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Homotopy-theoretically enriched categories of noncommutative motives

Jack Morava

Correspondence:
jack@math.jhu.edu
The Johns Hopkins University, 3400
North Charles Street, Baltimore,
Maryland 21218, USA

Abstract

Waldhausen's K -theory of the sphere spectrum (closely related to the algebraic K -theory of the integers) is naturally augmented as an S^0 -algebra, and so has a Koszul dual. Classic work of Deligne and Goncharov implies an identification of the rationalization of this (covariant) dual with the Hopf algebra of functions on the motivic group for their category of mixed Tate motives over \mathbb{Z} . This paper argues that the rationalizations of categories of noncommutative motives defined recently by Blumberg, Gepner, and Tabuada consequently have natural enrichments, with morphism objects in the derived category of mixed Tate motives over \mathbb{Z} . We suggest that homotopic descent theory lifts this structure to define a category of motives defined not over \mathbb{Z} but over the sphere ring-spectrum S^0 .

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1 Introduction

1.1 Building on earlier work going back at least three decades [26], Deligne and Goncharov have defined a \mathbb{Q} -linear Abelian rigid tensor category of mixed Tate motives over the integers of a number field: in particular, the category $\text{MT}_{\mathbb{Q}}(\mathbb{Z})$ of such motives over the rational integers. Its generators are tensor powers $\mathbb{Q}(n) = \mathbb{Q}(1)^{\otimes n}$ of a Tate object $\mathbb{Q}(1)$, inverse to the Lefschetz hyperplane motive (which can be regarded as a degree two shift of the complex

$$0 \rightarrow \mathbb{P}_1 \rightarrow \mathbb{P}_0 \rightarrow 0$$

in Voevodsky's derived category). We argue here that these objects are analogous to the (even-dimensional) cells of stable homotopy theory: in that, for example, the image

$$\mathbb{P}_n = \mathbb{Q}(0) \oplus \cdots \oplus \mathbb{Q}(-n)$$

of projective space in this category splits as a sum of terms resembling Lefschetz's hyperplane sections.

Deligne and Goncharov's definition ([27], § 1.6) depends on the validity of the Beilinson-Soulé vanishing conjecture for number fields, which implies that their category $\text{MT}_{\mathbb{Q}}(\mathbb{Z})$ can be characterized by a very simple spectral sequence with E_2 -term

$$\text{Ext}_{\text{MT}}^*(\mathbb{Q}(0), \mathbb{Q}(n)) \Rightarrow K(\mathbb{Z})_{2n-*} \otimes \mathbb{Q}$$

equal to zero if $* > 1$ or when $* = 0, n \neq 0$. Borel's theory of regulators [19,29] identifies the nonvanishing groups

$$K_{4k+1}(\mathbb{Z}) \otimes \mathbb{Q} \subset \mathbb{R}$$

with the subgroup of rational multiples of the conjecturally transcendental values $\zeta(1 + 2k)$ of the Riemann zeta function at odd positive integers.

To a homotopy theorist, this is strikingly reminiscent of Atiyah's interpretation of Adams' work on Whitehead's homomorphism

$$J_{n-1} : KO_n(*) = \pi_{n-1}O \rightarrow \lim_{m \rightarrow \infty} \pi_{m+n-1}(S^m) = \pi_{n-1}^S(*),$$

i.e., the effect on homotopy groups of the monoid map

$$O = \lim_{m \rightarrow \infty} O(m) \rightarrow \lim_{m \rightarrow \infty} \Omega^{m-1}S^{m-1} := Q(S^0).$$

The image of a Bott generator

$$KO_{4k}(*) \cong \mathbb{Z} \rightarrow \text{image } J_{4k-1} \cong (\frac{1}{2}\zeta(1 - 2k) \cdot \mathbb{Z})/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$$

under this homomorphism can be identified with the (rational) value of the zeta function at an odd *negative* integer.

Here is a quick sketch of this argument: A stable real vector bundle over S^{4k} is classified by its equatorial twist

$$\tilde{\alpha} : S^{4k-1} \xrightarrow{\alpha} O \longrightarrow Q(S^0),$$

which defines a stable cofibration

$$S^{4k-1} \xrightarrow{\tilde{\alpha}} S^0 \longrightarrow \text{Cof } \tilde{\alpha} \longrightarrow S^{4k} \longrightarrow \dots$$

Adams' *e*-invariant is the class of the resulting sequence

$$0 \longrightarrow KO(S^{4k}) \longrightarrow KO(\text{Cof } \tilde{\alpha}) \longrightarrow KO(S^0) \longrightarrow 0$$

in a group

$$\text{Ext}_{\text{Adams}}^1(KO(S^0), KO(S^{4k}))$$

of extensions of modules over the stable *KO*-cohomology operations. [*KO* is contravariant, while motives are covariant, making *KO* of a sphere analogous to a Tate object.] After profinite completion [4], these cohomology operations become an action of the group $\widehat{\mathbb{Z}}^\times$ of units in the profinite integers - in fact the action, through its abelianization, of the absolute Galois group of \mathbb{Q} - and the resulting group of extensions can be calculated in terms of generalized Galois cohomology as

$$\text{Ext}_{\widehat{\mathbb{Z}}^\times}^1(\widehat{\mathbb{Z}}(0), \widehat{\mathbb{Z}}(2k)) \cong H_c^1(\widehat{\mathbb{Z}}^\times, \widehat{\mathbb{Z}}(2k))$$

(where $u \in \widehat{\mathbb{Z}}^\times$ acts on $\widehat{\mathbb{Z}}(n)$ as multiplication by u^n). At an odd prime $p, H_c^1(\widehat{\mathbb{Z}}_p^\times, \widehat{\mathbb{Z}}_p(2k))$ is zero unless $2k = (p - 1)k_0$, when the group is cyclic of p -order $v_p(k_0) + 1$. By congruences of von Staudt and Clausen, this is the p -order of the Bernoulli quotient

$$\frac{B_{2k}}{2k} \in \mathbb{Q}/\mathbb{Z};$$

a global argument over \mathbb{Q} (i.e., using the Chern character ([1], § 7.1b)) refines this to a homomorphism

$$H_c^1(\widehat{\mathbb{Z}}^\times, \widehat{\mathbb{Z}}(2k)) \rightarrow \mathbb{Q}/\mathbb{Z}$$

which sends a generator of $KO(S^{4k})$ to the class of $\frac{1}{2}\zeta(1 - 2k)$. See also ([26], § 3.5).

