Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2009, Article ID 917175, 10 pages
doi:10.1155/2009/917175

Research Article

Fixed Point Theorems for Contractive Mappings in Complete G-Metric Spaces

Zead Mustafa[1] and Brailey Sims[2]

[1] *Department of Mathematics, The Hashemite University, P.O. Box 330127, Zarqa 13115, Jordan*
[2] *School of Mathematical and Physical Sciences, The University of Newcastle, NSW 2308, Australia*

Correspondence should be addressed to Zead Mustafa, zmagablh@hu.edu.jo

Received 31 December 2008; Accepted 7 April 2009

Recommended by Hélène Frankowska

We prove some fixed point results for mappings satisfying various contractive conditions on Complete G-metric Spaces. Also the Uniqueness of such fixed point are proved, as well as we showed these mappings are G-continuous on such fixed points.

1. Introduction

Metric spaces are playing an increasing role in mathematics and the applied sciences.

Over the past two decades the development of fixed point theory in metric spaces has attracted considerable attention due to numerous applications in areas such as variational and linear inequalities, optimization, and approximation theory.

Different generalizations of the notion of a metric space have been proposed by Gahler [1, 2] and by Dhage [3, 4]. However, HA et al. [5] have pointed out that the results obtained by Gahler for his 2 metrics are independent, rather than generalizations, of the corresponding results in metric spaces, while in [6]the current authors have pointed out that Dhage's notion of a D-metric space is fundamentally flawed and most of the results claimed by Dhage and others are invalid.

In 2003 we introduced a more appropriate and robust notion of a generalized metric space as follows.

Definition 1.1 (see [7]). Let X be a nonempty set, and let $G : X \times X \times X \to R^{+}$ be a function satisfying the following axioms:

(G_1) $G(x, y, z) = 0$ if $x = y = z$,

(G_2) $0 < G(x, x, y)$, forall $x, y \in X$, with $x \neq y$,

(G_3) $G(x,x,y) \leq G(x,y,z)$, forall $x,y,z \in X$, with $z \neq y$,

(G_4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$ (symmetry in all three variables),

(G_5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$, for all $x,y,z,a \in X$, (rectangle inequality).

Then the function G is called a *generalized metric*, or, more specifically a *G-metric* on X, and the pair (X,G) is called a *G-metric space*.

Example 1.2 (see [7]). Let (X,d) be a usual metric space, then (X,G_s) and (X,G_m) are G-metric space, where

$$\begin{aligned} G_s(x,y,z) &= d(x,y) + d(y,z) + d(x,z), \quad \forall\, x,y,z \in X, \\ G_m(x,y,z) &= \max\{d(x,y), d(y,z), d(x,z)\}, \quad \forall\, x,y,z \in X. \end{aligned} \tag{1.1}$$

We now recall some of the basic concepts and results for G-metric spaces that were introduced in ([7]).

Definition 1.3. Let (X,G) be a G-metric space, let (x_n) be a sequence of points of X, we say that (x_n) is *G-convergent* to x if $\lim_{n,m\to\infty} G(x,x_n,x_m) = 0$; that is, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x,x_n,x_m) < \epsilon$, for all $n,m \geq N$ (throughout this paper we mean by $\mathbb{N}$ the set of all natural numbers). We refer to x as the *limit* of the sequence (x_n) and write $x_n \xrightarrow{(G)} x$.

Proposition 1.4. *Let (X,G) be a G-metric space then the following are equivalent.*

(1) *(x_n) is G-convergent to x.*

(2) $G(x_n,x_n,x) \to 0$, *as* $n \to \infty$.

(3) $G(x_n,x,x) \to 0$, *as* $n \to \infty$.

Definition 1.5. Let (X,G) be a G-metric space, a sequence (x_n) is called G-Cauchy if given $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n,x_m,x_l) < \epsilon$, for all $n,m,l \geq N$ that is if $G(x_n,x_m,x_l) \to 0$ as $n,m,l \to \infty$.

