Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2008, Article ID 692713, 15 pages
doi:10.1155/2008/692713

Research Article

Almost Periodic Solutions of Nonlinear Discrete Volterra Equations with Unbounded Delay

Sung Kyu Choi and Namjip Koo

Department of Mathematics, Chungnam National University, Daejeon 305-764, South Korea

Correspondence should be addressed to Namjip Koo, njkoo@math.cnu.ac.kr

Received 30 June 2008; Revised 18 September 2008; Accepted 14 October 2008

Recommended by Mariella Cecchi

We study the existence of almost periodic solutions for nonlinear discrete Volterra equations with unbounded delay, as a discrete analogue of the results for integro-differential equations by Y. Hamaya (1993).

1. Introduction

Hamaya [1] discussed the relationship between stability under disturbances from hull and total stability for the integro-differential equation

$$x'(t) = f(t, x(t)) + \int_{-\infty}^{0} F(t, s, x(t+s), x(t))\,ds, \tag{1.1}$$

where $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and is almost periodic in t uniformly for $x \in \mathbb{R}^n$, and $F : \mathbb{R} \times (-\infty, 0] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and is almost periodic in t uniformly for $(s, x, y) \in (-\infty, 0] \times \mathbb{R}^n \times \mathbb{R}^n$. He showed that for a periodic integro-differential equation, uniform stability and stability under disturbances from hull are equivalent. Also, he showed the existence of an almost periodic solution under the assumption of total stability in [2].

Song and Tian [3] studied periodic and almost periodic solutions of discrete Volterra equations with unbounded delay of the form

$$x(n+1) = f(n, x(n)) + \sum_{j=-\infty}^{n} B(n, j, x(j), x(n)), \quad n \in \mathbb{Z}^+, \tag{1.2}$$

where $f : \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in $x \in \mathbb{R}^n$ for every $n \in \mathbb{Z}$, and for any $j, n \in \mathbb{Z}$ $(j \leq n)$, $B : \mathbb{Z} \times \mathbb{Z} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous for $x, y \in \mathbb{R}^n$. They showed that under some suitable conditions, if the bounded solution of (1.2) is totally stable, then it is an asymptotically almost periodic solution of (1.2), and (1.2) has an almost periodic solution. Also, Song [4] proved that if the bounded solution of (1.2) is uniformly asymptotically stable, then (1.2) has an almost periodic solution.

Equation (1.2) is a discrete analogue of the integro-differential equation (1.1), and (1.2) is a summation equation that is a natural analogue of this integro-differential equation. For the asymptotic properties of discrete Volterra equations, see [5].

In this paper, in order to obtain an existence theorem for an almost periodic solution of a discrete Volterra equations with unbounded delay, we will employ to change Hamaya's results in [1] for the integro-differential equation into results for the discrete Volterra equation.

2. Preliminaries

We denote by $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{R}^-$, respectively, the set of real numbers, the set of nonnegative real numbers, and the set of nonpositive real numbers. Let $\mathbb{R}^n$ denote n-dimensional Euclidean space.

Definition 2.1 (see [6]). A continuous function $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is called *almost periodic in $t \in \mathbb{R}$ uniformly for $x \in \mathbb{R}^n$* if for any $\varepsilon > 0$ there corresponds a number $l = l(\varepsilon) > 0$ such that any interval of length l contains a τ for which

$$\left| f(t+\tau, x) - f(t, x) \right| < \varepsilon \tag{2.1}$$

for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

Let $R^* = \mathbb{R}^- \times \mathbb{R}^n \times \mathbb{R}^n$ and let $F(t, s, x, y)$ be a function which is defined and continuous for $t \in \mathbb{R}$ and $(s, x, y) \in R^*$.

Definition 2.2 (see [9]). $F(t, s, x, y)$ is said to be *almost periodic in t uniformly for* $(s, x, y) \in R^*$ if for any $\varepsilon > 0$ and any compact set K^* in R^*, there exists an $L = L(\varepsilon, K^*) > 0$ such that any interval of length L contains a τ for which

$$\left| F(t+\tau, s, x, y) - F(t, s, x, y) \right| \leq \varepsilon \tag{2.2}$$

for all $t \in \mathbb{R}$ and all $(s, x, y) \in K^*$.

We denote by $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Z}^-$, respectively, the set of integers, the set of nonnegative integers, and the set of nonpositive integers.

Definition 2.3 (see [3]). A continuous function $f : \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}^n$ is said to be *almost periodic in $n \in \mathbb{Z}$ uniformly for $x \in \mathbb{R}^n$* if for every $\varepsilon > 0$ and every compact set $K \subset \mathbb{R}^n$, there corresponds an integer $N = N(\varepsilon, K) > 0$ such that among N consecutive integers there is one, here denoted p, such that

$$\left| f(n+p, x) - f(n, x) \right| < \varepsilon \tag{2.3}$$

for all $n \in \mathbb{Z}$, uniformly for $x \in \mathbb{R}^n$.

Definition 2.4 (see [3]). Let $Z^* = \mathbb{Z}^- \times \mathbb{R}^n \times \mathbb{R}^n$. A set $\Sigma \subset Z^*$ is said to be *compact* if there is a finite integer set $\Delta \subset \mathbb{Z}^-$ and compact set $\Theta \subset \mathbb{R}^n \times \mathbb{R}^n$ such that $\Sigma = \Delta \times \Theta$.