1.2 This paper proposes an analog of the theory of mixed Tate motives in the world of stable homotopy theory, based on Bökstedt’s theorem ([16], or more recently [14]) that the morphism

$$K(S^0) \rightarrow K(\mathbb{Z})$$

of ring-spectra (induced by the Hurewicz morphism

$$[1 : S^0 \rightarrow H\mathbb{Z}] \in H^0(S^0, \mathbb{Z})$$

becomes an isomorphism after tensoring with \mathbb{Q} . At this point, odd zeta-values enter differential topology [43]. To be more precise, we argue that (unlike $K(\mathbb{Z})$) $K(S^0)$ is naturally augmented as a ring-spectrum over S^0 , via the Dennis trace

$$\text{tr}_D : K(S^0) \rightarrow \text{THH}(S^0) \sim S^0$$

[70]. Current work [38,64] on descent in homotopy theory suggests the category of comodule spectra over the covariant Koszul dual

$$S^0 \wedge_{K(S^0)}^L S^0 := K(S^0)^\dagger$$

of $K(S^0)$ (or perhaps more conventionally, the category of module spectra over

$$\text{RHom}_{K(S^0)}(S^0, S^0)$$

as a natural candidate for a homotopy-theoretic analog of $\text{MT}_{\mathbb{Q}}(\mathbb{Z})$. This paper attempts to make this plausible after tensoring $K(S^0)$ with the rational field \mathbb{Q} .

1.3 Organization Koszul duality is a central concern of this paper; in its most classical form, it relates (graded) exterior and symmetric Hopf algebras. The first section below observes that the Hopf algebra of quasismetric functions is similarly related to a certain odd-degree square-zero augmented algebra. Stating this precisely (i.e., over \mathbb{Z}) requires comparison of the classical shuffle product ([31], Ch II) with the less familiar quasi-shuffle product [40,50]. I am especially indebted to Andrew Baker and Birgit Richter for explaining this to me.

The next section defines topologically motivated generators (quite different from those of Borel) for $K_*(S^0) \otimes \mathbb{Q}$. Work of Hatcher ([36], § 6.4), Waldhausen [69], and Bökstedt on pseudo-isotopy theory has been refined by Rognes [63] to construct an infinite-loop map

$$\omega : B(F/O) \rightarrow \text{Wh}(\ast) (\subset K(S^0))$$

(F being the monoid of homotopy self-equivalences of the stable sphere) which is a rational equivalence. This leads to the definition of a homotopy equivalence

$$\text{w} : (S^0 \vee \Sigma kO) \otimes \mathbb{Q} \rightarrow K(S^0) \otimes \mathbb{Q}$$

of ring-spectra, with a square-zero extension of the rational sphere spectrum on the left, which can then be compared with Borel’s calculations. Some of the work of Deligne and Goncharov is then summarized to construct a lift of this rational isomorphism to

an equivalence between the algebra $\mathcal{H}_{GT_{MT}^*}$ of functions on the motivic group of the Tannakian category $MT_{\mathbb{Q}}(\mathbb{Z})$ and the covariant Koszul dual $K(S^0)_*^{\dagger} \otimes \mathbb{Q}$.

The final section is devoted to applications: in particular to the ‘decategorification’ ([18], [49], § 4) of two-categories of ‘big’ (noncommutative) motives constructed by Blumberg, Gepner, and Tabuada [10,11], and to work of Kitchloo [46] on categories of symplectic analogs of motives. The objects in the categories of ‘big’ motives are themselves small stable ∞ -categories, with stable ∞ -categories of suitably exact functors between them as morphism objects. The (Waldhausen) K -theory spectra of these morphism categories define new categories enriched over the homotopy category of $K(S^0)$ -module spectra ([11], Corollary 4.13), having the original small stable categories as objects.

‘Rationalizing’ (tensoring the morphism objects in these homotopy categories with \mathbb{Q}) defines categories enriched over $K_*(S^0) \otimes \mathbb{Q}$ -modules, to which the Koszul duality machinery developed here can be applied. Under suitable finiteness hypotheses, this constructs categories of noncommutative motives enriched over the derived category $D_b(MT_{\mathbb{Q}}(\mathbb{Z}))$ of classical mixed Tate motives.

2 Quasisymmetric functions and Koszul duality

2.1 The fundamental example underlying this paper could well have appeared in Tate’s 1957 work [67] on the homology of local rings; but as far as I know, it is not in the literature, so I will begin with it: Let $E_* := E_*[e_{2k+1} \mid k \geq 0]$ be the primitively generated graded-commutative Hopf algebra over \mathbb{Z} with one generator in each odd degree, and let

$$\phi_E : E_* \rightarrow E_*/E_+ = \mathbb{Z}$$

be the quotient by its ideal E_+ of positive-degree elements; then

$$\text{Tor}_*^E(\mathbb{Z}, \mathbb{Z}) \cong P[x_{2(k+1)} \mid k \geq 0] \quad (:= \text{Sym}_*)$$

is a graded-commutative Hopf algebra with one generator in each even degree, canonically isomorphic to the classical algebra of symmetric functions with coproduct

$$\begin{aligned} \Delta x(t) &= x(t) \otimes x(t) \\ (x(t) &= \sum_{k \geq 0} x_{2k} t^k, \quad x_0 := 1). \end{aligned}$$

This is an instance of a very general principle: if $A_* \rightarrow k$ is an augmented commutative graded algebra (assuming for simplicity that k is a field), then

$$k \otimes_{A_*}^L k = \text{Tor}_*^A(k, k) := A_*^{\dagger}$$

is an augmented, graded-commutative Hopf algebra, with

$$\text{Ext}_A^*(k, k) := R\text{Hom}_A^*(k, k)$$

as its graded dual ([22], XVI § 6). More generally,

$$(A - \text{Mod}) \ni M \mapsto \text{Tor}_*^A(M, k) := M_*^{\dagger}$$

extends this construction to a functor taking values in a category of graded A_*^{\dagger} -comodules. This fascinated John Moore [41,60], and its implications have become quite important in representation theory [7,8]; more recently, the whole subject has been vastly generalized by the work of Lurie.