Proposition 1.6. *In a G-metric space (X,G), the following are equivalent.*

(1) *The sequence (x_n) is G-Cauchy.*

(2) *For every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n,x_m,x_m) < \epsilon$, for all $n,m \geq N$.*

Definition 1.7. Let (X,G) and (X',G') be G-metric spaces and let $f : (X,G) \to (X',G')$ be a function, then f is said to be *G-continuous at a point*$a \in X$ if given $\epsilon > 0$, there exists $\delta > 0$ such that $x,y \in X$; $G(a,x,y) < \delta$ implies $G'(f(a),f(x),f(y)) < \epsilon$. A function f is G-continuous on X if and only if it is G-continuous at all $a \in X$.

Proposition 1.8. *Let (X,G), (X',G') be G-metric spaces, then a function $f : X \to X'$ is G-continuous at a point $x \in X$ if and only if it is G-sequentially continuous at x; that is, whenever (x_n) is G-convergent to x, $(f(x_n))$ is G-convergent to $f(x)$.*

Proposition 1.9. *Let (X,G) be a G-metric space, then the function $G(x,y,z)$ is jointly continuous in all three of its variables.*

Definition 1.10. A G-metric space(X,G) is said to be *G-complete* (or a complete G-metric space) if every G-Cauchy sequence in (X,G) is G-convergent in (X,G).

2. The Main Results

We begin with the following theorem.

Theorem 2.1. *Let (X,G) be a complete G-metric space and let $T : X \to X$ be a mapping which satisfies the following condition, for all $x,y,z \in X$,*

$$G(T(x),T(y),T(z)) \le k\max\{G(x,y,z),G(x,T(x),T(x)),G(y,T(y),T(y)),\\ G(z,T(z),T(z)),G(x,T(y),T(y)),G(y,T(z),T(z)),G(z,T(x),T(x))\}, \tag{2.1}$$

where $k \in [0,1/2)$. Then T has a unique fixed point (say u) and T is G-continuous at u.

Proof. Suppose that T satisfies condition (2.1), let $x_0 \in X$ be an arbitrary point, and define the sequence (x_n) by $x_n = T^n(x_0)$, then by (2.1), we have

$$G(x_n,x_{n+1},x_{n+1}) \le k\max\{G(x_{n-1},x_n,x_n),G(x_{n-1},x_{n+1},x_{n+1}),G(x_n,x_{n+1},x_{n+1})\} \tag{2.2}$$

so,

$$G(x_n,x_{n+1},x_{n+1}) \le k\max\{G(x_{n-1},x_{n+1},x_{n+1}),G(x_{n-1},x_n,x_n)\}. \tag{2.3}$$

But, by (G5), we have

$$G(x_{n-1},x_{n+1},x_{n+1}) \le G(x_{n-1},x_n,x_n) + G(x_n,x_{n+1},x_{n+1}). \tag{2.4}$$

So, (2.3) becomes

$$\mathrm{G(x_n,x_{n+1},x_{n+1})} \le k\max\{G(x_{n-1},x_n,x_n)+G(x_n,x_{n+1},x_{n+1}),G(x_{n-1},x_n,x_n)\}. \tag{2.5}$$

So, it must be the case that

$$G(x_n,x_{n+1},x_{n+1}) \le k\{G(x_{n-1},x_n,x_n)+G(x_n,x_{n+1},x_{n+1})\}, \tag{2.6}$$

which implies

$$G(x_n,x_{n+1},x_{n+1}) \le \frac{k}{1-k}G(x_{n-1},x_n,x_n). \tag{2.7}$$

Let $q = k/1-k$, then $q<1$ and by repeated application of (2.7), we have

$$G(x_n, x_{n+1}, x_{n+1}) \leq q^n G(x_0, x_1, x_1). \tag{2.8}$$

Then, for all $n, m \in \mathbb{N}, n < m$, we have by repeated use of the rectangle inequality and (2.8) that

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + G(x_{n+2}, x_{n+3}, x_{n+3}) + \cdots + G(x_{m-1}, x_m, x_m) \\ &\leq \left(q^n + q^{n+1} + \cdots + q^{m-1}\right) G(x_0, x_1, x_1) \\ &\leq \frac{q^n}{1-q} G(x_0, x_1, x_1). \end{aligned} \tag{2.9}$$

