Supersymmetric AdS$_5$ solutions of massive IIA supergravity

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ABSTRACT: Motivated by a recently found class of AdS$_7$ solutions, we classify AdS$_5$ solutions in massive IIA, finding infinitely many new analytical examples. We reduce the general problem to a set of PDEs, determining the local internal metric, which is a fibration over a surface. Under a certain simplifying assumption, we are then able to analytically solve the PDEs and give a complete list of all solutions. Among these, one class is new and regular. These spaces can be related to the AdS$_7$ solutions via a simple universal map for the metric, dilaton and fluxes. The natural interpretation of this map is that the dual CFT$_6$ and CFT$_4$ are related by twisted compactification on a Riemann surface Σ_g. The ratio of their free energy coefficients is proportional to the Euler characteristic of Σ_g. As a byproduct, we also find the analytic expression for the AdS$_7$ solutions, which were previously known only numerically. We determine the free energy for simple examples: it is a simple cubic function of the flux integers.

Contents

1 Introduction

The study of supersymmetric conformal field theories (CFT) in four dimensions using holography is by now a venerable subject. Their holographic duals are AdS$_5$ solutions in either IIB supergravity or M-theory. A comprehensive analysis of supersymmetric AdS$_5$ solutions of IIB supergravity was carried out in [1]; these include the Freund-Rubin compactifications and the Pilch-Warner solution [2]. Analogous studies were performed for $\mathcal{N} = 1$ [3] and $\mathcal{N} = 2$ [4] supersymmetric AdS$_5$ backgrounds of M-theory, where new analytic solutions were found. AdS$_5$ solutions arising in M-theory usually have a higher-dimensional origin: they are compactifications ("twisted" in a certain way) of CFT's in six dimensions. Actually this latter CFT is essentially always the $(2,0)$ theory living on the world-volume of M5-branes, as in [5] (and in the more recent examples [6, 7]).

Recently, AdS$_7$ solutions in type II supergravity were classified [8]. A new infinite class of solutions was found in massive IIA: the internal space M_3 is always topologically an S^3, but its shape is not round — rather, it is a fibration of a round S^2 over an interval.[1] Both D6's and D8's can be present (and, a bit more exotically, O6's and O8's). The CFT duals of these solutions are $(1,0)$-supersymmetric theories, which were argued in [13] to be the ones obtained in [14, 15] from NS5-D6-D8 configurations (see also [16, 17] for earlier related theories). A similar class of $(1,0)$ theories can be found in F-theory [18, 19].

This prompts the question of whether these $(1,0)$ theories, when compactified on a Riemann surface, can also give rise to CFTs in four dimensions. If so, their duals should be AdS$_5$ solutions in massive IIA.

In this paper we classify AdS$_5$ solutions of massive IIA, and we find many analytic examples. The new (and physically sensible) ones are in bijective correspondence with the AdS$_7$ solutions; this strongly suggests that their dual CFT$_4$ are indeed twisted compactifications of the $(1,0)$ CFT$_6$. The correspondence is via a simple universal map, which was directly inspired by the map in [20] from AdS$_4$ to AdS$_7$ solutions. At the level of the metric it reads

$$
\begin{aligned}
e^{2A}(ds^2_{\text{AdS}_5} + ds^2_{\Sigma_g}) + dr^2 + e^{2A}v^2 ds^2_{S^2} \rightarrow \\
\sqrt{\frac{4}{3}}\left(\frac{4}{3}e^{2A}ds^2_{\text{AdS}_7} + dr^2 + \frac{v^2}{1+3v^2}e^{2A}ds^2_{S^2}\right),
\end{aligned}
\tag{1.1}
$$

where A, v are functions of r and Σ_g is a Riemann surface of genus $g \geq 2$. This map is so simple that it also allows us to find analytic expressions for the AdS$_7$ solutions. For

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[1]This Ansatz was also considered in [9–12], also in a non-supersymmetric setting.

example, the simplest massive AdS_5 solution has metric

$$ds^2 = \sqrt{\frac{3}{4} \frac{n_2}{F_0}} \left(\sqrt{\tilde{y}+2} \, (ds^2_{\mathrm{AdS}_5} + ds^2_{\Sigma_g}) + \frac{d\tilde{y}^2}{4(1-\tilde{y})\sqrt{\tilde{y}+2}} + \frac{1}{9} \frac{(1-\tilde{y})(\tilde{y}+2)^{3/2}}{2-\tilde{y}} ds^2_{S^2} \right) \quad (1.2)$$

with $\tilde{y} \in [-2, 1]$. Its AdS_7 "mother", obtained via the map (1.1), reads on the other hand

$$ds^2 = \frac{n_2}{F_0} \left(\frac{4}{3} \sqrt{\tilde{y}+2} \, ds^2_{\mathrm{AdS}_7} + \frac{d\tilde{y}^2}{4(1-\tilde{y})\sqrt{\tilde{y}+2}} + \frac{1}{3} \frac{(1-\tilde{y})(\tilde{y}+2)^{3/2}}{8 - 4\tilde{y} - \tilde{y}^2} ds^2_{S^2} \right). \quad (1.3)$$

Both these solutions have a stack of n_2 D6-branes at $\tilde{y} = 2$, and are regular elsewhere. The D6's can also partially or totally be replaced by several D8-branes, much like in a Myers effect [21]. (In a way, these solutions realize the vision of [22].) Such more complicated solutions are obtained by gluing together copies of (1.3), or sometimes also of a more complicated metric that we will see later on.

We start our analysis in complete generality. We use the time-honored trick of reducing the study of AdS_5 solutions to that of Minkowski$_4$ solutions whose internal space M_6 has a conical isometry. One can then use the general classification of [23], which uses generalized complex geometry on M_6. Due to the conical structure of M_6, the "pure spinor equations" of [23] become a certain new set of equations on M_5. (The idea of applying the pure spinor equations to AdS_5 solutions in this way goes back to [24], where it was applied to IIB solutions.) It is immediately seen that the only possibility that leads to solutions is that of an SU(2) structure on M_6 (where the pure spinors are of so-called type 1 and type 2), which means in turn that there is an identity structure on M_5.

The practical consequence of this is that we can determine the metric on M_5 in full generality. It is a fibration of a three-dimensional fiber M_3 over a two-dimensional space $\mathcal{C}$. The three-dimensional fiber also has a Killing vector, which is holographically dual to R-symmetry on the field theory side. The fluxes are also fully determined. The independent functions (one function a_2 in the metric, the warping A, and the dilaton ϕ) have to satisfy a total of six PDEs.

The problem simplifies dramatically once we impose what we will call the "compactification Ansatz". This consists in imposing that: 1) The metric of $\mathcal{C}$ is conformally related to that of a surface Σ, which does not depend on the coordinates of the three-dimensional space orthogonal to $\mathcal{C}$ inside M_5. The conformal factor is equal to the warping function e^{2A} in front of the AdS_5 metric; 2) neither A, nor the dilaton ϕ, nor the function a_2 entering the metric and fluxes, depend on the coordinates of Σ. Under this Ansatz, Σ has constant curvature[2] (and we can compactify it to produce a compact Riemann surface Σ_g); the PDEs reduce to only three. Moreover, these PDEs are all polynomial in one of the local coordinates on M_3. Thus they can be in fact reduced to a set of ODEs. At this point the analysis branches out in several possibilities; for each of those, only one ODE survives. In the massless case, there is a "generic case", which is the reduction to IIA of the BBBW solution [7, 26], and two special cases being the reduction of the $\mathcal{N} = 1$ Maldacena-Núñez solution [5] and the INST solution [27]. In the massive case, we get new solutions. Again there is a generic case and two special cases. In the generic case, we solve the ODE explicitly, but the solution appears not to be physically sensible: it has singularities which

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[2]For compactifications of $(2,0)$ theories, the fact that Σ has constant curvature was explained in [25].

we cannot interpret. The first special case, with positive curvature on Σ_g, again has singularities. The second, with constant negative curvature[3] on Σ_g, leads to physically sensible solutions. These latter ones are the main result of this paper.

Solving the ODE produces several solutions, of which (1.2) is the simplest. Without D8's, the most general solution has either two D6 stacks (unlike (1.2), which has one), or one D6 stack and one O6. As we already mentioned, there is also the possibility of introducing D8's, which can be done by gluing together copies of (1.2), of the Maldacena-Núñez solution, and possibly also of the more complicated solution we just mentioned. As we also anticipated, the map (1.1) can then be used to produce analytical expressions for all the AdS$_7$ solutions in [8, 13].

All these new explicit solutions are begging further investigation, particularly regarding their field theory interpretation. This might be the beginning of a correspondence between CFT$_6$ and CFT$_4$ similar to the celebrated class S theories [30] (although notice that we do not discuss Riemann surfaces with punctures here, as was done in [6]). A feature that those theories also had is that (at the supergravity level) the ratio of the number of degrees of freedom in four and six dimensions is proportional to $g-1$, just like for [6] (and for [7]); this is a simple consequence of the map (1.1). We compute the central charges for the CFT$_6$ in a couple of simple cases; for example, for a symmetric solution with two D8's. Along with the NSNS flux integer N, there is also another flux integer μ, which is basically the D6 charge of the D8's; the number of degrees of freedom is a simple cubic polynomial in N and μ, and agrees with an earlier approximate computation in [13]. It would be interesting to also compute contributions from stringy corrections, which we have not done here.

This paper is organized as follows. In section 2, we write the system of pure spinor equations relevant for supersymmetry. In section 3, we analyze the system: we determine the metric and fluxes in terms of a few functions, subject to a set of PDEs, which we summarize in subsection 3.4. In section 4 we introduce the compactification Ansatz, for which we are able to give a complete list of cases. One of these classes (apparently the only physically sensible one which was not already known) is then analyzed in 5 in more detail. The highlights of that analysis are the correspondence to AdS$_7$ in section 5.2, the explicit solutions in sections 5.5, 5.6, 5.7, and the preliminary field theory considerations in section 5.8. In appendix A we provide the proof of the existence of a Killing vector on M_5. In appendix B we consider an Ansatz simpler than the one in section 4; it reproduces a certain solution of [3]. Finally, in appendix C, we summarize the already known solutions which we recovered in our analysis.

2 The conditions for supersymmetry

In this section, we will derive a system of differential equations on forms in five dimensions that is equivalent to preserved supersymmetry for solutions of the type AdS$_5 \times M_5$. We will derive it by considering AdS$_5$ as a warped product of Mink$_4$ and $\mathbb{R}$. We will begin in section 2.1 by reviewing a system equivalent to supersymmetry for Mink$_4 \times M_6$. In section 2.2 we will then translate it to a system for AdS$_5 \times M_5$.

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[3]Compactifying on T^2 the NS5–D6–D8 configurations of [14, 15] and T-dualizing twice should lead to the NS5–D4–D6 system of [28]; the holographic dual to those solutions was found in [29].

2.1 Mink$_4 \times M_6$

Preserved supersymmetry for Mink$_4 \times M_6$ was found [23] to be equivalent to the existence on M_6 of an SU(3) $\times$ SU(3) structure satisfying a set of differential equations. The system is described by a pair of pure spinors

$$\phi_- \equiv e^{-A_6} \chi_1^+ \otimes \chi_2^{-\,\dagger}, \qquad \phi_+ \equiv e^{-A_6} \chi_1^+ \otimes \chi_2^{+\,\dagger}, \tag{2.1}$$

where the warping function A_6 is defined by

$$ds_{10}^2 = e^{2A_6} ds_{\text{Mink}_4}^2 + ds_{M_6}^2, \tag{2.2}$$

and the $\pm$ superscripts indicate the chirality of χ_1 and χ_2. The pure spinors ϕ_- and ϕ_+ can be expressed as a sum of odd and even forms respectively, via application of the Fierz expansion and the Clifford map

$$dx^{m_1} \wedge \cdots \wedge dx^{m_k} \to \gamma^{m_1 \dots m_k}. \tag{2.3}$$

The system of differential equations equivalent to supersymmetry for type IIA supergravity reads:

$$d_H\left(e^{2A_6 - \phi} \text{Re}\phi_-\right) = -\frac{c_-}{16} F, \tag{2.4a}$$

$$d_H\left(e^{3A_6 - \phi} \phi_+\right) = 0, \tag{2.4b}$$

$$d_H\left(e^{4A_6 - \phi} \text{Im}\phi_-\right) = -\frac{c_+ e^{4A_6}}{16} *_6 \lambda F. \tag{2.4c}$$

Here, ϕ is the dilaton, $d_H = d - H\wedge$ is the twisted exterior derivative and $c_\pm$ are constants such that

$$\|\chi_1\|^2 \pm \|\chi_2\|^2 = c_\pm e^{\pm A_6}. \tag{2.5}$$

F is the internal Ramond-Ramond flux which determines the external flux via self-duality:

$$F_{(10)} \equiv F + e^{6A_6} \text{vol}_4 \wedge *_6 \lambda F. \tag{2.6}$$

λ is an operator acting on a p-form F_p as $\lambda F_p = (-1)^{\left[\frac{p}{2}\right]} F_p$, where square brackets denote the integer part.

2.2 AdS$_5 \times M_5$

As we anticipated, we will now use the fact that anti-de Sitter space can be treated as a warped product of Minkowski space with a line. We would like to classify solutions of the type AdS$_5 \times M_5$. These in general will have a metric[4]

$$ds_{10}^2 = e^{2A} ds_{\text{AdS}_5}^2 + ds_{M_5}^2. \tag{2.7}$$

Since

$$ds_{\text{AdS}_5}^2 = \frac{d\rho^2}{\rho^2} + \rho^2 ds_{\text{Mink}_4}^2, \tag{2.8}$$

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[4]Here $ds_{\text{AdS}_5}^2$ is the unit radius metric on AdS$_5$.

ds_{10}^2 in equation (2.7) can be put in the form of equation (2.2) if we take

$$e^{A_6} = \rho e^A, \qquad ds_{M_6}^2 = \frac{e^{2A}}{\rho^2} d\rho^2 + ds_{M_5}^2. \tag{2.9}$$

In order to preserve the $SO(4,2)$ invariance of AdS_5, A should be a function of M_5. In addition, the fluxes F and H, which in subsection 2.1 were arbitrary forms on M_6, should now be forms on M_5. For IIA, $F = F_0 + F_2 + F_4 + F_6$; in order not to break $SO(4,2)$, we impose $F_6 = 0$.

Following the decomposition of the geometry of M_6 we wish to decompose the system of equations (2.4) so as to obtain the system equivalent to preserved supersymmetry for $AdS_5 \times M_5$. We start by decomposing the generators of $\text{Cliff}(6)$ as

$$\gamma_\rho^{(6)} = \frac{e^A}{\rho} 1 \otimes \sigma_1, \qquad \gamma_m^{(6)} = \gamma_m \otimes \sigma_2, \qquad m = 1, \ldots, 5 \tag{2.10}$$

where σ_1, σ_2 are the Pauli matrices and γ_m generate $\text{Cliff}(5)$. Accordingly, the chirality matrix $\gamma_7^{(6)} = 1 \otimes \sigma_3$ and the chiral spinors χ_1^+, χ_2^- are decomposed in terms of $\text{Spin}(5)$ spinors η_1, η_2 as

$$\chi_1^+ = \sqrt{\frac{\rho}{2}} \eta_1 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \chi_2^- = \sqrt{\frac{\rho}{2}} \eta_2 \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.11}$$

ϕ_- and ϕ_+ now read

$$\phi_- = \frac{1}{2} \left(\frac{e^A}{\rho} d\rho \wedge \psi_+^1 + i\psi_-^1 \right), \qquad \phi_+ = \frac{1}{2} \left(-i\frac{e^A}{\rho} d\rho \wedge \psi_-^2 + \psi_+^2 \right), \tag{2.12}$$

where

$$\psi^1 \equiv e^{-A} \eta_1 \otimes \eta_2^\dagger, \qquad \psi^2 \equiv e^{-A} \eta_1 \otimes \overline{\eta_2}. \tag{2.13}$$

The bar is defined as $\overline{\eta} \equiv (\eta^c)^\dagger \equiv (B\eta^*)^\dagger = -\eta^t B$, where B is a conjugation matrix that in five Euclidean dimensions can be taken to satisfy $B^* = B$, $B^t = -B$, $B^2 = BB^* = -1$. The subscripts plus and minus on ψ^1, ψ^2 refer to taking the even and odd form part respectively, in their expansion as forms. One should keep in mind here a comment about odd dimensions: the Clifford map (2.3) is not injective. Rather, a form ω and its cousin $*\lambda\omega$ are mapped to the same bispinor (recall the definition of λ right after (2.6)). Thus a bispinor can always be expressed both as an even and as an odd form, and in particular we have

$$\psi_-^{1,2} = *\lambda\psi_+^{1,2}. \tag{2.14}$$

Applying the decomposition (2.12) to equations (2.4) we obtain a *necessary and sufficient* system of equations for supersymmetric $AdS_5 \times M_5$ solutions:

$$d_H \left(e^{3A-\phi} \text{Re}\psi_+^1 \right) + 2e^{2A-\phi} \text{Im}\psi_-^1 = 0, \tag{2.15a}$$

$$d_H \left(e^{4A-\phi} \psi_-^2 \right) - 3ie^{3A-\phi} \psi_+^2 = 0, \tag{2.15b}$$

$$d_H \left(e^{4A-\phi} \text{Re}\psi_-^1 \right) = 0, \tag{2.15c}$$

$$d_H \left(e^{5A-\phi} \text{Im}\psi_+^1 \right) - 4e^{4A-\phi} \text{Re}\psi_-^1 = \frac{c_+}{8} e^{5A} * \lambda F. \tag{2.15d}$$

We also obtain the condition $c_- = 0$; it follows that the relation $\|\chi_1\|^2 \pm \|\chi_2\|^2 = c_\pm e^{\pm A_6}$ becomes

$$\|\eta_1\|^2 = \|\eta_2\|^2 = \frac{1}{2} c_+ e^A . \tag{2.16}$$

Henceforth, without loss of generality, we set $c_+ = 2$.