Definition 2.5. Let $B : \mathbb{Z} \times \mathbb{Z} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous for $x, y \in \mathbb{R}^n$, for any $j, n \in \mathbb{Z}$, $j \le n$. $B(n, j, x, y)$ is said to be *almost periodic in n uniformly for* $(j, x, y) \in Z^*$ if for any $\varepsilon > 0$ and any compact set $K^* \subset Z^*$, there exists a number $l = l(\varepsilon, K^*) > 0$ such that any discrete interval of length l contains a τ for which

$$\left|B(n+\tau, j, x, y) - B(n, j, x, y)\right| \le \varepsilon \tag{2.4}$$

for all $n \in \mathbb{Z}$ and all $(j, x, y) \in K^*$.

For the basic results of almost periodic functions, see [6–8].

Let $l^-(\mathbb{R}^n)$ denote the space of all $\mathbb{R}^n$-valued bounded functions on $\mathbb{Z}^-$ with

$$\|\phi\|_\infty = \sup_{n \in \mathbb{Z}^-} \left|\phi(n)\right| < \infty \tag{2.5}$$

for any $\phi \in l^-(\mathbb{R}^n)$.

Let $x : \{n \in \mathbb{Z} : n \le k\} \to \mathbb{R}^n$ for any integer k. For any $n \le k$, we define the notation $x_n : \mathbb{Z}^- \to \mathbb{R}^n$ by the relation

$$x_n(j) = x(n+j) \tag{2.6}$$

for $j \le 0$.

Consider the discrete Volterra equation with unbounded delay

$$\begin{aligned} x(n+1) &= f\left(n, x(n)\right) + \sum_{j=-\infty}^{n} B\left(n, j, x(j), x(n)\right), \quad n \in \mathbb{Z}^+, \\ &= f\left(n, x(n)\right) + \sum_{j=-\infty}^{0} B\left(n, n+j, x(n+j), x(n)\right), \end{aligned} \tag{2.7}$$

where $f : \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in $x \in \mathbb{R}^n$ for every $n \in \mathbb{Z}$ and is almost periodic in $n \in \mathbb{Z}$ uniformly for $x \in \mathbb{R}^n$, $B : \mathbb{Z} \times \mathbb{Z} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in $x, y \in \mathbb{R}^n$ for any $j \le n \in \mathbb{Z}$ and is almost periodic in n uniformly for $(j, x, y) \in Z^*$. We assume that, given $\phi \in l^-(\mathbb{R}^n)$, there is a solution x of (2.7) such that $x(n) = \phi(n)$ for $n \in \mathbb{Z}^-$, passing through $(0, \phi)$. Denote by this solution $x(n) = x(n, \phi)$.

Let K be any compact subset of $\mathbb{R}^n$ such that $\phi(j) \in K$ for all $j \le 0$ and $x(n) = x(n, \phi) \in K$ for all $n \ge 1$.

For any $\phi, \psi \in l^-(\mathbb{R}^n)$, we set

$$\rho(\phi, \psi) = \sum_{q=0}^{\infty} \frac{\rho_q(\phi, \psi)}{2^q\left[1 + \rho_q(\phi, \psi)\right]}, \tag{2.8}$$

where $\rho_q(\phi,\psi) = \max_{-q\le m\le 0}|\phi(m)-\psi(m)|,\ q\ge 0$. Then, ρ defines a metric on the space $l^-(\mathbb{R}^n)$. Note that the induced topology by ρ is the same as the topology of convergence on any finite subset of $\mathbb{Z}^-$ [3].

In view of almost periodicity, for any sequence $(n'_k)\subset\mathbb{Z}^+$ with $n'_k\to\infty$ as $k\to\infty$, there exists a subsequence $(n_k)\subset(n'_k)$ such that

$$f(n+n_k,x)\longrightarrow g(n,x) \tag{2.9}$$

uniformly on $\mathbb{Z}\times S$ for any compact set $S\subset\mathbb{R}^n$,

$$B(n+n_k,n+l+n_k,x,y)\longrightarrow D(n,n+l,x,y) \tag{2.10}$$

uniformly on $\mathbb{Z}\times S^*$ for any compact set $S^*\subset Z^*$, $g(n,x)$ and $D(n,n+l,x,y)$ are also almost periodic in n uniformly for $x\in\mathbb{R}^n$, and almost periodic in n uniformly for $(j,x,y)\in Z^*$, respectively. We define the *hull*

$$\begin{aligned} &H(f,B) \\ &\quad=\left\{(g,D):(2.9)\text{ and }(2.10)\text{ hold for some sequence }(n_k)\subset\mathbb{Z}^+\text{with }n_k\to\infty\text{ as }k\to\infty\right\}. \end{aligned} \tag{2.11}$$

Note that $(f,B)\in H(f,B)$ and for any $(g,D)\in H(f,B)$, we can assume the almost periodicity of g and D, respectively [3].

Definition 2.6 (see [3]). If $(g,D)\in H(f,B)$, then the equation

$$x(n+1)=g(n,x(n))+\sum_{j=-\infty}^{n}D(n,j,x(j),x(n)),\quad n\in\mathbb{Z}^+ \tag{2.12}$$

is called the *limiting equation* of (2.7).