Then, $\lim G(x_n, x_m, x_m) = 0$, as $n, m \to \infty$, since $\lim q^n/1-q\ G(x_0, x_1, x_1) = 0$, as $n, m \to \infty$. For $n, m, l \in \mathbb{N}$ (G5) implies that $G(x_n, x_m, x_l) \leq G(x_n, x_m, x_m) + G(x_l, x_m, x_m)$, taking limit as $n, m, l \to \infty$, we get $G(x_n, x_m, x_l) \to 0$. So (x_n) is G-Cauchy a sequence. By completeness of (X, G), there exists $u \in X$ such that (x_n) is G-converges to u. Suppose that $T(u) \neq u$, then

$$G(x_n, T(u), T(u)) \leq k \max \left\{ \begin{matrix} G(x_{n-1}, u, u), G(x_{n-1}, x_n, x_n), G(u, T(u), T(u)) \\ G(x_{n-1}, T(u), T(u)), G(u, x_n, x_n) \end{matrix} \right\}, \tag{2.10}$$

taking the limit as $n \to \infty$, and using the fact that the function G is continuous on its variables, we have $G(u, T(u), T(u)) \leq k\,G(u, T(u), T(u))$, which is a contradiction since $0 \leq k < 1/2$. So, $u = T(u)$. To prove uniqueness, suppose that $v \neq u$ is such that $T(v) = v$, then (2.1) implies that $G(u, v, v) \leq k \max\{G(u, v, v), G(v, u, u)\}$, thus $G(u, v, v) \leq kG(v, u, u)$ again by the same argument we will find $G(v, u, u) \leq kG(u, v, v)$, thus

$$G(u, v, v) \leq k^2 G(u, v, v) \tag{2.11}$$

which implies that $u = v$, since $0 \leq k < 1/2$. To see that T is G-continuous at u, let $(y_n) \subseteq X$ be a sequence such that $\lim(y_n) = u$, then

$$G(T(y_n), T(u), T(y_n)) \leq k \max \left\{ \begin{matrix} G(y_n, u, y_n), G(y_n, T(y_n), T(y_n)), \\ G(u, T(u), T(u)), G(y_n, T(u), T(u)), \\ G(u, T(y_n), T(y_n)) \end{matrix} \right\}, \tag{2.12}$$

and we deduce that

$$G(T(y_n), u, T(y_n)) \leq k \max \left\{ \begin{matrix} G(y_n, u, y_n), G(y_n, T(y_n), T(y_n)), \\ G(y_n, u, u) \end{matrix} \right\} \tag{2.13}$$

but (G5) implies that

$$G(y_n, T(y_n), T(y_n)) \leq G(y_n, u, u) + G(u, T(y_n), T(y_n)) \tag{2.14}$$

and (2.13) leads to the following cases,

(1) $G(T(y_n), u, T(y_n)) \leq kG(y_n, y_n, u)$,

(2) $G(T(y_n), u, T(y_n)) \leq kG(y_n, u, u)$,

(3) $G(T(y_n), u, T(y_n)) \leq qG(y_n, u, u)$.

In each case take the limit as $n \to \infty$ to see that $G(u, T(y_n), T(y_n)) \to 0$ and so, by Proposition 1.4, we have that the sequence $(T(y_n))$ is G-convergent to $u = Tu$, therefor Proposition 1.8 implies that T is G-continuous at u. □

Remark 2.2. If the G-metric space is bounded (that is, for some $M > 0$ we have $G(x, y, z) \leq M$ for all $x, y, z \in X$) then an argument similar to that used above establishes the result for $0 \leq k < 1$.