The stabilizer group $\mathcal{G} \in \mathrm{Spin}(5)$ of η_1 and η_2 can be either $\mathrm{SU}(2)$ or the identity group. In the next section we parametrize ψ^1, ψ^2 in terms of these structures. We will see however that only the identity case leads to supersymmetric solutions. An identity structure is actually a choice of vielbein; so we will end up parameterizing the ψ^1 and ψ^2 in terms of a vielbein.

2.3 Parametrization of ψ^1, ψ^2 and the identity structure

We first consider the case where there is only one spinor, $\eta_1 = \eta_2$ of norm $e^{\frac{A}{2}}$. In five dimensions it defines an $\mathrm{SU}(2)$ structure. This can be read off from the Fierz expansions of $\eta_1 \otimes \eta_1^\dagger$ and $\eta_1 \otimes \overline{\eta_1}$, which as remarked in (2.14) can be written both as even and as odd forms:

$$
\begin{aligned}
\psi_+^1 &= \frac{1}{4} e^{-ij} \,, & \psi_+^2 &= \frac{1}{4}\omega \,, \\
\psi_-^1 &= \frac{1}{4} v \wedge e^{-ij} \,, & \psi_-^2 &= v \wedge \omega \,.
\end{aligned}
\tag{2.17}
$$

Application of Fierz identities yields

$$v\eta_1 = \eta_1 \tag{2.18}$$

and the following set of algebraic constraints on the 1-form v and 2-forms j and ω:

$$
\begin{aligned}
\iota_v v &= 1 \,, & \iota_v j &= \iota_v \omega = 0 \\
j \wedge \omega &= 0 \,, & \omega \wedge \omega &= 0 \,, & \omega \wedge \overline{\omega} &= 2j \wedge j = \mathrm{vol}_4,
\end{aligned}
\tag{2.19}
$$

where vol_4 is the volume form on the four-dimensional subspace orthogonal to v. This set of forms and constraints define precisely an $\mathrm{SU}(2)$ structure in five dimensions.

In this case, however, the two-form part of (2.15b) tells us $\psi^2 = 0$, which is only possible for $\eta_1 = 0$. Hence, there are no supersymmetric $\mathrm{AdS}_5 \times M_5$ solutions in type IIA supergravity with an $SU(2)$ structure on M_5.

Let us then consider the case of two spinors η_1 and η_2, which as mentioned earlier define an identity structure. We can expand η_2 in terms of η_1 as

$$\eta_2 = a\eta_1 + a_0 \eta_1^c + \frac{1}{2} b\overline{w}\,\eta_1 \,, \tag{2.20}$$

where a, $a_0 \in \mathbb{C}$, $b \in \mathbb{R}$ and w is a complex vector that we normalize such that $w \cdot \overline{w} = 2$ (so that $\mathrm{Re}\,w$ and $\mathrm{Im}\,w$ are orthogonal and have norm 1). Also, by redefining if necessary $a \to a + \frac{b}{2}\overline{w} \cdot v$, $w \to w - (w \cdot v)v$ (which leaves (2.20) invariant, upon using (2.18)), we can assume

$$w \cdot v = 0 \,. \tag{2.21}$$

Now (2.16) implies

$$|a|^2 + |a_0|^2 + b^2 = 1 \,. \tag{2.22}$$

The identity structure is then spanned by v, w and

$$u \equiv \frac{1}{2} \iota_{\overline{w}} \omega \,, \tag{2.23}$$

in terms of which

$$\omega = w \wedge u \,, \qquad -ij = \frac{1}{2}(w \wedge \overline{w} + u \wedge \overline{u}) \,. \tag{2.24}$$

From (2.19) we now see that u is also orthogonal to v, as well as to w and $\overline{w}$; moreover, it satisfies $u \cdot \overline{u} = 2$. In other words,

$$\{v, \mathrm{Re}w, \mathrm{Im}w, \mathrm{Re}u, \mathrm{Im}u\} \tag{2.25}$$

are a vielbein.

We can now expand ψ^1 and ψ^2 in terms of this vielbein. We separate out their even and odd parts:

$$\psi_+^1 = \frac{1}{4}\overline{a} \exp\left[-ij + \frac{b}{\overline{a}}v \wedge w\right] \,, \qquad \psi_+^2 = -\frac{a_0}{4} \exp\left[-ij + \frac{u}{a_0} \wedge (a_+ w - bv)\right] \,, \tag{2.26}$$

$$\psi_-^1 = \frac{1}{4}(\overline{a}v + bw) \wedge e^{-ij} \,, \qquad \psi_-^2 = -\frac{1}{4}(a_0 v + bu) \wedge \exp\left[-ij + \frac{u}{a_0} \wedge (a_+ w - bv)\right] \,.$$

3 Analysis of the conditions for supersymmetry

Having obtained the expansions (2.26) of ψ^1, ψ^2 in terms of the identity structure on M_5, we can proceed with the study of the system (2.15). In section 3.1 we study the constraints imposed on the geometry of M_5 while in section 3.2 we obtain the expressions of the fluxes in terms of the geometry. The analysis in 3.1 is *local*.

3.1 Geometry

The equations of the system (2.15) which constrain the geometry of M_5 are (2.15a), (2.15b) and (2.15c) with the exception of the three-form part of (2.15a) which determines H. In the following study of these constraints, it is convenient to introduce the notation

$$a \equiv a_1 + ia_2 \,, \qquad k_1 \equiv \overline{a}v + bw \,, \qquad k_2 \equiv -bv + aw \,. \tag{3.1}$$

The zero form part of (2.15b), the one-form part of (2.15a), the two-form part of (2.15c) and the two-form part (2.15b) yield the following set of equations:

$$a_0 = 0 \,, \tag{3.2a}$$

$$d\left(e^{3A-\phi}a_1\right) + 2e^{2A-\phi}\mathrm{Im}k_1 = 0 \,, \tag{3.2b}$$

$$d\left(e^{4A-\phi}\mathrm{Re}k_1\right) = 0 \,, \tag{3.2c}$$

$$d\left(e^{4A-\phi}bu\right) - 3ie^{3A-\phi}u \wedge k_2 = 0 \,. \tag{3.2d}$$

It can then be shown that the higher-form parts of (2.15a), (2.15b) and (2.15c) follow from the above equations.

(3.2a) simplifies quite a bit (2.26), which now becomes

$$
\psi^1_+ = \frac{1}{4}\overline{a}\exp\left[-ij + \frac{b}{a}v\wedge w\right], \qquad \psi^2_+ = \frac{1}{4}(aw - bv)\wedge u\wedge e^{-ij},
$$
$$
\psi^1_- = \frac{1}{4}(\overline{a}v + bw)\wedge e^{-ij}, \qquad \psi^2_- = -\frac{1}{4}bu\wedge \exp\left[-ij - \frac{a}{b}v\wedge w\right].
$$

(3.3)

It is also interesting to see what the pure spinors $\phi_\pm$ on M_6 look like:

$$
\phi_+ = \frac{1}{4}E_1\wedge E_2\wedge \exp\left[\frac{1}{2}E_3\wedge \overline{E_3}\right], \qquad \phi_- = E_3\wedge \exp\left[\frac{1}{2}(E_1\wedge \overline{E_1} + E_2\wedge \overline{E_2})\right], \quad (3.4)
$$

where

$$
E_1 \equiv ie^A b\frac{d\rho}{\rho} + aw - bv, \qquad E_2 \equiv u, \qquad E_3 \equiv e^A\overline{a}\frac{d\rho}{\rho} + i(\overline{a}v + bw). \quad (3.5)
$$

(3.4) are the canonical forms of a type 1 — type 2 pure spinor pair (where the "type" of a pure spinor is the lowest form appearing in it); or, in other words, of a pure spinor pair associated with an SU(2) structure on M_6 (although remember that the structure on M_5 is the identity). It would be interesting to push this further, and to start an analysis similar to the one in [24]: in that paper, the language of generalized complex geometry is used to set up a generalized reduction procedure, which eventually leads to a set of four-dimensional equations.

Let us now go back to (3.2). Given (3.2a), equation (2.22) becomes

$$
a_1^2 + a_2^2 + b^2 = 1. \quad (3.6)
$$

Equations (3.2b) and (3.2c) can be integrated by introducing local coordinates y,

$$
y = -\frac{1}{2}e^{3A-\phi}a_1, \quad (3.7)
$$

and x such that

$$
\mathrm{Im}k_1 = e^{-2A+\phi}dy, \qquad \mathrm{Re}k_1 = e^{-4A+\phi}dx. \quad (3.8)
$$

M_5 possesses an abelian isometry generated by the Killing vector

$$
\xi \equiv \frac{1}{2}\left(\eta_1^\dagger \gamma^m \eta_2 - \eta_2^\dagger \gamma^m \eta_2\right)\partial_m = -e^A b(\mathrm{Re}k_2)^\sharp \quad (3.9)
$$

where $m = 1,\ldots,5$ and the $\sharp$ superscript denotes the vector dual to the one-form it acts on. A straightforward way to show that ξ is a Killing vector is to work directly with the supersymmetry variations (see appendix A) which yield $\nabla_{(m}\xi_{\nu)} = 0$ and $\mathcal{L}_\xi\phi = \mathcal{L}_\xi A = 0$, where ∇ is the Levi-Civita connection and $\mathcal{L}_\xi$ is the Lie derivative with respect to ξ. It would be interesting to show this directly using the language of generalized complex geometry, and to make contact with the analysis in [24].

Expressing w, v in terms of $\text{Re}k_2$, $\text{Re}k_1$, and $\text{Im}k_1$ we can write the metric on M_5 as

$$ds^2_{M_5} = ds^2_{\mathcal{C}} + (\text{Re}k_2)^2 + \frac{e^{-4A+2\phi}}{b^2}\left[(b^2+a_2^2)e^{-4A}dx^2 + (b^2+a_1^2)dy^2 + 2a_1a_2e^{-2A}dxdy\right], \quad (3.10)$$

where $ds^2_{\mathcal{C}} = u\bar{u}$, and $\mathcal{C}$ denotes the two-dimensional subspace spanned by u.

Let us introduce local coordinates x^I, $I = 1, 2, 3$ such that

$$ds^2_{\mathcal{C}} + (\text{Re}k_2)^2 = g_{IJ}(x^I, x, y)dx^I dx^J. \quad (3.11)$$

ϕ, A and a_2 are in principle functions of x^I, x and y. Given the fact that $\mathcal{L}_\xi \text{Re}k_1 = \mathcal{L}_\xi \text{Im}k_1 = 0$,[5] we can further introduce a coordinate $x^3 \equiv \psi$ adapted to the the Killing vector

$$\xi = 3\partial_\psi, \quad (3.12)$$

in terms of which

$$\text{Re}k_2 = -\frac{1}{3}e^A b D\psi, \qquad D\psi \equiv d\psi + \rho, \qquad \rho = \rho_i(x^i, x, y)dx^i, \quad (3.13)$$

where x^i, $i = 1, 2$ are local coordinates on $\mathcal{C}$. Thus

$$g_{IJ}(x^I, x, y)dx^I dx^J = (g_{\mathcal{C}})_{ij}(x^i, x, y)dx^i dx^j + \frac{1}{9}e^{2A}b^2 D\psi^2. \quad (3.14)$$

In addition, since ξ is a Killing vector and $\mathcal{L}_\xi \phi = \mathcal{L}_\xi A = 0$, A, ϕ and a_2 are independent of ψ.

The exterior derivative on M_5 can be decomposed as

$$d = d_2 + d\psi \wedge \partial_\psi + dx \wedge \partial_x + dy \wedge \partial_y, \quad (3.15)$$

where d_2 is the exterior derivative on $\mathcal{C}$. We can thus further refine equation (3.2d) as follows:

$$d_2 u = i\rho_0 \wedge u, \quad (3.16a)$$

$$\partial_\psi u = iu, \quad (3.16b)$$

$$\partial_x u = f_1 u, \quad (3.16c)$$

$$\partial_y u = f_2 u, \quad (3.16d)$$

where

$$\rho_0 \equiv \rho + *_2 d_2 \log\left(be^{4A-\phi}\right), \quad (3.17a)$$

$$f_1(x^i, x, y) \equiv -\partial_x \log\left(e^{4A-\phi}b\right) + \frac{3e^{-5A+\phi}a_2}{b^2}, \quad (3.17b)$$

$$f_2(x^i, x, y) \equiv -\partial_y \log\left(e^{4A-\phi}b\right) + \frac{3e^{-3A+\phi}a_1}{b^2}. \quad (3.17c)$$

$*_2$ is the Hodge star defined by $g_{\mathcal{C}}$, such that $*_2 u = -iu$. Integrability of equations (3.16) yields the constraints

$$\partial_y f_1 = \partial_x f_2 \quad (3.18)$$

---

[5]Deduced from $\iota_\xi \text{Re}k_1 = \iota_\xi \text{Im}k_1 = 0$ and equation (3.8).

and

$$\partial_x \rho_0 = - *_2 d_2 f_1 \,, \tag{3.19a}$$

$$\partial_y \rho_0 = - *_2 d_2 f_2 \,. \tag{3.19b}$$

We can write $ds_{\mathcal{C}}^2$ as

$$ds_{\mathcal{C}}^2 = e^{2\varphi(x^i,x,y)} \left(dx_1^2 + dx_2^2 \right) . \tag{3.20}$$

The Gaussian curvature or one-half the scalar curvature of $\mathcal{C}$, $\ell(x^i, x, y)$, is

$$\ell(x^i, x, y) = -e^{-2\varphi} \left(\partial_{x_1}^2 + \partial_{x_2}^2 \right) \varphi \,. \tag{3.21}$$

Equations (3.16b) and (3.16c), (3.16d) are solved by

$$u = e^{\varphi + i\psi}(dx_1 + idx_2) \,, \qquad \partial_x \varphi = f_1 \,, \qquad \partial_y \varphi = f_2 \,. \tag{3.22}$$

Equation (3.16a) then yields

$$\rho_0 = \partial_{x_2}\varphi dx_1 - \partial_{x_1}\varphi dx_2 \,, \tag{3.23}$$

and thus

$$d_2\rho_0 = \ell(x^i, x, y)\mathrm{vol}_{\mathcal{C}} \,. \tag{3.24}$$

Compatibility of (3.24) with (3.19a), (3.19b) requires that ℓ obey the equations

$$\partial_x \ell + 2f_1\ell = \Delta_2 f_1 \,, \tag{3.25a}$$

$$\partial_y \ell + 2f_2\ell = \Delta_2 f_2 \,, \tag{3.25b}$$

where $\Delta_2 \equiv d_2{}^\dagger d_2 + d_2 d_2{}^\dagger$. The last two equations also follow from (3.21), bearing in mind that $\Delta_2\varphi = -e^{-2\varphi}(\partial_{x_1}^2 + \partial_{x_2}^2)\varphi$.

3.2 Fluxes

In this section we give the expressions for the fluxes in terms of the geometry of M_5. In the following expressions we employ the notation

$$\begin{aligned}
\zeta_1 &\equiv \mathrm{Re}(ak_1)^\sharp = -2ye^A\partial_x - a_2 e^{2A-\phi}\partial_y \,, \\
\zeta_2 &\equiv \frac{1}{b^2}\mathrm{Im}(ak_1)^\sharp = a_2 e^{4A-\phi}\partial_x - 2ye^{-A}\partial_y \,.
\end{aligned} \tag{3.26}$$

The NSNS three-form flux H is given by the three-form part of equation (2.15a):

$$\begin{aligned}
H = {} & d\left(\frac{1}{6y}dx \wedge D\psi + \frac{1}{3}e^A\mathrm{Re}(ak_1) \wedge D\psi + \frac{e^{3A-\phi}a_2}{2y}\mathrm{vol}_{\mathcal{C}} \right) \\
& - \frac{1}{6y^2}dx \wedge dy \wedge D\psi + \frac{e^{-2A}}{y}dx \wedge \mathrm{vol}_{\mathcal{C}} + \frac{e^{3A-\phi}a_2}{2y^2}dy \wedge \mathrm{vol}_{\mathcal{C}} \,,
\end{aligned} \tag{3.27a}$$

where $\mathrm{Re}(ak_1) = -2ye^{-7A+2\phi}dx - a_2 e^{-2A+\phi}dy$.