For the compact set K in $\mathbb{R}^n$, $(p,P)\in H(f,B)$, $(q,Q)\in H(f,B)$, we define $\pi(p,q)$ and $\pi(P,Q)$ by

$$\begin{aligned} \pi(p,q) &= \sup\left\{|p(n,x)-q(n,x)| : n\in\mathbb{Z},\ x\in K\right\}, \\ \pi(P,Q) &= \sum_{N=1}^{\infty}\frac{\pi_N(P,Q)}{2^N[1+\pi_N(P,Q)]}, \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} \pi_N(P,Q) &= \sup\left\{|P(n,j,x,y)-Q(n,j,x,y)| : n\in\mathbb{Z},\ j\in[-N,0],\ x,y\in K\right\}, \\ \pi((p,P),(q,Q)) &= \max\left\{\pi(p,q),\pi(P,Q)\right\}, \end{aligned} \tag{2.14}$$

respectively. This definition is a discrete analogue of Hamaya's definition in [1].

3. Main results

Definition 3.1 (see [3]). A function $\phi : \mathbb{Z} \to \mathbb{R}^n$ is called *asymptotically almost periodic* if it is a sum of an almost periodic function ϕ_1 and a function ϕ_2 defined on $\mathbb{Z}$ which tends to zero as $n \to \infty$, that is $\phi(n) = \phi_1(n) + \phi_2(n),\ n \in \mathbb{Z}$.

It is known [8] that the decomposition $\phi = \phi_1 + \phi_2$ in Definition 3.1 is unique, and ϕ is asymptotically almost periodic if and only if for any integer sequence (τ_k') with $\tau_k' \to \infty$ as $k \to \infty$, there exists a subsequence $(\tau_k) \subset (\tau_k')$ for which $\phi(n + \tau_k)$ converges uniformly for $n \in \mathbb{Z}$ as $k \to \infty$.

Hamaya [9] proved that if the bounded solution $x(t)$ of the integro-differential equation (1.1) is asymptotically almost periodic, then $x(t)$ is almost periodic under the following assumption:

(H) for any $\varepsilon > 0$ and any compact set $C \subset \mathbb{R}^n$, there exists $S = S(\varepsilon, C) > 0$ such that

$$\int_{-\infty}^{-S} \left| F(t, s, x(t+s), x(t)) \right| ds \le \varepsilon, \quad t \in \mathbb{R}, \tag{3.1}$$

whenever $x(\sigma)$ is continuous and $x(\sigma) \in C$ for all $\sigma \le t$.

Also, Islam [10] showed that asymptotic almost periodicity implies almost periodicity for the bounded solution of the almost periodic integral equation

$$x(t) = f(t) + \int_{-\infty}^{t} F(t, s, x(s)) ds. \tag{3.2}$$

Throughout this paper, we impose the following assumptions.

(H1) For any $\varepsilon > 0$ and any $\tau > 0$, there exists an integer $M = M(\varepsilon, \tau) > 0$ such that

$$\sum_{j=-\infty}^{n-M} \left| B(n, j, x(j), x(n)) \right| < \varepsilon, \quad n \in \mathbb{Z}, \tag{3.3}$$

whenever $|x(j)| < \tau$ for all $j \le n$.

(H2) Equation (2.7) has a bounded solution $x(n) = x(n, \phi)$, that is, $|x(n)| \le c$ for some $c \ge 0$, passing through $(0, \phi)$, where $\phi \in l^-(\mathbb{R}^n)$.

Note that assumption (H1) holds for any $(g, D) \in H(f, B)$. Also, we assume that the compact set K in $\mathbb{R}^n$ satisfies $\varphi(j) \in K$ for all $j \le 0$ and $y(n) = y(n, \varphi) \in K$ for all $n \ge n_0$, where $y(n)$ is any solution of the limiting equation of (2.12) and (2.7) .

Theorem 3.2. *Under assumptions* (*H*1) *and* (*H*2), *if the bounded solution* $x(n)$ *is asymptotically almost periodic, then* (2.7) *has an almost periodic solution.*

Proof. Since $x(n)$ is asymptotically almost periodic, it has the decomposition

$$x(n) = p(n) + q(n), \tag{3.4}$$

where $p(n)$ is almost periodic in n and $q(n)\to 0$ as $n\to\infty$. Let (n_k) be a sequence such that $n_k\to\infty$ as $k\to\infty$, $p(n+n_k)\to p^*(n)$ as $k\to\infty$, and $p^*(n)$ is also almost periodic. We will prove that $p^*(n)$ is a solution of (2.7) for $n\geq 1$.

Note that, by almost periodicity,

$$f(n+n_k,x)\longrightarrow f^*(n,x) \tag{3.5}$$

uniformly on $\mathbb{Z}\times C$, where C is a compact set in $\mathbb{R}^n$, and

$$B(n+n_k,n+j+n_k,x,y)\longrightarrow B^*(n,n+j,x,y) \tag{3.6}$$

uniformly on $\mathbb{Z}\times K^*$, where K^* is a compact subset of $Z^*=\mathbb{Z}^-\times\mathbb{R}^n\times\mathbb{R}^n$.