Corollary 2.3. *Let (X, G) be a complete G-metric spaces and let $T : X \to X$ be a mapping which satisfies the following condition for some $m \in \mathbf{N}$ and for all $x, y, z \in X$:*

$$G(T^m(x), T^m(y), T^m(z)) \leq k \max \left\{ \begin{array}{c} G(x, y, z), G(x, T^m(x), T^m(x)), \\ G(y, T^m(y), T^m(y)), G(z, T^m(z), T^m(z)), \\ G(x, T^m(y), T^m(y)), G(y, T^m(z), T^m(z)), \\ G(z, T^m(x), T^m(x)), \end{array} \right\}, \tag{2.15}$$

where $k \in [0, 1/2)$, then T has a unique fixed point (say u), and T^m is G-continuous at u.

Proof. From the previous theorem, we have that T^m has a unique fixed point (say u), that is, $T^m(u) = u$. But $T(u) = T(T^m(u)) = T^{m+1}(u) = T^m(T(u))$, so $T(u)$ is another fixed point for T^m and by uniqueness $Tu = u$. □

Theorem 2.4. *Let (X, G) be a complete G-metric space, and let $T : X \to X$ be a mapping which satisfies the following condition for all $x, y, z \in X$:*

$$G(T(x), T(y), T(z)) \leq k \max \left\{ \begin{array}{c} [G(x, T(y), T(y)) + G(y, T(x), T(x))], \\ [G(y, T(z), T(z)) + G(z, T(y), T(y))], \\ [G(x, T(z), T(z)) + G(z, T(x), T(x))] \end{array} \right\}, \tag{2.16}$$

where $k \in [0, 1/2)$, then T has a unique fixed point (say u), and T is G-continuous at u.

Proof. Suppose that T satisfies the condition (2.16), let $x_0 \in X$ be an arbitrary point, and define the sequence (x_n) by $x_n = T^n(x_0)$, then by (2.16) we get

$$G(x_n, x_{n+1}, x_{n+1}) \le k \max \left\{ \begin{array}{c} [G(x_{n-1}, x_{n+1}, x_{n+1}) + G(x_n, x_n, x_n)], \\ [G(x_n, x_{n+1}, x_{n+1}) + G(x_n, x_{n+1}, x_{n+1})], \\ [G(x_{n-1}, x_{n+1}, x_{n+1}) + G(x_n, x_n, x_n)] \end{array} \right\} \\ = k \max\{G(x_{n-1}, x_{n+1}, x_{n+1}),\ 2G(x_n, x_{n+1}, x_{n+1})\}, \tag{2.17}$$

since $0 \le k < 1/2$, then it must be the case that

$$G(x_n, x_{n+1}, x_{n+1}) \le kG(x_{n-1}, x_{n+1}, x_{n+1}) \tag{2.18}$$

but from (G5), we have

$$G(x_{n-1}, x_{n+1}, x_{n+1}) \le G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1}), \tag{2.19}$$

so (2.18) implies that

$$G(x_n, x_{n+1}, x_{n+1}) \le \frac{k}{1-k}\ G(x_{n-1}, x_n, x_n), \tag{2.20}$$

let $q = k/1 - k$, then $q < 1$ and by repeated application of (2.20), we have

$$G(x_n, x_{n+1}, x_{n+1}) \le q^n\ G(x_0, x_1, x_1). \tag{2.21}$$

Then, for all $n, m \in \mathbb{N}$, $n < m$, we have, by repeated use of the rectangle inequality, $G(x_n, x_m, x_m) \le G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_{n+3}, x_{n+3}) + \cdots + G(x_{m-1}, x_m, x_m) \le (q^n + q^{n+1} + \cdots + q^{m-1}) G(x_0, x_1, x_1) \le q^n/1 - q\ G(x_0, x_1, x_1)$. So, $\lim G(x_n, x_m, x_m) = 0$, as $n, m \to \infty$ and (x_n) is G-Cauchy sequence. By the completeness of (X, G), there exists $u \in$ X such tha(x_n) is G-convergent to u.Suppose that $T(u) \neq u$, then

$$G(x_n, T(u), T(u)) \le k \max \left\{ \begin{array}{c} [G(x_{n-1}, T(u), T(u)) + G(u, x_n, x_n)], \\ [G(u, T(u), T(u)) + G(u, T(u), T(u))], \\ [G(x_{n-1}, T(u), T(u)) + G(u, x_n, x_n)] \end{array} \right\}. \tag{2.22}$$