The RR fluxes can be computed from equation (2.15d):

$$F_0 = -4e^{2A-2\phi}b^2\partial_y A - e^{-A}\iota_{\zeta_1}d(e^{A-\phi}a_2)\,, \tag{3.28a}$$

$$\begin{aligned}
F_2 &= \left[-4e^{-A-\phi}a_2 + 4e^{4A-2\phi}\partial_x A - e^{-5A}\iota_{\zeta_2}d(e^{5A-\phi}a_2)\right]\mathrm{vol}_{\mathcal{C}} \\
&\quad + \frac{1}{3}d(e^{A-\phi}a_2)\wedge D\psi + F_0\frac{1}{3}e^A\mathrm{Re}(ak_1)\wedge D\psi \\
&\quad - \frac{e^{-A}}{b^2}*_2 d_2(e^{A-\phi}a_2)\wedge\mathrm{Im}(ak_1) + 4e^{-4A}*_2 d_2 A\wedge dx\,,
\end{aligned} \tag{3.28b}$$

$$\begin{aligned}
F_4 &= \frac{1}{3}\left[e^{-6A}\partial_y(e^{5A-\phi}a_2)dx - e^{-2A}\partial_x(e^{5A-\phi}a_2)dy - 4e^{-2A}dy\right]\wedge d\psi\wedge\mathrm{vol}_{\mathcal{C}} \\
&\quad - \frac{1}{3}\left[4e^{-\phi}a_2 + e^{-4A}\iota_{\zeta_2}d(e^{5A-\phi}a_2)\right]\mathrm{Re}(ak_1)\wedge d\psi\wedge\mathrm{vol}_{\mathcal{C}} \\
&\quad - \frac{1}{3}e^{-10A+2\phi}*_2\left[d_2(e^{5A-\phi}a_2)\right]\wedge dx\wedge dy\wedge D\psi\,,
\end{aligned} \tag{3.28c}$$

where $\mathrm{Im}(ak_1) = a_2 e^{-4A+\phi}dx - 2ye^{-5A+2\phi}dy$.

The fluxes can also be computed from the expression

$$F = \mathcal{J}_+\cdot d_H(e^{-\phi}\mathrm{Im}\phi_-) \tag{3.29}$$

on M_6 [31]. The operator $\mathcal{J}_+\cdot$ is associated with the pure spinor ϕ_+, which can be found in (3.4):

$$\mathcal{J}_+\cdot = \frac{i}{2}\sum_{i=1}^{2}(E_i\wedge\overline{E_i}\llcorner - \overline{E_i}\wedge E_i\llcorner) + \frac{i}{2}(E_3\llcorner\overline{E_3}\llcorner + E_3\wedge\overline{E_3}\wedge)\,. \tag{3.30}$$

The degree of difficulty of computing the fluxes from (3.29) is proportional to the degree of the flux. The opposite is true for computing the fluxes from (2.15d).

3.3 Bianchi identities

In order to have a complete supersymmetric $\mathrm{AdS}_5\times M_5$ solution, apart from the conditions for supersymmetry (which imply the equations of motion [32]) the Bianchi identities of the fluxes need to be imposed. In this section we study the latter and the extra constraints that follow from their application.

We start with the Bianchi identity of H i.e. $dH = 0$. We find that it determines

$$d_2\rho = e^{-2A}\left[6 + 12y(\partial_y A - f_2) - 6e^{5A-\phi}a_2(\partial_x A - f_1) + 3\partial_x(e^{5A-\phi}a_2)\right]\mathrm{vol}_{\mathcal{C}}\,. \tag{3.31}$$

Next, we turn to the Bianchi identities of the RR fluxes. The Bianchi identity of F_0 just says that it is a constant. The Bianchi identity of F_2 is

$$dF_2 - F_0 H = 0\,. \tag{3.32}$$

The non-zero components on the left-hand side are the $dx \wedge \mathrm{vol}_\mathcal{C}$ and $dy \wedge \mathrm{vol}_\mathcal{C}$ components and imposing that they vanish yields the equations:

$$\partial_x \mathcal{Q} + 2f_1 \mathcal{Q} - \left[\frac{1}{3} \partial_x\big(e^{A-\phi}a_2\big) - \frac{F_0}{6y} \right] *_2 d_2\rho - F_0 \frac{e^{-2A}}{y} \tag{3.33a}$$

$$+ \Delta_2\big(e^{A-\phi}a_2\big)\frac{e^{-5A+\phi}a_2}{b^2} - \Delta_2(e^{-4A}) - d_2\big(e^{A-\phi}a_2\big) \cdot d_2\left(\frac{2e^{-5A+\phi}a_2}{b^2} \right) = 0 \,,$$

$$\partial_y \mathcal{Q} + 2f_2 \mathcal{Q} - \frac{1}{3} \partial_y\big(e^{A-\phi}a_2\big) *_2 d_2\rho - F_0 \frac{e^{3A-\phi}a_2}{2y^2} \tag{3.33b}$$

$$- \Delta_2\big(e^{A-\phi}a_2\big)\frac{2e^{-6A+2\phi}y}{b^2} + d_2\big(e^{A-\phi}a_2\big) \cdot d_2\left(\frac{4e^{-6A+2\phi}y}{b^2} \right) = 0 \,,$$

where

$$\mathcal{Q}(x^i,x,y) \equiv -4e^{-A-\phi}a_2 + 4e^{4A-2\phi}\partial_x A - e^{-5A}\iota_{\zeta_2}d\big(e^{5A-\phi}a_2\big) - F_0 \frac{e^{3A-\phi}a_2}{2y} \,. \tag{3.34}$$

Finally, the Bianchi identity of F_4

$$dF_4 - H \wedge F_2 = 0 \,, \tag{3.35}$$

is automatically satisfied.

3.4 Summary so far

So far, we have analyzed the constraints imposed by supersymmetry and the Bianchi identities without any Ansatz; let us summarize what we have obtained.

First of all, we have already determined the local form of the metric: (3.10), (3.13). Most notably, we see the emergence of a Killing vector ξ generating a U(1) isometry, and of a two-dimensional space $\mathcal{C}$. The geometry of $\mathcal{C}$ is constrained by (3.16). The S^1 upon which the U(1) acts is fibered over $\mathcal{C}$ with ρ being the connection of the fibration. The curvature of the connection is given by (3.31).

In fact the U(1) isometry is a symmetry of the full solution as it also leaves invariant the fluxes; the latter can be verified by computing the Lie derivative with respect to ξ of the fluxes' expressions as presented in section 3.2. This symmetry was to be expected: it is a U(1) R-symmetry corresponding to the R-symmetry of the dual $\mathcal{N} = 1$ field theory. The surface $\mathcal{C}$ is of less immediate interpretation, but already at this stage it seems to suggest that the field theory should be a compactification on $\mathcal{C}$ of a six-dimensional field theory. We will see later that this expectation is indeed borne out for the explicit solutions we will find.

We have also reduced the task of finding solutions to a set of partial differential equations on three functions: a_2, the dilaton ϕ, and the warp factor A, which in general depend on four variables i.e. the coordinates x^i, x, y. Supersymmetry equations alone give us (3.18), (3.25a), (3.25b). Moreover, the fluxes should satisfy the relevant Bianchi identities, which away from sources give the further equations (3.28a), (3.33a), (3.33b). Thus we have a total of six partial differential equations. Solving all of them might seem a daunting task, but we will see in the next section that they simplify dramatically with a simple Ansatz. This will allow us to find many explicit solutions.

4 A compactification Ansatz

We have reduced the general classification problem to a set of six PDEs. To simplify the problem, we will now make an Ansatz.

We assume that A, ϕ and a_2 are functions of x and y only, and that

$$ds_{\mathcal{C}}^2 = e^{2A}ds_{\Sigma}^2(x_1, x_2)\,. \tag{4.1}$$

In other words, the ten-dimensional metric becomes $ds_{10}^2 = e^{2A}(ds_{\mathrm{AdS}_5}^2 + ds_{\Sigma}^2) + ds_{M_3}^2$. It will soon follow that Σ has constant curvature; from now on we will assume it to be a compact Riemann surface Σ_g. For $g \geq 1$ this involves a quotient by a discrete subgroup, but since no functions depend on its coordinates, this presents no difficulty.

This Ansatz is motivated by the fact that most known solutions in eleven-dimensional supergravity (and hence in massless IIA) are of this type. We also have in mind our original motivation for this paper: finding solutions dual to twisted compactifications of CFT_6. If one wants to study a CFT_6 on $\mathbb{R}^4 \times \Sigma_g$ rather than on $\mathbb{R}^6$, one needs to replace $ds_{\mathrm{AdS}_7}^2 = \frac{d\rho^2}{\rho^2} + \rho^2 ds_{\mathbb{R}^6}^2$ with $\frac{d\rho^2}{\rho^2} + \rho^2(ds_{\mathbb{R}^4}^2 + ds_{\Sigma}^2)$ in the UV, and then look for a solution that represents the flow to the IR. Our Ansatz is basically that in the IR fixed point this metric is only modified in the ρ^2 term multiplying ds_{Σ}^2, which drops out and becomes a constant.

Whatever its origin, we will now see that this Ansatz is remarkably effective at simplifying the system of PDEs: we will be able to completely classify the resulting solutions. One particular case will be source of many solutions, which will be analyzed in section 5.

4.1 Simplifying the PDEs

(4.1) implies

$$f_1 = \partial_x A\,, \qquad f_2 = \partial_y A\,. \tag{4.2}$$

The integrability condition (3.18) is then satisfied trivially, while equations (3.19a) and (3.19b) yield $\partial_x \rho = \partial_y \rho = 0$ (in the present Ansatz $\rho_0 = \rho$).

Equations (3.24), (3.31) yield $\ell = e^{-2A}[6 + 3\partial_x(e^{5A-\phi}a_2)]$. (3.25a), (3.25b) are then solved by

$$e^{5A-\phi}a_2 = cx + \epsilon \qquad c = \mathrm{const.}\,, \tag{4.3}$$

where $\epsilon = \epsilon(y)$ is a function of y only. It follows that

$$\ell = e^{-2A}(6 + 3c) \tag{4.4}$$

i.e. the Gaussian curvature of Σ_g is equal to $6 + 3c$.

Given the definitions (3.17b), (3.17c), the equations $f_1 = \partial_x A$ and $f_2 = \partial_y A$ become

$$\partial_x\big(e^{10A-2\phi}b^2\big) = 6e^{5A-\phi}a_2\,, \tag{4.5a}$$

$$\partial_y\big(e^{10A-2\phi}b^2\big) = 6e^{7A-\phi}a_1\,. \tag{4.5b}$$

Recall that $a_1 = -2ye^{-3A+\phi}$ and $b^2 = 1 - a_1^2 - a_2^2$. Using (4.3) we can solve these for

$$e^{10A-2\phi} - 4y^2 e^{4A} = c(c+3)x^2 + 2(c+3)\epsilon x + \beta\,, \tag{4.6a}$$

$$e^{4A} = -\frac{\epsilon'}{2y}x - \frac{1}{12y}\big(\beta' - 2\epsilon\epsilon'\big)\,, \tag{4.6b}$$

where $\beta = \beta(y)$ is a function of y only, and a prime denotes differentiation with respect to y.

So far we have solved the differential equations imposed by supersymmetry; we now need to impose the Bianchi identities. First, the expression for F_0, (3.28a), becomes

$$e^{12A}F_0 = -[c(c+3)x^2 + 2(c+3)\epsilon x + \beta](e^{4A})' + e^{4A}\partial_y(cx+\epsilon)^2 + 2e^{8A}cy. \qquad (4.7)$$

Recalling (4.6b), we see that this equation is polynomial in x, of degree 3. In other words, we can view it as a set of four ODEs in y.

The Bianchi identities for F_2, (3.33), become

$$\partial_x^2\left(e^{6A-2\phi}\right) = 0, \qquad (4.8a)$$

$$\partial_y\partial_x\left(e^{6A-2\phi}\right) + F_0\frac{\epsilon'}{2y} = 0. \qquad (4.8b)$$

Substituting equations (4.3) and (4.6) in (4.8a) yields the differential equation

$$36(\epsilon')^2\beta = -(c+3)(\beta'-2\epsilon\epsilon')\left[c\beta' - 2(c+6)\epsilon\epsilon'\right]. \qquad (4.9)$$

Notice that the x dependence has dropped out from this equation. Concerning (4.8b), just as for (4.7), it can be written as a polynomial in x of degree 3, and viewed as four ODEs in y.

So we appear to have reduced the problem to four ODEs from (4.7), one from (4.8a) (which becomes (4.9)), and four from (4.8b), for a total of nine ODEs in y. However, many of these ODEs actually happen not to be independent from each other. For example, the x^3 component of both (4.7) and (4.8b) gives

$$4c(c+3)\left(\frac{\epsilon'}{y}\right)' + F_0\left(\frac{\epsilon'}{y}\right)^3 = 0, \qquad (4.10)$$

as well as the x^2 component of (4.8b).

To analyze the remaining ODEs, as a warm-up we will first look at the case $F_0 = 0$, where we will reproduce several known solutions. We will then look at the case $F_0 \neq 0$, which we will further split into a generic case where $\epsilon' \neq 0$, and a special case where $\epsilon' = 0$; both will give rise to new solutions.

4.2 $F_0 = 0$

For $F_0 = 0$, (4.10) becomes $c(c+3)(\epsilon' - y\epsilon'') = 0$. We can then have either $\epsilon' = y\epsilon''$, $c = -3$, or $c = 0$. In the $c = 0$ case, actually the x^2 coefficient of (4.7) gives again $\epsilon' = y\epsilon''$. So this case becomes a subcase of the $\epsilon' = y\epsilon''$ case.

- **Case 1:** $\epsilon' = y\epsilon''$. In this case we have

$$\epsilon = \frac{1}{2}c_1y^2 + c_2, \qquad c_1, c_2 = \text{const.}. \qquad (4.11)$$

  The x^3 component of (4.7) is (4.10), which we just looked at. The x^2 and x^1 components both require

$$\left(\frac{\beta'}{y}\right)' = 2\frac{c+3}{c+6}c_1y. \qquad (4.12)$$

The solution to this ODE is

$$\beta = \frac{c+6}{c+3}\frac{1}{4}c_1^2 y^4 + \frac{1}{2}c_3 y^2 + c_4 \qquad c_3,\, c_4 = \text{const..} \tag{4.13}$$

The x^0 component of (4.7) then gives

$$(2c_1 c_2 - c_3)(2(c+6)c_1 c_2 - cc_3) + \frac{36}{(c+3)}c_1^2 c_4 = 0\,. \tag{4.14}$$

Generically this can be solved for c_4. In this case, the transformation

$$x \to x + \frac{\delta}{c}, \qquad c_2 \to c_2 - \delta, \qquad \beta \to \beta + \frac{(3+c)(\delta^2 - 2\delta\epsilon)}{c} \tag{4.15}$$

leaves the solution invariant and δ can be chosen such that

$$\beta = \frac{c+6}{c+3}\epsilon^2. \tag{4.16}$$

This branch reproduces the solution obtained from reduction to ten dimensions of the BBBW AdS$_5$ solution of M-theory [7], as described in appendix C.4.

This however does not cover the case $c_1 = 0$. Treating this separately, we find that (4.14) leads to $c = 0$. This branch reproduces the INST solution [27], discussed in appendix C.2.

- **Case 2:** $c = -3$. In this case, the x^2 component of (4.7) gives $\epsilon' = 0$. With this, the whole of (4.7) gives

$$2\beta(\beta' - y\beta'') + y\beta'^2 = 0\,. \tag{4.17}$$

This equation is nonlinear, but if one defines $z = y^2/2$ it becomes $2\beta\partial_z^2\beta = (\partial_z\beta)^2$, which is easily solved by the square of a linear function; in other words, by

$$\beta = c_2(y^2 + 4c_1)^2\,, \qquad c_1,\, c_2 = \text{const..} \tag{4.18}$$

This case reproduces the solution obtained from reduction to ten dimensions of the Maldacena-Núñez AdS$_5$ solution of M-theory [5], described in appendix C.3.

4.3 $F_0 \neq 0$

We will divide the analysis in the generic case, where $c \neq 0$ and -3, and two special cases $c = 0$ or -3. Let us note that from (4.9), we see that $\epsilon' = 0$ implies either $c = 0$ or -3; in other words, if $c \neq 0$ and -3, then $\epsilon' \neq 0$. On the other hand, from (4.10), we see that $\epsilon' \neq 0$ implies $c \neq 0$ and -3; in other words, if $c = 0$ or -3, then $\epsilon' = 0$.