Let $x^k(n)=x(n+n_k),\ n+n_k\geq 0$. Then, we obtain

$$\begin{aligned} x(n+n_k+1)&=f(n+n_k,x(n+n_k))+\sum_{j=-\infty}^{n+n_k}B(n+n_k,j,x(j),x(n+n_k))\\ &=f(n+n_k,x^k(n))+\sum_{j=-\infty}^{n}B(n+n_k,j+n_k,x^k(j),x^k(n)). \end{aligned} \tag{3.7}$$

This implies that $x^k(n)$ is a solution of

$$x(n+1)=f(n+n_k,x(n))+\sum_{j=-\infty}^{n}B(n+n_k,j+n_k,x(j),x(n)). \tag{3.8}$$

For $n\leq 0,\ \ p^*(n)\in K$ since

$$\begin{aligned} |p(n+n_k)|&\leq|x(n+n_k)|+|q(n+n_k)|\\ &\leq c+|q(n+n_k)|,\quad n+n_k\geq 0. \end{aligned} \tag{3.9}$$

Moreover, for any $n\in\mathbb{Z}$, there exists a $k_0>0$ such that $n+n_k\geq 1$ for all $k\geq k_0$. Thus

$$x^k(n)=x(n+n_k)=p(n+n_k)+q(n+n_k)\longrightarrow p^*(n) \tag{3.10}$$

as $k\to\infty$ whenever $k\geq k_0$. Hence,

$$x^k(n+1)=f(n,x^k(n))+\sum_{j=-\infty}^{n}B(n,j,x^k(j),x^k(n)),\quad k\geq k_0. \tag{3.11}$$

Now, we show that

$$\sum_{j=-\infty}^{n}B(n,j,x^k(j),x^k(n))\longrightarrow\sum_{j=-\infty}^{n}B(n,j,p^*(j),p^*(n)), \tag{3.12}$$

as $k \to \infty$. Note that, for some $c > 0$, $|x^k(n)| \le c$ and $|p^*(n)| \le c$ for all $n \in \mathbb{Z}$ and $k \ge 1$. From (H1), there exists an integer $M > 0$ such that

$$\sum_{j=-\infty}^{n-M} \left|B(n,j,x^k(j),x^k(n))\right| < \varepsilon,$$
$$\sum_{j=-\infty}^{n-M} \left|B(n,j,p^*(j),p^*(n))\right| < \varepsilon \tag{3.13}$$

for any $\varepsilon > 0$. Then, we have

$$\begin{aligned}
&\left|\sum_{j=-\infty}^{n} B(n,j,x^k(j),x^k(n)) - \sum_{j=-\infty}^{n} B(n,j,p^*(j),p^*(n))\right| \\
&\quad \le \sum_{j=-\infty}^{n-M} \left|B(n,j,x^k(j),x^k(n))\right| + \sum_{j=-\infty}^{n-M} \left|B(n,j,p^*(j),p^*(n))\right| \\
&\qquad + \sum_{j=n-M+1}^{n} \left|B(n,j,x^k(j),x^k(n)) - B(n,j,p^*(j),p^*(n))\right| \\
&\quad \le 2\varepsilon + \sum_{j=n-M+1}^{n} \left|B(n,j,x^k(j),x^k(n)) - B(n,j,p^*(j),p^*(n))\right|
\end{aligned} \tag{3.14}$$

by (3.13).

Since $B(n,j,x,y)$ is continuous for $x, y \in \mathbb{R}^n$ and $x^k(n) \to p^*(n)$ on $[n-M, n]$ as $k \to \infty$, we obtain

$$\sum_{j=n-M+1}^{n} \left|B(n,j,x^k(j),x^k(n)) - B(n,j,p^*(j),p^*(n))\right| < \varepsilon. \tag{3.15}$$

It follows from the continuity of $f(n,x)$ that

$$\begin{aligned}
x^k(n+1) &= f(n,x^k(n)) + \sum_{j=-\infty}^{n} B(n,j,x^k(j),x^k(n)) \\
&\longrightarrow p^*(n+1) = f(n,p^*(n)) + \sum_{j=-\infty}^{n} B(n,j,p^*(j),p^*(n)),
\end{aligned} \tag{3.16}$$

as $k \to \infty$. Therefore, $p^*(n)$ is an almost periodic solution of (2.7) for $n \ge 1$. □

Remark 3.3. Recently Song [4] obtained a more general result than that of Theorem 3.2, that is, under the assumption of asymptotic almost periodicity of a bounded solution of (2.7), he showed the existence of an almost periodic solution of the limiting equation (2.12) of (2.7).

Total stability introduced by Malkin [11] in 1944 requires that the solution of $x'(t) = f(t,x)$ is "stable" not only with respect to the small perturbations of the initial conditions, but also with respect to the perturbations, small in a suitable sense, of the right-hand side of the equation [11]. Many results have been obtained concerning total stability [3, 7, 9, 12–15].