2.2.1 For our purposes it is the quotient

$$\varphi_{\tilde{E}} : E_* \rightarrow E_*/(E_+)^2 := \tilde{E}_*$$

of the exterior algebra above, by the ideal generated by products of positive-degree elements, which is relevant. This quotient is the square-zero extension

$$\tilde{E}_* = \mathbb{Z} \oplus \tilde{E}_+ = \mathbb{Z} \oplus \{e_{2k+1} \mid k \geq 0\}$$

of \mathbb{Z} by a graded module with one generator in each odd degree.

Proposition. After tensoring with \mathbb{Q} , the induced homomorphism

$$\varphi_*^{\tilde{E}} : \text{Tor}_*^E(\mathbb{Z}, \mathbb{Z}) \cong \text{Sym}_* \rightarrow \text{Tor}_*^{\tilde{E}}(\mathbb{Z}, \mathbb{Z}) \cong \mathbf{QS}\text{Sym}_*$$

of Hopf algebras is the inclusion of the graded algebra of rational symmetric functions into the algebra of rational quasi-symmetric functions, given the classical shuffle product \sqcup .

Proof. In this case the classical bar resolution

$$\mathbb{B}_*(\tilde{E}/\mathbb{Z}) = \tilde{E}_* \otimes_{\mathbb{Z}} (\oplus_{n \geq 0} \tilde{E}_+[1]^{\otimes n}) \otimes_{\mathbb{Z}} \mathbb{Z}$$

(of \mathbb{Z} as an \tilde{E}_* -module ([55] Ch X § 2.3, [31] Ch II, [24] § 2)) is, apart from the left-hand term, just the tensor algebra of the graded module $\tilde{E}_+[1]$ (obtained from \tilde{E}_+ by shifting the degrees of its generators up by one), with algebra structure defined by the shuffle product; but I will defer discussing that till § 2.3 below. Since \tilde{E}_* is a DGA with trivial differential and trivial product, the homology $\text{Tor}_*^{\tilde{E}}(\mathbb{Z}, \mathbb{Z})$ of the complex

$$\mathbb{Z} \otimes_{\tilde{E}} \mathbb{B}_*(\tilde{E}/\mathbb{Z}) = \oplus_{n \geq 0} (\tilde{E}_+[1])^{\otimes n}$$

with its resulting trivial differential is the algebra $\mathbf{QS}\text{Sym}_*$ on $\tilde{E}_+[1]$. [Tate, by the way, worked with a commutative noetherian local ring

$$\phi_A : A \rightarrow B = A/\mathfrak{m}_A = k$$

and studied $\text{Tor}_*^A(k, k)$, though not as a Hopf algebra; but in his calculations, he used what is visibly the resolution above.]

Remark. In fact under very general conditions ([38], § 3.1, [39], § 6.12), the bar construction associates to a morphism $\varphi : A \rightarrow B$ of suitable monoid objects, a pullback functor

$$L\varphi^* : M_* \mapsto M_* \otimes_A \mathbb{B}_*(A/B) \cong M_* \otimes_A^L B$$

from some (simplicial or derived) category of modules over $\text{Spec } A$ to a similar category of modules over $\text{Spec } B$, cf § 2.4 below. Here $\mathbb{B}_*(A/B)$ is a resolution of B as an A -module, corresponding to $B(A, B, A)$ in ([32], Prop 7.5), cf also ([15,31,33,51,53], ...). In the example above, regarding E_* as a DGA with trivial differential, we obtain a covariant functor from the bounded derived category of E_* -modules to the bounded derived category of graded modules with a *coaction* of the classical Hopf algebra of symmetric functions. However, the algebra of symmetric functions is canonically self-dual over \mathbb{Z} ([54], I § 4) and we can interpret this derived pullback as a functor to the bounded derived category of modules over the dual symmetric algebra.

2.2.2 In this paper I will follow K Hess ([38], § 2.2.23 - 2.2.28): a morphism

$$\varphi : A \rightarrow B$$

of monoids in a suitable (e.g., simplicially enriched ([38], § 3.16, § 5.3)) category of modules (perhaps over a differential graded algebra or a ring-spectrum) defines an A -bimodule **bialgebra**

$$B \wedge_A^L B := W(\varphi)$$

(analogous to an algebraic topologist’s Hopf algebraoid, though in general without antipode). In her framework the construction above is a descent functor: its target has a natural (‘Tannakian’) enrichment ([37], § 5.3, [44]) or lift $L\varphi^\dagger$ to a category of B -modules with compatible coaction by the descent coring $W(\varphi)$. Lurie’s work (e.g., [52], § 7.13), [53] § 5.2.2 - 5.2.3) provides the natural context for such constructions.

2.2.3 Completing the argument requires clarifying relations between the shuffle product \sqcup and the quasi-shuffle or ‘stuffle’ product $\overline{\sqcup}$. A shuffle of a pair $r, s \geq 1$ of integers is a partition of the set $\{1, \dots, r + s\}$ into disjoint subsets $a_1 < \dots < a_r$ and $b_1 < \dots < b_s$; such a shuffle defines a permutation

$$\sigma(1, \dots, r + s) = (a_1, \dots, a_r, b_1, \dots, b_s) .$$

The shuffle product on the tensor algebra $T^\bullet(V)$ of a module V is defined by

$$v_1 \cdots v_r \sqcup v_{r+1} \cdots v_{r+s} = \sum v_{\sigma(1)} \cdots v_{\sigma(r+s)} ,$$

with the sum taken over all shuffles of (r, s) . The deconcatenation coproduct

$$\Delta : T^\bullet(V) \rightarrow T^\bullet(V) \otimes T^\bullet(V)$$

sends $v_1 \cdots v_r$ to the sum

$$v_1 \cdots v_r \otimes 1 + \sum_{1 < i < r} v_1 \cdots v_i \otimes v_{i+1} \cdots v_r + 1 \otimes v_1 \cdots v_r .$$

The algebra $T^\bullet(V)$, with this (commutative but not cocommutative) Hopf structure, is sometimes called the cotensor (Hopf) algebra of V . The shuffle product is characterized by the identity

$$(v \cdot x) \sqcup (w \cdot y) = v \cdot (x \sqcup (w \cdot y)) + w \cdot ((v \cdot x) \sqcup y) ,$$

where $v, w \in V$ and $x, y \in T^\bullet(V)$. I will write \mathbf{QSymb}_* for the Hopf algebra $\text{Tor}_*^{\tilde{E}}(\mathbb{Z}, \mathbb{Z})$ of § 2.2.1, with \sqcup as product.