4.3.1 Generic case

We begin by analyzing (4.7) with the aid of (4.10) and (4.9). In particular, combining the last two we derive

$$(e^{4A})' = \frac{(\epsilon')^2}{8c(c+3)y^3}\left[F_0\epsilon' x + \frac{1}{6}F_0(\beta' - 2\epsilon\epsilon') - 4cy^2\right]\,. \tag{4.19}$$

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Substituting $(e^{4A})'$ as expressed in the above equation, and β as expressed in (4.9), in (4.7), the whole of the latter gives

$$\beta' = \frac{c+3}{c}2\epsilon\epsilon'.\tag{4.20}$$

(4.9) then actually fixes

$$\beta = \frac{c+3}{c}\epsilon^2.\tag{4.21}$$

Finally, (4.8b) follows from (4.9) and (4.10), which can be solved by quadrature. The solution is

$$\epsilon = -\frac{2\sqrt{2c(c+3)}}{3F_0^2}(F_0y - 2c_1)\sqrt{F_0y + c_1} + c_2, \qquad c_1, c_2 = \text{const.}.\tag{4.22}$$

This yields an AdS$_5$ solution which was not known before; we will analyze its features here. Let us define

$$\tilde{x} \equiv cx + \epsilon,\tag{4.23}$$

since this quantity will appear several times. We know already, from (4.3), that $e^{5A-\phi}a_2 = \tilde{x}$. By substituting β in (4.6a), we also find $e^{10A-2\phi}(1 - a_1^2) = \frac{c+3}{c}\tilde{x}^2$. Recalling (3.6), we also find $e^{5A-\phi}b = \sqrt{\frac{3}{c}}\tilde{x}$. Again from the expression of β and from (4.6b) we find A:

$$e^{4A} = -\frac{\epsilon'}{2cy}\tilde{x},\tag{4.24}$$

while (4.6a) determines ϕ:

$$e^{\phi} = \sqrt{c}\,\tilde{x}^{3/4}\left(\frac{-2cy}{\epsilon'}\right)^{-5/4}(-2y\epsilon' + (c+3)\tilde{x})^{-1/2}.\tag{4.25}$$

Finally, collecting everything and recalling the expression of the metric (3.10), we can write the metric for this solution as

$$ds^2 = e^{2A}ds_{\Sigma_g}^2 + e^{-8A+2\phi}\left(\frac{\tilde{x}^2}{3c}D\psi^2 + \frac{c+3}{c}dQ^2 + \frac{2y\epsilon' - (c+3)\tilde{x}}{2(c+3)y}\frac{\epsilon'}{c}dy^2\right)\tag{4.26}$$

$$= e^{2A}ds_{\Sigma_g}^2 + \sqrt{-\frac{c\epsilon'}{2y\tilde{x}}}\left(\frac{1}{-2y\epsilon' + (c+3)\tilde{x}}\left(\frac{\tilde{x}^2}{3c}D\psi^2 + \frac{c+3}{c}dQ^2\right) - \frac{\epsilon'}{2c(c+3)y}dy^2\right),$$

where $Q \equiv x + \frac{1}{c+3}\epsilon$.

Unfortunately, as it stands the metric (4.26) appears to be unphysical. To make M_5 compact, we should be able to find some locus where the coefficient of $D\psi^2$ vanishes. One way this could happen is if $\tilde{x} = 0$. However, this also leads to $e^A = 0$. Hence this cannot be a regular point. One might think about the presence of a brane (where e^A might legitimately go to zero), but the locus $\tilde{x} = 0$ appears to be codimension 2, and there are no such branes in IIA supergravity.

So we look for other loci where the coefficient of $D\psi^2$ might vanish. Notice that the coefficient $\sqrt{-\frac{c\epsilon'}{2y\tilde{x}}}$ cannot go to zero, since $\epsilon' = -y\frac{\sqrt{2c(c+3)}}{\sqrt{F_0y + c_1}}$. Nonetheless, we have the combination

$$-2y\epsilon' + 3\epsilon = \frac{c_1[2c(c+3)]^{1/2}}{F_0^2}\frac{F_0y + 2c_1}{\sqrt{F_0y + c_1}} + 3c_2.\tag{4.27}$$

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So the denominator $-2y\epsilon' + (c+3)\tilde{x}$ can go to infinity where $F_0 y + c_1 = 0$. However, on this locus $e^\phi \to 0$, and this locus cannot be regular. It also cannot correspond to the presence of branes, for the same reason noted above for the locus $\tilde{x} = 0$.

One last possibility, which we will not analyze here, would be to try to glue this solution to other solutions (perhaps ones with $F_0 = 0$) along a D8, much as we will do in the next section. As we will see, such gluing can happen along loci where $\sqrt{1 - a_1^2} e^{A-\phi}$ is constant; in our case this happens to be proportional to $\frac{y}{\epsilon'}$, which is a function of y. We leave the study of such a possibility for the future.

4.3.2 Special cases

- For $c = 0$, equation (4.10) is trivially satisfied, while the x^1 component of (4.7) yields $\epsilon = 0$. Then $a_2 = 0$ and this leads to the (unphysical) massive solution of appendix B.

- For $c = -3$, (4.10) is again trivially satisfied, while (4.7) yields the following ODE for β:

$$e^{12A} F_0 = -\beta \left(e^{4A} \right)' - 6e^{8A} y \,. \tag{4.28}$$

Using (4.6b) we see $e^{4A} = -\frac{\beta'}{12y}$. This ODE is nonlinear, and a little tougher than the ones we saw so far in this subsection. Hence we defer its further analysis to the next section. We will see there that it leads to many new AdS$_5$ solutions.

5 Compactification solutions

We will now analyze further the case we started considering in section 4.3.2. We will see that it corresponds to a compactification of the AdS$_7$ solutions considered in [8]. Moreover, we will be able to find the most general explicit solution, thus providing a new infinite class of AdS$_5$ solutions.

5.1 Metric and fluxes

In section 4.3.2, we found that there are AdS$_5$ solutions associated with solutions of the ODE (4.28). Replacing the expression of A given there, we have

$$\beta \left(y\beta'' - \beta' \right) = \frac{1}{2} y (\beta')^2 - \frac{F_0}{144y} (\beta')^3 \,. \tag{5.1}$$

This equation is non-linear; however, it can be rewritten as

$$(q_5^2)' = \frac{2}{9} F_0 \,, \qquad q_5 \equiv -\frac{4y\sqrt{\beta}}{\beta'} \,. \tag{5.2}$$

We will see later that q_5 has actually a useful physical interpretation (similar to the q of [8]): it will turn out to be related to D8-brane positions. In any case, the trick (5.2) allows us to solve the ODE (5.1): indeed we can write $16y^2 \frac{\beta}{(\beta')^2} = \frac{2}{9} F_0 (y - \hat{y}_0)$, which can now be integrated by quadrature.

We will postpone the detailed analysis of the solutions of (5.1) to sections 5.5 and 5.6. For the time being, in this subsection we will collect various features of the resulting AdS$_5$ solutions.

The internal metric for the class we are considering can be extracted from the general expression (3.10). However, at first its global meaning is not transparent. It proves useful to trade the coordinate x for a new coordinate θ, defined by

$$\cos\theta = \frac{-3x+\epsilon}{\sqrt{\beta}}\,. \tag{5.3}$$

The metric then becomes

$$ds_{M_5}^2 = e^{2A}ds_{\Sigma_g}^2 + ds_{M_3}^2\,, \qquad ds_{M_3}^2 = dr^2 + \frac{1}{9}e^{2A}(1-a_1^2)ds_{S^2}^2\,. \tag{5.4}$$

Here

$$ds_{S^2}^2 = d\theta^2 + \sin^2\theta D\psi^2 \tag{5.5}$$

is the metric of the round S^2, fibered over Σ_g, which is a Riemann surface of Gaussian curvature -3 (recalling (4.4), and $c=-3$) and hence $g \geq 2$; the new coordinate r is defined by

$$dr = \frac{e^{3A}}{\sqrt{\beta}}dy\,. \tag{5.6}$$

Moreover, from (3.7) and (4.6) we have

$$1 - a_1^2 = \frac{3\beta}{3\beta - y\beta'}\,, \qquad e^{4A} = -\frac{\beta'}{12y}\,, \qquad e^\phi = \frac{\sqrt{3}e^{5A}}{\sqrt{3\beta - y\beta'}}\,. \tag{5.7}$$

We can now remark that the q_5 defined in (5.2) is

$$q_5 \equiv e^{-\phi}R_{S^2} \equiv \frac{1}{3}e^{A-\phi}\sqrt{1-a_1^2} = -\frac{4y\sqrt{\beta}}{\beta'}\,. \tag{5.8}$$

$R_{S^2} = \frac{1}{3}e^A\sqrt{1-a_1^2}$ is the radius of the round S^2, as inferred from (5.4). The role of this particular combination of the radius and dilaton will become clearer in section 5.4.

From (5.8) and (5.7) we see that for the solution to make sense we must require

$$\beta \geq 0\,, \qquad -\frac{\beta'}{y} \geq 0\,. \tag{5.9}$$

We can now also obtain the fluxes, from the formulas in section 3.2. We have

$$F_2 = q_5\left[-(\mathrm{vol}_{S^2} + 3\cos\theta \mathrm{vol}_{\Sigma_g}) + \frac{1}{3}F_0 a_1 e^{A+\phi}\mathrm{vol}_{S^2}\right]\,, \tag{5.10}$$

where $\mathrm{vol}_{S^2} \equiv \sin\theta d\theta \wedge D\psi$. The four-form flux reads

$$F_4 = \frac{1}{3}\mathrm{vol}_{\Sigma_g} \wedge \left[\frac{2y\beta}{3\beta - y\beta'}\cos\theta \mathrm{vol}_{S^2} + \sin^2\theta D\psi \wedge dy\right]\,. \tag{5.11}$$

When $F_0 \neq 0$, we need not give an expression for H: as usual for massive IIA, it can be written as $H = dB$, where

$$B = \frac{F_2}{F_0} + b\,, \tag{5.12}$$

where b is a closed two-form. When $F_0 = 0$, the only solution in the class we are considering in this section is the Maldacena-Núñez solution; an expression for B is presented for that case in (C.21).

We can observe already now that the metric (5.4) and the flux (5.10) look related to those for the AdS_7 solutions in [8]; see (4.16) and (4.9) there. The expressions are very similar; one obvious difference is that the three-dimensional metric in (5.4) is fibered over Σ_g, and that the flux (5.10) has extra legs along Σ_g. Except for a few numerical factors, everything seems to correspond nicely; the role of x in [8] seems to be played here by a_1:

$$x \text{ in } [8] \to a_1 \text{ here.} \tag{5.13}$$

Actually this correspondence can be justified a little better. In [8], x is the zero-form part of $\text{Im}\psi_+^1$ (in that paper's notation), which is the calibration for a D6-brane extended along AdS_7. The analogue of this in our case would be a D6-brane extended along $\text{AdS}_5 \times \Sigma_g$; the relevant calibration is the part along $u \wedge \bar{u}$ of $\text{Im}\psi_+^1$ of the present paper. Looking at (2.26), we see that that is indeed $\text{Re}a = a_1$.

Motivated by this, in this section we will also use the name

$$x_5 \equiv a_1 \,. \tag{5.14}$$

This x_5 is meant to evoke the x in [8], and is not to be confused with the coordinate x we temporarily used in sections 3 and 4.

5.2 Correspondence with AdS_7

We will now show that solutions of the type considered in section 4.3.2 are in one-to-one correspondence with the AdS_7 solutions in [8]. The map we will find is directly inspired from a similar map from AdS_4 to AdS_7 found in [20]. It would be possible to present our new AdS_5 solutions perfectly independently from the map to AdS_7; in fact, in finding the analytic solutions the map does not help at all. However, the existence of the map tells us right away that infinitely many regular solutions do exist, and what data they depend on.

Let us start from (5.1). Using the definition (5.6), the expressions (5.7) and the expression $x_5 = a_1 = -2ye^{-3A+\phi}$ from (5.14), (3.7), we can see that

$$\partial_r \phi = \frac{1}{4} \frac{e^{-A}}{\sqrt{1-x_5^2}} \left(11x_5 - 2x_5^3 + \left(2x_5^2 - 5\right) F_0 e^{A+\phi}\right),$$

$$\partial_r x_5 = -\frac{1}{2} e^{-A} \sqrt{1-x_5^2} \left(4 - x_5^2 + x_5 F_0 e^{A+\phi}\right), \tag{5.15}$$

$$\partial_r A = \frac{1}{4} \frac{e^{-A}}{\sqrt{1-x_5^2}} \left(3x_5 - F_0 e^{A+\phi}\right).$$

Conversely, given a solution to this system, one may define $\beta = e^{10A-2\phi}(1 - x_5^2)$, $y = -\frac{1}{2}x_5 e^{3A-\phi}$ (with an eye to (4.6), (3.7), which correspond to (5.7)); if one then eliminates r from (5.15), the resulting equations imply $\beta' = -12ye^{4A}$ (the second in (5.7)), and (5.1). So the system (5.15) is in fact an equivalent way to characterize our solutions. It looks much more complicated than the original ODE (5.1). We write it because it bears an uncanny

resemblance with the system in [8, eq. (4.17)]: a few numerical factors have changed, and two new terms have appeared. This suggests that there might be a close relationship between solutions of one system and solutions of the other. This is in fact the case: to any solution (ϕ_5, x_5, A_5) of (5.15) one can associate a solution (ϕ_7, x_7, A_7) of [8, eq. (4.17)] given by

$$
e^{\phi_7} = \left(\frac{3}{4}\right)^{1/4} \frac{e^{\phi_5}}{\sqrt{1 - \frac{1}{4}x_5^2}}, \qquad e^{A_7} = \left(\frac{4}{3}\right)^{3/4} e^{A_5},
$$

$$
x_7 = \left(\frac{3}{4}\right)^{1/2} \frac{x_5}{\sqrt{1 - \frac{1}{4}x_5^2}}, \qquad r_7 = \left(\frac{4}{3}\right)^{1/4} r_5.
$$

(5.16)

Comparing (5.4) with [8, eq. (4.16)], we find that the map acts on the metrics as

$$
e^{2A_5} \left(ds_{\text{AdS}_5}^2 + ds_{\Sigma_g}^2\right) + dr_5^2 + \frac{1 - x_5^2}{9} e^{2A_5} ds_{S^2}^2 \rightarrow
$$
$$
\sqrt{\frac{4}{3}} \left(\frac{4}{3} e^{2A_5} ds_{\text{AdS}_7}^2 + dr_5^2 + \frac{e^{2A_5}}{12} \frac{1 - x_5^2}{1 - \frac{1}{4}x_5^2} ds_{S^2}^2\right).
$$

(5.17)

Conversely, to any solution (ϕ_7, x_7, A_7) of [8, eq. (4.17)], one can associate a solution (ϕ_5, x_5, A_5) of (5.15) given by

$$
e^{\phi_5} = \left(\frac{4}{3}\right)^{1/4} \frac{e^{\phi_7}}{\sqrt{1 + \frac{1}{3}x_7^2}}, \qquad e^{A_5} = \left(\frac{3}{4}\right)^{3/4} e^{A_7},
$$

$$
x_5 = \left(\frac{4}{3}\right)^{1/2} \frac{x_7}{\sqrt{1 + \frac{1}{3}x_7^2}}, \qquad r_5 = \left(\frac{3}{4}\right)^{1/4} r_7.
$$

(5.18)

This inverse map now acts on the metrics as

$$
e^{2A_7} ds_{\text{AdS}_7}^2 + dr_7^2 + \frac{1 - x_7^2}{16} e^{2A_7} ds_{S^2}^2 \rightarrow
$$
$$
\sqrt{\frac{3}{4}} \left(\frac{3}{4} e^{2A_7} \left(ds_{\text{AdS}_5}^2 + ds_{\Sigma_g}^2\right) + dr_7^2 + \frac{1}{12} \frac{1 - x_7^2}{1 + \frac{1}{3}x_7^2} e^{2A_7} ds_{S^2}^2\right).
$$

(5.19)

The simplicity of this map is basically a generalization of the simple Maldacena-Núñez solution [5], with the $1 + \frac{1}{3}x_7^2$ factor ultimately playing the role of the $\Delta = 1 + \sin^2\theta$ factor in [5].

One can also apply (5.16) directly to (5.7), and infer the expressions for the variables of the seven-dimensional solution:

$$
e^{A_7} = \frac{2}{3} \left(-\frac{\beta'}{y}\right)^{1/4}, \qquad x_7 = \sqrt{\frac{-y\beta'}{4\beta - y\beta'}}, \qquad e^{\phi_7} = \frac{(-\beta'/y)^{5/4}}{12\sqrt{4\beta - y\beta'}}.
$$

(5.20)

Moreover, $dr_7 = \left(\frac{3}{4}\right)^2 \frac{e^{3A_7}}{\sqrt{\beta}} dy$.

In [8], solving the system of ODEs in [8, eq. (4.17)] was only part of the problem. First, one had to take care of flux quantization; second, most solutions include D8's, and one must take care that supersymmetry be preserved also on top of them. We will see in

section 5.4 that the relevant conditions also map nicely under (5.16); that will lead us to conclude that there are infinitely many AdS_5 solutions, each one of them corresponding to the AdS_7 solutions in [8, 13]. Moreover, the map is quite simple: for example, it acts on the metrics as in (5.19).

5.3 Regularity analysis

We showed that solutions of (5.1) are in one-to-one correspondence with solutions of the system of ODEs relevant for AdS_7 solutions. However, (5.1) looks much simpler than that system; hence one may hope to learn more about both the AdS_5 and the AdS_7 solutions by studying it.

In this subsection we will see what boundary conditions on (5.1) have to be imposed in order to obtain compact and regular solutions.

We saw in (5.4) that the internal metric consists of an M_3 fibered over a Riemann surface Σ_g; M_3 is itself a fibration of S^2 over a one-dimensional space with coordinate r.