Definition 3.4 (see [1]). The bounded solution $x(t)$ of (1.1) is said to be *totally stable* if for any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that if $t_0 \geq 0$, $\rho(x_{t_0}, y_{t_0}) \leq \delta$ and $h(t)$ is any continuous function which satisfies $|h(t)| \leq \delta$ on $[t_0, \infty)$, then

$$\rho(x_t, y_t) < \varepsilon, \quad t \geq t_0, \tag{3.17}$$

where $y(t)$ is a solution of

$$x'(t) = f(t, x(t)) + \int_{-\infty}^{0} F(t, s, x(t+s)), x(t))ds + h(t), \tag{3.18}$$

such that $y_{t_0}(s) \in K$ for all $s \leq 0$. Here, $x_t : \mathbb{R}^- \to \mathbb{R}^n$ is defined by $x_t(s) = x(t+s)$ for any $x : (-\infty, A) \to \mathbb{R}^n$, $-\infty < A \leq \infty$.

Hamaya [1] defined the following stability notion.

Definition 3.5. The bounded solution $x(t)$ of (1.1) is said to be *stable under disturbances* from $H(f, F)$ with respect to K if for any $\varepsilon > 0$, there exists an $\eta = \eta(\varepsilon) > 0$ such that

$$\rho(x_t, y_t) < \varepsilon, \quad t \geq \tau, \tag{3.19}$$

whenever $(g, G) \in H(f, F)$, $\pi((f_\tau, F_\tau), (g, G)) \leq \eta$, and $\rho(x_\tau, y_\tau) \leq \eta$ for some $\tau \geq 0$, where $y(t)$ is a solution through (τ, y_τ) of the limiting equation

$$x'(t) = g(t, x(t)) + \int_{-\infty}^{0} G(t, s, x(t+s)), x(t))ds \tag{3.20}$$

of (1.1) such that $y_\tau(s) \in K$ for all $s \leq 0$.

The concept of stability under disturbances from hull was introduced by Sell [16, 17] for the ordinary differential equation. Hamaya proved that Sell's definition is equivalent to Hamaya's definition in [1]. Also, he showed that total stability implies stability under disturbances from hull in [1, Theorem 1]. To prove the discrete analogue for this result, we list definitions.

Definition 3.6 (see [3]). The bounded solution $x(n)$ of (2.7) is said to be *totally stable* if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that if $n_0 \geq 0$, $\rho(x_{n_0}, y_{n_0}) < \delta$ and $p(n)$ is a sequence such that $|p(n)| < \delta$ for all $n \geq n_0$, then

$$\rho(x_n, y_n) < \varepsilon, \quad n \geq n_0, \tag{3.21}$$

where $y(n)$ is any solution of

$$x(n+1) = f(n, x(n)) + \sum_{j=-\infty}^{n} B(n, j, x(j), x(n)) + p(n) \tag{3.22}$$

such that $y_{n_0}(j) \in K$ for all $j \in \mathbb{Z}^-$.

Definition 3.7. The bounded solution $x(n)$ of (2.7) is said to be *stable under disturbances* from $H(f, B)$ with respect to K if for any $\varepsilon > 0$, there exists an $\eta = \eta(\varepsilon) > 0$ such that if $\pi((f, B), (g, D)) \leq \eta$ and $\rho(x_{n_0}, y_{n_0}) \leq \eta$ for some $n_0 \geq 0$, then

$$\rho(x_n, y_n) < \varepsilon, \quad n \geq n_0, \tag{3.23}$$

where $y(n)$ is any solution of the limiting equation (2.12) of (2.7), which passes through (n_0, y_{n_0}) such that $y_{n_0}(j) \in K$ for all $j \in \mathbb{Z}^-$.

Theorem 3.8. *Under assumptions* $(H1)$ *and* $(H2)$, *if the bounded solution* $x(n)$ *of* (2.7) *is totally stable, then it is stable under disturbances from* $H(f, B)$ *with respect to* K.

Proof. Let $\varepsilon > 0$ be given and let $\delta = \delta(\varepsilon)$ be the number for total stability of $x(n)$. In view of (H1), there exists an $L = L(\delta(\varepsilon)/4, K) > 0$ such that

$$\sum_{j=-\infty}^{-L} \left| B(n, j, x(n+j), x(n)) \right| \leq \frac{\delta}{4} \tag{3.24}$$

whenever $|x(j)| \leq \tau$ for all $j \leq \tau$. Also, since $D \in H(B)$ satisfies (H1), we have

$$\sum_{j=-\infty}^{-L} \left| D(n, j, x(n+j), x(n)) \right| \leq \frac{\delta}{4} \tag{3.25}$$

whenever $|x(j)| \leq \tau$ for all $j \leq n$. We choose $N = N(\varepsilon) > 0$ such that $[-L, 0] \subset [-N, 0]$ and set

$$\eta(\varepsilon) = \max\left\{\delta'(\varepsilon), \frac{\delta(\varepsilon)}{4}\right\}, \qquad \delta' = \frac{\delta/4L}{2^N(1+\delta/4L)}. \tag{3.26}$$

Let $y(n)$ be any solution of the limiting equation (2.12), passing through (n_0, y_τ), $n_0 \geq 0$, such that $y_{n_0}(j) \in K$ for all $j \leq 0$. Note that $y(n) \in K$ for all $n \geq n_0$ by the assumption on K. We suppose that $\pi((f, B), (g, D)) \leq \eta$ and $\rho(x_{n_0}, y_{n_0}) \leq \eta$. We will show that $\rho(x_n, y_n) < \varepsilon$ for all $n \geq n_{n_0}$.