The closely related Hopf algebra \mathbf{QSymb}_* of quasi-symmetric functions over \mathbb{Z} , with the *quasi*-shuffle product $\overline{\sqcup}$, is perhaps most efficiently defined as dual to the free graded associative Hopf algebra

$$\mathbf{NSymb}^* := \mathbb{Z}\langle Z_{2(k+1)} \mid k \geq 0 \rangle$$

of noncommutative symmetric functions ([6], [22], § 4.1.F, [36]), with coproduct

$$\begin{aligned} \Delta Z(t) &= Z(t) \otimes Z(t) \\ Z(t) &= \sum_{k \geq 0} Z_{2k} t^k, \quad Z_0 = 1. \end{aligned}$$

More generally, if (V, \star) is a (graded) commutative algebra, the quasi-shuffle (or overlapping shuffle, or stuffle) product \sqcup on $T^\bullet(V)$ is a deformation ([40], § 6) of the shuffle product characterized by the identity

$$(v \cdot x) \sqcup (w \cdot y) = v \cdot (x \sqcup (w \cdot y)) + w \cdot ((v \cdot x) \sqcup y) + (v \star w) \cdot (x \sqcup y).$$

In particular, if we define an algebra structure on the graded vector space spanned by classes f_i dual to the Z_i 's by $f_i \star f_j = f_{i+j}$, we recover the quasi-shuffle product on the dual of NSymm^* .

The Lie algebra of primitives in NSymm^* is generated by the analogs of Newton's power functions ([23], § 4.1.F), and map under abelianization to the classical power function primitives in Symm^* ; dualizing yields a morphism of $\text{Symm}_* \rightarrow \text{QSymm}_*$, which rationalizes to the asserted inclusion. □

2.3 Remarks

i) The applications below will be based on a variant of \tilde{E}_* defined by generators in degree $4k + 1$, $k \geq 0$, rather than $2k + 1$. The corresponding free Lie algebras will then have generators in (homological) degree $-2(2k + 1)$. This doubling of topological degree relative to motivic weight is a familiar consequence of differing conventions.

ii) Hoffman ([40], Theorem 2.5) constructs an isomorphism

$$\text{exp} : \text{QSymm}_* \otimes \mathbb{Q} \rightarrow \text{QSymm}_* \otimes \mathbb{Q}$$

of graded Hopf algebras over the rationals, taking \sqcup to \sqcup ; so over \mathbb{Q} we can think of the morphism defined by the proposition as the inclusion of the symmetric functions in the quasisymmetric functions with the quasishuffle product.

iii) The rationalization $\text{NSymm}^* \otimes \mathbb{Q}$ is the (primitively generated) universal enveloping algebra $U(\mathfrak{f}^*)$ of the free Lie algebra \mathfrak{f}^* generated by the Z 's over \mathbb{Q} . By Poincaré-Birkhoff-Witt its modules can be regarded as representations of a pro-unipotent groupscheme $G_0(\mathfrak{f}^*)$ over \mathbb{Q} , or equivalently as comodules over the Hopf algebra $\text{QSymm}_* \otimes \mathbb{Q}$ of algebraic functions on that pro-unipotent group. If we interpret graded modules as representations of the multiplicative groupscheme in the usual way ([3], § 3.2.7), see also ([8], § 1.1.2), then we can regard these modules as representations of a proalgebraic groupscheme

$$1 \rightarrow G_0(\mathfrak{f}^*) \rightarrow G(\mathfrak{f}^*) := \mathbb{G}_m \times G_0(\mathfrak{f}^*) \rightarrow \mathbb{G}_m \rightarrow 1.$$

In a very helpful appendix, Deligne and Goncharov ([27], § A.15) characterize representations of $G(\mathfrak{f}^*)$ as graded \mathfrak{f}^* -modules, such that (if $\mathbb{Q}(n)$ denotes a copy of \mathbb{Q} in degree n)

$$\text{Ext}_{\text{Rep}(G(\mathfrak{f}^*))}^1(\mathbb{Q}(0), \mathbb{Q}(n)) = (\mathfrak{f}_{\text{ab}}^n)^\vee.$$

This is explained in more detail in ([35], § 8); we will return to this description below.

iv) The rational stable homotopy category is equivalent to the derived category of rational vector spaces, and the homotopy category of rational ring-spectra is equivalent to the homotopy category of DGAs: the Hurewicz map

$$[S^*, X_{\mathbb{Q}}] \cong \pi_*^S(X) \otimes \mathbb{Q} \rightarrow H_*(X, \mathbb{Q})$$

is an isomorphism. This leads to a convenient abuse of notation which may not distinguish the rationalization $X_{\mathbb{Q}}$ of a spectrum from its homology (or its homotopy). For example,

the rational de Rham algebra of forms on a reasonable space is a good model for the rational Spanier-Whitehead commutative ringspectrum $[X, S_{\mathbb{Q}}^0]$.

2.4 Finite-dimensional graded modules over a field k have a good duality functor

$$V_* \mapsto \text{Hom}_k(V_*, k) = (V^*)^\vee,$$

and a great deal of work on the homological algebra of augmented algebras $\phi : A \rightarrow k$ (and their generalizations) is formulated in terms of constructions generalizing

$$M_* \rightarrow \text{Hom}_k(M_* \otimes_A \mathbb{B}_*(A/k), k) \cong \text{Hom}_A(M_*, \mathbb{B}^*(A/k)^\vee) := \text{RHom}_A(M_*, k),$$

where $\mathbb{B}^*(A/k)^\vee$ is now essentially a cobar construction. This is the classical contravariant Koszul duality functor: see ([12,21], [30], § 4.22) for recent work in the context of modules over ring spectra.