To make M_3 compact, we can use the same logic as for the AdS_7 solutions of [8]. One might think of making it compact by periodically identifying r, but this doesn't work for the same reason as in [8, eq. (4.24)]: the quantity $y = -\frac{1}{2}e^{3A-\phi}x_5$ is monotonic — from (5.15) we see $\partial_r y = e^{2A-\phi}\sqrt{1-x_5^2}$, which is always positive; or also, directly from (5.6) we see $\frac{\partial y}{\partial r} = e^{-3A}\sqrt{\beta}$. So periodically identifying r is not an option. The other way to make M_3 compact is to make the S^2 shrink for two values of r, just like in [8]. This is what we will now devote ourselves to.

To make the S^2 shrink, we should make the coefficient $(1 - a_1^2)$ in (5.4) go to zero, which, recalling (5.7), can be accomplished by making β vanish. If β has a single zero,

$$\beta = \beta_1(y - y_0) + O(y - y_0)^2 \,, \tag{5.21}$$

the metric (5.4) near y_0 is proportional to

$$\frac{dy^2}{4(y - y_0)} + (y - y_0)ds_{S^2}^2 \,, \tag{5.22}$$

which in fact upon defining $r = \sqrt{y - y_0}$ turns into

$$dr^2 + r^2 ds_{S^2}^2 \,, \tag{5.23}$$

which is the flat metric on $\mathbb{R}^3$. Hence if β has a single zero at $y_0 \neq 0$ the metric is regular.

One might wonder what happens if β has a double zero:

$$\beta = \beta_2(y - y_0)^2 + O(y - y_0)^3 \,. \tag{5.24}$$

In this case, (5.4) is proportional to $\frac{dy^2}{\sqrt{y-y_0}} + (y-y_0)^{3/2}ds_{S^2}^2$, which upon defining $\rho = y - y_0$ turns into

$$\frac{1}{\sqrt{\rho}}\left(d\rho^2 + \rho^2 ds_{S^2}^2\right) \,; \tag{5.25}$$

we also have $e^A \sim \rho^{1/4}$, $e^\phi \sim \rho^{3/4}$. This is obviously not a regular point, but it is the local behavior appropriate for a D6 stack whose transverse directions are ρ and the S^2.

Higher-order zeros do not lead to anything of physical relevance, and in fact they would not lead to solutions, as we will see later. However, given that we have obtained boundary conditions corresponding to a regular point and to presence of a D6 stack, it is natural to wonder whether we can find boundary conditions corresponding to presence of an O6. This is realized when

$$\beta = \beta_0 + \beta_{1/2}\sqrt{y - y_0} + O(y - y_0); \tag{5.26}$$

in this case the metric is proportional to $(y - y_0)^{1/4}\left(\frac{dy^2}{y-y_0} + 16\alpha_0^2 ds_{S^2}^2\right)$, with $\alpha_0 \equiv \frac{\beta_{1/2}}{\beta_0}$. With the definition $\rho = \sqrt{y - y_0}$, this turns into

$$\sqrt{\rho}\left(d\rho^2 + 4\alpha_0^2 ds_{S^2}^2\right); \tag{5.27}$$

moreover, $e^A \sim \rho^{-1/4}$, $e^\phi \sim \rho^{-3/4}$. These are the appropriate behaviors for fields near the beginning of an O6 hole: to see this, one can start from the flat space O6 metric, given by $ds_\perp^2 = H^{1/2}(d\rho^2 + \rho^2 ds_{S^2}^2)$, $e^A = H^{1/4}$, $e^\phi \propto H^{3/4}$, $H = 1 - \frac{\rho_0}{\rho}$, and expand around $\rho = \rho_0$, which is indeed the boundary of the O6 hole.

This concludes our study of the physically relevant boundary conditions for the ODE (5.1); as it will turn out, these are the only ones which are actually realized in its solutions. Later in this section we will turn to the task of finding such solutions.

5.4 Flux quantization, D8 branes

Before we look at explicit solutions, we will discuss flux quantization. We will also introduce D8-branes in our construction, as was done in [8]. This subsection is in many ways similar to [8, section 4.8], which the reader may want to consult for more details.

We will start with some preliminary comments about the B field. In (5.12) we expressed it in terms of a closed two-form b. We will need this second term because the term $\frac{F_2}{F_0}$ in (5.12) will jump as we cross a D8 (since F_0 will jump there, by definition). More precisely, looking at F_2 we see that only the term proportional to $\mathrm{vol}_{S^2} + 3\cos\theta\mathrm{vol}_{\Sigma_g}$ jumps (since in the other term an F_0 cancels out). Thus we can limit ourselves to considering b of the form

$$b = b_0(\mathrm{vol}_{S^2} + 3\cos\theta\mathrm{vol}_{\Sigma_g}), \tag{5.28}$$

which is indeed closed (while $\mathrm{vol}_{S^2} = \sin\theta d\theta \wedge D\psi$ would not be, because of the presence of ρ). (5.12) now becomes

$$B = \left(b_0 - \frac{q_5}{F_0}\right)(\mathrm{vol}_{S^2} + 3\cos\theta\mathrm{vol}_{\Sigma_g}) + \frac{q_5}{3}x_5 e^{A+\phi}\mathrm{vol}_{S^2}. \tag{5.29}$$

At the poles, for regularity we should have that what multiplies vol_{S^2} should go to zero.

However, more precisely B should be understood as a "connection on a gerbe". Concretely, this means that it is not necessarily a globally well-defined two-form. On a chart intersection $U \cap U'$, $B_U - B_{U'}$ can be any closed two-form whose periods are integer multiples of $4\pi^2$ (known as a "large gauge transformation"). This translates into the requirement that the coefficient of vol_{S^2} in (5.29) should wind $\pi\times$ an integer number of times in going

from the north to the south pole. Alternatively, using Stokes' theorem, we see that the integral of H between r_N and r_S (the positions of the two poles) is

$$\int_{M_3} H = \int_{S^2} \int_{r_N}^{r_S} dr\, H = \int_{S^2} (B(r_N) - B(r_S))\,; \tag{5.30}$$

thus $\int_{M_3} H$ will be an integer multiple of $4\pi^2$, in agreement with flux quantization.

After these comments on the NSNS flux H, let us now consider the RR fluxes. First of all, the zero-form should satisfy $F_0 = \frac{n_0}{2\pi}$, $n_0 \in \mathbb{Z}$. For the higher forms, we should consider

$$\tilde{F}_2 \equiv F_2 - BF_0\,, \qquad \tilde{F}_4 \equiv F_4 - B \wedge F_2 + \frac{1}{2}B \wedge BF_0\,, \tag{5.31}$$

which are d-closed (unlike the original F_2 and F_4, which in our notation are $(d-H\wedge)$-closed). Flux quantization imposes that those should have integer periods. For the two-form we simply have

$$\tilde{F}_2 = -bF_0 = -b_0 F_0 (\mathrm{vol}_{S^2} + 3\cos\theta\, \mathrm{vol}_{\Sigma_g})\,. \tag{5.32}$$

Integrating this on the fiber S^2 and imposing that it is of the form $2\pi n_2$, $n_2 \in \mathbb{Z}$, we find

$$b_0 = -\frac{n_2}{2F_0} = -\pi\frac{n_2}{n_0}\,, \tag{5.33}$$

just like in [8]. A gauge transformation will change $b_0 \to b_0 + k\pi$, and simultaneously $n_2 \to n_2 - k$, so that (5.33) remains satisfied.

Near the north and south pole it is convenient to work in a gauge where B is regular; then $\int \tilde{F}_2 \to \int F_2$, and n_2 should be equal to the limit near the pole of $\left(b_0 - \frac{q_5}{F_0} + \frac{q_5}{3}x_5 e^{A+\phi}\right)$, the coefficient of vol_{S^2} in (5.29). For a regular point, n_2 near the pole is zero, and both $q_5 \to 0$ and $q_5 x_5 e^{A+\phi} \to 0$. For a stack of n_2 D6-branes, $q_5 x_5 e^{A+\phi} \to 0$, and $q_5 \to -\frac{n_2}{2}$. In section 5.3, we saw that presence of a D6 corresponds to a double zero in β, (5.24). The condition we just saw will then discretize the parameter β_2, giving

$$\beta_2 = \left(\frac{4y_0}{n_2}\right)^2\,. \tag{5.34}$$

An O6 point is different: $n_2 = \pm 1$ (depending on whether we are considering the north or south pole), $q_5 \to 0$, but $\frac{q_5}{3}x_5 e^{A+\phi}$ is non zero, and will have to tend to $-\frac{n_2}{2F_0}$. Again in section 5.3 we saw that an O6 corresponds in our class of solutions to the presence of a square root, (5.26). Flux quantization will then fix

$$\beta_0 = (18y_0)^2\,. \tag{5.35}$$

The four-form $\tilde{F}_4$ can now be written, after some manipulations, as

$$\tilde{F}_4 = \left(\frac{3}{F_0}\left(-q_5^2 + \frac{n_2^2}{4}\right)\cos\theta\, \mathrm{vol}_{S^2} + \frac{1}{3}\sin^2\theta D\psi \wedge dy\right) \wedge \mathrm{vol}_{\Sigma_g}\,. \tag{5.36}$$

Using (5.2) we can also write $\tilde{F}_4 = d\tilde{C}_3$, where

$$\tilde{C}_3 = \frac{3}{2F_0}\left(-q_5^2 + \frac{n_2^2}{4}\right)\sin^2\theta D\psi \wedge \mathrm{vol}_{\Sigma_g}\,. \tag{5.37}$$

If both poles are regular points, $\tilde{C}_3$ is a regular form. Indeed, as we saw, at such a pole we should have $n_2 = 0$ and $q_5 \to 0$. So the coefficient $\left(-q_5^2 + \frac{n_2^2}{4}\right)$ will actually go to zero at the pole. Now, using the fact that β has a single zero (5.21), from (5.6) and (5.8) we see that q_5 starts with a linear power in r. Hence we have

$$\tilde{C}_3 \sim r^2 \sin^2 \theta D\psi \wedge \mathrm{vol}_{\Sigma_g}. \tag{5.38}$$

Now, $r^2 \sin^2 \theta d\psi$, going from spherical to cartesian coordinates x^i, $i = 1, 2, 3$, is proportional to $x^1 dx^2 - x^2 dx^1$, and hence is regular. All in all, we conclude that $\tilde{F}_4$ does not have any non-zero periods, since it is exact. In presence of a D6 or O6 point, it is best to go back to (5.36). The space is topologically an S^3 fibration over Σ_g; standard topological arguments tell us that its cohomology is just the product of that of S^3 and that of Σ_g. As such it would have no four-cycles. Thus so far flux quantization for $\tilde{F}_4$ is not an issue.

We will now introduce D8-branes. We will consider them to be extended along all directions except r. Their treatment is very similar to [8] and we will be brief. The defining feature of a D8 stack is that the Romans mass F_0 jumps as we go across them. Let us call n_0 and n'_0 the flux integers on the two sides. Moreover, we will allow the D8's to have non-zero worldsheet flux, which can also be thought of as a smeared D6 charge. This will make the flux integer for $\tilde{F}_2$ jump as well; we will call n_2 and n'_2 its value on the two sides. The "slope" $\mu \equiv \frac{\Delta n_2}{\Delta n_0} \equiv \frac{n'_2 - n_2}{n'_0 - n_0}$ needs to be an integer. With this notation, imposing that (5.29) be continuous we find the condition

$$q_5|_{\mathrm{D8}} = \frac{1}{2} \frac{n'_2 n_0 - n_2 n'_0}{n'_0 - n_0} = \frac{1}{2}(-n_2 + \mu n_0) = \frac{1}{2}(-n'_2 + \mu n'_0). \tag{5.39}$$

This is to be read as a condition fixing the D8's position.

One might now also wonder whether the flux of $\tilde{F}_4$ along $\Sigma_g \times S^2$ might jump between D8's, as does the integral of $\tilde{F}_2$. But actually $\int_{S^2} \cos \theta \mathrm{vol}_{S^2} = 0$. So even in presence of D8's we need not worry about flux quantization for $\tilde{F}_4$.

Crucially, (5.39) is exactly the same condition that was found for D8-branes in [8, eq. (4.45)]. The function called q in that paper, which we will call q_7 here, is not exactly the same as our q_5 defined in (5.8): indeed $q_7 \equiv \frac{1}{4} e^{A_7 - \phi_7} \sqrt{1 - x_7^2}$. However, using the map (5.16), we see that the different overall factor is reabsorbed:[6]

$$q_5 = q_7. \tag{5.40}$$

So (5.39) fixes the D8's at exactly the same position in an AdS_5 solution and in its AdS_7 solution.

Since (5.39) was found by imposing that B should be continuous, it looks easy to impose the condition on flux quantization. As remarked earlier, by Stokes' theorem we can relate the integrality of H to the periodicity of the coefficient of vol_{S^2} in B. (This periodicity was expressed visually in several figures in [8, 13], as a dashed green line.) However, in

---

[6]Actually, the condition that the system (5.15) be mapped to the similar system [8, eq. (4.17)] for AdS_7 solutions only fixed the map (5.16) up to a constant. We fixed the constant so that (5.39) would look exactly equal to [8, eq. (4.45)].

presence of D8's one might encounter a region where $F_0 = 0$; generically such a region will exist (although there are also "limiting cases" where it does not exist; see [13, section 4.2]). In such a region, (5.12) (and hence (5.29)) cannot be used; we have to resort to (C.21). This allows to write a general expression for the integral of H, as shown in [13, eq. (4.7)].

Since we are going to simplify that formula for AdS$_7$ solutions, let us review it quickly here. To simplify things a bit, one derives first an expression for the integral in the "northern hemisphere", between $x_7 = 1$ and $x_7 = 0$; it can be shown that $x_7 = 0$ is in the massless region, where $F_0 = 0$. There might be many D8's; let D8$_n$ be the one right before the massless region, $\{n_{2,n}, n_{0,n} = 0\}$ the flux parameters right after it, and $\{n_{2,n-1}, n_{0,n-1}\}$ the ones right before it. Then we can divide the integral into a contribution from the massive region and one from the massless region:

$$
\begin{aligned}
\int_{\text{north}} H &= \int_{r_{\text{N}}}^{\text{D8}_n} H + \int_{\text{D8}_n}^{x=0} H \\
&= 4\pi \left[q_7 \left(\frac{x_7}{4} e^{A_7 + \phi_7} - \frac{1}{F_{0,n-1}} \right) - \frac{n_{2,n-1}}{2F_{0,n-1}} + \frac{3}{32} \frac{R^3}{n_{2,n}} \left(x_7 - \frac{x_7^3}{3} \right) \right]_{\text{D8}_n} \\
&= 4\pi \left[-\pi \mu_n + \frac{1}{4} q x_7 e^{A_7 + \phi_7} - \frac{1}{4} q_7 x_7 e^{A_7 + \phi_7} \frac{3 - x_7^2}{1 - x_7^2} \right]_{\text{D8}_n} \\
&= 4\pi \left[-\pi \mu_n + \frac{R^3}{16 n_{2,n}} x_7 \right]_{\text{D8}_n}.
\end{aligned}
\tag{5.41}
$$

We have used that for the massless solution $-8q_7 \frac{e^{A_7+\phi_7}}{1-x_7^2} = -2\frac{e^{2A_7}}{\sqrt{1-x_7^2}} = \frac{R^3}{n_2}$, where R is a constant. After this simplification, and putting together the contribution from $\int_{\text{south}} H$ from the "southern hemisphere", we can write

$$
N \equiv -\frac{1}{4\pi^2} \int H = (|\mu_n| + |\mu_{n+1}|) + \frac{1}{4\pi} e^{2A(x=0)} (|x_n| + |x_{n+1}|),
\tag{5.42}
$$

where x_n and x_{n+1} are the values of x_7 at the branes D8$_n$ and D8$_{n+1}$.[7]

To derive a similar expression for AdS$_5$ solutions, we follow a similar logic. It proves convenient to use from the very beginning (A_7, x_7, ϕ_7) variables, which are related to (A_5, x_5, ϕ_5) variables via (5.16). We can use (5.29) and (C.21), the latter of which is already expressed in terms of x_7. Some factors in the computation change, but remarkably the result turns out to be exactly the same as in (5.42). As a consequence, if the H flux quantization is satisfied for an AdS$_7$ solution, it is also satisfied for an AdS$_5$ solution, and viceversa.

So the conclusion of this section is that the flux quantization conditions and the constraints fixing the D8-brane positions are all precisely mapped by (5.16), in such a way that if they are satisfied for an AdS$_7$ solution they are also automatically satisfied for an AdS$_5$ solution. This proves that the map (5.16) produces infinitely many AdS$_5$ solutions.

5.5 The simplest massive solution

We will now start studying solutions to (5.1), and their associated physics. We have already indicated in (5.2) how to solve it analytically. However, in this section we will warm up

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[7]The μ_i and x_i before the massless region are positive, while those after the massless region are negative.

by a perturbative study, which we find instructive and which will allow us to isolate a particularly nice and useful solution.

