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On classical solutions of $4d$ supersymmetric higher spin theory

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Abstract: We present a simple construction of solutions to the supersymmetric higher spin theory based on solutions to bosonic theories. We illustrate this for the case of the Didenko-Vasiliev solution in arXiv:0906.3898, for which we have found a striking simplification where the higher-spin connection takes the vacuum value. Studying these solutions further, we check under which conditions they preserve some supersymmetry in the bulk, and when they are compatible with the boundary conditions conjectured to be dual to certain 3d SUSY Chern-Simons-matter theories. We perform the analysis for a variety of theories with $\mathcal{N}=2$, $\mathcal{N}=3$, $\mathcal{N}=4$ and $\mathcal{N}=6$ and find a rich spectrum of 1/4, 1/3 and 1/2-BPS solutions.

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Contents

1 Introduction

Vasiliev's theory of higher-spins [1] consistently describes the interaction of fields of any positive spin, and apart from its intrinsic mathematical interest, it was first investigated as a possible contender for an unbroken phase of string theory. The theory has seen a surge of interest in recent years due to the fact that it can consistently be formulated on negatively curved spaces and thus a ripe candidate to study the holographic principle [2–7].

One of the most appealing features of higher spin holography is that it does not require the full machinery of string theory and in particular supersymmetry. The bulk side is a theory of higher spins, including the graviton. Of course, there are disadvantages as well, with no quantum definition of the theory as of yet. Still, though supersymmetry is not required, it is easy to incorporate and allows to holographically study supersymmetric boundary theories. It is also useful in order to see how string theory and higher spin theory are related.

In particular, a very wide range of supersymmetric higher spin holographic duals were proposed in [8]. These are four dimensional bulk theories dual to 3d supersymmetric Chern-Simons-matter theories. One example even covers a limit of ABJ theory [9], giving a new perspective for studying the relationship of Vasiliev's theory to string theory.

Exact solutions to Vasiliev's higher spin equations in 4-dimensions are scarce. Apart from the vacuum solution, the first exact solution was given by Sezgin and Sundell in [10]. A few years later, Didenko and Vasiliev obtained in [11] an exact solution with many similarities to an extremal black hole. Iazeolla and Sundell then generalized some aspect of the construction and obtained six families of exact solutions in [12]. More recently, a new family of solutions was found by Gubser and Song in [13]. In this paper we wish to study the classical solutions of the supersymmetric higher spin theories.

Classical solutions of interacting theories like gravity and non-abelian gauge theory provide important reference points for the understanding of the theories. One can study the fluctuations about them and consider them as alternative vacua. In particular for supersymmetric theories, solutions which preserve a large fraction of supersymmetry may be insensitive to quantum effects and followed from weak to strong coupling, like D-branes in string theory. In the holographic setting they can be studied by both a weakly coupled gauge theory and a weakly curved gravitational theory.

With this in mind, we study the embedding of classical solutions of bosonic higher spin theory into the supersymmetric theory. After presenting some general features of such embeddings, we focus on the case of the solution of Didenko and Vasiliev (DV) [11]. By studying the solution in a particular coordinate system and choosing a specific time-like Killing vector which appears in the solution, we note a remarkable simplification to the form of the solution — one of the master fields of the theory takes the same value as on the AdS vacuum. We also generalized the solution to more general bosonic higher spin theories, by allowing a non-zero parity breaking phase θ_0.[1]

This allows us to study such solutions in the supersymmetric higher spin theory, and as already noted in [11], preserving supersymmetry requires embedding two different (but closely related) bosonic solutions into the supersymmetric theory. In theories with extended supersymmetry we find a rich structure of solutions which preserve different fractions of the supersymmetries of the bulk AdS_4 vacuum.

To define the higher spin theory unambiguously requires choosing boundary conditions for the fields of lowest spin, as was advocated in [9]. Depending on the boundary conditions, the same bulk theory is dual to many different 3d vector Chern-Simons theories with varying amount of supersymmetry. We therefore examine the asymptotics of the DV solution and its possible embeddings in the proposed holographic duals of certain 3d theories with $\mathcal{N} = 2$, $\mathcal{N} = 3$, $\mathcal{N} = 4$ and $\mathcal{N} = 6$ supersymmetry. We find an intricate structure of families of 1/4-BPS, 1/3-BPS and 1/2-BPS solutions depending on varying numbers of free parameters. This structure of the solutions is quite nicely correlated to the global symmetries of the holographic duals, and we comment briefly on how to match them to the spectrum of protected operators in the dual theory.

We tried to organize the paper in a natural logical order and included most technical details (including the review and simplification of the DV solution) in appendices.

2 Supersymmetric higher spin theory: a brief review

Our aim here is to give a very short introduction to Vasiliev's theory of higher-spin, omitting a lot of subtleties for the sake of clarity. More details can be found in the various existing reviews such as [14–16].

2.1 Master fields of the theory

We present in this section the higher-spin theory following [8]. The theory consistently describes the interactions of an infinite tower of real massless fields in AdS_4 carrying integer (and in the presence of supersymmetry also half integer) spin with a scalar field of mass $m^2 = -2\lambda^{-2}$, where λ is the AdS curvature radius. The fields are packaged into the master fields (W, B, S), which all depend on the following sets of variables

[1] The solutions that are found in [12] already take into account the phase θ_0, but are presented in a different gauge.

- x: the space-time coordinates.
- $(y_\alpha, \bar{y}_{\dot{\alpha}})$: internal bosonic coordinates. Roughly speaking, the coefficients in the expansion of the master fields in these coordinates contain the dynamical (or auxiliary) fields. They are collectively denoted Y.
- $(z_\alpha, \bar{z}_{\dot{\alpha}})$: similar in nature to the Y coordinates, but have a very different role as they are introduced to turn on interactions in an explicitly gauge-invariant way. They are collectively denoted Z.
- ϑ^i: variables satisfying a Clifford algebra $\left\{\vartheta^i, \vartheta^j\right\} = 2\delta^{ij}$. Taking $i = 1, \cdots, n$ turns every field into a 2^n component superfield. We recover the bosonic theory by setting $n = 0$.

The master fields have the following content

- $W(Y, Z|x, \vartheta)$: is the higher-spin connection, containing the massless higher-spin gauge fields of spin $s \geq 1$, as well as auxiliary fields. It is a space-time one-form.
- $B(Y, Z|x, \vartheta)$: contains the curvature of the fields, such as the Weyl tensor and its higher-spin generalisation, as well as the massive scalar, massless fermion and Maxwell field. It is a space-time zero-form.
- $S(Y, Z|x, \vartheta)$: is also introduced to turn on interactions, and is purely auxiliary. It is a space-time zero-from, but a one-form in Z-space

$$S = S_\alpha dz^\alpha + \bar{S}_{\dot{\alpha}} d\bar{z}^{\dot{\alpha}} \tag{2.1}$$

One can combine the the two one-forms into a single field

$$\mathcal{A} = W + S\,, \tag{2.2}$$

It is required that $\{dx^\mu, dz^\alpha\} = \left\{dx^\mu, d\bar{z}^{\dot{\alpha}}\right\} = 0$ and we will also define $dz^2 = \epsilon_{\alpha\beta} dz^\alpha dz^\beta$ and $d\bar{z}^2 = \epsilon_{\dot{\alpha}\dot{\beta}} d\bar{z}^{\dot{\alpha}} d\bar{z}^{\dot{\beta}}$.

2.2 Equations of motion

The Vasiliev equations for interacting higher spin fields are

$$\mathcal{F} \equiv d\mathcal{A} - \mathcal{A} \wedge_\star \mathcal{A} = -f_\star\left(B \star v\right) dz^2 - \bar{f}_\star\left(B \star \bar{v}\Gamma\right) d\bar{z}^2\,, \tag{2.3}$$

$$dB - \mathcal{A} \star B + B \star \pi(\mathcal{A}) = 0\,. \tag{2.4}$$

This deceptively elegant form requires all the following definitions

- Multiplication is performed using the star product (see appendix A.2)

$$\Phi(Y, Z) \star \Theta(Y, Z) = \Phi(Y, Z) \exp\left[-\epsilon^{\alpha\beta}\left(\overleftarrow{\partial}_{y^\alpha} + \overleftarrow{\partial}_{z^\alpha}\right)\left(\overrightarrow{\partial}_{y^\beta} - \overrightarrow{\partial}_{z^\beta}\right) + \text{c.c.}\right] \Theta(Y, Z)\,. \tag{2.5}$$

- A function $f(X) = 1 + Xe^{i\theta(X)}$ determining the interactions of the theory. A-priori f can be a general function, but it was shown in [8, 17] that after field redefinitions and imposing reality conditions it can be written in this form with θ an even function of X (and all products and exponentiation have to be done with the star product, hence the subscript f_*).

- The Kleiniens $\upsilon = e^{z_\alpha y^\alpha}$, $\bar{\upsilon} = e^{\bar{z}_{\dot{\alpha}} \bar{y}^{\dot{\alpha}}}$ which satisfy

$$\begin{aligned} \upsilon \star \upsilon &= 1\,, \qquad \upsilon \star \Phi(Y,Z) \star \upsilon = \Phi(-\gamma Y, -\gamma Z)\,, \\ \bar{\upsilon} \star \bar{\upsilon} &= 1\,, \qquad \bar{\upsilon} \star \Phi(Y,Z) \star \bar{\upsilon} = \Phi(\gamma Y, \gamma Z)\,, \\ \gamma(\circ_\alpha) &= \circ_\alpha\,, \qquad\qquad \gamma(\bar{\circ}_{\dot{\alpha}}) = -\bar{\circ}_{\dot{\alpha}} \end{aligned} \tag{2.6}$$

- The generalized twist operators π and $\bar{\pi}$ acting by

$$\begin{aligned} \pi\left(\Phi(Y,Z,dZ)\right) &= \Phi(-\gamma Y, -\gamma Z, -\gamma dZ)\,, \\ \bar{\pi}\left(\Phi(Y,Z,dZ)\right) &= \Phi(\gamma Y, \gamma Z, \gamma dZ) \end{aligned} \tag{2.7}$$

  One should be careful with the fact that for a 1-form in $(z_\alpha, \bar{z}_{\dot{\alpha}})$, this generalized twist operator is equivalent to conjugation by υ *and* the flipping of the sign of dz^α.

- The chirality operator $\Gamma = i^{\frac{n(n-1)}{2}} \vartheta^1 \vartheta^2 \dots \vartheta^n$.

The equations of motion (2.3), (2.4) are invariant under a very large set of gauge transformations

$$\delta\mathcal{A} = d\epsilon - [\mathcal{A}, \epsilon]_\star\,, \qquad \delta B = \epsilon \star B - B \star \pi\left(\epsilon\right)\,. \tag{2.8}$$

where the gauge parameter $\epsilon(Y,Z|x,\vartheta)$ is a zero-form which satisfies the same reality conditions and truncations as W.

In components, the equations of motion (2.3)–(2.4) read

$$dW - W \wedge_\star W = 0\,, \tag{2.9a}$$
$$dB - W \star B + B \star \pi(W) = 0\,, \tag{2.9b}$$
$$dS_\alpha - [W, S_\alpha]_\star = 0\,, \qquad d\bar{S}_{\dot{\alpha}} - \left[W, \bar{S}_{\dot{\alpha}}\right]_\star = 0\,, \tag{2.9c}$$
$$B \star \pi(S_\alpha) + S_\alpha \star B = 0\,, \qquad B \star \pi(\bar{S}_{\dot{\alpha}}) - \bar{S}_{\dot{\alpha}} \star B = 0\,, \tag{2.9d}$$
$$S_\alpha \star S^\alpha = 2 f_\star\left(B \star \upsilon\right)\,, \qquad \bar{S}_{\dot{\alpha}} \star \bar{S}^{\dot{\alpha}} = 2\bar{f}_\star(B \star \bar{\upsilon}\Gamma)\,, \qquad [S_\alpha, \bar{S}_{\dot{\alpha}}]_\star = 0\,, \tag{2.9e}$$

and the gauge transformations take the form

$$\delta W = d\epsilon - [W, \epsilon]_\star\,, \quad \delta B = \epsilon \star B - B \star \pi\left(\epsilon\right)\,, \quad \delta S_\alpha = [\epsilon, S_\alpha]_\star\,, \quad \delta \bar{S}_{\dot{\alpha}} = \left[\epsilon, \bar{S}_{\dot{\alpha}}\right]_\star\,. \tag{2.10}$$

2.3 Spin-statistics theorem

The construction of the superfields above is based on bosonic variables Y, Z carrying half spin and fermionic variables ϑ^i which are scalars. To satisfy the spin-statistics theorem we must project on to half of the components of all the fields such that the number of Y, Z is equal to the number of ϑ^i modulo 2. The correct condition to be imposed on the fields is

$$W(Y,Z|x,\vartheta) = \Gamma W(-Y,-Z|x,\vartheta)\Gamma\,, \tag{2.11}$$

$$B(Y,Z|x,\vartheta) = \Gamma B(-Y,-Z|x,\vartheta)\Gamma\,, \tag{2.12}$$
$$\epsilon(Y,Z|x,\vartheta) = \Gamma\epsilon(-Y,-Z|x,\vartheta)\Gamma\,, \tag{2.13}$$

where ϵ designates a higher-spin gauge parameter. Consistency with the equations of motion imposes (this depends on the choice of reality conditions)

$$S_\alpha(Y,Z|x,\vartheta) = -\Gamma S_\alpha(-Y,-Z|x,\vartheta)\Gamma\,, \tag{2.14}$$
$$\bar{S}_{\dot\alpha}(Y,Z|x,\vartheta) = -\Gamma\bar{S}_{\dot\alpha}(-Y,-Z|x,\vartheta)\Gamma\,. \tag{2.15}$$

These conditions can alternatively be written using the properties of the kleinians as follows

$$[v\bar{v}\Gamma, W]_\star = [v\bar{v}\Gamma, B]_\star = [v\bar{v}\Gamma, \epsilon]_\star = \{v\bar{v}\Gamma, S\}_\star = 0\,. \tag{2.16}$$

This ensures that functions which are even functions of the ϑ^i are even functions in Y and Z, while functions which are odd functions of the ϑ^i are odd functions in Y, Z. This is due to the fact that the matrix Γ (anti-)commutes with an (odd) even number of ϑ^i. This means that now we can consider fermions in the theory while respecting the spin-statistics theorem.

It should be noted that this supersymmetric extension is different from the ones introduced by other authors. In [18] extensions are constructed out of the minimal bosonic theory. Grassmann odd variables are also introduced, but some of them have a very distinct role, as they introduce back the fields of odd spin. In [19], the fermions are also truncated away but fields of odd spin are kept. The fermions are reintroduced with a grading inside the tensored matrix algebra. Here the supersymmetric extension is constructed from the bosonic theory with fields of all spin, even and odd while keeping the fermions from the start.

2.4 Generalized reality conditions

For the equations (2.3)–(2.4) to consistently describe the interactions of real massless higher spin fields, one has to impose adequate reality conditions on the master fields.[2]

It should be noted that different signatures imply different isometry groups and by extension different higher-spin algebras. Notions of spinors will vary in subtle ways, and so will the representation in terms of oscillators. In this work, we are mainly interested in the Lorentzian signature, with the convention where we use mostly minus signs. As explained in detail in [20], choosing a different signature puts constraints on the consistent reality projections one can choose, and even on the Vasiliev equations, especially in the interacting part. We will now present the reality conditions that we will be using and which are identical to the ones in [8].

First we define a natural generalisation of the complex conjugation to the master fields with

$$(y_\alpha)^\dagger = \bar{y}_{\dot\alpha}\,, \qquad (z_\alpha)^\dagger = \bar{z}_{\dot\alpha}\,, \qquad (dz_\alpha)^\dagger = d\bar{z}_{\dot\alpha}\,, \qquad (\vartheta^i)^\dagger = \vartheta^i \tag{2.17}$$

[2]Note that different conventions can be used for the reality conditions. This can be consistently tracked down to a different choice for the canonical commutation relations between the spinor variables (sometimes a factor i is introduced as in [15]). See appendix A.3.

