\nu} = \xi^\gamma \partial_\gamma \tilde{D}_{\mu\nu} - \xi^\gamma \partial_{(\mu} \tilde{D}_{\nu)\gamma} - \xi^\gamma \partial_{(\mu} \tilde{\Lambda}_{\nu)\gamma}. \tag{19}$$

Given that at the linearised level we can treat ξ as a constant, this result shows that this supersymmetry commutator produces a gauge transformation as in Eq. (13), with parameter

$$\Lambda_\mu = -\xi^\gamma (\tilde{D}_{\mu\gamma} + \tilde{\Lambda}_{\mu\gamma}), \tag{20}$$

as well as translations. This proves that one can close the supersymmetry algebra of minimal supergravity in four dimensions on the dual graviton at the linearised level.

3 Inclusion of matter

Matter couples to gravity via its energy–momentum tensor

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa^2 T_{\mu\nu}. \tag{21}$$

One can retain non-linear matter while linearising gravity. At lowest non-vanishing order in the graviton, matter and gravity decouple and one is left with the sum of a free spin two field and the remaining, possibly self-interacting, matter propagating on a Minkowski background. In this situation one can dualise the graviton as before since there are no matter contributions in the defining equations. This trivial dualisation is, however, not satisfactory from the point of view of the recently proposed infinite-dimensional symmetries [5] where the dual graviton should bear some marks of the matter present in the theory.[3]

Repeating the steps that led to (6) in the matter coupled action yields again the duality relation (5), but now (4) is replaced by

$$\partial_{[\mu_1} Y_{\mu_2\ldots\mu_{D-1}],\rho} = \kappa^2 \tilde{T}_{\mu_1\ldots\mu_{D-1},\rho}, \tag{22}$$

[3] Indeed, in the example of $D = 11$ supergravity one would expect from the structure of the E_{11} coset element that the dual graviton transforms non-trivially under the gauge transformations of the three-form potential and its dual six-form and that these transformations cannot be completely removed by field redefinitions.

where the right hand side in (22) is dual to the energy momentum tensor $T_{\mu\nu}$:

$$\tilde{T}_{\mu_1\ldots\mu_{D-1},\rho} = \frac{(-1)^{D-2}}{(D-2)!}\epsilon_{\mu_1\ldots\mu_{D-1}}{}^{\sigma}T_{\sigma\rho}. \tag{23}$$

The symmetry of $T_{\mu\nu}$ implies that the trace of the dual energy–momentum tensor vanishes. Now, since the r.h.s. of (22) is no longer zero we are not immediately led to the introduction of a dual graviton $D_{\mu_1\ldots\mu_{D-3},\rho}$; the integrability condition has changed. If, however, the dual of the energy–momentum tensor satisfies

$$\tilde{T}_{\mu_1\ldots\mu_{D-1},\rho} = -\partial_{[\mu_1}M_{\mu_2\ldots\mu_{D-1}],\rho}\ , \tag{24}$$

which is equivalent to

$$T^{\lambda}{}_{\rho} = \frac{(-1)^{D-2}}{D-1}\epsilon^{\mu_1\ldots\mu_{D-1}\lambda}\partial_{\mu_1}M_{\mu_2\ldots\mu_{D-1},\rho}\ , \tag{25}$$

we can define an improved Y by

$$Y_{\mu_1\ldots\mu_{D-2},\rho} \to Y_{\mu_1\ldots\mu_{D-2},\rho} + M_{\mu_1\ldots\mu_{D-2},\rho}. \tag{26}$$

This improved Y then satisfies the standard integrability relation and gives rise to the dual graviton as before. This improvement is only useful if M has a local expression in the matter fields and their duals. In other words, the introduction of a dual graviton in the presence of matter is equivalent to peeling one derivative off the dual energy momentum tensor in (24).

A similar conclusion is reached by studying the approach via the Riemann tensor. To obtain the Einstein equation as an integrability condition from (9) one requires that S gives rise to the energy–momentum contribution from the matter sector. This requires that there exists a tensor $\tilde{M}$ which plays the same role with respect to $\tilde{Y}$ as M to Y in (26):

$$\tilde{Y}_{\mu_1\ldots\mu_{D-2}\,\rho} \to \tilde{Y}_{\mu_1\ldots\mu_{D-2}\,\rho} + \tilde{M}_{\mu_1\ldots\mu_{D-2}\,\rho}\ , \tag{27}$$

which again leads to the problem of finding a local expression $\tilde{M}$ such that the Einstein equation arises from (9).