Taking the limit as $n \to \infty$, and using the fact that the function G is continuous in its variables, we get

$$G(u, T(u), T(u)) \le k \max\{2G(u, T(u), T(u)), G(u, T(u), T(u))\}, \tag{2.23}$$

since $0 \le k < 1/2$, this contradiction implies that $u = T(u)$.To prove uniqueness, suppose that $v \ne u$ such that $T(v) = v$, then

$$G(u,v,v) \le k \max \left\{ \begin{array}{l} [G(u,v,v) + G(v,u,u)], \\ [G(v,v,v) + G(v,v,v)], \\ [G(u,v,v) + \ G(v,u,u)] \end{array} \right\}, \tag{2.24}$$

so we deduce that $G(u,v,v) \le k[G(u,v,v) + G(v,u,u)]$. This implies that $G(u,v,v) \le (k/1-k)G(v,u,u)$ and by repeated use of the same argument we will find $G(v,u,u) \le (k/1-k)G(u,v,v)$. Therefor we get $G(u,v,v) \le (k/1-k)^2 G(v,u,u)$, since $0 < k/1-k < 1$, this contradiction implies that $u = v$. To show that T is G-continuous at u let $(y_n) \subseteq X$ be a sequence such that $\lim(y_n) = u$ in (X,G), then

$$G(T(y_n),T(u),T(u)) \le k \max \left\{ \begin{array}{c} [G(y_n,T(u),T(u)) + G(u,T(y_n),T(y_n))], \\ [G(u,T(u),T(u)), G(u,T(u),T(u))] \\ [G(y_n,T(u),T(u)) + G(u,T(y_n),T(y_n))] \end{array} \right\}. \tag{2.25}$$

Thus, (2.25) becomes

$$G(T(y_n),u,u) \le k[G(y_n,u,u) + G(u,T(y_n),T(y_n))] \tag{2.26}$$

but by (G5) we have $G(u,T(y_n),T(y_n)) \le 2G(T(y_n),u,u)$, therefor (2.26) implies that $G(T(y_n),u,u) \le kG(y_n,u,u) + 2kG(T(y_n),u,u)$ and we deduce that

$$G(T(y_n),u,u) \le \frac{k}{1-2k} G(y_n,u,u). \tag{2.27}$$

Taking the limit of (2.27) as $n \to \infty$, we see that $G(T(y_n),u,u) \to 0$ and so, by Proposition 1.8, we have $T(y_n) \to u = Tu$ which implies that T is G-continuous at u. □

Corollary 2.5. *Let (X,G) be a complete G-metric space, and let $T : X \to X$ be a mapping which satisfies the following condition for some $m \in \mathbb{N}$ and for all $x,y,z \in X$:*

$$G(T^m(x),T^m(y),T^m(z)) \le k \max \left\{ \begin{array}{c} [G(x,T^m(y),T^m(y)) + G(y,T^m(x),T^m(x))], \\ [G(y,T^m(z),T^m(z)) + \ G(z,T^m(y),T^m(y))], \\ [G(x,T^m(z),T^m(z)) + G(z,T^m(x),T^m(x))] \end{array} \right\}, \tag{2.28}$$

where $k \in [0,1/2)$, then T has a unique fixed point (say u), and T^m is G-continuous at u.

Proof. The proof follows from the previous theorem and the same argument used in Corollary 2.3. □

Theorem 2.6. *Let (X,G) be a complete G-metric space, and let $T : X \to X$ be a mapping which satisfies the following condition, for all $x, y \in X$,*

$$G(T(x),T(y),T(y)) \leq k\max\left\{\begin{array}{c}[G(y,T(y),T(y)) + G(x,T(y),T(y))], \\ [2G(y,T(x),T(x))]\end{array}\right\}, \tag{2.29}$$

where $k \in [0,1/3)$, then T has a unique fixed point, say u, and T is G-continuous at u.