For any functions Φ, Θ of $(Y, Z|x, \vartheta)$, we then obtain from the definition (2.5)

$$(\Phi \star \Theta)^\dagger = \Theta^\dagger \star \Phi^\dagger. \tag{2.18}$$

We then define a second operator τ (denoted ι in [8]) which reverses the order of the ϑ^i and otherwise acts by

$$\tau\left[\Phi(Y, Z, dZ|x, \vartheta)\right] = \Phi(iY, -iZ, -idZ|x, \tau[\vartheta])\,. \tag{2.19}$$

In this way, for any two functions Φ, Θ of $(Y, Z|x, \vartheta)$, we obtain

$$\tau\left(\Phi \star \Theta\right) = \tau(\Theta) \star \tau(\Phi)\,, \tag{2.20}$$

where it should be noted that the order of the functions is exchanged. Also we have that $\tau\left(\Gamma\right)^\dagger = \Gamma^{-1} = \Gamma$.

Following [8] we impose the *non-minimal* reality conditions

$$\tau(W)^\dagger = -W\,, \qquad \tau(S)^\dagger = -S\,, \qquad \tau(B)^\dagger = \bar{v} \star B \star \bar{v}\Gamma = \Gamma v \star B \star v. \tag{2.21}$$

This differs from the *minimal* reality conditions where in addition to the condition above, the fields are also assumed to be invariant under complex conjugation alone.

2.5 Extended higher-spin symmetry

One convenient way to add degrees of freedom to the theory is to tensor the higher-spin algebra with another algebra. There are some restrictions as to the nature of the algebras one can choose, which were discussed very early on by Konstein and Vasiliev in [19]. This procedure amounts to promoting all the fields of the theory to matrices[3] $\Phi(Y, Z|x, \vartheta) \to \Phi_i{}^j(Y, Z|x, \vartheta)$, where $i, j = 1 \ldots M$. One can choose $\mathrm{U}(M)$ as the tensored matrix algebra, and this results in having the Maxwell field becoming a $\mathrm{U}(M)$ gauge field. This extension is called in [8] Vasiliev's theory with $\mathrm{U}(M)$ Chan-Paton factors.

2.6 Higher-spin holography

Higher spin theories provide a realization of the holographic principle, as first introduced in [3] and [2] for the 4-dimensional case. We present here a brief review and refer the reader to the original papers and [6–8] for more details.

As is usual in the AdS_{d+1}/CFT_d correspondence, fields in the bulk are dual to operators in the boundary theory. In particular for a scalar field of mass m, the dual operator has dimension

$$\Delta_\pm = \frac{d \pm \sqrt{d^2 + 4m^2\lambda^2}}{2}\,. \tag{2.22}$$

This can be seen from the solution of the Klein-Gordon equation, which near the boundary of AdS ($r \to 0$ in the metric (B.3)) takes the form

$$C(r, x) = \frac{a}{r^{\Delta_-}} + \frac{b}{r^{\Delta_+}} + \ldots \tag{2.23}$$

[3]Complex conjugation is replaced by the usual hermitian conjugation with respect to the matrix indices.

For most fields the dual operator has dimension Δ_-, but for small enough m, there is an ambiguity, where enforcing $b = 0$ corresponds to a dual theory an operator of dimension Δ_- while the $a = 0$ boundary conditions gives a theory with operator of dimension Δ_+.

Furthermore, generic choices of boundary conditions will break the higher spin symmetry. It is argued in [8] that for fields with spins higher than 3/2, generic boundary conditions preserve the higher-spin symmetry.[4] The problem lies with the fields of spin 0, 1/2 and 1.

In the minimal parity preserving bosonic theory there is one real massless field for every even spin and in addition one real scalar field with mass $m^2 = -2\lambda^{-2}$. Plugging this mass and $d = 3$ in (2.22), we find that the dual operator has $\Delta_- = 1$ and $\Delta_+ = 2$. This leads to the four simplest holographic duals with either choice of dimension and with the phase $\theta(X)$ in the interaction term $f(X) = 1 + Xe^{i\theta(X)}$ equal to a constant 0 or $\pi/2$. These four choices are dual to either the free or critical, bosonic or fermionic $O(N)$ vector models.

This can be generalized to the case of non-minimal theories, which contain fields of all non-negative integer spins. Then one has to take care of the fact that the theory contains a gauge field of spin 1, whose boundary conditions will be severely constrained if higher-spin symmetry is to be preserved. In particular, the case we shall focus on in this paper, of supersymmetric parity violating theories was studied in [8]. It was argued that the introduction of a parity breaking phase $\theta(X) = \theta_0$ corresponds to gauging the vector model and including a Chern-Simons term at level k such that $\theta_0 = N/k$ is the 't Hooft coupling. We will return to the question of the boundary conditions on the scalar and vector fields in section 5.

3 Embedding bosonic solutions

The equations of motion of the supersymmetric higher spin theory involve all the the fields of the theory including the fermionic fields and several copies of the bosonic ones. A simple organizing principle is to separate the fields into eigenstates of the chirality operator $\Gamma = i^{\frac{n(n-1)}{2}} \vartheta^1 \vartheta^2 \dots \vartheta^n$

$$\begin{aligned} W &= \Gamma^+ W_+ + \Gamma^- W_- \,, \\ B &= \Gamma^+ B_+ + i\Gamma^- B_- \,, \qquad \Gamma^\pm = \frac{1 \pm \Gamma}{2} \,, \\ S &= \Gamma^+ S_+ + \Gamma^- S_- \,, \end{aligned} \tag{3.1}$$

The extra i in the decomposition of B is due to the fact that it is in the twisted adjoint representation and it will simplify the equations of motion and reality conditions below. Each of the fields $\Phi_\pm$ are superfields which are given by an expansion in ϑ^i restricted of course to half the possible terms. Each of those superfields can then be further separated into bosonic and fermionic parts, where the bosonic part has the same matter content as the field of the bosonic theory with $\mathrm{U}(2^{n/2-1})$ Chan-Paton factors (generated by $\Gamma^+ \vartheta^{i_1} \cdots \vartheta^{i_{2k}}$). The cases discussed below are those with $n = 2, 4, 6$ and hence Chan-Paton factors U(1), U(2) and U(4) respectively.

[4]There is however an unresolved puzzle concerning this statement, for more details see the end of page 34 of [8]. We thank S. Giombi for bringing this point to our attention.

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When considering classical solutions we can set all the fermionic fields to zero and restrict to the two sets of bosonic fields in $W_\pm$, $B_\pm$ and $S_\pm$. The main observation is that the projectors $\Gamma^\pm$ either act trivially or annihilate these bosonic fields leading to very simple equations of motion. The flatness equation for W (2.9a) becomes

$$\Gamma^\pm(dW - W \wedge_\star W) = \Gamma^\pm W_\pm - \Gamma^\pm W_\pm \wedge_\star \Gamma^\pm W_\pm = \Gamma^\pm (dW_\pm - W_\pm \wedge_\star W_\pm)\,. \tag{3.2}$$

The equations project onto separate bosonic equations for W_+ and W_-. This holds also for the equations (2.9b)–(2.9d).

Equations (2.9e) require a bit more care, due to the explicit appearance of Γ in them. They become

$$\begin{aligned} &S_{+\alpha} \star S^\alpha_+ = 2f_\star(B_+ \star v)\,, \quad \bar{S}_{+\dot\alpha} \star \bar{S}^{\dot\alpha}_+ = 2\bar{f}_\star(B_+ \star \bar{v})\,, \quad\;\;\; [S_{+\alpha}, \bar{S}_{+\dot\alpha}]_\star = 0\,, \\ &S_{-\alpha} \star S^\alpha_- = 2f_\star(iB_- \star v)\,, \quad \bar{S}_{-\dot\alpha} \star \bar{S}^{\dot\alpha}_- = 2\bar{f}_\star(-iB_- \star \bar{v})\,, \quad [S_{-\alpha}, \bar{S}_{-\dot\alpha}]_\star = 0\,. \end{aligned} \tag{3.3}$$

The equation for $S_+, \bar{S}_+$ are the usual equations as in the bosonic theory, but the equation for $\bar{S}_-$ is different due to the extra sign in $\bar{f}$. Recall that $f(X) = 1 + Xe^{i\theta(X)}$. If we assume[5] that θ is a power series in X^4, then

$$f_*(iB_-) = 1 + B_- \star e_\star^{i(\theta(B_-)+\pi/2)}\,, \qquad \bar{f}_*(-iB_-) = 1 + B_- \star e_\star^{-i(\theta(B_-)+\pi/2)}\,, \tag{3.4}$$

Thus the equation for $S_-, \bar{S}_-$ is the same as the bosonic equation with a shift in the phase $\theta(X) \to \theta(X) + \pi/2$.

As mentioned before, the simplest two choices for f are given by $\theta(X) = 0$ and $\theta(X) = \pi/2$. These two examples are the only ones which do not break parity. They correspond respectively to the so-called type-A (with parity-even B-field) and type-B (with parity-odd B-field) theories. Classical solutions of a parity invariant supersymmetric theory can therefore incorporate solutions of both the type-A and type-B bosonic theories.

3.1 Matrix factors

As shown, any solution of the supersymmetric equation can be written in terms of solutions of the bosonic theory with $\mathrm{U}(M)$ Chan-Paton factors. We would like to discuss here how to construct a solution to the bosonic theory with Chan-Paton factors, based on the solutions to the abelian theory. Unlike the previous discussion, where we found all possible solutions, in this case we will take a simple ansatz and will not find the most general solution. The ansatz is

$$\begin{aligned} W_i{}^j(Y,Z|x) &= w_i{}^j \tilde{W}(Y,Z|x)\,, \\ B_i{}^j(Y,Z|x) &= b_i{}^j \tilde{B}(Y,Z|x)\,, \\ S_{\alpha,i}{}^j(Y,Z|x) &= s_i{}^j \tilde{S}_\alpha(Y,Z|x)\,, \\ \bar{S}_{\dot\alpha,i}{}^j(Y,Z|x) &= \bar{s}_i{}^j \tilde{\bar{S}}_{\dot\alpha}(Y,Z|x)\,. \end{aligned} \tag{3.5}$$

[5]This is not a very strong assumption, since it is known that $\theta(X)$ can be written as a power series in X^2 but mostly assumed to be X independent, for which a new argument was proposed in [21].

We take w, b, s and $\bar{s}$ to be matrix pre-factors for the classical solution given by $\tilde{W}(Y,Z|x)$, $\tilde{B}(Y,Z|x)$, $\tilde{S}_\alpha(Y,Z|x)$ and $\tilde{\bar{S}}_{\dot{\alpha}}(Y,Z|x)$. We deduce the reality conditions that apply on the matrix factors

$$w^\dagger = w\,,\; b^\dagger = b\,,\; s^\dagger = \bar{s}. \tag{3.6}$$

The equations of motion (2.9a)–(2.9e) will be solved for this very general ansatz only if all the terms in each equation are proportional to each other[6]

$$\begin{aligned} &w^2 = w\,, \qquad wb = bw = b\,, \quad ws = sw = s\,, \quad w\bar{s} = \bar{s}w = \bar{s}\,, \\ &s^2 = \bar{s}^2 = b\,, \quad s\bar{s} = \bar{s}s\,. \end{aligned} \tag{3.7}$$

All the matrices commute, so can be simultaneously diagonalized. w is a projector which acts trivially on an m dimensional subspace and annihilates an $M-m$ dimensional one. s has to vanish on the same space as w and may have up to m non-zero eigenvalues. b is then determined by $b = s^2$.

When the entries in the matrices are all made of a unique solution we saw that the matrices can all be simultaneously diagonalized. In that case there is no reason to restrict them to be proportional to a single solution of the bosonic theory and we can construct a more general solution made of different solutions along the diagonal.

3.2 Supersymmetry invariance

Global symmetries (including supersymmetry) of a classical solution of higher spin theory are represented by gauge symmetries which leave the solution invariant. The prime example is given by the global symmetries of the vacuum solution (B.1)–(B.2). First, the equation $\delta S_0 = 0$ tells us that the gauge parameter ϵ is independent of the Z variables. The B field transforms homogeneously and thus gives no additional constraints. Finally with the equation $\delta W_0 = 0$ we conclude that the global symmetries of the vacuum are generated by the gauge parameter $\epsilon(Y|x,\vartheta)$ satisfying

$$d\epsilon - [W_0, \epsilon]_\star = 0. \tag{3.8}$$

There may be solutions to this equation of the form $\epsilon(Y|x,\vartheta) = R_{ij}\vartheta^i\vartheta^j$ for $i \neq j$. From (3.8) and the reality condition $\tau(\epsilon(\vartheta))^\dagger = -\epsilon(\vartheta)$ we deduce that R_{ij} is a real space-time independent parameter. These solutions correspond to the generators of the R-symmetry of the theory, which will be broken in later sections upon introduction of boundary conditions.

The second type of solutions corresponds to gauge parameters that are proportional to Y. These are spinorial, and hence supersymmetry generators. Let us write these gauge parameters as

$$\epsilon(Y|x,\vartheta) = \Xi_\alpha(x,\vartheta)y^\alpha + i\bar{\Xi}_{\dot{\alpha}}(x,\vartheta)\bar{y}^{\dot{\alpha}}\,, \tag{3.9}$$

where the reality condition sets $\bar{\Xi}_{\dot{\alpha}} = \Xi^\dagger_\alpha$ and they are odd functions of the ϑ^i due to the truncation (2.13). If we replace the components of W_0 by the vierbein and connection

[6] We assume here that the interacting phase is a constant.

1-form of the AdS_4 metric using (B.1)–(B.2), we can rewrite (3.8) as

$$\tilde{\nabla}\begin{pmatrix}\Xi\\ \bar{\Xi}\end{pmatrix} \equiv \left(d - \frac{i}{2}\omega_{ab}\gamma^{ab} + \frac{i}{\sqrt{2}}h_a\gamma^a\right)\begin{pmatrix}\Xi\\ \bar{\Xi}\end{pmatrix} = 0\,, \tag{3.10}$$

where we split the equation into its y^α and $\bar{y}^{\dot{\alpha}}$ components. This is nothing but the Killing spinor equation in AdS background.

For AdS_4, the solution to the spinor part of (3.10) is well known [22], and is parametrized by four independent constants, as we summarize in appendix C.2. Since there are 2^{n-1} odd functions of ϑ^i, there are a total of 2^{n+1} real supersymmetry parameters for the AdS_4 background.

We can thus express the parameter ϵ (3.9) as linear combinations of the Killing spinors of AdS_4 $\psi^I = \psi^I_\alpha y^\alpha + i\bar{\chi}^I_{\dot{\alpha}}\bar{y}^{\dot{\alpha}}$

$$\epsilon(Y|x,\vartheta) = \psi^I(Y|x)\xi^I(\vartheta)\,, \tag{3.11}$$

with $I = 1, 2, \bar{1}, \bar{2}$. Due to the reality condition, the various ξ^I are related through

$$(\xi^i)^\dagger = \xi^{\bar{i}}\,, \tag{3.12}$$

with $i = 1, 2$. To see this, we write

$$\tau\left(\psi^i\right)^\dagger = \tau\left(\psi^i_\alpha y^\alpha + i\bar{\chi}^i_{\dot{\alpha}}\bar{y}^{\dot{\alpha}}\right)^\dagger = -i\left(\bar{\psi}^i_{\dot{\alpha}}\bar{y}^{\dot{\alpha}} - i\chi^i_\alpha y^\alpha\right) = -\left(\chi^i_\alpha y^\alpha + i\bar{\psi}^i_{\dot{\alpha}}\bar{y}^{\dot{\alpha}}\right) = -\psi^{\bar{i}}\,, \tag{3.13}$$

where we have used the properties (C.13), so that

$$\tau(\xi^i\psi^i + \xi^{\bar{i}}\psi^{\bar{i}})^\dagger = -(\xi^i)^\dagger\psi^{\bar{i}} - (\xi^{\bar{i}})^\dagger\psi^i = -(\xi^i\psi^i + \xi^{\bar{i}}\psi^{\bar{i}}) \tag{3.14}$$

where the last equality comes from the reality condition.