In this paper, however, we work instead (following [34,60]) with the covariant functor $M_* \mapsto M_*^\dagger$ defined as in § 2.2.2, but regarded as mapping modules over an augmented algebra A to comodules over a coaugmented coalgebra A^\dagger . In particular, in the case of our main example (over \mathbb{Q}) Hess’s hypotheses ([38], § 5.3) are satisfied, and we have the

Corollary. The Hess-Koszul-Moore functor

$$L\varphi_E^\dagger : D_b(\tilde{E}_* \otimes \mathbb{Q} - \text{Mod}) \rightarrow D_b(\mathbf{QSymb}_* \otimes \mathbb{Q} - \text{Comod})$$

is an equivalence of symmetric monoidal categories.

Proof. The point is that, in the context of graded commutative augmented algebras A over a field k , the functor $L\varphi^*$ is monoidal, in the sense that

$$L\varphi^*(M_0) \otimes_k L\varphi^*(M_1) := (M_0 \otimes_A \mathbb{B}(A/k)) \otimes_k k \otimes_k (M_1 \otimes_A \mathbb{B}(A/k)) \otimes_k k$$

is homotopy-equivalent to

$$L\varphi(M_0 \otimes_A M_1) := ((M_0 \otimes_A M_1) \otimes_A \mathbb{B}(A/k)) \otimes_k k$$

via the morphism

$$(M_0 \otimes_A \mathbb{B}(A/k)) \otimes_k M_1 \rightarrow M_0 \otimes_A M_1$$

induced by the homotopy equivalence of $\mathbb{B}(A/k)$ with k as an A -module. This then lifts to an equivalence $L\varphi^\dagger$ of comodules, cf [58,68].

In the terminology of § 2.2.3iii, the composition

$$\text{PBW} \circ \text{exp}^* \circ L\varphi_E^\dagger := L\Phi_E^\dagger$$

thus defines an equivalence of the derived category of $\tilde{E}_* \otimes \mathbb{Q}$ -modules with the derived category of $G(\mathfrak{f}^*)$ -representations. A similar argument identifies the bounded derived category of modules over $E_* \otimes \mathbb{Q}$ with the bounded derived category of representations of the graded abelianization $G(\mathfrak{f}_{\text{ab}}^*)$ of $G(\mathfrak{f}^*)$: in other words, of graded modules over $\text{Sym}^* \otimes \mathbb{Q}$.

Remarks. The quite elementary results above were inspired by groundbreaking work of Baker and Richter [6], who showed that the integral cohomology of $\Omega\Sigma\mathbb{C}P_+^\infty$ is isomorphic to \mathbf{QSymm}^* as a Hopf algebra. Indeed, the E_2 -term

$$\mathrm{Tor}^{H^*(Z)}(H^*(X), H^*(Y)) \Rightarrow H^*(X \times_Z Y)$$

of the Eilenberg-Moore spectral sequence for the fiber product

$$\begin{array}{ccc} \Omega\Sigma\mathbb{C}P_+^\infty & \cdots \twoheadrightarrow & P\Sigma\mathbb{C}P_+^\infty \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma\mathbb{C}P_+^\infty \end{array}$$

is the homology of the bar construction on the algebra $H^*(\Sigma\mathbb{C}P_+^\infty)$, which is a square-zero extension of \mathbb{Z} . The spectral sequence collapses for dimensional reasons but has nontrivial multiplicative extensions (connected to the fact that $H^*(\mathbb{C}P^\infty)$ is polynomial (cf § 2.2.2, [5,50])).

In view of Proposition 3.2.1 below, Proposition 2.2.1 can be rephrased as the algebraically similar assertion that the Künneth spectral sequence ([32], IV § 4.1)

$$\mathrm{Tor}^{H_*(K(S^0), \mathbb{Q})}(H_*(S^0, \mathbb{Q}), H_*(S^0, \mathbb{Q})) \Rightarrow H_*(K(S^0)^\dagger, \mathbb{Q})$$

for ring-spectra collapses. In this case the algebra structure on $H_*(K(S^0), \mathbb{Q})$ is trivial, resulting in the shuffle algebra \mathbf{QSymm}_* . Note that although these spectral sequences look algebraically similar, one is concentrated in positive, the other in negative, degrees. If or how they might be related, e.g., via the cyclotomic trace (cf § 4.1), seems quite mysterious to the author.

I am deeply indebted to Baker and Richter for help with this, and with many other matters. I am similarly indebted to John Rognes for patient attempts to educate me about the issues in the section following.

3 Geometric generators for $K_*(S^0) \otimes \mathbb{Q}$

3.1 Stable smooth cell bundles are classified by a space (i.e., simplicial set)

$$\mathrm{colim}_{n \rightarrow \infty} B\mathrm{Diff}(\mathbb{D}^n),$$

where $\mathrm{Diff}(\mathbb{D}^n)$ is the group of diffeomorphisms of the *closed* n -disk (which are *not* required to fix the boundary^a). Following ([72], § 1.2, § 6.1), there is a fibration

$$\mathcal{H}_{\mathrm{Diff}}(S^{n-1}) \rightarrow B\mathrm{Diff}(\mathbb{D}^n) \rightarrow B\mathrm{Diff}(\mathbb{D}^{0n}),$$

where \mathbb{D}^{0n} is the *open* disk, and $\mathcal{H}_{\mathrm{Diff}}(S^{n-1})$ is the simplicial set of smooth h -cobordisms of a sphere with itself [71]. The homomorphism $O(n) \rightarrow \mathrm{Diff}(\mathbb{D}^{0n})$ is a homotopy equivalence, while the constructions of ([72], § 3.2) define a system

$$\mathrm{colim}_{n \rightarrow \infty} B\mathcal{H}_{\mathrm{Diff}}(S^{n-1}) \rightarrow \mathrm{Wh}(\ast) = \Omega^\infty \tilde{K}(S^0)$$

of maps to the fiber of the Dennis trace, which becomes a homotopy equivalence in the limit. It follows that the K -theory groups

$$\mathrm{colim}_{n \rightarrow \infty} \pi_* B\mathrm{Diff}(\mathbb{D}^n) := K_*^{\mathrm{cell}}$$

(of smooth cell bundles over a point) satisfy

$$K_i^{\mathrm{cell}} \otimes \mathbb{Q} \cong \mathbb{Q}^2 \text{ if } i = 4k > 0$$

and are zero for other positive i , cf e.g., ([42], p 7).