In section 5.3 we studied the boundary conditions for the ODE (5.1). We can now proceed to study it in the neighborhood of such a solution. We will do so by assuming analytic behavior around y_0: $\beta = \sum_{k=1}^{\infty} \beta_k (y - y_0)^k$, by plugging this Taylor expansion in (5.1), and solving order by order.

Already at order zero we find

$$\left(\beta_1 - \frac{72 y_0^2}{F_0} \right) \beta_1^2 = 0 \,. \tag{5.43}$$

The first branch, $\beta_1 = \frac{72 y_0^2}{F_0}$, lets β have a single zero, which as we saw after (5.21) corresponds to a regular point. The second branch, $\beta_1 = 0$, makes β have a double zero, which as we saw after (5.24) corresponds to a D6. In this section we will use the first branch, leaving the second for section 5.6.

Continuing to solve (5.1) perturbatively after having set $\beta_1 = \frac{72 y_0^2}{F_0}$, we find a nice surprise: the perturbative expansion stops after three iterations. This leads to a very simple solution to (5.1):

$$\beta = \frac{8}{F_0} (y - y_0)(y + 2y_0)^2 \,. \tag{5.44}$$

This has the desired single zero at $y = y_0$, and it also has a double zero at $y = -2y_0$, signaling that M_3 has a D6 stack there. These are the qualitative features one expects from the solution in [8, section 5.2]; in that paper, that solution was argued to exist (along with many others, which we shall discuss in due course) on numerical grounds — see in particular figure 3 in that paper. It would also be possible to find (5.44) by finding the general solution, and imposing the presence of a simple zero; we will see this in section 5.6.

(5.44) looks superficially very similar to (4.18). Taking $c_1 = -y_0^2/4$, we see that (4.18) has *two* double zeros, at $y = \pm y_0$, corresponding to two D6 stacks. This is indeed correct for that massless solution: the two D6 stacks are generated by the reduction from eleven dimensions, in a similar way as in [8, section 5.1]. Notice also that the massless limit of (5.44), on the other hand, does not exist, since F_0 appears there in the denominator.

Now that we have obtained one solution of (5.44), we can pause to explore what the resulting AdS$_5$ solution looks like; moreover, using the map (5.16), we can also produce an AdS$_7$ solution which will indeed be the one found numerically in [8, section 5.2].

The conditions (5.9) give us two possibilities:

$$\{y_0 < 0, \ F_0 > 0, \ y \in [y_0, -2y_0]\} \quad \text{or} \quad \{y_0 > 0, \ F_0 < 0, \ y \in [-2y_0, y_0]\} \,. \tag{5.45}$$

We will assume the first possibility. One can then write the metric and fields most conveniently in terms of

$$\tilde{y} \equiv \frac{y}{y_0} \,, \tag{5.46}$$

which then has to belong to $[-2, 1]$. We have

$$ds^2_{M_5} = e^{2A} ds^2_{\Sigma_g} + \sqrt{-\frac{y_0}{8F_0}} \left(\frac{d\tilde{y}^2}{(1 - \tilde{y})\sqrt{\tilde{y} + 2}} + \frac{4}{9} \frac{(1 - \tilde{y})(\tilde{y} + 2)^{3/2}}{2 - \tilde{y}} ds^2_{S^2} \right), \tag{5.47}$$

and

$$e^{4A} = -2\frac{y_0}{F_0}(2 + \tilde{y}), \qquad e^{2\phi} = \sqrt{-\frac{1}{2y_0 F_0^3}\frac{(\tilde{y}+2)^{3/2}}{2-\tilde{y}}}.$$ (5.48)

We also need to implement flux quantization, which in this case is the statement that the D6 stack at the $\tilde{y} = -2$ point has an integer number n_2 of D6-branes. This constraint was discussed right below (5.33). From (5.8) and (5.44) we find $q_5 = \frac{1}{3}\sqrt{2F_0(y - y_0)}$, which implies

$$y_0 = -\frac{3}{8}\frac{n_2^2}{F_0}.$$ (5.49)

We did not replace this constraint in (5.47), as we did in (1.2), because later we will glue pieces of it together with other metrics and with itself, and in that context the parameter y_0 will be fixed by flux quantization a bit differently.

The AdS$_7$ solutions can now be found easily by applying the map (5.16), and in particular its action on the metric, (5.17). The internal metric on M_3 is

$$ds_{M_3}^2 = \sqrt{-\frac{y_0}{6F_0}}\left(\frac{d\tilde{y}^2}{(1-\tilde{y})\sqrt{\tilde{y}+2}} + \frac{4}{3}\frac{(1-\tilde{y})(\tilde{y}+2)^{3/2}}{8 - 4\tilde{y} - \tilde{y}^2}ds_{S^2}^2\right),$$ (5.50)

and

$$e^{4A} = -\left(\frac{4}{3}\right)^3 2\frac{y_0}{F_0}(\tilde{y}+2), \qquad e^{2\phi} = \sqrt{-\frac{6}{y_0 F_0^3}\frac{(\tilde{y}+2)^{3/2}}{8 - 4\tilde{y} - \tilde{y}^2}}.$$ (5.51)

(5.50) and (5.51) give analytically the solution found numerically in [8, section 5.2]. The flux F_2 can be read off from the expression $F_2 = q(\frac{x_7}{4}F_0 e^{A+\phi} - 1)\text{vol}_{S^2}$ in [8, eq. (4.42)]:

$$F_2 = \frac{k}{\sqrt{3}}\frac{(1-\tilde{y})^{3/2}(\tilde{y}+4)}{8 - 4\tilde{y} - \tilde{y}^2}\text{vol}_{S^2}.$$ (5.52)

For both the AdS$_5$ and AdS$_7$ solutions, from (5.49) we can see that, making n_2 large, curvature and string coupling become as small as one wishes. This guarantees that the supergravity approximation is applicable. Similar limits can be taken for the solutions that we will present later. (This was shown in general in [13, section 4.1].)

5.6 General massive solution

Let us now go back to (5.43) and see what happens if we use the branch $\beta_1 = 0$. This means that β has a double zero, which corresponds to presence of a D6 stack at $y = -2y_0$.

The perturbative expansion for (5.1) now does not truncate anymore. It is possible to go to higher order, guess an expression for the k-th term β_k in the Taylor expansion $\beta = \sum_{k=1}^{\infty} \beta_k(y - y_0)^k$, and resum this guess. (This is in fact the way we originally proceeded.) At this point it is of course much easier to use the trick explained below (5.2), and find the general solution directly. Assuming $y_0 > 0$, it reads

$$\beta = \frac{y_0^3}{b_2^3 F_0}\left(\sqrt{\hat{y}} - 6\right)^2\left(\hat{y} + 6\sqrt{\hat{y}} + 6b_2 - 72\right)^2,$$ (5.53)

where

$$\hat{y} \equiv 2b_2 \left(\frac{y}{y_0} - 1 \right) + 36 \,, \qquad b_2 \equiv \frac{F_0}{y_0} \beta_2 \,. \tag{5.54}$$

This solution now depends on the two parameters y_0 and b_2, rather than just one as (5.44), and we expect it to be the most general solution to (5.1). To see whether this is true, let us analyze its features and compare them to what we expect from the qualitative study in [8, section 5.2].

(5.53) has zeros in $\hat{y} = 36$ (which corresponds to $y = y_0$) and for $b_2 < 12$ also in $\hat{y} = \sqrt{-3 + \sqrt{81 - 6b_2}}$. Also, at $\hat{y} = 0$ it has a point where it behaves as $\beta \sim \beta_0 + \sqrt{\hat{y}} + O(\hat{y})$, which up to translation is the same as in (5.26), which corresponds to an O6 point. Taking also into account the constraints in (5.9), we find two possibilities, and one special case between them.

- If $b_2 < 12$, the solution is defined in the interval $\hat{y} \in \left[\sqrt{-3 + \sqrt{81 - 6b_2}}, 36 \right]$; there are two double zeros at both extrema. This represents a solution with two D6 stacks at both ends, but where the numbers of D6s are not the same on the two sides (unlike for (4.18)). Under the map (5.16) to AdS$_7$, it becomes a solution that was briefly mentioned at the end of [8, section 5.2]; in terms of the graph in figure 3(b) in that paper, its path would come from below and miss the green dot on the top side from the left, so as to end up in a D6 asymptotics on the top side as well.

- If $b_2 > 12$, the solution is defined for $\hat{y} \in [0, 36]$; there is a double zero at $\hat{y} = 36$, and an O6 singularity (see (5.26)) at $\hat{y} = 0$. This represents a solution with one D6 stack at one end, and one O6 at the other extremum. Under the map to AdS$_7$, it becomes another solution that was briefly mentioned in [8, section 5.2]; in terms of the graph in figure 3(b) in that paper, its path would come from below and miss the green dot on the top side from the right, so as to end up in an O6 asymptotics on the top side.

- In the limiting case, $b_2 = 12$, the solution is again defined for $\hat{y} \in [0, 36]$; under the map to AdS$_7$ we expect to find the case where (again referring to [8, figure 3(b)]) we hit the green dot at the top, which should correspond to having a regular point. Indeed in this case (5.53) reduces to

$$\beta = \frac{y_0^3}{1728 F_0} \hat{y}(\hat{y} - 36)^2 \,, \tag{5.55}$$

which has a double zero in $\hat{y} = 36$ and a single zero in $\hat{y} = 0$; it is essentially (5.44). It would have been possible to obtain (5.44) this way, but we chose to highlight it in a subsection by itself because of its simplicity.

So the solution (5.53) has the features we expected from the qualitative analysis in [8, section 5.2].

We record also here some data of the corresponding solutions. For the AdS_5 solution, the metric, warping and dilaton read

$$ds^2_{M_5} = e^{2A}ds^2_{\Sigma_g} + \frac{y_0^{5/4}d\hat{y}^2}{4(b_2^5 F_0^3 \hat{y}^3 \beta)^{1/4}} + \frac{(b_2^7 F_0 \hat{y})^{1/4}}{18y_0^{7/4}} \frac{\beta^{3/4}ds^2_{S^2}}{2(b_2-18)^2 + 18(b_2-12)\sqrt{\hat{y}} - (b_2-18)\hat{y}},$$

$$e^{8A} = \frac{b_2\beta}{F_0 y_0 \hat{y}}, \qquad e^{8\phi} = \frac{b_2^{11}\beta^3}{16 F_0^3 y_0^{11} \hat{y}^3 \left(2(b_2-18)^2 + 18(b_2-12)\sqrt{\hat{y}} - (b_2-18)\hat{y}\right)^4}. \tag{5.56}$$

The AdS_7 solution reads

$$ds^2_{M_3} = \frac{y_0^{5/4}d\hat{y}^2}{4(b_2^5 F_0^3 \hat{y}^3 \beta)^{1/4}} + \frac{(b_2^7 F_0 \hat{y})^{1/4}}{3y_0^{7/4}} \frac{\beta^{3/4}ds^2_{S^2}}{12(b_2-18)^2 + 144(b_2-12)\sqrt{\hat{y}} - 12(b_2-18)\hat{y} - \hat{y}^2},$$

$$e^{8A} = \frac{2^{12}b_2\beta}{3^6 F_0 y_0 \hat{y}}, \quad e^{8\phi} = \frac{144 b_2^{11}\beta^3}{F_0^3 y_0^{11} \hat{y}^3 \left(-12(b_2-18)^2 - 144(b_2-12)\sqrt{\hat{y}} + 12(b_2-18)\hat{y} + \hat{y}^2\right)^4}. \tag{5.57}$$

Finally, flux quantization can be taken into account by using (5.33), (5.34) and the expansion of β around its zeros (or around its zero and its square root point, for the O6–D6 case). We obtain two equations, which discretize the two parameters b_2 and y_0. The expressions are not particularly inspiring (especially in the D6–D6 case) and we will not give them here.

5.7 Some solutions with D8's

We will now show two simple examples of solutions with D8-branes. These will be the ones studied numerically in [8, section 5.3]; here we will give their analytic expressions. We will simply have to piece together solutions we have already studied; all we will have to work out is the position of the D8's.

The first example is a solution with only one D8 stack. This can be obtained by gluing two metrics of the type (5.47). We will assume

$$y_0 < 0, \qquad F_0 > 0; \qquad y_0' > 0, \qquad F_0' < 0. \tag{5.58}$$

Following the logic in [8, section 5.3], the flux quantization conditions can be satisfied by taking for example the two-form flux integer after the D8 stack to vanish, $n_2' = 0$, $n_2 = \mu(n_0' - n_0)$, $\mu \in \mathbb{Z}$, and

$$n_0' = n_0\left(1 - \frac{N}{\mu}\right), \tag{5.59}$$

where $N = \frac{1}{4\pi^2}\int H$ is the NSNS flux integer. (Recall that $F_0 = \frac{n_0}{2\pi}$, and similarly for F_0'.) As usual the metric can be written as $ds^2_{M_5} = e^{2A}ds^2_{\Sigma_g} + ds^2_{M_3}$, and putting together two

copies of (5.47) we can write[8]

$$
ds^2_{M_3} = \begin{cases} \dfrac{1}{\sqrt{8F_0}} \left(\dfrac{dy^2}{(y-y_0)\sqrt{-2y_0-y}} + \dfrac{4}{9} \dfrac{(y-y_0)(-2y_0-y)^{3/2}}{-y_0(y-2y_0)} ds^2_{S^2} \right), & y_0 < y < y_{D8}; \\[4mm] \dfrac{1}{\sqrt{-8F'_0}} \left(\dfrac{dy^2}{(y'_0-y)\sqrt{2y'_0+y}} + \dfrac{4}{9} \dfrac{(y'_0-y)(2y'_0+y)^{3/2}}{y'_0(2y'_0-y)} ds^2_{S^2} \right), & y_{D8} < y < y'_0. \end{cases}
$$
(5.60)

We reverted to using y rather than $\hat{y}$, so as to be able to use the same coordinate before and after the D8 stack. Imposing that A and ϕ (or, equivalently, that β and β') be continuous across the D8 stack, we get

$$
y_0 = \frac{1}{2} \frac{2F_0 - F'_0}{F_0 + F'_0} y_{D8}, \qquad y'_0 = \frac{1}{2} \frac{2F'_0 - F_0}{F_0 + F'_0} y_{D8}.
$$
(5.61)

We also have to impose (5.39), which fixes

$$
y_{D8} = y_0 + \frac{9(F'_0)^2 n_2^2}{8F_0(F_0 - F'_0)^2},
$$
(5.62)

which together with (5.61) and (5.59) gives

$$
y_0 = -\frac{3}{2} F_0 \pi^2 (N^2 - \mu^2), \quad y'_0 = \frac{3}{2} F_0 \pi^2 (N-\mu)(2N-\mu), \quad y_{D8} = 3F_0 \pi^2 (N-2\mu)(N-\mu).
$$
(5.63)

One can also obtain the corresponding AdS$_7$ solution. This can be done using the map (5.19) on (5.60). Alternatively, we can just write one copy of (5.50) for $y_0 < y < y_{D8}$, and a second copy of (5.50), formally obtained by $y \to -y$, $y_0 \to -y'_0$, $F_0 \to -F'_0$. This provides the analytic expression of the solution in [8, figure 4].

We can also consider a configuration with two D8 stacks. We will take it to by symmetric, in the sense that the flux integers before the first D8 stack will be $(n_0, 0)$, between the two stacks $(0, n_2 = -k < 0)$, and after the second stack $(-n_0, 0)$. This corresponds to [8, figure 5]. Again we will assume $y_0 < 0$; the positions of the two D8 stacks will be $y_{D8} < 0$ and $y_{D8'} = -y_{D8} > 0$. We will give only the AdS$_7$ internal metric:

$$
ds^2_{M_3} = \begin{cases} \dfrac{1}{\sqrt{6F_0}} \left(\dfrac{dy^2}{(y-y_0)\sqrt{-2y_0-y}} + \dfrac{4}{3} \dfrac{(y-y_0)(-2y_0-y)^{3/2}}{8y_0^2 - 4yy_0 - y^2} ds^2_{S^2} \right), & y_0 < y < y_{D8}; \\[4mm] \dfrac{24^4 R^6 dy^2 + (9^2 R^6 - 32^2 y^2)^2 ds^2_{S^2}}{3 \cdot 6^5 (9^2 R^6 - 32^2 y^2)^{1/2}}, & y_{D8} < y < -y_{D8}; \\[4mm] \dfrac{1}{\sqrt{6F_0}} \left(\dfrac{dy^2}{(-y_0-y)\sqrt{-2y_0+y}} + \dfrac{4}{3} \dfrac{(-y_0-y)(-2y_0+y)^{3/2}}{8y_0 + 4yy_0 - y^2} ds^2_{S^2} \right) & -y_{D8} < y < -y_0; \end{cases}
$$
(5.64)

---

[8]The sign differences between the expression before and after the D8 have to do with the simplification of factors involving $\sqrt{F_0^2} = |F_0|$ from applying (5.7) to (5.44).

the metric in the middle region is the known massless metric in [8, eq. (5.4)], with the change of coordinate (C.20).