In the remainder of this paper we will study the Didenko-Vasiliev solution, which as we show in appendix B, has $W = W_0$ — the same connection as the vacuum AdS solution. These solutions have non-trivial values for the B and S fields, so to check supersymmetry we will therefore have to impose that they are invariant under the gauge transformations (3.9). Plugging this into (2.10) gives

$$\psi^I \star B\,\xi^I b - B \star \pi(\psi^I)\,b\xi^I = 0\,, \tag{3.15}$$

$$\psi^I \star S\,\xi^I s - S \star \psi^I\,s\xi^I = 0\,. \tag{3.16}$$

The solutions to these equations determine the bulk supersymmetries of the solutions and further conditions need to be checked to verify that they are not broken by the boundary conditions, see section 5.

4 Supersymmetric embedding of the Didenko-Vasiliev solution

We turn our attention now to the Didenko-Vasiliev (DV) solution of the bosonic theory [11], whose construction is outlined in appendix B. The presentation in the appendix goes beyond

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that in [11], first we generalize it to a bosonic theory with arbitrary parity breaking phase θ. Second, we obtain a striking simplification by choosing a particular Killing vector and a particular Lorentz frame such that the quantity $\kappa_{\alpha\beta}/r$ defined in (B.12) is space-time independent. We find that the W master field simply takes the AdS_4 background value W_0. The full solution is given by (B.52)–(B.55).

In the original paper [11] this solution was also embedded into a supersymmetric theory with the same amount of supersymmetry as the one with $n = 2$ (though in a different formalism) and it was checked that it preserves 1/4 of the supercharges in the bulk.

We implement now our formalism from section 3 and find all such embeddings of the bosonic solution into the supersymmetric theory with $n = 2, 4$. We then check under what conditions those solutions are BPS and whether they are compatible with boundary conditions conjectured in [8] to be dual to different supersymmetric 3d-theories.

We saw in section 3, that one can diagonally embed any bosonic solution into the supersymmetric theory with the restriction that the solution in the block projected to by Γ^- is in the theory with the phase shifted by $\pi/2$. As mentioned above, the DV solution can be easily adapted to arbitrary phase. To find BPS embeddings we shall consider two solutions with different Killing matrices differing by sign; K and $-K$ [11].

In a basis where all the matrices are diagonal our embedding has $W = W_0$, or $w = 1$ in (3.5) (the case where w has vanishing eigenvalues does not seem particularly interesting). The fields B and S are diagonal with every entry involving either of the Killing matrices $\pm K$ and a continuous parameter b.

To be specific, the ansatz is (see appendix B for the ingredients of the solution)

$$\begin{aligned} W(Y,Z|x) &= W_0 \,, \\ B(Y,Z|x) &= b\,[\eta_p F_K + \eta_m F_{-K}] \star \delta(y) \,, \\ S_\alpha(Y,Z|x) &= z_\alpha + s\big[\eta_p F_K \sigma_\alpha(a,K|x) + \eta_m F_{-K}\sigma_\alpha(a,-K|x)\big] \,, \\ \bar{S}_{\dot\alpha}(Y,Z|x) &= \bar{z}_{\dot\alpha} + \bar{s}\big[\eta_p F_K \bar{\sigma}_{\dot\alpha}(\bar{a},K|x) + \eta_m F_{-K}\bar{\sigma}_{\dot\alpha}(\bar{a},-K|x)\big] \,, \end{aligned} \tag{4.1}$$

where η_p, η_m are orthogonal projectors. The matrices (including η_p, η_m) are diagonal, and we also define

$$\begin{aligned} b &\equiv b_+ + ib_- \equiv \mathrm{diag}(b_{+,1}, b_{+,2}, ib_{-,1}, ib_{-,2}) \,, \\ s &\equiv e^{i\theta_0} s_+ + e^{i(\theta_0+\pi/2)} s_- \,, \\ \bar{s} &\equiv e^{-i\theta_0}\bar{s}_+ + e^{-i(\theta_0+\pi/2)}\bar{s}_- \,, \end{aligned} \tag{4.2}$$

where the $\pm$ subscript indicates as usual the fact that the matrix is an eigenstate of $\Gamma^\pm$, and where all the parameters in b are real. The equations of motion then impose the following constraints on the matrix factors

$$s_\pm = b_\pm \,, \qquad bs = sb \,, \qquad s\bar{s} = \bar{s}s \,, \tag{4.3}$$

where we have omitted the equations obtained from the barred sector which can be deduced from the unbarred sector by complex conjugation and using the reality conditions which read for the matrix factors in (4.1)

$$b_\pm^\dagger = b_\pm \,, \qquad s^\dagger = \bar{s}. \tag{4.4}$$

One can of course rotate this solution to a different basis, where it is not diagonal.

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4.1 The BPS equations for the DV solution

Following the discussion in section 3, our ansatz (4.1) is BPS if it solves the equations (3.15) and (3.16) for non-trivial gauge parameters of the form

$$\epsilon(Y|x,\vartheta) = \psi^I(Y|x)\xi^I(\vartheta)\,, \tag{4.5}$$

where ψ^I are the Killing spinors of AdS_4 given in appendix C.2.

Note that the DV solution is based on a black hole solution of supergravity with a Killing matrix K_{AB} discussed in appendix B.3. The B and S fields are proportional to the star product projectors F_K (B.25). In particular F_K projects the four Y_A coordinates to a two-dimensional subspace

$$(\Pi_{-A}{}^B Y_B)\star F_K = F_K\star(\Pi_{+A}{}^B Y_B) = 0\,, \qquad \Pi_{\pm AB} = \frac{1}{2}(\epsilon_{AB}\pm iK_{AB}) \tag{4.6}$$

In fact, since K_{AB} is a bilinear in the AdS_4 Killing spinors, F_K projects the four Killing spinors onto a two-dimensional subspace. We discuss in detail the properties of the projector $\Pi_\pm$ and the Killing matrix K_{AB} (B.12) and how it is related to the Killing spinors in appendix C.5. Here we write for convenience the following properties we will be using. We have

$$\psi^i = \Pi_+\psi^i\,, \qquad \psi^{\bar{i}} = \Pi_-\psi^{\bar{i}}\,, \tag{4.7}$$

which gives

$$\psi^i\star F_K = F_K\star\psi^{\bar{i}} = 0\,, \qquad F_K\star\psi^i = 2F_K\psi^i\,, \qquad \psi^{\bar{i}}\star F_K = 2\psi^{\bar{i}}F_K\,. \tag{4.8}$$

for $i = 1, 2$. Using these properties, we can simplify the BPS equation (3.15) as follows

$$\psi^i F_{-K}\xi^i\eta_m b + \psi^{\bar{i}}F_K\xi^{\bar{i}}\eta_p b - F_K\psi^i\eta_p b\xi^i - F_{-K}\psi^{\bar{i}}\eta_m b\xi^{\bar{i}} = 0\,. \tag{4.9}$$

This equation is satisfied if

$$\xi^i\eta_m b = \eta_p b\xi^i = 0\,. \tag{4.10}$$

Such an equation is easy to solve, as it is purely algebraic. We take b to be generic, with no zero eigenvalues on the space that $\eta_p + \eta_m$ projects onto (the value of b on the orthogonal space is not important, as it is always projected out). For each nonzero entry in η_m, the corresponding column of ξ^i has to vanish and for each nonzero entry in η_p, the corresponding line of ξ^i has to vanish. Clearly, both the i-th column and the i-th line for a fixed i cannot be crossed out simultaneously, since $\eta_p\eta_m = 0$. The BPS equation (3.16) is then automatically satisfied, as properties similar to (4.8) are satisfied when $F_{\pm K}$ is replaced by $F_{\pm K}f(a)$, where $f(a)$ is any holomorphic (or anti-holomorphic) function of a (see appendix B and (B.39)).

4.2 $n = 2$

The supersymmetric theory with $n = 2$ is particularly simple in the sense that the two $\Gamma^\pm$ subspaces are of rank 1. We thus have two independent eigenvalues. If both are non-zero, we have $\eta_p + \eta_m = 1$. Then we see that the only configuration with a non-trivial ξ^i which still satisfy (4.10) is (up to exchanging the role of F_K and F_{-K})

$$\eta_p = \mathrm{diag}(1,0)\,, \quad \eta_m = \mathrm{diag}(0,1) \tag{4.11}$$

and preserves the supersymmetry with ϑ^i-content

$$\xi^i = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}\,, \tag{4.12}$$

and this corresponds to a half-BPS configuration, where in this section we include all the supersymmetries of the bulk for this counting. This will change when we impose boundary conditions which break some supersymmetries in the next section. In particular, if both eigenvalues are to be non-zero, this means that the B-field cannot contain only F_K (or only F_{-K}) and still be BPS. If we impose only one of the two eigenvalues to be non-vanishing, then we only need to cross either a line or a column, and the preserved supersymmetry will have a ϑ^i-content where only one off-diagonal entry is non-zero, leading again to half-BPS configurations.

Finally, the case where both eigenvalues of b vanish simply corresponds to the AdS_4 vacuum which is maximally supersymmetric.

4.3 $n = 4$

We now turn to the $n = 4$ case, where b has four eigenvalues. Again, starting with the configuration where none of the eigenvalues vanish, the most supersymmetric configuration is given by (again, up to the exchange of the role of F_K and F_{-K})

$$\eta_p = \mathrm{diag}(1,1,0,0)\,, \qquad \eta_m = \mathrm{diag}(0,0,1,1)\,, \tag{4.13}$$

as it preserves the supserymmetries with ϑ^i-content

$$\xi^i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{pmatrix}\,, \tag{4.14}$$

which gives a half-BPS configuration. There are other configurations where B has four non-vanishing eigenvalues, which preserve 1/4 of the supersymmetries. In a basis where b is diagonal they are

$$\eta_p = \mathrm{diag}(1,0,1,0)\,, \qquad \eta_m = \mathrm{diag}(0,1,0,1)\,, \qquad \xi^i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix}\,. \tag{4.15}$$

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Configurations where $\eta_p + \eta_m$ is not the identity will generically preserve more supersymmetries. If we consider only three non-vanishing eigenvalues, we can obtain configurations that preserve a different amount of supersymmetry, such as the 3/8-BPS

$$\eta_p = \mathrm{diag}(1,0,0,0)\,, \qquad \eta_m = \mathrm{diag}(0,1,0,1)\,, \qquad \xi^i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix}. \tag{4.16}$$

Now turning on to only two non-vanishing eigenvalues, we obtain the following two cases. The 1/2-BPS

$$\eta_p = \mathrm{diag}(1,0,0,0)\,, \qquad \eta_m = \mathrm{diag}(0,1,0,0)\,, \qquad \xi^i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & * & * \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix}, \tag{4.17}$$

and the 5/8-BPS

$$\eta_p = \mathrm{diag}(1,0,0,0)\,, \qquad \eta_m = \mathrm{diag}(0,0,0,1)\,, \qquad \xi^i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{pmatrix}. \tag{4.18}$$

Finally if the B-field has only one non-vanishing eigenvalue, we preserve 3/4 of the supercharges

$$\eta_p = \mathrm{diag}(1,0,0,0)\,, \qquad \eta_m = \mathrm{diag}(0,0,0,0)\,, \qquad \xi^i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & * & * \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{pmatrix}. \tag{4.19}$$

5 Compatibility of solutions with boundary conditions

Thus far we have studied the embeddings of the DV solution into different supersymmetric extensions of higher spin theory with $n = 2, 4$. We now turn to study the asymptotics of the solutions. It was proven in [23] that theories with higher spin symmetry in 4d are holographically dual to free field theories. To describe interacting theories one needs to break the higher spin symmetry, which can be achieved by imposing different boundary conditions on the fields.

The boundary conditions for the supersymmetric theories were studied in [8] where the fields of spin $(0, \frac{1}{2}, 1)$ can break higher-spin symmetries as well as supersymmetry. Furthermore, different boundary conditions were related there to different supersymmetric 3d CFTs. To study the classical solutions we need to concern ourselves only with the boundary conditions of the bosonic fields — the scalar and vector components of B, whose fall-off near the boundary of AdS can take the form

$$B^{(0)} = \frac{1}{r}\left(\Gamma^+ \cos\gamma + i\Gamma^- \sin\gamma\right)\tilde{f}_1 + \frac{1}{r^2}\left(\Gamma^- \cos\gamma + i\Gamma^+ \sin\gamma\right)\tilde{f}_2 + O\!\left(\frac{1}{r^3}\right), \tag{5.1}$$

$$B^{(1)} = \frac{1}{r^2}\left[e^{i\beta}F_{\alpha\beta}y^\alpha y^\beta + \Gamma e^{-i\beta}\bar{F}_{\dot{\alpha}\dot{\beta}}\bar{y}^{\dot{\alpha}}\bar{y}^{\dot{\beta}}\right] + O\!\left(\frac{1}{r^3}\right), \tag{5.2}$$

where $\tilde{f}_{1,2}$, F, $\bar{F}$ are functions[7] of space-time and of the ϑ^i and the choice of asymptotics was imposed in [8] through them.

All our solutions are based on the DV solution and the asymptotics will be governed by the expansion of (4.1) using[8] (B.54)

$$B = \frac{4}{r}\,b(\eta_p + \eta_m) - \frac{2i}{r^2}\,b(\eta_p - \eta_m)\left(\frac{\kappa_{\alpha\beta}}{r}y^\alpha y^\beta + \frac{\kappa_{\dot{\alpha}\dot{\beta}}}{r}\bar{y}^{\dot{\alpha}}\bar{y}^{\dot{\beta}}\right) + O(Y^4)\,. \tag{5.3}$$

In particular we see that the scalar piece does not have a $1/r^2$ component, so we'll have to set $\tilde{f}_2 = 0$ throughout, and will rename $\tilde{f}_1$ as $\tilde{f}$.

In this section we study the subsets of supersymmetries preserved by the different boundary conditions proposed in [8] and see whether they can be preserved by the field configurations satisfying the equations of motion. In this sense, the counting of the preserved supersymmetries differs from the one of the previous section. The boundary conditions correspond to specific choices for $(\beta, \gamma, \tilde{f}_{1,2})$ which in turn constrain our ansatz (4.1). One important point to note here is that the following equation

$$4b(\eta_p + \eta_m) = \left(\Gamma^+ \cos\gamma + i\Gamma^- \sin\gamma\right)\tilde{f} \tag{5.4}$$

obtained from (5.1) and (5.3) tells us that the boundary conditions that are expressed through constraints on $\tilde{f}$ generically impose some relation between the two chiral parts $b_\pm$ of b.

The constraints imposed on $(\beta, \gamma, \tilde{f}_{1,2})$ are expressed in [8] in terms of commutation relations with the ϑ^i. To make the link with the previous section explicit, we choose a specific matrix representation for the ϑ^i, that we write down in appendix D.

In the previous section we worked in a basis where the master fields were diagonal. In this section we will choose a particular basis for the supersymmetry generators that preserve the boundary conditions (and β, γ, $\tilde{f}$), so we need in principle to repeat the analysis of section 4 in the specific bases. That requires to solve the BPS equation (4.10) for a general linear combination of the supersymmetries preserved by the imposed boundary conditions. In practice, we were able to choose the supersymmetry generators in the examples below such that they are compatible with the basis of solutions in section 4.