The resulting parallel manifestations of classical zeta-values in algebraic geometry, and in algebraic and differential topology, seem quite remarkable, and I am arguing here that they have a unified origin in the fibration

$$\Omega \text{Wh}(\ast) \longrightarrow B\text{Diff}(\mathbb{D}) \longrightarrow BO$$

with odd negative zeta-values originating in the J -homomorphism to $Q(S^0)$ on the right, and odd positive zeta-values originating in pseudoisotopy theory through $K(S^0)$ on the left. The adjoint functors B and Ω account for the shift of homological dimension by two, from $K_{4k-1}(\mathbb{Z})$ (where $\zeta(1 - 2k)$ lives) to $K_{4k+1}(\mathbb{Z})$ (where $\zeta(1 + 2k)$ lives).

One can hope that this provocative fact might someday provide a basis for a theory of *smooth* motives (conceivably involving the functional equation of the zeta-function), but at the moment, even the multiplicative structure of $K_*^{\text{cell}} \otimes \mathbb{Q}$ is obscure to me.

3.2.1 Work of Rognes [63], sharpening earlier constructions of Hatcher ([36], § 6.4), Waldhausen, and Bökstedt, provides geometrically motivated generators for $K(S^0)_* \otimes \mathbb{Q}$ by defining a rational infinite-loop equivalence

$$\tilde{w} : B(F/O) \rightarrow \text{Wh}(\ast) .$$

Here F is the monoid of homotopy self-equivalences of the stable sphere [56]; I will write f/O for the spectrum defined by the infinite loop space F/O . One of my many debts to this paper’s referees is the construction of a rational equivalence

$$(S^0 \vee \Sigma kO) \otimes \mathbb{Q} \rightarrow (S^0 \vee \Sigma f/O) \otimes \mathbb{Q}$$

of ring-spectra (with simple multiplication) via the zigzag

$$BO = \ast \times_O EO \longleftarrow F \times_O EO \longrightarrow F \times_O \ast = F/O$$

of maps of infinite loopspaces; together with Rognes’s construction, this defines an equivalence

$$w : (S^0 \vee \Sigma kO) \otimes \mathbb{Q} \rightarrow K(S^0) \otimes \mathbb{Q}$$

of rational ring-spectra (alternately: of DGAs with trivial differentials and product structure).

Proposition. The resulting homomorphism

$$w_* : \mathbb{Q} \oplus kO_*[1] \otimes \mathbb{Q} \rightarrow K_*(S^0) \otimes \mathbb{Q}$$

presents the rationalization of $K(S^0)$ as a square-zero extension of \mathbb{Q} by an ideal

$$\mathbb{Q}\{\sigma v^k \mid k \geq 1\}$$

($|v| = 4$) with trivial multiplication. □

3.2.2 Writing $S^0[X_+]$ for the suspension spectrum of a space X emphasizes the similarity of that construction to the free abelian group generated by a set. The equivalence

$$\text{Maps}_{\text{Spaces}}(\Omega^\infty Z_+, \Omega^\infty Z_+) = \text{Maps}_{\text{Spectra}}(S^0[\Omega^\infty Z_+], Z)$$

sends the identity map on the left to a stabilization morphism

$$S^0[\Omega^\infty Z_+] \rightarrow Z$$

of spectra: for example, if $Z = \Sigma kO$ then $\Omega^\infty Z$ is the Bott space SU/SO , and the extension

$$S^0[SU/SO_+] \rightarrow S^0 \vee \Sigma kO$$

of stabilization by the collapse map $SU/SO \rightarrow S^0$ to a map of ring-spectra (with the target regarded as a square-zero extension) is the product-killing quotient

$$e_{4k+1} \mapsto \sigma v^k : H_*(SU/SO, \mathbb{Q}) = E(e_{4k+1} \mid k \geq 1) \otimes \mathbb{Q} \rightarrow \mathbb{Q} \oplus \mathbb{Q} \left\{ \sigma v^k \mid k \geq 1 \right\} .$$

3.2.3 The Künneth spectral sequence

$$\mathrm{Tor}^{H_*(SU/SO, \mathbb{Q})}(H_*(S^0, \mathbb{Q}), H_*(S^0, \mathbb{Q})) \Rightarrow H_*((S^0[SU/SO_+]^\dagger, \mathbb{Q}))$$

([32], IV § 4.1) for the rational homology of $S^0 \wedge_{S^0[SU/SO_+]}^L S^0$ collapses, yielding an isomorphism of its target with the algebra of symmetric functions on generators of degree $4k+2$, $k \geq 0$. It is algebraically isomorphic to the Rothenberg-Steenrod spectral sequence

$$\mathrm{Tor}^{H_*(SU/SO, \mathbb{Q})}(\mathbb{Q}, \mathbb{Q}) \Rightarrow H_*(\mathrm{Sp}/\mathrm{SU}, \mathbb{Q}) ,$$

([59], § 7.4) for $B(SU/SO)$, allowing us to identify $S^0_{\mathbb{Q}}[\mathrm{Sp}/\mathrm{SU}_+]$ with the covariant Koszul dual of $S^0_{\mathbb{Q}}[SU/SO_+]$. The composition

$$S^0_{\mathbb{Q}}[\mathrm{Sp}/\mathrm{SU}_+] = (S^0 \wedge_{S^0[SU/SO_+]}^L S^0)_{\mathbb{Q}} \rightarrow (S^0 \wedge_{S^0 \vee \Sigma kO}^L S^0)_{\mathbb{Q}} \rightarrow (S^0 \wedge_{K(S^0)} S^0)_{\mathbb{Q}} = K(S^0)_{\mathbb{Q}}^\dagger$$

represents the abelianization map

$$G(f^*) \rightarrow G(f_{\mathrm{ab}}^*) (= \mathrm{Spec} H_*(\mathrm{Sp}/\mathrm{SU}, \mathbb{Q}))$$

of § 2.4 above.

3.2.4 Remarks

- 1) v^2 is twice the Bott periodicity class.
- 2) The arguments above are based on the equivalence, over the rationals, of $K(S^0)$ and $K(\mathbb{Z})$. In a way this is analogous to the isomorphism between singular (Betti) and algebraic de Rham (Grothendieck) cohomology of algebraic varieties. Nori ([48], Theorem 6) formulates the theory of periods in terms of functions on the torsor of isomorphisms between these theories; from this point of view, zeta-values appear as functions on $\mathrm{Spec} (K(\mathbb{Z})_* K(S^0))$, viewed as a torsor relating arithmetic geometry to differential topology.