For every $n \geq n_0$, we set

$$p(n) = g(n, y(n)) - f(n, y(n)) + \sum_{j=-\infty}^{0} D(n, j, y(n+j), y(n)) - \sum_{j=-\infty}^{0} B(n, j, y(n+j), y(n)). \tag{3.27}$$

Then, $y(n)$ is a solution of the perturbation

$$x(n+1) = f(n, x(n)) + \sum_{j=-\infty}^{0} B(n, j, x(n+j), x(n)) + p(n) \tag{3.28}$$

such that $y_{n_0}(j) \in K$ for all $j \in \mathbb{Z}^-$. We claim that $|p(n)| \leq \delta$ for all $n \geq n_0$. From

$$\pi((f, B), (g, D)) = \max\{\pi(f, g), \pi(B, D)\} = \max\left\{\delta', \frac{\delta}{4}\right\}, \tag{3.29}$$

we have

$$\pi(f, g) = \sup\{|f(n, x) - g(n, x)| : n \in \mathbb{Z},\ x \in K\} \leq \frac{\delta}{4}. \tag{3.30}$$

Thus

$$|g(n, y(n)) - f(n, y(n))| \leq \frac{\delta}{4}, \tag{3.31}$$

when $y(n) \in K$ for $n \geq n_0$. Since

$$\pi(B, C) = \sum_{N=1}^{\infty} \frac{\pi_N(B, D)}{2^N [1 + \pi_N(B, D)]} \leq \eta = \max\left\{\delta', \frac{\delta}{4}\right\}, \tag{3.32}$$

we obtain

$$\frac{\pi_N(B, D)}{2^N [1 + \pi_N(B, D)]} \leq \delta' = \frac{\delta/4L}{2^N(1 + \delta/4L)}, \tag{3.33}$$

and thus

$$\pi_N(B, D) = \sup\{|B(n, m, x, y) - D(n, m, x, y)| : n \in \mathbb{Z},\ m \in [-N, 0],\ x, y \in K\} \leq \frac{\delta}{4L}. \tag{3.34}$$

This implies that

$$|D(n, m, y(n+m), y(n)) - B(n, m, y(n+m), y(n))| \leq \frac{\delta}{4L}, \tag{3.35}$$

where $m \in [-L, 0] \subset [-N, 0]$, as long as $y(n) \in K$. Therefore, we have

$$\begin{aligned}
&\left|\sum_{m=-\infty}^{0} D(n,m,y(n+m),y(n)) - \sum_{m=-\infty}^{0} B(n,m,y(n+m),y(n))\right| \\
&\quad\le \left|\sum_{m=-\infty}^{-L} D(n,m,y(n+m),y(n)) + \sum_{m=-L}^{0} D(n,m,y(n+m),y(n))\right. \\
&\qquad \left. - \sum_{m=-\infty}^{-L} B(n,m,y(n+m),y(n)) - \sum_{m=-L}^{0} B(n,m,y(n+m),y(n))\right| \\
&\quad\le \sum_{m=-\infty}^{L} \left|D(n,m,y(n+m),y(n))\right| + \sum_{m=-\infty}^{-L} \left|B(n,m,y(n+m),y(n))\right| \\
&\qquad + \sum_{m=-L}^{0} \left|D(n,m,y(n+m),y(n)) - B(n,m,y(n+m),y(n))\right| \\
&\quad\le \frac{\delta}{4} + \frac{\delta}{4} + \frac{\delta}{4L}L \\
&\quad= \frac{3\delta}{4}
\end{aligned} \tag{3.36}$$

as long as $y(n) \in K$. Consequently, we obtain that $|p(n)| \le \delta$ for all $n \ge n_0$. Since $x(n)$ is totally stable, we have

$$\rho(x_n, y_n) < \varepsilon, \quad n \ge n_0. \tag{3.37}$$

This shows that $x(n)$ is stable under disturbances from $H(f, B)$ with respect to K. □

Remark 3.9. Yoshizawa [15, Lemma 5] proved that the total stability of a bounded solution of the functional differential equation $x'(t) = f(t, x_t)$ implies the stability under disturbances from hull. For a similar result for the integro-differential equation (1.1), see [1, Theorem 1].

Yoshizawa showed the existence of asymptotically almost periodic solution by stability under disturbances from hull for the nonlinear differential equation $x'(t) = f(t, x)$ and the functional differential equation $x'(t) = f(t, x_t)$ in [7, Theorem 12.4] and [15, Theorem 5], respectively.

Also, as the discrete case, Zhang and Zheng [18, Theorem 3.2] obtained the similar result for the functional difference equation $x(n+1) = f(n, x_n)$. For the discrete Volterra equation (2.7), we get the following result.

Theorem 3.10. *Under assumptions* $(H1)$ *and* $(H2)$, *if the bounded solution* $x(n)$ *of* (2.7) *is stable under disturbances from* $H(f, B)$ *with respect to* K, *then* $x(n)$ *is asymptotically almost periodic.*

Proof. For any sequence $(n_k) \subset \mathbb{Z}$ with $n_k \to \infty$ as $k \to \infty$, let $w(n) = x^k(n) = x(n + n_k)$. Then, $x^k(n)$ is a solution of (3.8) passing through $(0, x_0^k)$ where $x_0^k(s) = x_{n_k}(s)$ for all $s \le 0$, as in the proof of Theorem 3.2. We claim that $x^k(n)$ is stable under disturbances from $H(f_{n_k}, B_{n_k})$ with respect to K for $(\varepsilon, \eta(\varepsilon))$.