We now have three unknowns: R, y_0, y_{D8}. Continuity of β and β' this time only imposes one condition; we then have (5.39) and the condition (5.42). We get

$$y_0 = -\frac{9}{4}k\pi(N - \mu)\,, \quad y_{D8} = -\frac{9}{4}k\pi(N - 2\mu)\,, \quad R^6 = \frac{64}{3}k^2\pi^2(3N^2 - 4\mu^2)\,, \qquad (5.65)$$

where in this case $\mu = \frac{k}{n_0}$. Notice that the in this case the bound in [13, eq. (4.10)] (which can also be found by (5.42)) implies $N > 2\mu$.

It would now be possible to produce solutions with a larger number of D8's. It is in fact possible to introduce an arbitrary number of them, although there are certain constraints on their numbers and their D6 charges [13, section 4]. The most general solution can be labeled by the choice of two Young diagrams; there is also a one-to-one correspondence with the brane configurations in [14, 15]. One can in fact think of the AdS$_7$ solutions as a particular near-horizon limit of the brane configurations. For more details, see [13]. For these more general solutions, we expect to have to glue together not only pieces of the solution in subsection 5.5 and of the massless solution, but also pieces of the more complicated solution in 5.6.

5.8 Field theory interpretation

In this section we have found infinitely many new AdS$_5$ solutions in massive IIA, and we have established that they are in one-to-one correspondence with the AdS$_7$ solutions of [8, 13].

It is easy to guess the field theory interpretation of this correspondence. Recall first the Maldacena-Núñez $\mathcal{N} = 2$ solutions [5]. The original AdS$_7 \times S^4$ solution of M-theory has an SO(5) R-symmetry; when one compactifies on a Riemann surface Σ_g, one "mixes" the SO(2) of local transformations on Σ_g with an SO(2) $\subset$ SO(5) subgroup; the commutant SO(2)×SO(3)≅U(2) remains as the R-symmetry of the resulting $\mathcal{N} = 2$ CFT$_4$. This is reflected in the form of the metric of the S^4, that gets distorted (except for the directions protected by the R-symmetry).

In similar $\mathcal{N} = 1$ solutions [5, 7], the SO(2) is embedded in SO(5) in a more intricate way, so that its commutant is a U(1), which is indeed the R-symmetry of an $\mathcal{N} = 1$ theory.

For us, the CFT$_6$ has only $(1, 0)$ supersymmetry, and thus its R-symmetry is already only SU(2). The twisting is very similar to the usual one in [5]: it is signaled by the fact that the ψ coordinate is fibered over the Riemann surface Σ_g.

When we mix this with the SO(2) of local transformations on Σ_g, the commutant is only a U(1). So in principle there is no symmetry protecting the shape of the internal S^2 in the AdS$_7$ solutions; indeed the metric (5.4) does not have SO(3) isometry, because the ψ direction is fibered over Σ_g. What is a bit surprising is that the breaking is not more severe: (5.5) might have become considerably more complicated, with $\sin\theta$ for example being replaced by a different function. Likewise, in the fluxes, one can see that there is no SO(3) symmetry: the $\cos\theta$ in front of vol$_{\Sigma_g}$, for example, breaks it. Still, there are various nice vol$_{S^2}$ terms which were not guaranteed to appear.

In any case, we interpret our solutions as the twisted compactification of the CFT_6 dual to the AdS_7 solutions in [8, 13]. Recently, there has been a lot of progress in understanding such compactifications for the $(2,0)$ theories [6, 7, 30], and it would be very interesting to extend those results to our AdS_5 solutions. Here, we will limit ourselves to pointing out a couple of preliminary results about the number of degrees of freedom.

A common way of estimating the number of degrees of freedom using holography in any dimension is to introduce a cut-off in AdS, and estimate the Bekenstein-Hawking entropy (see for example [33, section 3.1.3]). This leads to $\frac{R_{\mathrm{AdS}_7}^5}{G_{\mathrm{N},7}}$ in AdS_7, and to $\frac{R_{\mathrm{AdS}_5}^3}{G_{\mathrm{N},5}}$ in AdS_5, where $G_{\mathrm{N},d}$ is Newton's constant in d dimensions. The latter can be computed as $\frac{1}{g_s^2}\mathrm{vol}_{10-d}$. In a warped compactification with non-constant dilaton, both R_{AdS} and g_s are non-constant, and should be integrated over the internal space. In our case, for AdS_7 this leads to

$$\mathcal{F}_{0,6} \equiv \int_{M_3} e^{5A_7 - 2\phi_7} \mathrm{vol}_3 \tag{5.66}$$

and for AdS_5 to $\mathcal{F}_{0,4} \equiv \int_{M_5} e^{5A_5 - 2\phi_5} \mathrm{vol}_5$. These can be thought of as the coefficient in the thermal partition function, $\mathcal{F} = \mathcal{F}_{0,d} V T^d$, where V is the volume of space and T is temperature. These computations however are basically the same for the coefficients in the Weyl anomaly, at least at leading order (i.e. in the supergravity approximation).

As a consequence of our map (5.16), $\mathcal{F}_{0,6}$ and $\mathcal{F}_{0,4}$ are related. Taking into account the transformation of the volume form according to (5.17), we find

$$\mathcal{F}_{0,4} = \left(\frac{3}{4}\right)^4 \mathcal{F}_{0,6} \mathrm{Vol}(\Sigma_g) \,. \tag{5.67}$$

The volume of Σ_g can be easily computed using Gauss-Bonnet and the fact that its scalar curvature equals -6: we get

$$\mathrm{Vol}(\Sigma_g) = \frac{4}{3}\pi(g-1) \,. \tag{5.68}$$

So the ratio of degrees of fredom in four and six dimensions is universal, in that it depends only on g and not on the precise $(1,0)$ theory we are considering in our class. This is reminiscent of what happens for compactifications of the $(2,0)$ theory; see e.g. [6, eq. (2.8)], or [7, eq. (2.22)].

We have not computed $\mathcal{F}_{0,6}$ in full generality for the $(1,0)$ theories. This would now be possible in principle, since the analytic expressions are now known. One first example is the solution in section 5.5. The corresponding brane configuration according to the identification in [13] consists in k D6's ending on $N = \frac{k}{n_0}$ NS5-branes; see figure 1a. We get

$$\mathcal{F}_{0,6} = \frac{512}{45} k^2 \pi^4 N^3 \,, \tag{5.69}$$

which reassuringly goes like N^3. (By way of comparison, for the massless case one gets $\mathcal{F}_{0,6} = \frac{128}{3} k^2 \pi^4 N^3$.)

We also computed $\mathcal{F}_{0,6}$ for the solution (5.64), which has two D8's and a massless region between them. The corresponding brane configuration would be N NS5-branes in the middle with $k = \mu n_0$ D6's sticking out of them, ending on n_0 D8-branes both on the
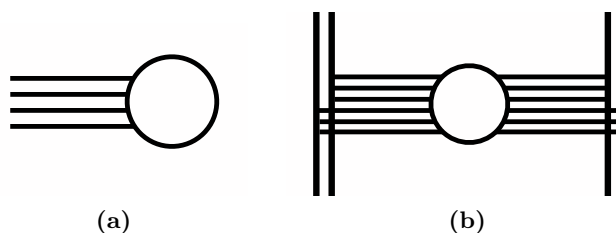
Figure 1. Brane configurations for two sample theories. The circles represent stacks of N NS5-branes; the horizontal lines represent D6-branes; the vertical lines represent D8-branes. In the second case, on each side we have $n_0 = 2$ D8-branes; $|\mu| = 3$ D6-branes end on each, for a total of $k = n_0|\mu| = 6$.

left and on the right; see figure 1b. This case was considered in [13, section 5], where approximate expressions for $\mathcal{F}_{0,6}$ were computed, using perturbation theory around the massless limit. Using (5.64) we can now obtain the exact result:

$$\mathcal{F}_{0,6} = \frac{128}{3}k^2\pi^4\left(N^3 - 4N\mu^2 + \frac{16}{5}\mu^3\right). \tag{5.70}$$

This agrees with [13, section 5], but is now exact. Recall that $\mu = \frac{k}{n_0}$; since this number can be large, the second and third term are also large, and are not competing with stringy corrections. Using (5.67), and comparing with the $(2,0)$ theory to fix the proportionality factors, we get that for the CFT$_4$ theory $a = c = \frac{1}{3}(g-1)\left(N^3 - 4N\mu^2 + \frac{16}{5}\mu^3\right)$. Stringy corrections will modify this result with terms linear in N and probably in μ.

6 Conclusions

We have classified supersymmetric AdS$_5 \times M_5$ solutions of massive type IIA supergravity, and we have found a large class of new analytic solutions.

The general classification, obtained in section 3, is summarized in section 3.4. We reduced the supersymmetry equations to six PDE's. A solution to this system completely determines the bosonic fields — metric, dilaton, and fluxes. The geometry of M_5 is given by a fibration of a three-dimensional manifold M_3 over a two-dimensional space $\mathcal{C}$.

We found an Ansatz that makes the PDE system solvable. As described in section 4, it consists in relating the metric on $\mathcal{C}$ to the warping function A. We recover in this way several known massless solutions: the BBBW [7, 26], Maldacena-Núñez [5], and INST [27] solutions. More interestingly, we find new analytic solutions. One, described in section 4.3.1, is unfortunately non-compact. Another class, however, is compact and globally well-defined.

This new class, analyzed in section 5, consists of infinitely many new solutions, which preserve eight supercharges in five dimensions and are in one-to-one correspondence with the AdS$_7 \times S^3$ type IIA backgrounds classified in [8]. We have explicitly described the map between the former and the latter. The geometry of the fibre M_3 inside M_5 is a certain modification of the "distorted" S^3 of the AdS$_7$ compactifications, whereas the base $\mathcal{C}$ is a Riemann surface with constant negative curvature and genus $g > 1$. An S^2 inside

M_3 is twisted over $\mathcal{C}$, breaking the SU(2)-isometry of $M_3 \cong S^3$ to U(1); this bears out the field-theoretic expectation of having a U(1) R-symmetry for the dual four-dimensional $\mathcal{N} = 1$ SCFT.

Importantly, we have been able to find analytic expressions for all these AdS$_5$ solutions. Then, by means of the aforementioned one-to-one correspondence, we have obtained analytic versions of all the AdS$_7$ solutions in [8] (which were previously known only numerically). Thanks to the analytic expressions for the fields on the gravity side, we have computed explicitly the free energy for some examples of four-dimensional $\mathcal{N} = 1$, and six-dimensional $\mathcal{N} = (1, 0)$ SCFT's at large N (using the AdS/CFT dictionary). It would be very interesting to find a field theory description of these theories, perhaps along the lines of [30].

Acknowledgments

A Supersymmetry variations and the Killing vector

Setting to zero the type IIA supersymmetry variations (of gravitinos and dilatinos) yields the following set of equations[9]

$$0 = \left(\nabla_M + \frac{1}{4} H_M \right) \epsilon_1 + \frac{e^\phi}{16} \lambda(F) \Gamma_M \epsilon_2 \,, \tag{A.1a}$$

$$0 = \left(\nabla_M - \frac{1}{4} H_M \right) \epsilon_2 + \frac{e^\phi}{16} F \, \Gamma_M \epsilon_1 \,, \tag{A.1b}$$

$$0 = \left(\nabla - \partial\phi + \frac{1}{4} H \right) \epsilon_1 \,, \tag{A.1c}$$

$$0 = \left(\nabla - \partial\phi - \frac{1}{4} H \right) \epsilon_1 \,, \tag{A.1d}$$

where suppressed indices are contracted with antisymmetric products of gamma matrices and ϵ_1, ϵ_2 are Spin(1, 9) Majorana-Weyl spinors of opposite chirality.

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[9]The first two equations follow from setting the gravitino variation $\delta\psi_M$ to zero, while the last two equations follow from $\Gamma^M \delta\psi_M - \delta\lambda = 0$ where λ is the dilatino.

We wish to obtain a set of differential and algebraic equations for the Spin(5) spinors η_1, η_2 and so we decompose the the generators of Cliff(1, 9) as

$$\Gamma_\mu = e^A \gamma_\mu^{(1,4)} \otimes 1 \otimes \sigma_3 \qquad \Gamma_i = 1 \otimes \gamma_m^{(5)} \otimes \sigma_1 \,, \tag{A.2}$$

where $\mu = 0, \ldots, 4$, $m = 1, \ldots, 5$ and σ_1 and σ_3 are the Pauli matrices; $\gamma_\mu^{(1,4)}$ generate Cliff(1, 4) and $\gamma_m^{(5)}$ Cliff(5). Accordingly, the chirality matrix Γ_{11} and the intertwiner B_{10} relating Γ_M and Γ_M^*, are decomposed as

$$\Gamma_{11} = 1 \otimes 1 \otimes \sigma_2 \,, \qquad B_{10} = B_{1,4} \otimes B_5 \otimes \sigma_1 \,. \tag{A.3}$$

Furthermore, the supersymmetry parameters ϵ_1, ϵ_2 split as

$$\epsilon_1 = (\zeta \otimes \eta_1 + \zeta^c \otimes \eta_1^c) \otimes \theta \,, \tag{A.4a}$$

$$\epsilon_2 = (\zeta \otimes \eta_2 + \zeta^c \otimes \eta_2^c) \otimes \theta^* \,, \tag{A.4b}$$

where $\eta_{1,2}^c = B_5 \eta_{1,2}^*$ and $\zeta^c = B_{1,4} \zeta^*$. ζ is a Spin(1, 4) spinor obeying the AdS$_5$ Killing spinor equation

$$\nabla_\mu \zeta = \frac{1}{2} \gamma_\mu \zeta \,, \tag{A.5}$$

while θ obeys $\sigma_2 \theta = \theta$ and $\sigma_1 \theta = \theta^*$.

Applying the above decomposition, the equations (A.1) become

$$0 = \left(\nabla_i + \frac{1}{4} H_i \right) \eta_1 + \frac{e^\phi}{16} \lambda(F) \gamma_m^{(5)} \eta_2 \,, \tag{A.6a}$$

$$0 = \left(\nabla_i - \frac{1}{4} H_i \right) \eta_2 + \frac{e^\phi}{16} F \gamma_m^{(5)} \eta_1 \,, \tag{A.6b}$$

$$0 = \left(\frac{i}{2} e^{-A} - \frac{1}{2} \partial A \right) \eta_1 - \frac{e^\phi}{16} \lambda(F) \eta_2 \,, \tag{A.6c}$$

$$0 = \left(\frac{i}{2} e^{-A} + \frac{1}{2} \partial A \right) \eta_2 + \frac{e^\phi}{16} F \eta_1 \,, \tag{A.6d}$$

$$0 = \left(\frac{5i}{2} e^{-A} - \nabla - \frac{5}{2} \partial A + \partial \phi - \frac{1}{4} H \right) \eta_1 \,, \tag{A.6e}$$

$$0 = \left(\frac{5i}{2} e^{-A} + \nabla + \frac{5}{2} \partial A - \partial \phi - \frac{1}{4} H \right) \eta_2 \,. \tag{A.6f}$$

Using equations (A.6a) and (A.6b) it is straightforward to show that $\xi \equiv \frac{1}{2} (\eta_1^\dagger \gamma^m \eta_2 - \eta_2^\dagger \gamma^m \eta_2) \partial_m$ satisfies

$$\nabla_{(m} \xi_{n)} = 0 \,, \tag{A.7}$$

i.e. that ξ is a Killing vector, while equations (A.6c) and (A.6d) yield $\mathcal{L}_\xi A = 0$. That $\mathcal{L}_\xi \phi = 0$ follows from the algebraic equations obtained from (A.6e) and (A.6f) afer eliminating ∇, using (A.6a) and (A.6b).[10]

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[10]These conditions also follow directly from setting the dilatino variation $\delta\lambda$ to zero.

B   A simple Ansatz

We assume that ϕ and A are functions of y only and that $g_{\mathcal{C}}$ is independent of x i.e. $f_1 = 0$. From equation (3.17b) it follows that $a_2 = 0$. The metric becomes

$$ds_{M_5}^2 = ds_{\mathcal{C}}^2 + \frac{1}{9}e^{2A}b^2 D\psi^2 + e^{-8A+2\phi}dx^2 + \frac{e^{-4A+2\phi}}{b^2}dy^2\,, \tag{B.1}$$

where now $b^2 = 1 - a_1^2$. Equation (3.18) is satisfied trivially while equations (3.19a) and (3.19b) yield $\partial_x\rho = \partial_y\rho = 0$ (in the present Ansatz $\rho_0 = \rho$). $A(y)$ and $\phi(y)$ are subject to the differential equations coming from the Bianchi identities of F_0 and F_2, and equation (3.25b),

$$\partial_y\ell + 2f_2\ell = 0\,. \tag{B.2}$$

ℓ is determined by (3.24) and (3.31) to be $\ell = 6e^{-2A} + 12e^{-2A}y(\partial_y A - f_2)$.

We first look at the Bianchi identity of F_2; it yields:

$$F_0(\partial_y A - f_2) = 0\,, \tag{B.3}$$

so either $F_0 = 0$ or $f_2 = \partial_y A$. We consider the two cases $F_0 = 0$ and $F_0 \neq 0$ separately.