We have investigated, in a variety of cases related to supergravity systems with hidden symmetries, the relation (24) for the dual energy–momentum tensor to obtain local expressions for M and $\tilde{M}$. For simplicity we present the analysis in $D=4$ with gravity coupled to a single Maxwell field A_μ with the covariant energy–momentum tensor

$$T_{\mu\nu} = F_{\mu\sigma_1}F_{\nu}{}^{\sigma_1} - \frac{1}{4}g_{\mu\nu}F_{\sigma_1\sigma_2}F^{\sigma_1\sigma_2}. \tag{28}$$

In lowest order the dual energy–momentum tensor (23) then takes the form

$$\tilde{T}_{\mu_1\mu_2\mu_3,\rho} = \frac{3}{4} F_{\rho[\mu_1} \tilde{F}_{\mu_2\mu_3]} - \frac{3}{4} \tilde{F}_{\rho[\mu_1} F_{\mu_2\mu_3]} + \frac{3}{2} \eta_{\rho[\mu_1} F_{\mu_2}{}^{\sigma} \tilde{F}_{\mu_3]\sigma}. \tag{29}$$

Since $T^{\mu}{}_{\mu} = 0$ here we also have the constraint that $\tilde{T}_{[\mu_1\mu_2\mu_3,\rho]} = 0$. According to (24) we make the ansatz

$$\begin{aligned} M_{\mu_1\mu_2,\rho} =\; & \alpha_1 A_{[\mu_1} \partial_{\mu_2]} \tilde{A}_\rho + \alpha_2 A_{[\mu_1} \partial_{|\rho|} \tilde{A}_{\mu_2]} + \alpha_3 A_\rho \partial_{[\mu_1} \tilde{A}_{\mu_2]} + \beta_1 \tilde{A}_{[\mu_1} \partial_{\mu_2]} A_\rho \\ & + \beta_2 \tilde{A}_{[\mu_1} \partial_{|\rho|} A_{\mu_2]} + \beta_3 \tilde{A}_\rho \partial_{[\mu_1} A_{\mu_2]} + \gamma_1 \eta_{\rho[\mu_1} A_{\mu_2]} \partial^\nu \tilde{A}_\nu \\ & + \gamma_2 \eta_{\rho[\mu_1} A^\nu \partial_{\mu_2]} \tilde{A}_\nu + \gamma_3 \eta_{\rho[\mu_1} A^\nu \partial_{|\nu|} \tilde{A}_{\mu_2]} + \gamma_4 \eta_{\rho[\mu_1} \tilde{A}_{\mu_2]} \partial^\nu A_\nu \\ & + \gamma_5 \eta_{\rho[\mu_1} \tilde{A}^\nu \partial_{\mu_2]} A_\nu + \gamma_6 \eta_{\rho[\mu_1} \tilde{A}^\nu \partial_{|\nu|} A_{\mu_2]}, \end{aligned} \tag{30}$$

without any restrictions on the real coefficients α_i, β_i γ_i.[4] The terms with coefficients α_i and β_i are needed to reproduce the first two terms in (29) whereas the γ_i terms in the ansatz correspond to the third term in (29). Taking a curl of (30) through $\partial_{[\mu_1} M_{\mu_2\mu_3],\rho}$ and demanding that all terms combine into covariant field strengths after dualisation implies for α_i and β_i that

$$\alpha_1 + \beta_3 = 0, \quad \alpha_3 + \beta_1 = 0, \quad \alpha_2 = 0, \quad \beta_2 = 0 \tag{31}$$

and all $\gamma_i = 0$. Any $M_{\mu_1\mu_2,\rho}$ satisfying this condition leads to $\partial_{[\mu_1} M_{\mu_2\mu_3],\rho} = 0$ which implies $\tilde{T}_{\mu_1\mu_2\mu_3,\rho} = 0$. Therefore one cannot recover the matter coupled Einstein equations from a dual formulation in this way.[5]

Turning to the introduction of the dual graviton via the dualised Riemann tensor as in (8) one can again use the ansatz (30) for $\tilde{M}_{\mu_1\mu_2,\rho}$. Now the matter coupled Einstein equation should arise as in (9), which leads to the following condition between $\tilde{M}$ and the energy–momentum tensor:

$$\frac{1}{2} \epsilon^{\mu\sigma_1\sigma_2\sigma_3} \partial_{\sigma_3} \tilde{M}_{\nu\sigma_1,\sigma_2} = T^{\mu}{}_{\nu}. \tag{32}$$