[7] F and $\bar{F}$ are of course related via the reality condition imposed on B.

[8] One should note that the DV solutions are obtained in a gauge in which they take the deceivingly simple form of a perturbation around the AdS_4 vacuum. However this gauge is different from the so-called *physical* gauge used in [8] for which there is a prescription to linearise the theory and to make contact with Fronsdal's theory. The complete form of the DV solutions are not known in this physical gauge. At first order in b however, the master fields B coincide in these two gauges, and this allows us to use (5.1) with (5.3) for our discussion. Starting from the next order in b, due to the non-linearity of the equations of motion there will most likely be additional terms both in the $1/r$ and $1/r^2$ branches.

5.1 $\mathcal{N} = 2$

We start with the bulk theories which are conjectured to be dual to $\mathcal{N} = 2$ Chern-Simons vector models with matter. We consider two types of boundary conditions that are dual to two different such theories.

5.1.1 SU(2) flavour symmetry

The first theory preserves the supersymmetries (3.11) with the ϑ^i-dependence restricted to $(\vartheta^1, \Gamma\vartheta^1)$.[9] It is the $n = 4$ supersymmetric extended Vasiliev theory with the boundary conditions

$$\beta = \gamma = \theta_0 \,, \qquad \left[\vartheta^1, \tilde{f}\right] = 0 \,, \tag{5.5}$$

which is conjectured in [8] to be dual to the $\mathcal{N} = 2$ Chern-Simons vector model with two fundamental chiral multiplets. The SU(2) flavour symmetry rotating the two chiral multiplets is identified with the SO(3) symmetry of rotations in ϑ^j, for $j = 2, 3, 4$.

To solve equation (4.10) we note that ξ^i is a linear combination of $\Gamma^\pm\vartheta^1$. For either of the signs we get the equations

$$\Gamma^\mp \eta_m b = \Gamma^\pm \eta_p b = 0 \,. \tag{5.6}$$

Since $\Gamma^\pm$ are orthogonal projectors, non-trivial solutions exist only when ξ^i equals either of the two, and not a general linear combination. For example, if $\xi^i = \Gamma^+\vartheta^1$, then η_m is an eigenstate of Γ^- while η_p is an eigenstate of Γ^+. Another way to say this is to note that if we write ξ^i in (4.14) in terms of ϑ^i (see appendix D), then it is a linear combination of $\Gamma^-\vartheta^i$, so in particular preserves $\Gamma^-\vartheta^1$, but none of the other examples will also preserve $\Gamma^+\vartheta^1$ (or a different pair $\Gamma^\pm\vartheta^i$, since we may be in a different basis). So indeed for the choice of $\Gamma^-\vartheta^1$ we find η_p and η_m as in (4.13) which is half BPS.

Next, the second equation in (5.5) tells us that $\tilde{f}$ is generated by 1, $\vartheta^2\vartheta^3$, $\vartheta^2\vartheta^4$ and $\vartheta^3\vartheta^4$. Since $\tilde{f}$ cannot be an eigenstate of $\Gamma^\pm$, equation (5.4) implies that b_+ and b_- are proportional to each other (with proportionality constant $\tan\theta$) and therefore the solution is given by four arbitrary parameters, a generic (not necessarily diagonal) b_+. Lastly, $B^{(1)}$ is not constrained further by (5.5).

We conclude that we have found a four-parameter family of half-BPS configurations to the theory with boundary conditions (5.5).

5.1.2 U(1) × U(1) flavour symmetry

In the second case with $\mathcal{N} = 2$ SUSY, the boundary conditions preserve the two supersymmetries generated by $(\vartheta^1, \vartheta^2)$ and are given by

$$\beta = \theta_0 \,, \qquad \gamma = \theta_0 P_{1,\vartheta^3\vartheta^4} \,, \qquad \tilde{f} \in \operatorname{span}\left\{1, \vartheta^3\vartheta^4, \vartheta^3\vartheta^1, \vartheta^3\vartheta^2, \vartheta^4\vartheta^1, \vartheta^4\vartheta^2\right\} \,, \tag{5.7}$$

where $P_{1,\vartheta^3\vartheta^4}$ projects onto the subspace spanned by $1, \vartheta^3\vartheta^4$. It is conjectured in [8] that the dual boundary theory is the $\mathcal{N} = 2$ Chern-Simons vector model with one fundamental

[9] Here we should note that in [8], the supersymmetry generators are parametrized by two constant spinors Λ_0 and Λ_- which are functions of the ϑ^i. Only the structure of Λ_0 has been explicitly given, and we here assume that Λ_- is generated by the same ϑ^i as Λ_0.

and one anti-fundamental chiral matter, with $\mathrm{U}(1)\times\mathrm{U}(1)$ flavour symmetry corresponding to the components of the bulk vector gauge field proportional to $1, \vartheta^3\vartheta^4$. The last statement in (5.7) means that each monomial generating $\tilde{f}$ commutes with either ϑ^1 and/or ϑ^2. The projector in γ in (5.7) allows us to rewrite (5.4) in the following way

$$4b(\eta_p+\eta_m) = \left(\Gamma^+\cos\theta_0 + i\Gamma^-\sin\theta_0\right)\tilde{f}(1,\vartheta^3\vartheta^4) + \Gamma^+\tilde{f}'(\vartheta^3\vartheta^1,\vartheta^3\vartheta^2,\vartheta^4\vartheta^1,\vartheta^4\vartheta^2) \qquad (5.8)$$

However, $\mathrm{span}\{\vartheta^3\vartheta^1,\vartheta^3\vartheta^2,\vartheta^4\vartheta^1,\vartheta^4\vartheta^2\} = \mathrm{span}\{\Gamma^\pm\vartheta^3\vartheta^1,\Gamma^\pm\vartheta^3\vartheta^2\}$, and the Γ^+ factor in front of f' restricts its dependence to $f'(\Gamma^+\vartheta^3\vartheta^1,\Gamma^+\vartheta^3\vartheta^2)$. We conclude that b has at most four real independent parameters.

To make the link with section 4, we note that the ξ^i in (4.14) all involve a chiral projection $\Gamma^-\vartheta^i$. These cannot be made out of the pair ϑ^1,ϑ^2 (or any rotations of them). On the other hand, the ξ^i in (4.15) is equal to $\vartheta^1 - i\vartheta^2$ (for appropriate values of the stars), so this configuration will be half-BPS (it is easy to see that it does not preserve the second supersymmetry).

Imposing (5.7) restricts through (5.4) the parameter space of the solutions. We find that $b(\eta_p+\eta_p)$ has two real parameters. We thus find a two-parameter family of half-BPS configuration for the theory with boundary conditions (5.7).

5.1.3 A one-parameter family of $\mathcal{N}=2$ theories

The boundary conditions just discussed above are in fact a special point in a one-parameter family of $\mathcal{N}=2$ theories preserving the same set of supersymmetries. The boundary conditions are similar to (5.7) with γ now having an additional term

$$\gamma = \theta_0 P_{1,\vartheta^3\vartheta^4} + \tilde{\alpha}P_{\vartheta^2\vartheta^4,\vartheta^1\vartheta^4}\,, \qquad (5.9)$$

where $\tilde{\alpha}$ is a real parameter. By taking $\tilde{\alpha}$, we recover (5.7),while taking $\tilde{\alpha}=\theta_0$ yields a theory with supersymmetry enhanced to $\mathcal{N}=3$ described in the next section. The discussion for preservation of supersymmetry is the same as before. The difference is that (5.4) now reads

$$\begin{aligned}4b(\eta_p+\eta_m) = &\left(\Gamma^+\cos\theta_0 + i\Gamma^-\sin\theta_0\right)\tilde{f}(1,\vartheta^3\vartheta^4) + \Gamma^+\tilde{f}'(\vartheta^3\vartheta^1,\vartheta^3\vartheta^2)\\ &+\left(\Gamma^+\cos\tilde{\alpha} + i\Gamma^-\sin\tilde{\alpha}\right)\tilde{f}''(\vartheta^2\vartheta^4,\vartheta^1\vartheta^4)\end{aligned}$$

The situation is similar as to the one for $\tilde{\alpha}=0$, as we find the same number of parameters.

5.2 $\mathcal{N}=3$

The boundary conditions given by

$$\beta=\gamma=\theta_0\,, \qquad \tilde{f}\in\mathrm{span}\{1,\vartheta^1\vartheta^4,\vartheta^2\vartheta^4,\vartheta^3\vartheta^4\}\,, \qquad (5.10)$$

preserve the supersymmetries generated by $(\vartheta^1,\vartheta^2,\vartheta^3)$, and yield a bulk theory conjectured to be dual to the $\mathcal{N}=3$ Chern-Simons vector model with a single fundamental hypermultiplet.

As in the last example, there is no chiral supersymmetry preserved by the boundary conditions, which excludes solutions of the type (4.13). But we can look at solutions which preserve the same supercharge as in the previous case, $\xi^i = \vartheta^1 - i\vartheta^2$, which corresponds to (4.15). It is easy to verify that no other linear combination of ϑ^1, ϑ^2 and ϑ^3 are preserved, so these configurations are 1/3-BPS.

As before, the second equation in (5.10) reduces the parameter space to two real parameters. This leads to a two-parameter family of 1/3-BPS configuration for the theory with boundary conditions (5.10).

5.2.1 A one-parameter family of $\mathcal{N} = 3$ theories

The boundary conditions just discussed are in fact a special point in a one parameter family of $\mathcal{N} = 3$ theories preserving the same set of supersymmetries. The boundary conditions are

$$\beta = \theta_0(1 - P_\Gamma) + \tilde{\beta} P_\Gamma\,, \quad \gamma = \theta_0 P_1 + \tilde{\beta} P_{\vartheta^1\vartheta^4,\vartheta^2\vartheta^4,\vartheta^3\vartheta^4}\,, \quad \tilde{f} \in \operatorname{span}\{1, \vartheta^1\vartheta^4, \vartheta^2\vartheta^4, \vartheta^3\vartheta^4\}\,, \tag{5.11}$$

where $\tilde{\beta}$ is a real parameter. At $\tilde{\beta} = \theta_0$, we recover (5.10), while at $\tilde{\beta} = 0$, the supersymmetry is enhanced to $\mathcal{N} = 4$ and corresponds to the case described in the next section. With these boundary conditions, (5.4) reads

$$\begin{aligned} 4b(\eta_p + \eta_m) = &\left(\Gamma^+ \cos\theta_0 + i\Gamma^- \sin\theta_0\right) \tilde{f}(1) \\ &+ \left(\Gamma^+ \cos\tilde{\beta} + i\Gamma^- \sin\tilde{\beta}\right) \tilde{f}(\vartheta^1\vartheta^4, \vartheta^2\vartheta^4, \vartheta^3\vartheta^4) \end{aligned} \tag{5.12}$$

We again find that the space of 1/3-BPS solutions has two real parameters.

5.3 $\mathcal{N} = 4$

The boundary conditions given by

$$\beta = \theta_0\,(1 - P_\Gamma)\,, \qquad \gamma = \theta_0 P_1\,, \qquad P_\Gamma \tilde{f} = 0\,, \tag{5.13}$$

preserve the supersymmetries generated by $(\vartheta^1, \vartheta^2, \vartheta^3, \vartheta^4)$, and yield a bulk theory conjectured to be dual to the $\mathcal{N} = 4$ Chern-Simons quiver theory with gauge group $\mathrm{U}(N)_k \times \mathrm{U}(1)_{-k}$ and a single bi-fundamental hypermultiplet.

The presence of the projector P_1 in the second equation of (5.13) restricts greatly the type of field configurations that we can have. We obtain

$$4b = \left(\Gamma^+ \cos\theta_0 + i\Gamma^- \sin\theta_0\right) \tilde{f}(1) + \Gamma^+ \tilde{f}'(\vartheta^1\vartheta^2, \vartheta^1\vartheta^3, \vartheta^1\vartheta^4)\,. \tag{5.14}$$

We can now repeat the procedure of the previous subsections. As in the last two cases, the configurations (4.13) are excluded by the absence of supersymmetry parameters of the form $\Gamma^\pm\vartheta^i$, so we focus again on (4.15). Clearly this preserves $\vartheta_1 - i\vartheta_2$ and is 1/4-BPS.

For $\theta_0 \neq 0$ equation (5.14) requires $\Gamma^- b$ to be proportional to the identity, so in (4.15) we need to set $b_{-,1} = b_{-,2}$ (4.2). $\Gamma^+ b$ is then also related to them. The remaining free parameter is $b_{+,1} - b_{+,2}$, so there are overall two parameters for this 1/4-BPS solution.

A special case is when the lower right block of b vanishes, so $b_{-,1} = b_{-,2} = 0$. This corresponds to case (4.17), which is compatible with two of our preserved supercharges, $\vartheta^1 - i\vartheta^2$ and $\vartheta^3 - i\vartheta^4$. In this case we have only one parameter, since (5.14) now enforces $b_{+,1} = -b_{+,2}$.

We conclude that for these boundary conditions there is a two-parameter family of embeddings of the DV-solution which are 1/4-BPS and a one-parameter family which is 1/2-BPS.

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5.4 $\mathcal{N} = 6$

We will now consider the bulk theory with $n = 6$ extended supersymmetry. The boundary conditions

$$\beta = \theta_0(1 - P_\Gamma) - \theta_0 P_\Gamma\,, \qquad \gamma = \theta_0 P_{1,\vartheta^i\vartheta^j}\,, \qquad P_{\Gamma,\vartheta^i\vartheta^j\Gamma}\tilde{f} = 0\,, \tag{5.15}$$

where $\vartheta^i\vartheta^j$ stands for all such terms with $i, j = 1, \ldots, 6$, preserve the supersymmetries generated by $\vartheta^1, \vartheta^2, \vartheta^3, \vartheta^4, \vartheta^5, \vartheta^6$. Upon adding a $\mathrm{U}(M)$ Chan-Paton factor, it is proposed in [8] that the dual theory is the $\mathrm{U}(N)_k \times \mathrm{U}(M)_{-k}$ ABJ model in the large N, k, fixed M limit. We can then proceed as before and write (5.4) as

$$b(\eta_p + \eta_m) = \frac{1}{4}\left(\Gamma^+ \cos\theta_0 + i\Gamma^- \sin\theta_0\right)\tilde{f}(1, \vartheta^i\vartheta^j) \tag{5.16}$$

where we have used the last equation in (5.15) which tells us that $\tilde{f}$ is only spanned by $1, \vartheta^i\vartheta^j$.

Now for this $n = 6$ case we will work the other way around. Say we want to find the configurations which preserve the supersymmetry $\vartheta^1 - i\vartheta^2$. Then the most general configuration (that is the one with the greatest number of parameters) is given by (as long as b, η_p, η_m are diagonal)

$$\eta_p = \mathrm{diag}(1, 0, 1, 0, 1, 0, 1, 0)\,, \quad \eta_m = \mathrm{diag}(0, 1, 0, 1, 0, 1, 0, 1) \tag{5.17}$$

which is thus 1/6-BPS. We can then consider degenerate cases by taking some of the non-zero entries above to zero. We then find that the following configuration

$$\eta_p = \mathrm{diag}(1, 0, 0, 0, 1, 0, 0, 0)\,, \quad \eta_m = \mathrm{diag}(0, 1, 0, 0, 0, 1, 0, 0) \tag{5.18}$$

preserves $\vartheta^1 - i\vartheta^2$ and $\vartheta^3 - i\vartheta^4$, and so is 1/3-BPS. Then the configuration

$$\eta_p = \mathrm{diag}(1, 0, 0, 0, 0, 0, 0, 0)\,, \quad \eta_m = \mathrm{diag}(0, 0, 0, 0, 0, 1, 0, 0) \tag{5.19}$$

preserves $\vartheta^1 - i\vartheta^2$, $\vartheta^3 - i\vartheta^4$ and $\vartheta^5 - i\vartheta^6$, and is thus half-BPS. However this last case is not compatible with (5.16).