Proof. Suppose that T satisfies the condition (2.29). Let $x_0 \in X$ be an arbitrary point, and define the sequence (x_n) by $x_n = T^n(x_0)$, then by (2.29), we have

$$G(x_n,x_{n+1},x_{n+1}) \leq k\max\left\{\begin{array}{c}[G(x_n,x_{n+1},x_{n+1}) + G(x_{n-1},x_{n+1},x_{n+1})], \\ [2G(x_n,x_n,x_n)],\end{array}\right\}, \tag{2.30}$$

thus $G(x_n,x_{n+1},x_{n+1}) \leq kG(x_n,x_{n+1},x_{n+1}) + kG(x_{n-1},x_{n+1},x_{n+1})$ and so

$$G(x_n,x_{n+1},x_{n+1}) \leq \frac{k}{1-k}\ G(x_{n-1},x_{n+1},x_{n+1}). \tag{2.31}$$

But by (G5) we have

$$G(x_{n-1},x_{n+1},x_{n+1}) \leq G(x_{n-1},x_n,x_n) + G(x_n,x_{n+1},x_{n+1}). \tag{2.32}$$

Let $p = k/1-2k$, then $p \in [0,1)$ since $k \in [0,1/3)$ and from (2.31) we deduce that

$$G(x_n,x_{n+1},x_{n+1}) \leq pG(x_{n-1},x_n,x_n). \tag{2.33}$$

Continuing this procedure we get $G(x_n,x_{n+1},x_{n+1}) \leq p^nG(x_0,x_1,x_1)$. Then, for all $n,m \in \mathbb{N}, n < m$, we have by repeated use of the rectangle inequality that $G(x_n,x_m,x_m) \leq G(x_n,x_{n+1},x_{n+1}) + G(x_{n+1},x_{n+2},x_{n+2}) + G(x_{n+2},x_{n+3},x_{n+3}) + \cdots + G(x_{m-1},x_m,x_m) \leq (p^n + p^{n+1} + \cdots + p^{m-1})\ G(x_0,x_1,x_1) \leq p^n/1-p\ \ G(x_0,x_1,x_1)$.Thus, $\lim\ G(x_n,x_m,x_m) = 0$, as $n,m \to \infty$, so, (x_n) is G-Cauchy a sequence. By completeness of (X,G), there exists $u \in X$ such that (x_n) is G-convergent to u. Suppose that $T(u) \neq u$, then

$$G(x_n,T(u),T(u)) \leq k\max\left\{\begin{array}{c}[G(u,T(u),T(u)) + G(x_{n-1},T(u),T(u))], \\ [2G(u,x_n,x_n)]\end{array}\right\}, \tag{2.34}$$

taking the limit as $n \to \infty$, and using the fact that the function G is continuous in its variables, we obtain $G(u,T(u),T(u)) \leq 2kG(u,T(u),T(u))$. Since $0 < k < 1/3$ this is a contradiction so, $u = T(u)$. To prove uniqueness, suppose that $v \neq u$ is such that $T(v) = v$, then

$$G(u,v,v) \leq k\max\left\{\begin{array}{c}[G(v,v,v) + G(u,v,v)], \\ [2G(v,u,u)],\end{array}\right\}, \tag{2.35}$$

thus $G(u,v,v) \leq k \max\{G(u,v,v), 2G(v,u,u)\}$ and we deduce that

$$G(u,v,v) \leq 2kG(v,u,u). \tag{2.36}$$

By the same argument we get

$$G(v,u,u) \leq 2k\ G(u,v,v), \tag{2.37}$$

hence, $G(u,v,v) \leq 4k^2\ G(u,v,v)$ which implies that $u = v$ (since $0 \leq k < 1/3 \Rightarrow 0 \leq 4k^2 < 1$). To show that T is G-continuous at u, let $(y_n) \subseteq X$ be a sequence such that $\lim y_n = u$, then

$$G(T(u), T(y_n), T(y_n)) \leq k \max\left\{\begin{array}{c} [G(y_n, T(y_n), T(y_n)) + G(u, T(y_n), T(y_n))], \\ [2G(y_n, T(u), T(u))] \end{array}\right\}, \tag{2.38}$$

therefore, (2.38) implies two cases.