Consider the limiting equation

$$x(n+1) = g(n, x(n)) + \sum_{j=-\infty}^{0} D(n, j, x(n+j), x(n)), \tag{3.38}$$

where $(g, D) \in H(f_{n_k}, B_{n_k})$. Assume that

$$\pi((f_{n_k}, B_{n_k}), (g, D)) \leq \eta,$$
$$\rho(w_\tau, y_\tau) \leq \eta \tag{3.39}$$

for some $\tau \geq 0$, where $y(n)$ is any solution of (3.38). We will show that $\rho(w_n, y_n) < \varepsilon$ for all $n \geq \tau$.

Putting $z(n) = y(n - n_k)$, $z(n)$ is a solution of

$$x(n+1) = g(n - n_k, x(n)) + \sum_{j=-\infty}^{0} D(n - n_k, j, x(n+j), x(n)) \tag{3.40}$$

passing through $(\tau + n_k, y_\tau)$ such that $z_{\tau+n_k}(s) = y_\tau(s)$ for all $s \leq 0$. If we set $(h, E) = (g_{-n_k}, D_{-n_k}) \in H(f, B)$, then $z(n)$ is a solution of

$$x(n+1) = h(n, x(n)) + \sum_{j=-\infty}^{0} E(n, j, x(n+j), x(n)). \tag{3.41}$$

Since

$$\pi((f_{n_k}, B_{n_k}), (g, D)) = \max\{\pi(f_{n_k}, g), \pi(B_{n_k}, D)\} \leq \eta, \tag{3.42}$$

we have

$$\pi((f, B), (h, E)) = \pi((f,B), (g_{-n_k}, D_{-n_k})) \leq \eta,$$
$$\rho(x_{\tau+n_k}, z_{\tau+n_k}) = \rho(w_\tau, y_\tau) \leq \eta. \tag{3.43}$$

Since $x(n)$ is stable under disturbances from $H(f, B)$, we obtain

$$\rho(x_n, z_n) < \varepsilon, \quad n \geq \tau + n_k, \tag{3.44}$$

that is,

$$\rho(w_n, y_n) < \varepsilon, \quad n \geq \tau. \tag{3.45}$$

This shows that $w(n) = x^k(n)$ is stable under disturbances from $H(f_{n_k}, B_{n_k})$ with respect to K for $(\varepsilon, \eta(\varepsilon))$.

Now, from the almost periodicity, there exists a subsequence of (n_k), which we denote by (n_k) again, such that $f(n+n_k,x)$ converges uniformly on $\mathbb{Z}\times K$ and $B(n+n_k,j,x,y)$ converges uniformly on $\mathbb{Z}\times T\times K\times K$, where T is a compact subset of $\mathbb{Z}^-$, as $k\to\infty$. It follows that for any $\varepsilon>0$, there exists a $k_1(\varepsilon)>0$ such that $k,m\ge k_1$ implies

$$\begin{gathered}\left|f(n+n_k,x)-f(n+n_m,x)\right|<\eta,\quad n\in\mathbb{Z},\ x\in K,\\ \left|B(n+n_k,j,x,y)-B(n+n_m,j,x,y)\right|<\frac{\eta}{2}\end{gathered}\tag{3.46}$$

for all $n\in\mathbb{Z},\ j\in[-N,0],\ x,y\in K$, where N is a positive integer such that

$$\sum_{j=N+1}^{\infty}\frac{1}{2^j}<\frac{\eta}{2}.\tag{3.47}$$

Since

$$\begin{aligned}\pi(B_{n_k},B_{n_m})&\le\sum_{j=1}^{N}\frac{\pi_j(B_{n_k},B_{n_m})}{2^j[1+\pi_j(B_{n_k},B_{n_m})]}+\sum_{j=N+1}^{\infty}\frac{1}{2^j}\\&\le\sum_{j=1}^{N}\frac{\pi_j(B_{n_k},B_{n_m})}{2^j}+\frac{\eta}{2}\\&<\eta,\end{aligned}\tag{3.48}$$

we have

$$\pi((f_{n_k},B_{n_k}),(f_{n_m},B_{n_m}))<\eta\tag{3.49}$$

whenever $k,m\ge k_1$. We can assume that $x^k(n)$ converges uniformly on any compact interval in $\mathbb{Z}^-$. Thus, there exists a $k_2(\varepsilon)>0$ such that $\rho(x_0^k,x_0^m)<\eta$ whenever $k,m\ge k_2$. To show that $x(n)$ is asymptotically almost periodic, we will show that

$$\left|x^k(n)-x^m(n)\right|<\varepsilon,\tag{3.50}$$

if $k,m\ge k_0(\varepsilon)=\max\{k_1(\varepsilon),k_2(\varepsilon)\}$, where $x^m(n)$ is a solution of