Without making any assumptions on the symmetry of $\tilde{M}_{\mu_1\mu_2,\rho}$ one finds a one-parameter family of non-trivial solutions represented by

$$\alpha_1 = -\alpha_3 = \frac{1}{15}, \quad \alpha_2 = 1, \quad \beta_1 = -\beta_3 = \frac{1}{3}, \quad \beta_2 = \frac{1}{5}. \tag{33}$$

All coefficients can be rescaled by the same constant. However, insisting on the irreducibility condition of the dual graviton (which automatically holds in the approach

[4] Demanding that $M_{\mu_1\mu_2,\rho}$ comes from the dual graviton requires that $M_{[\mu_1\mu_2,\rho]} = 0$, or $\alpha_1 - \alpha_2 + \alpha_3 = \beta_1 - \beta_2 + \beta_3 = 0$ but we relax this condition for the moment.

[5] Allowing for a term which is a total ρ derivative as in (10) there are additional possibilities and there is a solution which gives the first two terms in (29). The third term cannot be accounted for in this way.

through the dualised Riemann tensor), removes this solution. This difficulty was already anticipated in [8].

The result of the explicit analysis above can be summarized in the following way [16]. If we could find a solution for M in (25) [or $\tilde{M}$ in (32)] the energy–momentum tensor would be defined in terms of a local improvement term, and would be conserved independently of the equations of motion. This is clearly undesirable.

4 Discussion

In both approaches to the dual graviton we found that there is no satisfactory way of coupling linearised gravity to matter and then describing both the gravity and the matter sector using dual variables in a local and covariant way. This is reminiscent of the findings of [17,18] where it was also argued that the coupling of linearised gravity to dynamical matter sources induces a non-linear completion of the gravity sector. Treating the gravity sector non-linearly, one is however immediately faced with the problem of the obstructions established in [14] when trying to maintain locality and covariance. One possible way out then is to abandon covariance [13], see also [19].

One of the motivations for this work was to add the dual graviton to the supersymmetry algebra in eleven dimensions in the same spirit as was done for the dual matter fields in $D = 10$ maximal supergravity theories in [20,21]. The supersymmetry algebra in $D = 10$ closes on the dual matter fields if one imposes appropriate duality equations which imply the dynamical matter equations of supergravity. This computation can also be done using algebraic correspondences [22] and it is therefore tempting to use the same techniques to derive the supersymmetry rules of the dual graviton coupled to matter in maximal supergravity. If successful, this would reveal the way the dual graviton transforms under the $A_{(3)}$ and $A_{(6)}$ gauge transformations as required by supersymmetry, which could then be compared to the predictions of, e.g., E_{11}. Whereas the dual graviton of pure minimal supergravity in $D = 4$ can be included in the supersymmetry algebra if one linearises and uses a duality relation of the type (10) (see Sect. 2) we find that in $D = 11$ matter enters the duality relation in such a way that it no longer gives rise to the correct, gauge invariant Einstein-matter equations. Phrased differently, the supersymmetry algebra can be closed on the dual graviton in maximal supergravity (and the answer agrees with the algebraic considerations) but the duality relation is not an equivalent reformulation of the Einstein equation. This result is in agreement with the non-existence of a dual graviton coupled to matter using the approach we outlined in Sect. 3.

Finally, we discuss some possible resolutions of this apparent difficulty in addition to abandoning Lorentz covariance which was already mentioned. A possible but trivial resolution is to fully decouple the matter and the gravity sector (as suggested by a κ expansion of the equations) and treat them as sums of free fields.[6] One should keep in mind that there are (at least) two ways to introduce the dual graviton, as presented

[6] This is what happens also in Kaluza–Klein reduction. Linearised pure gravity in D dimensions admits a dual graviton. After dimensional reduction to $D - 1$ dimensions this gives again dual gravitation but the Kaluza–Klein scalar and vector do not couple to gravity in $D - 1$ dimensions.

in Sect. 1. Additional possibilities or combinations might be envisaged, and it would be useful to understand the precise relation between these different approaches. The way the Einstein equations were constructed from the tentative dual graviton involved very specific choices of taking derivatives, cf. (4) and (9). Since the dual graviton is a mixed symmetry tensor there be might other curvatures one could construct from it which then give the Bianchi identities and field equations of the original theory. However, this has to be done in such a way that the assumptions of the generalised Poincaré lemma [8] are satisfied, and we could not find any non-trivial solution this way. This leads us to conclude that the requirement of a local and covariant expression for M $(\tilde{M})$ cannot be maintained.

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