6 Discussion

We have developed the tools to study embeddings of solutions of bosonic higher spin theory into its supersymmetric extensions and implemented them for the case of the DV solution. In the process we also simplified the solution and generalized it to arbitrary parity violating phase. The final result of our study is presented in the preceding section, where we checked which of the possible embeddings are compatible with the boundary conditions and supersymmetries of theories conjectured to be dual to several different 3d Chern-Simons vector models.

One of the theories we considered has an SU(2) flavour symmetry and $\mathcal{N} = 2$ supersymmetry. We found 1/2-BPS solutions parametrized by an arbitrary 2×2 Hermitian

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matrix. The matrix structure should be associated to a U(2) flavour symmetry (the center of which is normally gauged, in the field theory dual). Thus ignoring the center and choosing a Cartan there is a single parameter.

In the case with $\mathrm{U}(1) \times \mathrm{U}(1)$ symmetry there are two parameters for the 1/2-BPS solution, which matches with a single free parameter for each element of the Cartan.

This theory can be enhanced (on both sides of the duality) to $\mathcal{N} = 3$. We find though the exact same set of BPS solutions, which are now 1/3-BPS.

Lastly we consider a theory with $\mathcal{N} = 4$ supersymmetry. Again we find a 2-parameter family of solutions with the same number of supersymmetries as before, which in this case are 1/4-BPS. If we restrict to a subspace of these solutions we find a one dimensional family of 1/2-BPS solutions.

We should mention that part of this analysis was carried out (in a somewhat different supersymmetric formalism) in [11], and there it was found that the solution preserves only 1/4 of the bulk supersymmetries. We do not understand the reason for the discrepancy.

The DV solutions have $\mathrm{SO}(3) \times \mathbb{R}$ global symmetry [11] so it is natural to identify them with local operators in the dual field theory and ask whether we have found the higher spin dual of all such 1/2-BPS operators.

The holographic duality of 4d higher spin theories has been studied at the level of matching of perturbative spectrum and correlation functions. Normally it is very hard to match a classical asymptotically AdS bulk solution with a state in the field theory, since the spectrum of high-dimension operators in an interacting field theory is very complicated. Studying BPS protected operators eliminates this problem, as they can be rather easily identified and classified and their dimensions are fixed by their charges. Our analysis allows the first identification of solutions of higher spin theory with states in the dual CFT (other than the vacuum).

Indeed, a rudimentary examination of the index for some of these theories [24] indicates that the dimension of the space of 1/2-BPS operators does match the Cartan of the flavour group, as we have found. More precisely, the three-dimensional index as defined in [24] is given by a trace formula which gets contributions from states that preserve a single supercharge labelled Q (and an associated superconformal generator $Q^\dagger$). The trace then contains fugacities that correspond to the Cartans of the subalgebra of the full superconformal algebra commuting with $Q, Q^\dagger$, as well as the Cartans for the flavours. In the spirit of the work done in [25], the index can be refined further by enhancing supersymmetry, such that the contributions to the trace come from states that preserve at least half of the supersymmetries. In the three-dimensional case, the only remaining fugacities are then the Cartans of the flavours. It would be worthwhile studying this in more detail. Finding a precise match should then lead to a quantization of the space of solutions to match the discrete dimensions of operators in the dual field theory.

There are obvious generalizations of our analysis. First, a generalization which we have mentioned but have not analysed in detail, are theories with extra Chan-Paton factors. These are dual to more general Chern-Simons models, whose degrees of freedom are rectangular matrices, not just single-column vector models.

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Lastly one can study the embeddings of other classical solutions, like those of [10, 12, 13], or find new solutions and embed them.

Supersymmetric higher spin theories provide a fertile ground for deeper understanding of the higher spin holographic duality.

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A Notations and conventions

A.1 Spinors

The word *spinor* here designates complex 2-dimensional vectors belonging to the fundamental or anti-fundamental representation of the group $Sl(2,\mathbb{C})$ (SU(2)) when using a Lorentzian (Euclidian) metric. Indices are raised and lowered in the following way

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta\,, \qquad \psi_\alpha = \psi^\beta\epsilon_{\beta\alpha}, \tag{A.1}$$

where similar rules apply for dotted spinors, and where

$$\begin{aligned}\epsilon^{12} &= -\epsilon^{21} = \epsilon_{12} = -\epsilon_{21} = 1\\ \epsilon_{\alpha\beta}\epsilon^{\beta\gamma} &= -\delta_\alpha^\gamma\\ \epsilon^{\alpha\beta} &= \bar\epsilon^{\dot\alpha\dot\beta}\end{aligned} \tag{A.2}$$

In Lorentzian metric, we choose the explicit form for the hermitian soldering form

$$\sigma^0 = \frac{I}{\sqrt{2}}\,, \qquad \sigma^1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad \sigma^2 = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \qquad \sigma^3 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{A.3}$$

so that we have the following useful relations

$$\begin{aligned}(\bar\sigma^a)^{\dot\alpha\beta} &\equiv \epsilon^{\dot\alpha\dot\gamma}\epsilon^{\beta\delta}(\sigma^a)_{\delta\dot\gamma}\,, & \mathrm{Tr}[\sigma^a\bar\sigma^b] &= \eta^{ab}\,,\\ (\sigma^a)_{\alpha\dot\beta}(\bar\sigma_a)^{\dot\gamma\delta} &= \delta_\alpha^\delta\delta_{\dot\beta}^{\dot\gamma}\,, & (\sigma^a)_{\alpha\dot\beta}(\sigma_a)_{\gamma\dot\delta} &= \epsilon_{\alpha\gamma}\epsilon_{\dot\beta\dot\delta}\,,\\ (\sigma^a\bar\sigma^b + \sigma^b\bar\sigma^a)_\alpha^{\ \beta} &= \eta^{ab}\delta_\alpha^\beta\,.\end{aligned} \tag{A.4}$$

The maps from multispinors to tensors of SO(3,1) are then given by

$$T^{\nu_1\ldots\nu_s}_{\mu_1\ldots\mu_n} = (\sigma^{\nu_1})_{\alpha_1\dot\alpha_1}\cdots(\sigma^{\nu_s})_{\alpha_s\dot\alpha_s}(\sigma_{\mu_1})^{\dot\beta_1\beta_1}\cdots(\sigma_{\mu_n})^{\dot\beta_n\beta_n}T^{\dot\alpha_1\alpha_1\ldots\dot\alpha_s\alpha_s}_{\beta_1\dot\beta_1\ldots\beta_n\dot\beta_n}\,, \tag{A.5}$$

$$T^{\dot{\alpha}_1\alpha_1\ldots\dot{\alpha}_s\alpha_s}_{\beta_1\dot{\beta}_1\ldots\beta_n\dot{\beta}_n} = (\sigma_{\nu_1})^{\dot{\alpha}_1\alpha_1}\cdots(\sigma_{\nu_s})^{\dot{\alpha}_s\alpha_s}(\sigma^{\mu_1})_{\beta_1\dot{\beta}_1}\cdots(\sigma^{\mu_n})_{\beta_n\dot{\beta}_n}T^{\nu_1\ldots\nu_s}_{\mu_1\ldots\mu_n}. \tag{A.6}$$

The ones we will be particularly interested in are

$$\begin{aligned} h^a &= (\bar{\sigma}^a)^{\dot{\gamma}\alpha}h_{\alpha\dot{\gamma}}\,, \quad \omega_{ab} = -i\left((\sigma_{ab})^{\dot{\beta}}{}_{\dot{\alpha}}\omega^{\dot{\alpha}}{}_{\dot{\beta}} - (\sigma_{ab})_{\alpha}{}^{\beta}\omega_{\beta}{}^{\alpha}\right)\,, \\ h_{\alpha\dot{\beta}} &= h^a(\sigma_a)_{\alpha\dot{\beta}}\,, \quad \omega_{\alpha}{}^{\beta} = -\frac{i}{2}\omega_{ab}(\sigma^{ab})_{\alpha}{}^{\beta}\,, \qquad \omega^{\dot{\alpha}}{}_{\dot{\beta}} = \frac{i}{2}\omega_{ab}(\bar{\sigma}^{ab})^{\dot{\alpha}}{}_{\dot{\beta}}, \end{aligned} \tag{A.7}$$

where we have defined the following hermitian matrices

$$(\sigma^{ab})_{\alpha}{}^{\beta} \equiv \frac{i}{2}(\sigma^a\bar{\sigma}^b - \sigma^b\bar{\sigma}^a)_{\alpha}{}^{\beta}, \qquad (\bar{\sigma}^{ab})^{\dot{\beta}}{}_{\dot{\alpha}} \equiv \frac{i}{2}(\bar{\sigma}^a\sigma^b - \bar{\sigma}^b\sigma^a)^{\dot{\beta}}{}_{\dot{\alpha}}, \tag{A.8}$$

which satisfy the Lorentz algebra

$$\left[\sigma^{ab}, \sigma^{cd}\right] = i\left(\eta^{bc}\sigma^{ad} - \eta^{ac}\sigma^{bd} + \eta^{ad}\sigma^{bc} - \eta^{bd}\sigma^{ac}\right). \tag{A.9}$$

as well as

$$(\sigma_{ab})_{\alpha}{}^{\beta}(\sigma^{ab})_{\gamma}{}^{\delta} = \delta^{\delta}_{\alpha}\delta^{\beta}_{\gamma} - \epsilon_{\alpha\gamma}\epsilon^{\beta\delta}\,, \qquad (\bar{\sigma}_{ab})^{\dot{\beta}}{}_{\dot{\alpha}}(\sigma^{ab})_{\gamma}{}^{\delta} = 0. \tag{A.10}$$

We can then define the four dimensional γ matrices

$$\gamma^a = \begin{pmatrix} 0 & \sigma^a \\ \bar{\sigma}^a & 0 \end{pmatrix}, \qquad \gamma^{ab} \equiv \frac{i}{2}\left[\gamma^a, \gamma^b\right] = \begin{pmatrix} \sigma^{ab} & 0 \\ 0 & \bar{\sigma}^{ab} \end{pmatrix}. \tag{A.11}$$

with

$$\left\{\gamma^a, \gamma^b\right\} = \eta^{ab}. \tag{A.12}$$

A.2 Star-product formulae

The equations of motion (2.3)–(2.4) are written in this compact form with the use of a star-product defined as

$$\begin{aligned} \Phi(Y,Z)\star\Theta(Y,Z) &= \Phi(Y,Z)\exp\left[-\epsilon^{\alpha\beta}\big(\overleftarrow{\partial}_{y^\alpha} + \overleftarrow{\partial}_{z^\alpha}\big)\big(\overrightarrow{\partial}_{y^\beta} - \overrightarrow{\partial}_{z^\beta}\big) + \text{c.c.}\right]\Theta(Y,Z) \\ &= \int d^2u d^2\bar{u} d^2v d^2\bar{v}\, \Phi(Y{+}U, Z{+}U)\Theta(Y{+}V, Z{-}V)e^{\left(u_\alpha v^\alpha + \bar{u}_{\dot{\alpha}}\bar{v}^{\dot{\alpha}}\right)} \end{aligned} \tag{A.13}$$

where the integral in the second line is implicitly normalized and the integration contour chosen in such a way that

$$\Phi\star 1 = 1\star\Phi = 1. \tag{A.14}$$

More precisely, this means that we assume the following representation of the delta function

$$\int du^2 dv^2 e^{u_\alpha v^\alpha} = \int du^2\delta^2(u)\,, \tag{A.15}$$

and similarly for the dotted variables. This is equivalent to integrating over the real axis rotated by a certain angle. We see that if u_1 and v_2 are rotated by $\pi/4$, then we obtain a valid representation of the delta function. However, the rotation angle is not entirely fixed, as can be seen from the fact that we could have chosen u_1 and v_2 to be both rotated by $-\pi/4$ instead. Our choice will be the following

$$\begin{aligned} &u_1\,,\,v_2 \quad \text{rotated by } -\frac{\pi}{4} \\ &u_2\,,\,v_1 \quad \text{rotated by } \frac{\pi}{4} \end{aligned} \tag{A.16}$$

and similarly for the dotted variables.

With this definition, we have

$$[y_\alpha, y_\beta]_\star = -2\epsilon_{\alpha\beta}\,, \qquad [\bar{y}_{\dot{\alpha}}, \bar{y}_{\dot{\beta}}]_\star = -2\epsilon_{\dot{\alpha}\dot{\beta}}\,, \qquad [y_\alpha, \bar{y}_{\dot{\beta}}]_\star = 0\,, \tag{A.17}$$

$$[z_\alpha, z_\beta]_\star = 2\epsilon_{\alpha\beta}\,, \qquad [\bar{z}_{\dot{\alpha}}, \bar{z}_{\dot{\beta}}]_\star = 2\epsilon_{\dot{\alpha}\dot{\beta}}\,, \qquad [z_\alpha, \bar{z}_{\dot{\beta}}]_\star = 0\,, \tag{A.18}$$

as well as the following properties which will be used in appendix B

$$v = \delta(y) \star \delta(z)\,, \qquad \delta(y) \star \delta(y) = \delta(z) \star \delta(z) = 1. \tag{A.19}$$

Note that the bilinears

$$L_{\alpha\beta} = \frac{1}{2}\{y_\alpha, y_\beta\}\,, \qquad \bar{L}_{\dot{\alpha}\dot{\beta}} = \frac{1}{2}\{\bar{y}_{\dot{\alpha}}, \bar{y}_{\dot{\beta}}\}\,, \qquad P_{\alpha\dot{\beta}} = y_\alpha \bar{y}_{\dot{\beta}}\,, \tag{A.20}$$

give an oscillator representation of the algebra $so(3,2) \simeq sp(4)$, which is the unique finite dimensional subalgebra of the higher spin algebra spanned by general polynomials in Y of the form

$$P(y,\bar{y}) = \sum_{n,m=0}^{\infty} P^{\alpha_1\ldots\alpha_n,\dot{\alpha}_1\ldots\dot{\alpha}_m} y_{\alpha_1}\ldots y_{\alpha_n} \bar{y}_{\dot{\alpha}_1}\ldots\bar{y}_{\dot{\alpha}_m}. \tag{A.21}$$

In perturbation theory, it is often needed to evaluate the star product of W_0 (B.1) with an arbitrary function. If Φ is a zero-form, we obtain

$$\Big[W_{\alpha\beta}y^\alpha y^\beta, \Phi(Y,Z)\Big]_\star = -4W_{\alpha\beta}\epsilon^{\alpha\gamma}\epsilon^{\beta\delta}\left\{y_\delta \partial_{y^\gamma} + \partial_{y^\gamma}\partial_{z^\delta}\right\}\Phi(Y,Z) \tag{A.22}$$

$$\Big[W_{\dot{\alpha}\dot{\beta}}\bar{y}^{\dot{\alpha}}\bar{y}^{\dot{\beta}}, \Phi(Y,Z)\Big]_\star = -4W_{\dot{\alpha}\dot{\beta}}\epsilon^{\dot{\alpha}\dot{\gamma}}\epsilon^{\dot{\beta}\dot{\delta}}\left\{\bar{y}_{\dot{\delta}} \partial_{\bar{y}^{\dot{\gamma}}} + \partial_{\bar{y}^{\dot{\gamma}}}\partial_{\bar{z}^{\dot{\delta}}}\right\}\Phi(Y,Z) \tag{A.23}$$

$$\Big[W_{\alpha\dot{\beta}}y^\alpha \bar{y}^{\dot{\beta}}, \Phi(Y,Z)\Big]_\star = -2W_{\alpha\dot{\beta}}\epsilon^{\alpha\gamma}\epsilon^{\dot{\beta}\dot{\delta}}\Big\{\bar{y}_{\dot{\delta}}\partial_{y^\gamma} + y_\gamma \partial_{\bar{y}^{\dot{\delta}}} + \Big(\partial_{z^\gamma}\partial_{\bar{y}^{\dot{\delta}}} + \partial_{y^\gamma}\partial_{\bar{z}^{\dot{\delta}}}\Big)\Big\}\Phi(Y,Z) \tag{A.24}$$