3.3 The Tannakian category of mixed Tate motives over \mathbb{Z} constructed by Deligne and Goncharov is equivalent to the category of linear representations of the motivic group $\mathrm{GT}_{\mathrm{MT}}$ of that category (thought to be closely related to Drinfel'd's prounipotent version of the Grothendieck-Teichmüller group ([2], § 25.9.4; [28]; [73] § 6.1, Prop 9.1)). At the end of a later paper, Goncharov describes the Hopf algebra $\mathcal{H}_{\mathrm{GT}_{\mathrm{MT}}^*}$ of functions on this motivic group in some detail: in particular ([35], § 8.2 Theorem 8.2, § 8.4 exp (110)) he identifies it as the cotensor algebra $T^\bullet(K_{\mathbb{Q}}\mathbb{Z})$, where

$$K_{\mathbb{Q}}\mathbb{Z} := \bigoplus_{n \geq 1} K_{2n-1}(\mathbb{Z}) \otimes \mathbb{Q}$$

(regarded as a graded module with K_{2n-1} situated in degree n).

The composition of the pseudo-isotopy map w_* of § 3.2.1 with Waldhausen’s isomorphism $K(S^0) \otimes \mathbb{Q} \cong K(\mathbb{Z}) \otimes \mathbb{Q}$ identifies the free graded Lie algebra on $K_{\mathbb{Q}}\mathbb{Z}$ with the free Lie algebra f_* of § 2.3iii above, yielding an isomorphism

$$G(f_*) \rightarrow GT_{MT}$$

of proalgebraic groups. Corollary 2.4 then implies the

Theorem. The composition

$$w_* \circ L\Phi^\dagger : D_b(K_*(S^0) \otimes \mathbb{Q} - \text{Mod}) \rightarrow D_b(GT_{MT} - \text{Mod}) ,$$

defines an equivalence of the homotopy category of rational $K(S^0)$ -module spectra with the derived category of mixed Tate motives over \mathbb{Z} . □

4 Some applications

This section discusses some applications of the preceding discussion. The first paragraph below is essentially an acknowledgement of ignorance about topological cyclic homology. The second discusses some joint work in progress [47] with Nitu Kitchloo. The setup and ideas are entirely his; the section below sketches how Koszul duality seems to fit in with them. I am indebted to Kitchloo for generously sharing these ideas with me.

The third paragraph summarizes some of the work of Blumberg, Gepner, and Tabuada mentioned in the ‘Introduction’, concerned with a program for constructing enriched decategorifications of their approach to generalized motives as small stable ∞ -categories.

4.1 Topological cyclic homology ([13,17], ...) is a powerful tool for the study of the algebraic K -theory of spaces, and its role in these matters deserves discussion here; but at the moment, there are technical obstructions to telling a coherent story. The current state of the art defines local invariants $TC(X;p)$ for a space at each prime p (closely related to the homotopy quotient of the suspension of the free loop space of X), whereas the theory of mixed Tate motives over integer rings is intrinsically global. For example, the topological cyclic homology of a point looks much like the p -completion of an ad hoc geometric model

$$TC^{\text{geo}}(S^0) \sim S^0 \vee \Sigma CP_{-1}^\infty$$

([9,57], [62], § 3) with

$$H_*(TC^{\text{geo}}(S^0), \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}\{\sigma t^k \mid k \geq -1\} .$$

The (rational) Koszul dual of this object defines a proalgebraic groupscheme associated to a free graded Lie algebra roughly twice as big as f_* , i.e., with generators in topological degree $-2k$ rather than $-2(2k + 1)$. A similar group appears in work of Connes and Marcolli ([25], Prop 5.4) on renormalization theory, and topological cyclic homology is plausibly quite relevant to that work; but because the global arithmetic properties of topological cyclic homology are not yet well understood, it seems premature to speculate further here; this remark is included only to signal this possible connection to physics.

4.2 Example: Kitchloo [46] has defined a rigid monoidal category $s\mathbb{S}$ with symplectic manifolds (M, ω) as objects, and stable equivalence classes of oriented Lagrangian correspondences as morphisms. It has a fiber functor which sends such a manifold (endowed with a compatible almost-complex structure) to a Thom spectrum

$$s\Omega(M) = U/SO(T_M)^{-\zeta}$$

constructed from the U/SO -bundle of Lagrangian structures on its stable tangent space. An Eilenberg-Moore spectral sequence with

$$E_2 = \text{Tor}_*^{H^*(BU)}(H^*(M), H^*(BSO))$$

([59], § 7.4) computes $H^*(U/SO(T_M))$, and away from the prime two, the equivariant Borel cohomology

$$H_{U/SO}^*(s\Omega(M)) := H^*(s\Omega(M) \times_{U/SO} E(U/SO))$$

is naturally isomorphic to $H^*(M)$.

The functor $s\Omega(-)$ has many of the formal properties of a homology theory; for example, when M is a point, $s\Omega := s\Omega(*)$ is a ring-spectrum [65], and $s\Omega(-)$ takes values in the category of $s\Omega$ -modules. Moreover, when V is compact oriented, with the usual symplectic structure on its cotangent space,

$$s\Omega(T^*V) \sim [V, s\Omega]$$

([46], § 2.6) defines a cobordism theory of Lagrangian maps (in the sense of Arnol'd) to V .