Case 1. $G(u, T(y_n), T(y_n)) \leq 2kG(y_n, u, u)$.

Case 2. $G(u, T(y_n), T(y_n)) \leq (k/1-k)G(y_n, T(y_n), T(y_n))$.

But, by (G5) we have $G(y_n, T(y_n), T(y_n)) \leq G(y_n, u, u) + G(u, T(y_n), T(y_n))$, so case 2 implies that $G(u, T(y_n), T(y_n)) \leq\ pG(y_n, u, u)$. In each case taking the limit as $n \to \infty$, we see that $G(u, T(y_n), T(y_n)) \to 0$ and so, by Proposition 1.8, we have $T(y_n) \to u = Tu$ which implies that T is G-continuous at u. □

Corollary 2.7. *Let (X,G) be a complete G-metric spaces, and let $T : X \to X$ be a mapping which satisfies the following condition for some $m \in \mathbb{N}$ and for all $x, y \in X$:*

$$G(T^m(x), T^m(y), T^m(y)) \leq k \max\left\{\begin{array}{c} [G(y, T^m(y), T^m(y)) +\ G(x, T^m(y), T^m(y))], \\ [2G(y, T^m(x), T^m(x))] \end{array}\right\}, \tag{2.39}$$

where $k\ \in [0, 1/3)$, then T has a unique fixed point, say u, and T^m is G-continuous at u.

Proof. The proof follows from the previous theorem and the same argument used in Corollary 2.3. The following theorem has been stated in [8] without proof, but this can be proved by using Theorem (2.6) as follows. □

Theorem 2.8 (see [8]). *Let (X, G) be a complete G-metric space and let $T : X \to X$ be a mapping which satisfies the following condition, for all $x, y, z \in X$,*

$$G(T(x), T(y), T(z)) \leq k \max \left\{ \begin{array}{c} [G(z, T(x), T(x)) + G(y, T(x), T(x))], \\ [G(y, T(z), T(z)) + G(x, T(z), T(z))], \\ [G(x, T(y), T(y)) + G(z, T(y), T(y))] \end{array} \right\}, \tag{2.40}$$

where $k \in [0, 1/3)$, then T has a unique fixed point, say u, and T is G-continuous at u.

Proof. Setting $z = y$ in condition (2.40), then it reduced to condition (2.29), and the proof follows from Theorem (2.6). □

References

[1] S. Gähler, "2-metrische Räume und ihre topologische Struktur," *Mathematische Nachrichten*, vol. 26, no. 1–4, pp. 115–148, 1963.

[2] S. Gähler, "Zur geometric 2-metriche räume," *Revue Roumaine de Mathématiques Pures et Appliquées*, vol. 11, pp. 665–667, 1966.

[3] B. C. Dhage, "Generalised metric spaces and mappings with fixed point," *Bulletin of the Calcutta Mathematical Society*, vol. 84, no. 4, pp. 329–336, 1992.

[4] B. C. Dhage, "Generalized metric spaces and topological structure. I," *Analele Ştiinţifice ale Universităţii Al. I. Cuza din Iaşi. Serie Nouă. Matematică*, vol. 46, no. 1, pp. 3–24, 2000.

[5] K. S. Ha, Y. J. Cho, and A. White, "Strictly convex and strictly 2-convex 2-normed spaces," *Mathematica Japonica*, vol. 33, no. 3, pp. 375–384, 1988.

[6] Z. Mustafa and B. Sims, "Some remarks concerning *D*-metric spaces," in *Proceedings of the International Conference on Fixed Point Theory and Applications*, pp. 189–198, Valencia, Spain, July 2004.

[7] Z. Mustafa and B. Sims, "A new approach to generalized metric spaces," *Journal of Nonlinear and Convex Analysis*, vol. 7, no. 2, pp. 289–297, 2006.

[8] Z. Mustafa, *A new structure for generalized metric spaces—with applications to fixed point theory*, Ph.D. thesis, The University of Newcastle, Callaghan, Australia, 2005.