If Φ is a one-form, it will appear in the equations in anti-commutators, and we can use the above formula without changing signs, and replacing $[\cdot,\cdot]_\star \to \{\cdot,\cdot\}_\star$. If Φ transforms in the so-called twisted adjoint representation, we are led to evaluate quantities of the form $\Phi \star \pi(W_0) - W_0 \star \Phi$. We obtain

$$\begin{aligned} \Phi \star \pi(W_0) - W_0 \star \Phi = &-\Big[W_{\alpha\beta}y^\alpha y^\beta, \Phi(y,\bar{y},z)\Big]_\star - \Big[W_{\dot{\alpha}\dot{\beta}}\bar{y}^{\dot{\alpha}}\bar{y}^{\dot{\beta}}, \Phi(y,\bar{y},z)\Big]_\star \\ &+ W_{\alpha\dot{\beta}}\Big[-2y^\alpha \bar{y}^{\dot{\beta}} - 2\epsilon^{\alpha\gamma}\bar{y}^{\dot{\beta}}\partial_{z^\gamma} - 2\epsilon^{\dot{\beta}\dot{\gamma}}y^\alpha \partial_{\bar{z}^{\dot{\gamma}}} - 2\epsilon^{\alpha\gamma}\epsilon^{\dot{\beta}\dot{\delta}}\Big(\partial_{y^\gamma}\partial_{\bar{y}^{\dot{\delta}}} + \partial_{z^\gamma}\partial_{\bar{z}^{\dot{\delta}}}\Big)\Big]\Phi(Y,Z). \end{aligned} \tag{A.25}$$

A.3 Conventions in the literature

We will here briefly describe different conventions used by other authors. Setting aside conventions used for spinors and soldering forms, the most fundamental choice of convention is usually made at the level of the definition of the canonical commutation relations of the auxiliary spinor variables. We have opted for

$$[y_\alpha, y_\beta]_\star = -[z_\alpha, z_\beta]_\star = -2\epsilon_{\alpha\beta}\,, \qquad \left[\bar{y}_{\dot\alpha}, \bar{y}_{\dot\beta}\right]_\star = -\left[\bar{z}_{\dot\alpha}, \bar{z}_{\dot\beta}\right]_\star = -2\epsilon_{\dot\alpha\dot\beta}. \tag{A.26}$$

In many places, one will find the following definitions

$$[y_\alpha, y_\beta]_\star = -[z_\alpha, z_\beta]_\star = 2i\epsilon_{\alpha\beta}\,, \qquad \left[\bar{y}_{\dot\alpha}, \bar{y}_{\dot\beta}\right]_\star = -\left[\bar{z}_{\dot\alpha}, \bar{z}_{\dot\beta}\right]_\star = 2i\epsilon_{\dot\alpha\dot\beta}. \tag{A.27}$$

With this choice, the star-product (2.5) becomes

$$f(Y,Z)\star g(Y,Z) = f(Y,Z)\exp i\left[\epsilon^{\alpha\beta}\left(\overleftarrow{\partial}_{y^\alpha}+\overleftarrow{\partial}_{z^\alpha}\right)\left(\overrightarrow{\partial}_{y^\beta}-\overrightarrow{\partial}_{z^\beta}\right)+\text{c.c.}\right]g(Y,Z). \tag{A.28}$$

Hermitian conjugation can of course still be defined, but in a less trivial way. Looking again at the simplest case of the AdS vacuum solution, we see that the definition (B.1)–(B.2) is no longer valid, in the sense that the components do no longer correspond to the ones that satisfy the first order formulation of the Einstein equations. To circumvent this, authors usually expand the field W as (see [15])

$$W(x,y,\bar{y}) = W_\mu(x,y,\bar{y})dx^\mu = \sum_{\substack{n,m=0\\ n+m=even}}^{\infty}\frac{1}{2in!m!}W_{\alpha_1\ldots\alpha_n,\dot\alpha_1\ldots\dot\alpha_m}y^{\alpha_1}\cdots y^{\alpha_n}\bar{y}^{\dot\alpha_1}\cdots\bar{y}^{\dot\alpha_m}\,, \tag{A.29}$$

where the important point is the presence of the factor i. Then one can identify the components of the bilinears in the oscillators with the Lorentz connection and vielbein as we did, as long as the reality condition is chosen to be $(W)^\dagger = -W$. For the fully non-linear theory, with the Z dependence included, one has to modify the way the oscillators transform

$$(y^\alpha)^\dagger = \bar{y}^{\dot\alpha}\,, \qquad (z^\alpha)^\dagger = -\bar{z}^{\dot\alpha}\,, \tag{A.30}$$

so that for two functions f, g of $(Y, Z|x)$ we then obtain

$$(f\star g)^\dagger = g^\dagger\star f^\dagger\,, \tag{A.31}$$

where it should be noted that the order of the functions has been inverted. Once more one has to introduce appropriate minus signs when manipulating forms, as the hermitian conjugation is blind to the wedge product. For consistency with the equations of motion involving the field B, this in turn imposes that $B^\dagger = v\star B\star v$. Finally, we would like to point out some possible differences in the definitions of the master field S. Indeed, S_0 can be taken to be identically zero, in which case the differential operator in (2.3) should be replaced as follows

$$d = dx^\mu\frac{\partial}{\partial x^\mu} \quad\to\quad \hat{d}\equiv dx^\mu\frac{\partial}{\partial x^\mu}+dz^\alpha\frac{\partial}{\partial z^\alpha}+d\bar{z}^{\dot\alpha}\frac{\partial}{\partial\bar{z}^{\dot\alpha}}. \tag{A.32}$$

B The Didenko-Vasiliev solution

We review here the construction of the Didenko-Vasiliev solution [11]. We start with the AdS_4 vacuum solution, then the extremal AdS-black-hole solution. Then we present the solution of the free higher-spin theory and finally the full solution to the interacting theory employed in the main text.

B.1 The vacuum solution

The AdS_4 vacuum solution in the interacting theory is given by

$$W_0 = -\frac{1}{4}\left(\omega_{\alpha\beta}y^\alpha y^\beta + \omega_{\dot\alpha\dot\beta}\bar y^{\dot\alpha}\bar y^{\dot\beta} - \sqrt{2}h_{\alpha\dot\beta}y^\alpha\bar y^{\dot\beta}\right), \tag{B.1}$$

$$B_0 = 0\,, \qquad S_0 = z_\alpha dz^\alpha + \bar z_{\dot\alpha}d\bar z^{\dot\alpha}. \tag{B.2}$$

One can define a perturbation scheme around the vacuum solution, and the obtained linearised equations of motion are equivalent to the ones written by Fronsdal in [26] for real massless fields propagating in AdS_4 (see [15] for an explicit derivation).

In this paper we will be using global coordinates for AdS_4

$$ds^2 = g^0_{\mu\nu}dx^\mu dx^\nu \equiv (1+\lambda^{-2}r^2)dt^2 - \frac{1}{1+\lambda^{-2}r^2}dr^2 - r^2(d\theta^2+\sin^2\theta\, d\varphi^2)\,. \tag{B.3}$$

From here onwards we set the AdS scale $\lambda \to 1$ for simplicity. We choose the following explicit expressions for the vielbeins

$$h^0 = \sqrt{1+r^2}\,dt\,, \qquad h^1 = \frac{1}{\sqrt{1+r^2}}\,dr\,, \qquad h^2 = r\,d\theta\,, \qquad h^3 = r\sin\theta\, d\varphi\,, \tag{B.4}$$

and obtain the connection one-forms

$$\omega_{01} = r\,dt\,, \quad \omega_{12} = \sqrt{1+r^2}\,d\theta\,, \quad \omega_{13} = \sqrt{1+r^2}\sin\theta\, d\varphi\,, \quad \omega_{23} = \cos\theta\, d\varphi\,, \tag{B.5}$$

with all others are zero. The connections and vielbein $\omega_{\alpha\beta}$, $\omega_{\dot\alpha\dot\beta}$, $h_{\alpha\dot\beta}$ appearing in (B.1) are then obtained using appendix A.1.

B.2 Black-holes in AdS_4

The simplest black holes in 4-dimensions are characterized by an $SO(3)\times\mathbb{R}$ symmetry. The black-hole solution in AdS_4 can be written in the so-called Kerr-Schild form (see [27] and [28])

$$g_{\mu\nu} = g^0_{\mu\nu} - \frac{2M}{r}k_\mu k_\nu\,, \qquad g^{\mu\nu} = g^{0\mu\nu} + \frac{2M}{r}k^\mu k^\nu\,, \tag{B.6}$$

with

$$k_\mu dx^\mu = dt - \frac{dr}{1+r^2}. \tag{B.7}$$

Using this definition, one can define the following traceless completely symmetric tensors

$$\phi_{\mu_1\ldots\mu_s} = \frac{2M}{r}k_{\mu_1}\ldots k_{\mu_s}\,, \tag{B.8}$$

which satisfy the equations of motion of a massless spin-s field in a AdS background

$$D^{\mu}D_{\mu}\phi_{\nu(s)} - sD_{\mu}D_{\nu}\phi^{\mu}_{\nu(s-1)} = -2(s-1)(s+1)\phi_{\nu(s)}. \tag{B.9}$$

From these facts we learn a number of things which will be directly transposed in the construction of the fully interacting theory. First, a black-hole solution in AdS_4 can be written as a one loop perturbation. The perturbation is constructed from a vector k which in turn generates an infinite tower of massless higher-spin fields. Finally, it should be noted that this Kerr vector k can be expressed in terms of the killing vector $V = \sqrt{2}\partial/\partial t$ and the associated Killing two-form $\kappa_{\alpha\beta}, \kappa_{\dot{\alpha}\dot{\beta}}$ through this simple formula

$$k_{\alpha\dot{\alpha}} = \frac{1}{1+r^2}\left(v_{\alpha\dot{\alpha}} - \frac{\kappa_{\alpha}{}^{\beta}v_{\beta\dot{\alpha}}}{r}\right). \tag{B.10}$$

In the next section we explain in detail the role of these quantities.

B.3 Killing matrix

The main ingredients in the construction of the Didenko-Vasiliev (DV) solution are the Killing vectors and Killing two-forms of the vacuum. From (2.10), we see that a global symmetry is generated by a Z-independent gauge parameter $\epsilon = \epsilon(Y|x)$ satisfying

$$d\epsilon - [W_0, \epsilon]_{\star} = 0\,. \tag{B.11}$$

If we take ϵ to be bilinear in Y and identify $v_{\alpha\dot{\beta}}$ with a Killing vector V_{μ}, and $\kappa_{\alpha\beta}$, $\kappa_{\dot{\alpha}\dot{\beta}}$ with the self and anti-self dual parts of the Killing two-form $V_{\mu\nu} = D_{[\mu}V_{\nu]}$ associated with the same Killing vector, (B.11) reduces to the Killing equation with the identification

$$K_{AB} = \begin{pmatrix} \sqrt{2}\kappa_{\alpha\beta} & v_{\alpha\dot{\beta}} \\ v_{\alpha\dot{\beta}} & \sqrt{2}\kappa_{\dot{\alpha}\dot{\beta}} \end{pmatrix}\,, \tag{B.12}$$

to which we will be referring to as the Killing matrix. In Sp(4) notations, (B.11) can be written in a compact way as

$$D_0\left(K_{AB}Y^AY^B\right) = 0. \tag{B.13}$$

In what follows we will be using a particular time-like Killing vector defined as

$$V = V^{\mu}\frac{\partial}{\partial x^{\mu}} = \sqrt{2}\frac{\partial}{\partial t} \tag{B.14}$$

The various factors in the definition of the Killing matrix in terms of the Killing vector and Killing two-forms have been chosen so that we have the following identity

$$K_A{}^B K_B{}^C = -\delta_A^C\,, \tag{B.15}$$

which reads in components

$$\kappa_{\alpha}{}^{\beta}\kappa_{\beta\gamma} \equiv -\epsilon_{\alpha\gamma}\kappa^2\,, \qquad\qquad \kappa^2 \equiv \frac{1}{2}\kappa_{\alpha\beta}\kappa^{\alpha\beta} = \det(\kappa_{\alpha\beta})\,, \tag{B.16}$$

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$$2\kappa_\alpha{}^\beta\kappa_{\beta\gamma} + v_\alpha{}^{\dot\beta}v_{\dot\beta\gamma} = -\epsilon_{\alpha\gamma}\,, \qquad 2\kappa_{\dot\alpha}{}^{\dot\beta}\kappa_{\dot\beta\dot\gamma} + v_{\dot\alpha}{}^{\beta}v_{\beta\dot\gamma} = -\epsilon_{\dot\alpha\dot\gamma}\,, \tag{B.17}$$

$$\kappa_\alpha{}^\beta v_{\beta\dot\gamma} + v_\alpha{}^{\dot\beta}\kappa_{\dot\beta\dot\gamma} = 0\,, \qquad \kappa_{\dot\alpha}{}^{\dot\beta}v_{\dot\beta\gamma} + v_{\dot\alpha}{}^{\beta}\kappa_{\beta\gamma} = 0. \tag{B.18}$$

If we contract the second set of these equations, we obtain

$$2\kappa^2 + v^2 = 2\bar\kappa^2 + v^2 = 1\,, \tag{B.19}$$

where we have defined $v^2 \equiv \frac{1}{2}v_{\alpha\dot\beta}v^{\alpha\dot\beta} = \frac{1}{2}V_\mu V^\mu$ which is consistent with $\kappa^2 = \bar\kappa^2$. More explicitly, we have

$$v^2 = 1 + r^2\,, \qquad \kappa^2 = -\frac{r^2}{2}. \tag{B.20}$$

We will also be needing explicit expressions for $\kappa_{\alpha\beta}$

$$\begin{aligned} &\kappa_{\alpha\beta} = \frac{\sqrt{2}}{2}r\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}_{\alpha\beta}\,, \qquad \kappa_{\dot\alpha\dot\beta} = \frac{\sqrt{2}}{2}r\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}_{\dot\alpha\dot\beta}\,,\\ &v_{\alpha\dot\beta} = \sqrt{1+r^2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}_{\alpha\dot\beta}\,. \end{aligned} \tag{B.21}$$

Of course, it is straightforward to recover (B.16)–(B.19) from (B.21).

The linearised equations of motion are given by

$$\begin{aligned} D_0\Omega &\equiv d\Omega - W_0 \wedge_\star \Omega - \Omega \wedge_\star W_0\\ &= h^{\gamma\dot\alpha}\wedge h_\gamma{}^{\dot\beta}\partial_{\dot\alpha}\partial_{\dot\beta}C(0,\bar y|x) + h^{\alpha\dot\gamma}\wedge h^{\beta}{}_{\dot\gamma}\partial_\alpha\partial_\beta C(y,0|x)\,, \end{aligned} \tag{B.22}$$

$$\tilde D_0 C \equiv dC - W_0 \star C + C \star \pi(W_0) = 0\,, \tag{B.23}$$

where the master fields Ω, C correspond to the Z-independent part of (W, B). They are solved by the following ansatz

$$C = bF_K \star \delta(y)\,, \tag{B.24}$$

with

$$F_K \equiv 4\exp\left(\frac{i}{2}K_{AB}Y^AY^B\right). \tag{B.25}$$

This solution has very interesting properties

$$F_K \star \delta(y) = F_K \star \delta(\bar y)\,, \qquad F_K \star F_K = F_K. \tag{B.26}$$

By performing the star-product explicitly, we obtain

$$C = \frac{4b}{r}\exp\left[\frac{i}{2\kappa^2}\left(\kappa_{\alpha\beta}y^\alpha y^\beta + \kappa_{\dot\alpha\dot\beta}\bar y^{\dot\alpha}\bar y^{\dot\beta} + 2i\kappa_{\alpha\gamma}v^\gamma{}_{\dot\beta}y^\alpha\bar y^{\dot\beta}\right)\right]. \tag{B.27}$$

We can read off the components of the generalised higher-spin Weyl tensors

$$C_{\alpha(2s)} = \frac{b}{s!2^{s-2}r}\left(\frac{i\kappa_{\alpha\alpha}}{\kappa^2}\right)^s\,, \qquad \bar C_{\dot\alpha(2s)} = \frac{b}{s!2^{s-2}r}\left(\frac{i\kappa_{\dot\alpha\dot\alpha}}{\kappa^2}\right)^s. \tag{B.28}$$

This corresponds at the spin two level to a Petrov type-D Weyl tensor, describing a black hole.