$$x(n+1)=f(n+n_m,x(n))+\sum_{j=-\infty}^{0}B(n+n_m,j,x(n+j),x(n))\tag{3.51}$$

such that $x_0^m(s)\in K$ for all $s\le 0$ and $(f_{n_m},b_{n_m})\in H(f_{n_k},B_{n_k})=H(f,B)$. Since

$$\begin{gathered}\pi((f_{n_k},B_{n_k}),(f_{n_m},B_{n_m}))<\eta,\\ \rho(x_0^k,x_0^m)<\varepsilon\end{gathered}\tag{3.52}$$

whenever $k, m \geq k_0$, we have

$$\rho(x_n^k, x_n^m) < \varepsilon, \quad n \geq 0, \ k, m \geq n_0 \tag{3.53}$$

from the fact that $x^k(n)$ is stable under disturbances from $H(f_{n_k}, B_{n_k})$ with respect to K. Consequently, we obtain

$$\left|x(n+n_k) - x(n+n_m)\right| \leq \sup_{s\in[-1,0]} \left|x(n+n_k+s) - x(n+n_m+s)\right| \tag{3.54}$$

whenever $k, m \geq k_0$. Therefore, $x(n)$ is asymptotically almost periodic. □

Finally, in view of Theorems 3.10 and 3.2, we obtain the following.

Corollary 3.11. *Under assumptions* $(H1)$ *and* $(H2)$ *if the bounded solution* $x(n)$ *of* (2.7) *is stable under disturbances from* $H(f, B)$ *with respect to* K, *then* (2.7) *has an almost periodic solution.*

Remark 3.12. Song and Tian obtained the result for the existence of almost periodic solution to (2.7) by showing that if the bounded solution $x(n)$ of (2.7) is totally stable, then it is an asymptotically almost periodic solution in [3, Theorem 4.4]. Note that total stability implies stability under disturbances from hull for (2.7) in view of Theorem 3.8.

Acknowledgments

The authors would like to thank the referees for their helpful comments and suggestions which led to an important improvement of original manuscript. This work was supported by the Second Stage of Brain Korea 21 Project in 2008.

References

[1] Y. Hamaya, "Stability property for an integrodifferential equation," *Differential and Integral Equations*, vol. 6, no. 6, pp. 1313–1324, 1993.

[2] Y. Hamaya, "Periodic solutions of nonlinear integrodifferential equations," *Tohoku Mathematical Journal*, vol. 41, no. 1, pp. 105–116, 1989.

[3] Y. Song and H. Tian, "Periodic and almost periodic solutions of nonlinear discrete Volterra equations with unbounded delay," *Journal of Computational and Applied Mathematics*, vol. 205, no. 2, pp. 859–870, 2007.

[4] Y. Song, "Asymptotically almost periodic solutions of nonlinear Volterra difference equations with unbounded delay," *Journal of Difference Equations and Applications*, vol. 14, no. 9, pp. 971–986, 2008.

[5] S. K. Choi and N. J. Koo, "Asymptotic property of linear Volterra difference systems," *Journal of Mathematical Analysis and Applications*, vol. 321, no. 1, pp. 260–272, 2006.

[6] C. Corduneanu, *Almost Periodic Functions*, Chelsea, New York, NY, USA, 2nd edition, 1989.

[7] T. Yoshizawa, *Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions*, Applied Mathematical Sciences, vol. 14, Springer, New York, NY, USA, 1975.

[8] C. Zhang, *Almost Periodic Type Functions and Ergodicity*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2003.

[9] Y. Hamaya, "Total stability property in limiting equations of integrodifferential equations," *Funkcialaj Ekvacioj*, vol. 33, no. 2, pp. 345–362, 1990.

[10] M. N. Islam, "Almost periodic solutions of nonlinear integral equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 30, no. 2, pp. 865–869, 1997.

[11] I. Malkin, "Stability in the case of constantly acting disturbances," *Prikladnaya Matematika i Mekhanika*, vol. 8, pp. 241–245, 1944.

[12] P. Anderson and S. R. Bernfeld, "Total stability of scalar differential equations determined from their limiting functions," *Journal of Mathematical Analysis and Applications*, vol. 257, no. 2, pp. 251–273, 2001.
[13] Y. Hino and S. Murakami, "Total stability and uniform asymptotic stability for linear Volterra equations," *Journal of the London Mathematical Society*, vol. 43, no. 2, pp. 305–312, 1991.
[14] X. Liu and S. Sivasundaram, "Stability of nonlinear systems under constantly acting perturbations," *International Journal of Mathematics and Mathematical Sciences*, vol. 18, no. 2, pp. 273–278, 1995.
[15] T. Yoshizawa, "Asymptotically almost periodic solutions of an almost periodic system," *Funkcialaj Ekvacioj*, vol. 12, pp. 23–40, 1969.
[16] G. R. Sell, "Nonautonomous differential equations and topological dynamics. I. The basic theory," *Transactions of the American Mathematical Society*, vol. 127, pp. 241–262, 1967.
[17] G. R. Sell, "Nonautonomous differential equations and topological dynamics. II. Limiting equations," *Transactions of the American Mathematical Society*, vol. 127, pp. 263–283, 1967.
[18] S. Zhang and G. Zheng, "Almost periodic solutions of delay difference systems," *Applied Mathematics and Computation*, vol. 131, no. 2-3, pp. 497–516, 2002.