B.1 $F_0 = 0$

In this case, from the expression (3.28a) for F_0 we conclude that $\partial_y A = 0$ i.e. A is constant which without loss of generality we set to zero. F_2 is zero, as can be seen from its expression (3.28c). We thus need to solve equation (B.2). This yields the ODE

$$\partial_y f_2 + 2f_2^2 = 0\,, \tag{B.4}$$

which is solved by

$$f_2 = \frac{1}{2}\frac{c}{cy - k}\,, \qquad c,\, k = \text{const.}. \tag{B.5}$$

Recalling the definition (3.17c) of f_2, equation (B.5) is in turn solved for

$$e^{2\phi} = \frac{k - cy}{2(c_1 - ky^2)}\,, \qquad c_1 = \text{const.}. \tag{B.6}$$

Equations (3.16) are then solved by

$$u = e^{i\psi}\sqrt{2(k - cy)}\,\widehat{u}(x_1, x_2)\,. \tag{B.7}$$

Substituting (B.5) and (B.7) in (3.31) yields

$$d_2\rho = 12k\,\text{vol}_\Sigma\,, \tag{B.8}$$

where Σ is the surface spanned by $\widehat{u}$. Its Gaussian curvature is thus $12k$.

This solution was first discovered by Gauntlett, Martelli, Sparks and Waldram [3] (see appendix C.1), and it is the T-dual of the $AdS_5 \times Y^{p,q}$ solution in type IIB supergravity.

B.2 $F_0 \neq 0$

In this case $f_2 = \partial_y A$; equations (3.16) are solved by

$$u = e^{i\psi} e^A \, \widehat{u}(x_1, x_2) \,. \tag{B.9}$$

$\ell = 6e^{-2A}$ obeys equation (B.2) automatically and

$$d_2 \rho = 6\mathrm{vol}_\Sigma \,. \tag{B.10}$$

Substituting f_2 in (3.17b) gives

$$e^{4A} = -\frac{1}{12y} \partial_y \beta \,, \tag{B.11}$$

where $\beta(y) \equiv e^{10A-2\phi} b^2$. The Bianchi identity of F_0 becomes then an ODE for ϵ:

$$e^{12A} F_0 = -\beta \, \partial_y e^{4A} \,. \tag{B.12}$$

The situation appears promising: we have reduced the problem to the ODE (B.12). However, as we will now see, one cannot obtain physical compact solutions to this system.

Let us introduce the coordinate $\tilde{y}$ by $d\tilde{y} = \frac{e^{-2A+\phi}}{b} dy$, so that the metric (B.1) contains $d\tilde{y}^2$. (B.11) now reads

$$F_0 = 16e^{-\phi} b^2 \partial_{\tilde{y}} A \,. \tag{B.13}$$

In order to obtain a compact solution, we should have the factor in front of the S^1 in (B.1), namely $e^A b$, go to zero for some $y = y_0$. For a regular point, this is impossible: since A and ϕ should go to constant at y_0, we should have b go to zero; but from (B.13) we see that this is in contradiction with $F_0 \neq 0$. We might think of having a singularity corresponding to a brane, but since only an S^1 would shrink at $y = y_0$, such a brane would be codimension-2; there are no such objects in IIA supergravity.

C   Recovered solutions

In this appendix we discuss a set of known, supersymmetric $\mathrm{AdS}_5 \times M_5$ solutions of type IIA supergravity with zero Romans mass, which we recovered in our analysis. Two of them descend from AdS_5 solutions of M-theory, whose reduction to ten dimensions we present. We focus on the geometry of the solutions, as the fluxes are determined by it. We aim to adhere to the notation of the original papers; whenever there is overlap with notation used in the main body of the paper, we add a hatˆ.

There are more supersymmetric AdS_5 solutions in IIA [29, 34–36] that should be particular cases of our general classification of section 3. These are outside the compactification Ansatz of section 4.

C.1 The Gauntlett-Martelli-Sparks-Waldram (GMSW) solution

The metric on M_5 reads

$$ds^2_{M_5} = \frac{k - cy}{6m^2} ds^2_{C_k} + e^{-6\lambda} \sec^2 \zeta + \frac{1}{9m^2} \cos^2 \zeta D\psi^2 + e^{-6\lambda} dx_3^2 \,, \qquad (C.1)$$

where

$$e^{6\lambda} = \frac{2m^2(\hat{a} - ky^2)}{k - cy}\,, \qquad \cos^2 \zeta = \frac{\hat{a} - 3ky^2 + 2cy^2}{\hat{a} - ky^2} \,. \qquad (C.2)$$

The dilaton is given by $e^{-2\phi} = e^{6\lambda}$.

$\hat{a}$, c are constants, $k = 0, \pm 1$ and m^{-1} is the radius of AdS$_5$. C_k is a Riemann surface of unit radius; it is a sphere S^2, a torus T^2 or a hyperbolic space H^2 for $k = 1, 0$ or -1 respectively. The GMSW solution is the reduction to ten dimensions of an AdS$_5 \times M_6$ solution of M-theory, where M_6 is a fibration of S^2 over $C_k \times T^2$ and the reduction is along an $S^1 \in T^2$.

The solution is the one recovered in subsection B.1. The constants c and k are identified with the corresponding of B.1, while $\hat{a} = c_1$. The coordinate x_3 is related to x via $x_3 = -x$; a minus is introduced for matching the expressions of the fluxes. Finally, in B.1 $m = 1$.

C.2 The Itsios-Núñez-Sfetsos-Thompson (INST) solution

The INST solution [27] was discovered by nonabelian T-dualizing the AdS$_5 \times T^{1,1}$ solution in type IIB supergravity. The metric on M_5 reads

$$ds^2_{M_5} = \lambda_1^2 ds^2_{S^2} + \frac{\lambda_2^2 \lambda^2}{\Delta} x_1^2 D\psi^2 + \frac{1}{\Delta}\left[(x_1^2 + \lambda^2 \lambda_2^2)dx_1^2 + (x_2^2 + \lambda_2^4)dx_2^2 + 2x_1 x_2 dx_1 dx_2\right], \;\; (C.3)$$

where

$$\Delta = \lambda_2^2 x_1^2 + \lambda^2(x_2^2 + \lambda_2^4)\,, \qquad \lambda_1^2 = \lambda_2^2 = \frac{1}{6}\,, \qquad \lambda^2 = \frac{1}{9}\,, \qquad (C.4)$$

and

$$ds^2_{S^2} = d\theta_1^2 + \sin^2 \theta_1 \phi_1^2\,, \qquad \rho = \cos \theta_1 d\phi_1 \,. \qquad (C.5)$$

The dilaton is given by $e^{-2\phi} = \Delta$.

The INST solution fits into the $c_1 = 0$ branch of the first case of subsection 4.2 for $c_3 = -12$ (achieved by setting the constant warp factor to zero) and $\epsilon = c_2 = \lambda \lambda_2^2$. Σ_g is S^2 of radius $\frac{1}{\sqrt{6}}$. The coordinate transformation relating x_1, x_2 to x, y is:

$$x_1^2 = -36y^2 + 36\epsilon x + 6c_4 - 6\epsilon^2\,, \qquad x_2 = 6y \,. \qquad (C.6)$$

C.3 The Maldacena-Núñez solution

We write the metric of the $\mathcal{N} = 1$ Maldacena-Núñez solution [5] in the form presented in [3]:

$$e^{-2\lambda} ds^2_{11} = ds^2_{\text{AdS}_5} + \frac{1}{3} ds^2_{H^2} + e^{-6\lambda} \sec^2 \zeta dy^2 + \frac{1}{9m^2} \cos^2 \zeta \left((d\psi + \tilde{P})^2 + ds^2_{S^2}\right), \quad (C.7)$$

where

$$e^{6\lambda} = \hat{a} + y^2\,, \qquad \cos^2 \zeta = \frac{\hat{a} - 3y^2}{\hat{a} + y^2} \,, \qquad (C.8)$$

and m^{-1} is the radius of AdS$_5$. The metrics on H^2 and S^2 are

$$ds^2_{H^2} = \frac{dX^2 + dY^2}{Y^2}\,, \qquad ds^2_{S^2} = d\theta^2 + \cos^2\theta d\nu^2\,, \tag{C.9}$$

while the connection of the fibration of ψ is

$$\tilde{P} = -\cos\theta d\nu - \frac{dX}{Y}\,. \tag{C.10}$$

C.3.1 Reduction to ten dimensions

We reduce the Maldacena-Núñez solution to ten dimensions, along ν. In order to do so, we rewrite the part of $ds^2_{M_6}$ involving $d\psi$ or $d\nu$ as

$$\frac{1}{9m^2}\cos^2\zeta \left[(d\nu + A_1)^2 + \sin^2\theta D\psi^2\right]\,, \tag{C.11}$$

where

$$A_1 = -\cos\theta D\psi\,, \qquad \rho = -\frac{dX}{Y}\,. \tag{C.12}$$

Reducing along $d\nu$ yields then

$$ds^2_{10} = e^{2A}ds^2_{\mathrm{AdS}_5} + ds^2_{M_5}\,, \tag{C.13}$$

where

$$e^{-2A}ds^2_{M_5} = \frac{1}{3}ds^2_{H^2} + e^{-6A+2\phi}\sec^2\zeta dy^2 + \frac{1}{9m^2}\cos^2\zeta \left(d\theta^2 + \sin^2\theta D\psi^2\right)\,. \tag{C.14}$$

Furthermore,

$$\phi = \frac{3}{4}\log\left(\frac{1}{9m^2}e^{2\lambda}\cos^2\zeta\right)\,, \qquad A = \lambda + \frac{1}{3}\phi\,. \tag{C.15}$$

The reduced Maldacena-Núñez solution fits into the second case of section 4.2, for $\epsilon = 0$ (achieved by by a $x \to x + \frac{\epsilon}{3}$ shift), $c_1 = -\frac{\hat{a}}{12}$ and $c_2 = 1$. Σ_g is H^2 of radius $\frac{1}{\sqrt{3}}$. In our conventions $m = 1$. The coordinate transformation relating x to y, θ is:

$$x = -\frac{1}{9}(\hat{a} - 3y^2)\cos\theta\,. \tag{C.16}$$

C.3.2 AdS$_7$ variables

For our discussion in the main text, it is useful to also include two parameters R and k which are usually set to one. If we use the slightly awkward-looking

$$\beta = \frac{4}{k^2}\left(y^2 - \frac{3^4}{2^{10}}R^6\right)^2 \tag{C.17}$$

the corresponding solution, using (5.4) and (5.7), is

$$ds^2_{M_5} = e^{2A}ds^2_{\sigma_g} + \frac{1}{3^{3/2}k}\frac{64dy^2}{\sqrt{9^2R^6 - 32^2y^2}} + \frac{(9^2R^6 - 32^2y^2)^{3/2}}{16(3^5R^6 + 32^2y^2)}\,, \tag{C.18}$$

$$e^{4A} = \frac{9^2R^6 - 32^2y^2}{3 \cdot 2^8 k^2}\,, \qquad e^{4\phi} = \frac{(9^2R^6 - 32^2y^2)^3}{2 \cdot 6^3 k^6(3^5R^6 + 32^2y^2)^2}\,. \tag{C.19}$$

These again look messy, but upon using the map (5.16) and defining an angle α via

$$\cos\alpha \equiv \frac{32}{9R^3}y \tag{C.20}$$

turn into the expressions for the metric, A and ϕ of the massless AdS$_7$ solution, obtained by reducing AdS$_7 \times S^4/\mathbb{Z}_k$ to IIA supergravity: see [8, section 5.1].

In the main text we will need an expression for the B field of the AdS$_5$ solution. We give it directly in terms of x_7, which is related to (5.14) via (5.16):

$$B = \frac{R^3}{48k}x_7\frac{(5-x_7^2)}{1+\frac{1}{3}x_7^2}\mathrm{vol}_{S^2} + \frac{1}{\sqrt{3}}\frac{x_7}{\sqrt{1-x_7^2}}\cos\theta\mathrm{vol}_{\Sigma_g}\,. \tag{C.21}$$

This is similar to the one given for the AdS$_7$ solution in [8, eq. (5.8)].

C.4 The Bah-Beem-Bobev-Wecht (BBBW) solution

The metric of the BBBW [7] solution is

$$ds_{11}^2 = e^{2\lambda}\left[ds_{\mathrm{AdS}_5}^2 + e^{2\nu+2\hat{A}(x_1,x_1)}\left(dx_1^2 + dx_2^2\right)\right] + e^{-4\lambda}ds_{M_4}^2\,, \tag{C.22}$$

where $ds_{\mathrm{AdS}_5}^2$ is the unit radius metric on AdS$_5$, and $\hat{A}(x_1,x_2)$ is the conformal factor of the constant curvature metric on the Riemann surface $\widehat{\Sigma}_g$ of genus g, obeying

$$(\partial_{x_1}^2 + \partial_{x_2}^2)\hat{A} + \kappa e^{2\hat{A}} = 0\,. \tag{C.23}$$

The constant κ is the Gaussian curvature of the Riemann surface which is set to 1, 0 or -1 for the sphere S^2, the torus T^2 or a hyperbolic surface respectively. ν is a real constant. The metric $ds_{M_4}^2$ is

$$ds_{M_4}^2 = \left(1 + \frac{4y^2}{qf}\right)dy^2 + \frac{qf}{k}\left(dq + \frac{12yk}{qf}dy\right)^2 + \frac{\hat{a}_1^2}{4}\frac{fk}{q}(d\chi + V)^2 + \frac{qf}{9}(d\psi + \hat{\rho})^2\,. \tag{C.24}$$

The metric functions are

$$e^{6\lambda} = qf + 4y^2\,, \qquad f(y) \equiv 1 + 6\frac{\hat{a}_2}{\hat{a}_1}y^2\,, \qquad k(q) \equiv \frac{\hat{a}_2}{\hat{a}_1}q^2 + q - \frac{1}{36}\,, \tag{C.25}$$

while the one-forms which determine the fibration of the ψ and χ directions are given by

$$\hat{\rho} = (2-2g)V - \frac{1}{2}\left(\hat{a}_2 + \frac{\hat{a}_1}{2q}\right)(d\chi + V)\,, \qquad dV = \frac{\kappa}{2-2g}e^{2\hat{A}}dx_1 \wedge dx_2\,. \tag{C.26}$$

The constants $\hat{a}_1$, $\hat{a}_2$ are fixed as

$$\hat{a}_1 \equiv \frac{2(2-2g)e^{2\nu}}{\kappa}\,, \qquad \hat{a}_2 \equiv 2(2-2g)\left(1 - \frac{6e^{2\nu}}{\kappa}\right)\,. \tag{C.27}$$

C.4.1 Reduction to ten dimensions

We reduce the BBBW solution to ten dimensions, along χ. In order to do so, we rewrite the part of $ds^2_{M_4}$ involving $d\psi$ or $d\chi$ as

$$h_1^2(d\chi + A_1)^2 + h_2^2 D\psi^2 \,, \tag{C.28}$$

where

$$h_1^2(y, q) \equiv \frac{\hat{a}_1^2}{4}\frac{fk}{q} + \frac{1}{4}\frac{qf}{9}\left(\hat{a}_2 + \frac{\hat{a}_1}{2q}\right)^2 \,, \tag{C.29a}$$

$$h_2^2(y, q) \equiv \frac{qf}{9} - \frac{1}{4}\left(\frac{qf}{9}\right)^2\left(\hat{a}_2 + \frac{\hat{a}_1}{2q}\right)^2 \,, \tag{C.29b}$$

and

$$\rho = (2 - 2g)V \,, \qquad A_1 = V - \frac{qf}{9}\frac{1}{2}\left(\hat{a}_2 + \frac{\hat{a}_1}{2q}\right)h_1^{-1}D\psi \,. \tag{C.30}$$

Reducing along $d\chi$ yields then

$$ds^2_{10} = e^{2A}\left[ds^2_{\mathrm{AdS}_5} + e^{2\nu + 2\hat{A}}(dx_1^2 + dx_2^2)\right] + ds^2_{M_3} \,, \tag{C.31}$$

where

$$e^{4A - 2\phi}ds^2_{M_3} = \left(1 + \frac{4y^2}{qf}\right)dy^2 + \frac{qf}{k}\left(dq + \frac{12yk}{qf}dy\right)^2 + h_2^2 D\psi^2 \,. \tag{C.32}$$

Furthermore,

$$\phi = \frac{3}{2}(\log h_1 - 2\lambda) \,, \qquad A = \frac{1}{2}\log h_1 \,. \tag{C.33}$$

The reduced BBBW solution fits into the generic branch of the first case of subsection 4.2 for $c_1 = \frac{9\hat{a}_1 + \hat{a}_2}{108}$, $c_2 = \frac{(9\hat{a}_1 + \hat{a}_2)\hat{a}_2}{18\hat{a}_1}$ and $c = \frac{\hat{a}_2}{3\hat{a}_1}$. The coordinate transformation relating x to y, q is:

$$x = -\frac{\hat{a}_1(18\hat{a}_1 + \hat{a}_2 + 18\hat{a}_2 q)}{36\hat{a}_2}\left(1 + 6\frac{\hat{a}_2}{\hat{a}_1}y^2\right) \,. \tag{C.34}$$

Certain generalizations of the BBBW class of solutions have also appeared [37, 38]. It would be interesting to reduce these to solutions of IIA supergravity and verify that they fit in our classification of section 3.

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