The composition

$$s\Omega(M) \rightarrow s\Omega(M) \wedge M_+ \rightarrow s\Omega(M) \wedge B(U/SO)_+$$

(defined by the map $M \rightarrow B(U/SO)$ which classifies the bundle $U/SO(T_M)$ of Lagrangian frames on M) makes $s\Omega(M)$ a comodule over the Hopf spectrum

$$\text{THH}(s\Omega) \cong s\Omega \wedge B(U/SO)_+ \sim s\Omega[\text{Sp}/U_+]$$

(the analog, in this context, of an action of the abelianization $G(\mathfrak{f}_{\text{ab}}^*)$ ([47], § 4)). The Hopf algebra counit

$$\left[1 : S_{\mathbb{Q}}^0[U/SO_+] \rightarrow S_{\mathbb{Q}}^0 \right] \in H^0(U/SO, \mathbb{Q})$$

provides, via the Thom isomorphism, an augmentation

$$\left[s\Omega_{\mathbb{Q}} \rightarrow S_{\mathbb{Q}}^0 \right] \in H^*(s\Omega, \mathbb{Q}) \cong H^*(U/SO, \mathbb{Q}) .$$

Proposition. The covariant Koszul dual

$$s\Omega(M)_{\mathbb{Q}}^{\dagger} := s\Omega(M) \wedge_{s\Omega_{\mathbb{Q}}}^L S_{\mathbb{Q}}^0$$

is a comodule over

$$S_{\mathbb{Q}}^0 \wedge_{s\Omega_{\mathbb{Q}}}^L S_{\mathbb{Q}}^0 \sim S_{\mathbb{Q}}^0 \wedge_{S_{\mathbb{Q}}^0[U/SO_+]}^L S_{\mathbb{Q}}^0 \sim S^0[\text{Sp}/U_+] ;$$

by naturality its *contravariant* Koszul dual

$$\text{RHom}_{s\Omega_{\mathbb{Q}}}(S_{\mathbb{Q}}^0, s\Omega_{\mathbb{Q}}^{\dagger}(M)) \cong s\Omega_{\mathbb{Q}}(M)$$

inherits an $s\Omega_{\mathbb{Q}}[\text{Sp}/U_+]$ - coaction: equivalently, an action of the abelianized Grothendieck-Teichmüller group $G(\mathfrak{f}_{\text{ab}}^*)$.

Remarks. i) It seems likely that this coaction agrees with the $\mathrm{THH}(s\Omega)$ -coaction described above.

ii) If $M = T^*V$ is a cotangent bundle, we have an isomorphism

$$s\Omega_{\mathbb{Q}}^{\dagger}(M) \cong H_*(V, \mathbb{Q}) .$$

iii) The sketch above is proposed as an analog, in the theory of geometric quantization, to work ([48], § 4.6.2, § 8.4) of Kontsevich on deformation quantization. A version of the Grothendieck-Teichmüller group acts on the Hochschild cohomology

$$\mathrm{HH}_{\mathbb{C}}^*(M) := \mathrm{Ext}_{\mathcal{O}_{M \times M}}^*(\mathcal{O}_M, \mathcal{O}_M) \cong \bigoplus H^*(M, \Lambda^* T_M)$$

of a complex manifold (defined in terms of coherent sheaves of holomorphic functions on $M \times M$). If M is Calabi-Yau, its tangent and cotangent bundles can be identified, resulting in an action of the abelianized Grothendieck-Teichmüller group on the Hodge cohomology of M .

Note that $\mathrm{Sp}/\mathrm{U} \sim B\mathbb{T} \times \mathrm{Sp}/\mathrm{SU}$ splits. The action on $s\Omega$ of the two-dimensional cohomology class carried by $B\mathbb{T}$ does not seem to come from a $K(S^0)^{\dagger}$ coaction but rather from variation of the symplectic structure. This may be related to Kontsevich’s remarks (just after Theorems 7 and 9) about Euler’s constant.

4.3 Example Marshaling the forces of higher category theory, Blumberg, Gepner, and Tabuada [10] have developed a beautiful approach to the study of noncommutative motives, defining symmetric monoidal categories \mathcal{M} (there are several interesting variants ([10], § 6.7, § 8.10)) whose objects are small stable ∞ -categories (e.g., of perfect complexes of quasicoherent sheaves of modules over a scheme, or of suitably small modules over the Spanier-Whitehead dual ring-spectrum $[X, S^0]$ of a finite complex). The morphism objects

$$\mathrm{Mor}_{\mathcal{M}}(\mathcal{A}, \mathcal{B})$$

in these constructions are K -theory spectra of categories of exact functors between \mathcal{A} and \mathcal{B} ; this defines spectral enrichments over the homotopy category of $K(S^0)$ -modules ([11], Corollary 1.11).

The arguments of this paper imply that covariant Koszul duality, as outlined above, defines versions of these categories with morphism objects

$$\mathrm{Mor}_{\mathrm{Ho}(\mathcal{M})}(\mathcal{A}, \mathcal{B})_{\mathbb{Q}}^{\dagger} \in K(S^0)_{\mathbb{Q}}^{\dagger} - \mathrm{Comod}$$

which, under suitable finiteness conditions, may be regarded as enriched over $D_b(\mathrm{MT}_{\mathbb{Q}}(\mathbb{Z}))$. They suggest the existence of categories $\mathrm{Ho}(\mathcal{M})^{\dagger}$ with morphism objects

$$\mathrm{Mor}_{\mathrm{Ho}(\mathcal{M})}(\mathcal{A}, \mathcal{B}) \in K(S^0)^{\dagger} - \mathrm{Comod}$$

which rationalize to the categories described above. This seems to fit well with recent work [66] on Kontsevich’s conjecture on noncommutative motives over a field ([61], § 4.4). The theory of cyclotomic spectra [13] suggests the existence of related constructions from that point of view, but (as noted in § 4.1) their arithmetic properties are not yet very well understood.

Recently F Brown, using earlier work of Zagier [74], has shown that the algebra $\mathcal{H}_{\mathrm{GT}_{\mathrm{MT}}}^*$ is isomorphic to a polynomial algebra

$$\mathbb{Q}[\zeta^m(w) \mid w \in \mathrm{Lyndon}\{2, 3\}] := \mathbb{Q}[\zeta^m]$$

of motivic polyzeta values indexed by certain Lyndon words (cf ([20], § 3 exp 3.6, § 8): working with motivic polyzetas avoids questions of algebraic independence of numerical polyzetas). This suggests the category $Ho(\mathcal{M})_{\mathbb{Q}[\zeta^m]}^\dagger$, with morphism objects

$$\mathrm{Mor}_{Ho(\mathcal{M})}(\mathcal{A}, \mathcal{B})_{\mathbb{Q}}^\dagger \otimes_{\mathrm{GTMT}} \mathbb{Q}[\zeta^m]$$

as a convenient ‘untwisted’ $\mathbb{Q}[\zeta^m]$ -linear category of noncommutative motives.

Endnote

^aThis paper was inspired by Graeme Segal’s description of such objects as ‘blancmanges’.

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