B.4 Didenko-Vasiliev solution for the purely bosonic theory

The solution to the non-linear equations (2.9a)–(2.9e) was obtained in [11] by starting from the solution of the linearised theory and studying the corrections order by order in b. It turns out that in a similar way to what happens for the Kerr-Schild ansatz for classical GR, there are no higher order corrections beyond the linearised part.

B.4.1 Induced star-product and Fock space

To adapt the solution of the free theory to the interacting one, the idea is to use an ansatz for the linearised part of the master fields of the form $F_K \star f(Z|x)$ where the Z-dependent part is explicitly factored out. If one defines A_A by

$$F_K \star Z_A \equiv F_K\, A_A\,, \qquad A_A \equiv (a_\alpha, \bar{a}_{\dot\alpha}) \equiv Z_A + iK_A{}^B Y_B\,, \qquad [A_A, A_B] = 4\epsilon_{AB}. \tag{B.29}$$

then in fact the star product of F_K with any function of Z can be expressed without the star product in terms of A

$$F_K \star \phi(Z|x) = F_K\, \phi(A|x)\,. \tag{B.30}$$

Furthermore, it can be shown that functions of this form define a subalgebra as

$$(F_K\phi_1) \star (F_K\phi_2) = F_K\,(\phi_1 * \phi_2)\,, \tag{B.31}$$

where the star-product $*$ is associative and takes the form

$$(\phi_1 * \phi_2)(A) = \int d^4u\, \phi_1(A + 2U_+)\phi_2(A - 2U_-)e^{2U_{+A}U_-^A}. \tag{B.32}$$

where we have defined $U_{\pm A} = \Pi_{\pm A}{}^B U_B$ in terms of the projectors

$$\Pi_{\pm AB} = \frac{1}{2}\left(\epsilon_{AB} \pm iK_{AB}\right)\,, \qquad \Pi_{\pm A}{}^B \Pi_{\pm B}{}^C = \Pi_{\pm A}{}^C\,, \qquad \Pi_{\pm A}{}^B \Pi_{\mp B}{}^C = 0. \tag{B.33}$$

We can then obtain the following formulae for any function $\phi(a)$ holomorphic in a

$$[a_\alpha, \phi(a)]_* = 2\partial_{a^\alpha}\phi(a)\,, \qquad \{a_\alpha, \phi(a)\}_* = 2\left(a_\alpha + i\kappa_\alpha{}^\beta \partial_{a^\beta}\right)\phi(a)\,, \tag{B.34}$$

where similar equations hold for functions that depend only on $\bar{a}$. The star-product (B.32) possesses Kleinien operators $\mathcal{K}$ and $\bar{\mathcal{K}}$, defined as follows

$$F_K \star \delta(z) = F_K\mathcal{K}\,, \qquad F_K \star \delta(\bar{z}) = F_K\bar{\mathcal{K}}. \tag{B.35}$$

Their explicit expressions are

$$\mathcal{K} = \frac{1}{r}\exp\left[\frac{i\kappa_{\alpha\beta}}{2\kappa^2}a^\alpha a^\beta\right]\,, \qquad \bar{\mathcal{K}} = \frac{1}{r}\exp\left[\frac{i\kappa_{\dot\alpha\dot\beta}}{2\kappa^2}\bar{a}^{\dot\alpha}\bar{a}^{\dot\beta}\right]\,, \tag{B.36}$$

and it is then straightforward to show that

$$\begin{aligned} \mathcal{K} * \mathcal{K} = \bar{\mathcal{K}} * \bar{\mathcal{K}} = 1\,, \qquad \{\mathcal{K}, a_\alpha\}_* = \left\{\bar{\mathcal{K}}, \bar{a}_{\dot\alpha}\right\}_* = 0\,, \\ \left[\mathcal{K}, \bar{\mathcal{K}}\right]_* = [\mathcal{K}, \bar{a}_{\dot\alpha}]_* = \left[\bar{\mathcal{K}}, a_\alpha\right]_* = 0. \end{aligned} \tag{B.37}$$

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As for the action of F_K on the Y coordinates, it is easy to see that it annihilates half of them

$$Y_{-A} \star F_K = F_K \star Y_{+A} = 0\,, \tag{B.38}$$

Using the closure (B.31), we can immediately see that this projection holds true for any member of the subalgebra

$$(F_K\,\phi) \star Y_{+A} = (F_K\,\phi) \star F_K \star Y_{+A} = 0. \tag{B.39}$$

B.4.2 Solving the non-linear equations of motion

We have thus seen that by choosing to factor out the Z-dependence by looking at functions of the form $F_K \star \phi(Z|x)$ we are led to study a subalgebra of the full higher-spin algebra. This can be used to simplify the equations of motions. More precisely, we take the following ansatz for the master fields of the theory

$$W = W_0 + F_K\left[\Omega(a|x) + \bar{\Omega}(\bar{a}|x)\right]. \tag{B.40}$$

$$S_\alpha = z_\alpha + F_K \sigma_\alpha(a|x)\,, \qquad \bar{S}_{\dot{\alpha}} = \bar{z}_{\dot{\alpha}} + F_K \bar{\sigma}_{\dot{\alpha}}(\bar{a}|x)\,, \tag{B.41}$$

while the B-field is kept identical to (B.24)

$$B = bF_K \star \delta(y). \tag{B.42}$$

This ansatz reduces the equations (2.9d) and (2.9e) to the following set of equations

$$\begin{aligned} &[\varsigma_\alpha(a|x), \varsigma_\beta(a|x)]_* = 2\epsilon_{\alpha\beta}\left(1 + e^{i\theta} b\mathcal{K}\right), \quad && [\varsigma_\alpha(a|x), \bar{\varsigma}_{\dot{\alpha}}(\bar{a}|x)]_* = 0\,, \\ &\{\mathcal{K}, \varsigma_\alpha(a|x)\}_* = 0\,, && \left\{\varsigma_\alpha(a|x), \bar{\mathcal{K}}\right\}_* = 0\,, \\ &\mathcal{Q}\Omega - \Omega \wedge_* \Omega = 0\,, && \mathcal{Q}\varsigma_\alpha - [\Omega, \varsigma_\alpha]_* = 0\,, \end{aligned} \tag{B.43}$$

with similar equations in the barred sector and where we have defined

$$\varsigma_\alpha = a_\alpha + \sigma_\alpha(a|x)\,, \qquad \bar{\varsigma}_{\dot{\alpha}} = \bar{a}_{\dot{\alpha}} + \bar{\sigma}_{\dot{\alpha}}(\bar{a}|x)\,, \tag{B.44}$$

$$\mathcal{Q} = \left(\hat{d} - \frac{i}{2} d\kappa^{\alpha\beta}\partial_{a^\alpha}\partial_{a^\beta}\right), \tag{B.45}$$

where $\hat{d}$ acts as the standard exterior derivative but leaves invariant $a, \bar{a}$ i.e., $\hat{d}a = \hat{d}\bar{a} = 0$. The simplified set of equations (B.43) can be solved exactly by analogy with the standard perturbative analysis, taking here b to be the infinitesimal parameter. We obtain

$$\sigma_\alpha(a|x) = \frac{be^{i\theta}}{r}\pi_\alpha^{+\beta} a_\beta \int_0^1 dt \exp\left(\frac{it}{2}\frac{\kappa_{\alpha\beta}}{\kappa^2} a^\alpha a^\beta\right), \tag{B.46}$$

$$\bar{\sigma}_{\dot{\alpha}}(\bar{a}|x) = \frac{be^{-i\theta}}{r}\pi_{\dot{\alpha}}^{+\dot{\beta}} \bar{a}_{\dot{\beta}} \int_0^1 dt \exp\left(\frac{it}{2}\frac{\kappa_{\dot{\alpha}\dot{\beta}}}{\kappa^2} \bar{a}^{\dot{\alpha}} \bar{a}^{\dot{\beta}}\right), \tag{B.47}$$

with a new set of projectors

$$\pi^\pm_{\alpha\beta} = \frac{1}{2}\left(\epsilon_{\alpha\beta} \pm \frac{\kappa_{\alpha\beta}}{\sqrt{-\kappa^2}}\right), \qquad \pi^\pm_{\dot{\alpha}\dot{\beta}} = \frac{1}{2}\left(\epsilon_{\dot{\alpha}\dot{\beta}} \pm \frac{\kappa_{\dot{\alpha}\dot{\beta}}}{\sqrt{-\kappa^2}}\right). \tag{B.48}$$

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Because of the introduction of the above projectors, the $O(b^2)$ terms vanish. Then one can note that with the ansatz for W, (2.9b) is trivially satisfied. At first order in b, the second equation of (B.43) reads

$$\frac{\partial}{\partial a^\alpha}\Omega = -\frac{1}{2}\mathcal{Q}\sigma^{\pm}_{\alpha}\,, \tag{B.49}$$

and is solved by

$$\Omega(a) = f_0 + O(b^2)\,, \tag{B.50}$$

where we use the fact that $\kappa_{\alpha\beta}/r$ is a constant. As argued in [11], the $O(b^2)$ terms vanish, and the first equation in the last line of (B.43) then imposes

$$df_0 = 0\,, \tag{B.51}$$

and we take $f_0 = 0$.

B.4.3 The solution

We can now give the solution to the full non-linear equations of motion

$$S_\alpha = z_\alpha + F_K\frac{e^{i\theta}b}{r}\pi^{+\beta}_{\alpha}a_\beta\int_0^1 dt\exp\left(-\frac{it\kappa_{\alpha\beta}}{2r^2}a^\alpha a^\beta\right) \tag{B.52}$$

$$\bar{S}_{\dot\alpha} = \bar{z}_{\dot\alpha} + F_K\frac{e^{-i\theta}b}{r}\pi^{+\dot\beta}_{\dot\alpha}\bar{a}_{\dot\beta}\int_0^1 dt\exp\left(-\frac{it\kappa_{\dot\alpha\dot\beta}}{2r^2}\bar{a}^{\dot\alpha}\bar{a}^{\dot\beta}\right) \tag{B.53}$$

$$B = \frac{4b}{r}\exp\left[-\frac{i}{2r^2}\left(\kappa_{\alpha\beta}y^\alpha y^\beta + \kappa_{\dot\alpha\dot\beta}\bar{y}^{\dot\alpha}\bar{y}^{\dot\beta} + 2i\kappa_{\alpha\gamma}v^{\gamma}{}_{\dot\beta}y^\alpha\bar{y}^{\dot\beta}\right)\right] \tag{B.54}$$

$$W = W_0 = -\frac{1}{4}\left(\omega_{\alpha\beta}y^\alpha y^\beta + \omega_{\dot\alpha\dot\beta}\bar{y}^{\dot\alpha}\bar{y}^{\dot\beta} - \sqrt{2}h_{\alpha\dot\beta}y^\alpha\bar{y}^{\dot\beta}\right). \tag{B.55}$$

It is then straightforward to show that the solution obtained satisfies the bosonic non-minimal reality projection which is obtained from (2.21) by removing the Γ. The bosonic symmetries of the solution are discussed in detail in [11]. There it is shown that the higher-spin symmetries are broken to a subalgebra, whose unique finite dimensional subalgebra is $su(2)\oplus gl(1)$, thus hinting at yet another similarity with a black hole solution.

As advertised in the main text, the master-field W (B.55) is equal to the vacuum value W_0, and is thus much simpler than the expression given in [11]. This can be traced back to our choice of Killing vector, for which the quantity $\kappa_{\alpha\beta}/r$ is a space-time independent constant.

It should also be noted that this solution generalizes slightly the one presented in [11] as it accommodates for an arbitrary parity breaking phase θ_0. As previously mentioned, the DV solution was generalized in [12], by working in a different gauge and using an infinite family of projectors similar to (B.33). Furthermore, the authors of [12] considered an arbitrary real even function for the interaction ambiguity $\theta(X) = \theta_0 + \theta_2 X^2 + \ldots$, instead of just a phase. We show here that this can also be done in the gauge of [11]. Indeed, using

$$B\star v = bF_K\star\delta(y)\star v = bF_K\star\delta(z) = bF_k\mathcal{K} \tag{B.56}$$

and since $\mathcal{K}$ squares to one under the induced star-product (B.32) and F_K is a projector we obtain

$$\begin{aligned} f(B\star v) &= 1 + B\star v\star \exp_\star[i\theta(B\star v)] = 1 + B\star v\star \exp_\star[i\theta(bF_K\mathcal{K})] \\ &= 1 + B\star v\star F_K \exp[i\theta(b)] = 1 + B\star v \exp[i\theta(b)] \end{aligned} \tag{B.57}$$

This means that this particular solution also solves the equations of motion in the general case after a redefinition of θ_0.

C Killing spinors in global AdS_4

C.1 Sp(4) matrices and notations

Here we will try to make clear a few things about notations. We define an Sp(4) matrix the following way

$$M_{AB} = \begin{pmatrix} M_{\alpha\beta} & M_{\alpha\dot\beta} \\ M_{\dot\alpha\beta} & M_{\dot\alpha\dot\beta} \end{pmatrix}. \tag{C.1}$$

Then whenever we use it in expressions involving the star-product and the Y, Z, we will manipulate the quantity $M_{AB}Y^AY^B$ with $Y^A = (y^\alpha, \bar y^{\dot\alpha})$. As an example, the covariant constancy condition for such a matrix reads $D_0(M_{AB}Y^AY^B) = 0$.

Now we define its action on Y^A (or on a spinor $\xi^A = (\xi^\alpha, i\bar\chi^{\dot\alpha})$) as $M_A{}^BY_B$. However, it will prove to be easier to manipulate M_{AB} as a usual matrix when acting on spinors. Then one has to be careful how to go from an expression involving the star-product to a matrix expression. To see how this works, we will work with the example of imposing that a spinor is covariantly constant. We define $\epsilon = \xi_\alpha y^\alpha + i\bar\chi_{\dot\alpha}\bar y^{\dot\alpha}$. Note here the i in the definition. The equation we want to consider is thus $D_0\epsilon = 0$. This reads in components

$$\begin{aligned} d\xi_\alpha + \omega_\alpha{}^\beta\xi_\beta + \frac{i}{\sqrt2}h_{\alpha\dot\beta}\bar\chi^{\dot\beta} &= 0\,, \\ d\bar\chi^{\dot\alpha} - \omega^{\dot\alpha}{}_{\dot\beta}\bar\chi^{\dot\beta} + \frac{i}{\sqrt2}h^{\dot\alpha\beta}\xi_\beta &= 0. \end{aligned} \tag{C.2}$$

We then represent the spinor ϵ as a column vector that we denote $\tilde\epsilon$ as follows

$$\tilde\epsilon = \begin{pmatrix} \xi_\alpha \\ \bar\chi^{\dot\alpha} \end{pmatrix} \tag{C.3}$$

where one should note the absence of i, and the position of the indices. Using this notation, the Killing spinor equation reads:

$$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \left(d - \frac{i}{2}\omega_{ab}\gamma^{ab} + \frac{i}{\sqrt2}h_a\gamma^a \right) \tilde\epsilon = 0. \tag{C.4}$$

Now we will consider the case of a general Sp(4) matrix M_{AB} again, and see how it acts on the column vector $\tilde\epsilon$ (C.3). We have

$$\xi_AY^A \to M_A{}^B\xi_BY^A = \left(M_\alpha{}^\beta\xi_\beta + M_\alpha{}^{\dot\beta}i\bar\chi_{\dot\beta}\right)y^\alpha + \left(M_{\dot\alpha}{}^{\dot\beta}i\bar\chi_{\dot\beta} + M_{\dot\alpha}{}^\beta\xi_\beta\right)\bar y^{\dot\alpha} \tag{C.5}$$

Now if we raise and lower indices accordingly to see what this gives on a spinor in the notation of (C.3), we obtain

$$\begin{pmatrix}\xi_\alpha\\ \bar{\chi}^{\dot\alpha}\end{pmatrix}\to\tilde{M}\begin{pmatrix}\xi_\alpha\\ \bar{\chi}^{\dot\alpha}\end{pmatrix}=\begin{pmatrix}1&0\\0&i\end{pmatrix}\begin{pmatrix}M_\alpha{}^\beta\xi_\beta-M_{\alpha\dot\beta}i\bar{\chi}^{\dot\beta}\\ -M^{\dot\alpha}{}_{\dot\beta}i\bar{\chi}^{\dot\beta}+M^{\dot\alpha\beta}\xi_\beta\end{pmatrix}=\begin{pmatrix}M_\alpha{}^\beta&-iM_{\alpha\dot\beta}\\ -iM^{\dot\alpha\beta}&-M^{\dot\alpha}{}_{\dot\beta}\end{pmatrix}\begin{pmatrix}\xi_\beta\\ \bar{\chi}^{\dot\beta}\end{pmatrix}\tag{C.6}$$

So we learn that the matrix

$$\tilde{M}=\begin{pmatrix}M_\alpha{}^\beta&-iM_{\alpha\dot\beta}\\ -iM^{\dot\alpha\beta}&-M^{\dot\alpha}{}_{\dot\beta}\end{pmatrix}\tag{C.7}$$

acts on $\tilde{\epsilon}$ and gives back a spinor of the same form.

C.2 The Killing spinors of global AdS_4

The AdS_4 Killing spinor equation is given by

$$\left(d-\frac{i}{2}\omega_{ab}\gamma^{ab}+\frac{i}{\sqrt{2}}h_a\gamma^a\right)\epsilon=0\,,\tag{C.8}$$

where ϵ is a 4-component spinor. We obtain

$$\begin{aligned}\partial_t\epsilon&=i\left(r\gamma^{01}-\frac{1}{\sqrt{2}}\sqrt{1+r^2}\gamma^0\right)\epsilon=\frac{i}{\sqrt{2}}\gamma^0\left(i\sqrt{2}r\gamma^1-\sqrt{1+r^2}\right)\epsilon\,,\\ \partial_r\epsilon&=\frac{i}{\sqrt{2}\sqrt{1+r^2}}\gamma^1\epsilon\,,\\ \partial_\theta\epsilon&=i\left(\sqrt{1+r^2}\gamma^{12}+\frac{1}{\sqrt{2}}r\gamma^2\right)\epsilon\,,\\ \partial_\varphi\epsilon&=i\left(\sqrt{1+r^2}\sin\theta\gamma^{13}+\cos\theta\gamma^{23}+\frac{1}{\sqrt{2}}r\sin\theta\gamma^3\right)\epsilon\,.\end{aligned}\tag{C.9}$$

The solution to these equations is given by

$$\epsilon=\Omega\epsilon_0\,,\qquad\Omega=e^{\frac{i\rho}{\sqrt{2}}\gamma^1}e^{-\frac{it}{\sqrt{2}}\gamma^0}e^{i\theta\gamma^{12}}e^{i\varphi\gamma^{23}}\,,\tag{C.10}$$

with $\sinh\rho=r$ and ϵ_0 is an arbitrary constant spinor.

An important point is that if $\epsilon=\xi_\alpha y^\alpha+i\bar{\chi}_{\dot\alpha}\bar{y}^{\dot\alpha}$ is a Killing spinor, then so is $\epsilon'=\chi_\alpha y^\alpha+i\bar{\xi}_{\dot\alpha}\bar{y}^{\dot\alpha}$, where we define $\bar{\chi}_{\dot\alpha}=(\chi_\alpha)^\dagger$ and $\bar{\xi}_{\dot\alpha}=(\xi_\alpha)^\dagger$. With this in mind, a convenient choice of constant spinors leads to the four Killing spinors ϵ_I

$$\epsilon_1=\frac{\Omega}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix},\quad\epsilon_2=\frac{\Omega}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\1\end{pmatrix},\quad\epsilon_{\bar{2}}=\frac{\Omega}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\\0\end{pmatrix},\quad\epsilon_{\bar{1}}=\frac{\Omega}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}.\tag{C.11}$$

If we write each spinor as in (C.3)

$$\epsilon_i\equiv\begin{pmatrix}\xi_{i,\alpha}\\ \bar{\chi}_i^{\dot\alpha}\end{pmatrix},\tag{C.12}$$

then we obtain

$$\begin{pmatrix} \chi_{1,\alpha} \\ \bar{\xi}_1^{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} \xi_{\bar{1},\alpha} \\ \bar{\chi}_{\bar{1}}^{\dot{\alpha}} \end{pmatrix}, \qquad \begin{pmatrix} \chi_{2,\alpha} \\ \bar{\xi}_2^{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} \xi_{\bar{2},\alpha} \\ \bar{\chi}_{\bar{2}}^{\dot{\alpha}} \end{pmatrix}. \tag{C.13}$$

Furthermore, we have

$$\bar{\epsilon}_1\epsilon_1 = \bar{\epsilon}_2\epsilon_2 = -\bar{\epsilon}_{\bar{2}}\epsilon_{\bar{2}} = -\bar{\epsilon}_{\bar{1}}\epsilon_{\bar{1}} = 1. \tag{C.14}$$

$$\bar{\epsilon}_i\epsilon_j = 0\,, \quad i \neq j. \tag{C.15}$$

where we have defined $\bar{\epsilon}_i = \epsilon_i^\dagger \sqrt{2}\gamma^0$.

C.3 Killing vectors of AdS_4

In this section we will find explicitly the Killing vectors of global AdS_4 and show that they can be put into two distinct categories. The splitting corresponds to the conformal and non-conformal isometries of the boundary $R \times S^2$.

First recall that AdS_4 can be seen as the embedding in $\mathbb{R}^{2,3}$ of the following hyperboloid

$$X_0^2 + X_4^2 - \sum_{i=1}^{3} X_i^2 = 1 \tag{C.16}$$

From this it is easy to see that the isometries of AdS_4 are generated by the following Killing vectors

$$L_{ab} = X_a \frac{\partial}{\partial X^b} - X_b \frac{\partial}{\partial X^a} \tag{C.17}$$

If we want to write them using the coordinates of global AdS_4 where the metric takes the form

$$ds^2 = \left[\cosh^2\rho\, dt^2 - d\rho^2 - \sinh^2\rho\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right], \tag{C.18}$$

We need to make the following change of variables

$$X_0 + iX_4 = e^{it}\cosh\rho\,, \qquad X_i = n_i \sinh\rho \tag{C.19}$$

with $n_i = (\cos\theta, \sin\theta\cos\varphi, \sin\theta\sin\varphi)$. We obtain a first set of four Killing vectors which do not depend on ρ nor t

$$L_{04} = \partial_t\,, \qquad L_{32} = \partial_\varphi\,, \qquad L_{21} + iL_{31} = e^{i\varphi}\left(\partial_\theta + i\cot\theta\,\partial_\varphi\right). \tag{C.20}$$

These are also Killing vectors of the boundary $R \times S^2$. The other six Killing vectors of AdS_4 are

$$\begin{aligned} L_{01} + iL_{14} &= e^{it}\left(i\cos\theta\tanh\rho\,\partial_t + \cos\theta\,\partial_\rho - \coth\rho\sin\theta\,\partial_\theta\right), \\ L_{02} + iL_{24} &= e^{it}\Bigg(i\cos\varphi\sin\theta\tanh\rho\,\partial_t + \cos\varphi\sin\theta\,\partial_\rho \\ &\qquad + \cos\theta\cos\varphi\coth\rho\,\partial_\theta - \frac{\coth\rho}{\sin\theta}\sin\varphi\,\partial_\varphi\Bigg), \\ L_{03} + iL_{34} &= e^{it}\Bigg(i\sin\theta\sin\varphi\tanh\rho\,\partial_t + \sin\theta\sin\varphi\,\partial_\rho \\ &\qquad + \cos\theta\coth\rho\sin\varphi\,\partial_\theta + \frac{\cos\varphi\coth\rho}{\sin\theta}\,\partial_\varphi\Bigg), \end{aligned} \tag{C.21}$$

and in the $\rho \to \infty$ become the *conformal* Killing vectors of the boundary manifold.

C.4 Killing vectors from Killing spinors

It is well known that Killing vectors can be obtained as the bilinears $\bar{\epsilon}'\gamma^\mu\epsilon$ where ϵ, ϵ' are two independent Killing spinors and $\bar{\epsilon} = \sqrt{2}\epsilon^\dagger\gamma^0$. We now give explicit expressions for the Killing vectors (C.20) in terms of the Killing spinors ϵ_1, ϵ_2 defined in (C.11)

$$\begin{aligned} L^\mu_{04} &= \frac{1}{\sqrt{2}}\bar{\epsilon}_1\gamma^\mu\epsilon_1 + \frac{1}{\sqrt{2}}\bar{\epsilon}_2\gamma^\mu\epsilon_2\,, \quad L^\mu_{32} = -\frac{1}{\sqrt{2}}\bar{\epsilon}_1\gamma^\mu\epsilon_2 - \frac{1}{\sqrt{2}}\bar{\epsilon}_2\gamma^\mu\epsilon_1\,, \\ L^\mu_{31} &= \frac{i}{\sqrt{2}}\bar{\epsilon}_1\gamma^\mu\epsilon_2 - \frac{i}{\sqrt{2}}\bar{\epsilon}_2\gamma^\mu\epsilon_1\,, \quad L^\mu_{21} = -\frac{1}{\sqrt{2}}\bar{\epsilon}_1\gamma^\mu\epsilon_1 + \frac{1}{\sqrt{2}}\bar{\epsilon}_2\gamma^\mu\epsilon_2\,, \end{aligned} \tag{C.22}$$

C.5 The DV Killing matrix from Killing spinors

A central ingredient in [11] is an Sp(4) matrix which is AdS covariantly constant and which squares to minus one, i.e., from which we can construct a projector. The projector is then a key element in simplifying the equations of motion, as well as in proving that the solution is BPS.

Using the notations of the previous subsection, V_μ of (B.14) is

$$V_{\gamma\dot{\delta}} = V^a\,(\sigma_a)_{\gamma\dot{\delta}} = (\bar{\epsilon}_1\gamma^a\epsilon_1 + \bar{\epsilon}_2\gamma^a\epsilon_2)\,(\sigma_a)_{\gamma\dot{\delta}} \tag{C.23}$$

Noting that:

$$\bar{\epsilon}_i\gamma^a\epsilon_i = \begin{pmatrix}\chi_i^\alpha & \bar{\xi}_{i,\dot{\alpha}}\end{pmatrix}\begin{pmatrix}0 & (\sigma^a)_{\alpha\dot{\beta}} \\ (\bar{\sigma}^a)^{\dot{\alpha}\beta} & 0\end{pmatrix}\begin{pmatrix}\xi_{i,\beta} \\ \bar{\chi}_i^{\dot{\beta}}\end{pmatrix} = \chi_i^\alpha\,(\sigma^a)_{\alpha\dot{\beta}}\,\bar{\chi}_i^{\dot{\beta}} + \bar{\xi}_{i,\dot{\alpha}}\,(\bar{\sigma}^a)^{\dot{\alpha}\beta}\,\xi_{i,\beta}. \tag{C.24}$$

So that

$$V_{\gamma\dot{\delta}} = \sum_{i=1,2}\left(\chi_i^\alpha\,(\sigma^a)_{\alpha\dot{\beta}}\,\bar{\chi}_i^{\dot{\beta}} + \bar{\xi}_{i,\dot{\alpha}}\,(\bar{\sigma}^a)^{\dot{\alpha}\beta}\,\xi_{i,\beta}\right)(\sigma_a)_{\gamma\dot{\delta}} = \sum_{i=1,2}\left(\chi_{i,\gamma}\bar{\chi}_{i,\dot{\delta}} + \xi_{i,\gamma}\bar{\xi}_{i,\dot{\delta}}\right) \tag{C.25}$$

Now if we define

$$\varkappa_{i\bar{i},AB} \equiv 2\epsilon_{i,A}\epsilon_{\bar{i},B} = \begin{pmatrix}\xi_{i,\alpha}\chi_{i,\beta} + \chi_{i,\alpha}\xi_{i,\beta} & \xi_{i,\alpha}i\bar{\xi}_{i,\dot{\beta}} + \chi_{i,\alpha}i\bar{\chi}_{i,\dot{\beta}} \\ \chi_{i,\alpha}i\bar{\chi}_{i,\dot{\beta}} + \xi_{i,\alpha}i\bar{\xi}_{i,\dot{\beta}} & i\bar{\chi}_{i,\dot{\alpha}}i\bar{\xi}_{i,\dot{\beta}} + i\bar{\xi}_{i,\dot{\alpha}}i\bar{\chi}_{i,\dot{\beta}}\end{pmatrix}\,, \tag{C.26}$$

then we see that the diagonal entries are reminiscent of the time-like Killing vectors expressed in terms of the components of the basis of Killing spinors. The corresponding operators which acts on the Killing spinors basis (C.11) is

$$P_i \equiv \epsilon_i\bar{\epsilon}_i + \epsilon_{\bar{i}}\bar{\epsilon}_{\bar{i}}. \tag{C.27}$$

However one should note that for this particular Killing vector, there is a sum over $i = 1, 2$. This amounts to taking $P_1 + P_2$. Indeed, this yields a matrix which is by construction covariantly constant, and which also squares to the identity, since $(P_1 + P_2)^2 = \epsilon_1\bar{\epsilon}_1 + \epsilon_2\bar{\epsilon}_2 - \epsilon_{\bar{1}}\bar{\epsilon}_{\bar{1}} - \epsilon_{\bar{2}}\bar{\epsilon}_{\bar{2}} = 1$, when acting on the space spanned by (C.11). In terms of Sp(4) matrices, this is $\varkappa_{1\bar{1},AB} + \varkappa_{2\bar{2},AB}$. If we look at the explicit content of this matrix, it is straightforward to relate it to the Killing matrix K_{AB} defined in (B.12)

$$\varkappa_{1\bar{1},AB} + \varkappa_{2\bar{2},AB} = iK_{AB}\,, \tag{C.28}$$

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just as expected. This allows us to see clearly how K_{AB}, as well as the projector $\Pi_{\pm,AB}$ defined in (B.33) act on our basis of Killing spinors (C.11). We see using (C.13) and (C.14) that Π_+ projects on the space spanned by ϵ_1, ϵ_2, while Π_- projects on the space spanned by $\epsilon_{\bar{1}}, \epsilon_{\bar{2}}$.

D Representations of the Clifford algebra

We choose the following matrix representation for the ϑ^i in the $n = 4$ case

$$\begin{aligned} \vartheta^1 &= \frac{1+i}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \vartheta^2 = \frac{1+i}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \\ \vartheta^3 &= \frac{1+i}{\sqrt{2}} \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \vartheta^4 = \frac{1+i}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \end{aligned} \tag{D.1}$$

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