PUBLISHED FOR SISSA BY SPRINGER

RECEIVED: *June 16, 2014*
REVISED: *September 20, 2014*
ACCEPTED: *September 25, 2014*
PUBLISHED: *October 22, 2014*

Global and local properties of AdS_2 higher spin gravity

K.B. Alkalaev

I.E. Tamm Department of Theoretical Physics, P.N. Lebedev Physical Institute, Leninsky ave. 53, 119991 Moscow, Russia
Moscow Institute of Physics and Technology, Dolgoprudnyi, 141700 Moscow region, Russia

E-mail: alkalaev@lpi.ru

ABSTRACT: Two-dimensional BF theory with infinitely many higher spin fields is proposed. It is interpreted as the AdS_2 higher spin gravity model describing a consistent interaction between local fields in AdS_2 space including gravitational field, higher spin partially-massless fields, and dilaton fields. We carry out analysis of the frame-like and the metric-like formulation of the theory. Infinite-dimensional higher spin global algebras and their finite-dimensional truncations are realized in terms of $o(2,1) - sp(2)$ Howe dual auxiliary variables.

KEYWORDS: Higher Spin Gravity, Higher Spin Symmetry

ARXIV EPRINT: 1404.5330

doi:10.1007/JHEP10(2014)122

Contents

1 Introduction

In the recent years, higher spin gauge theories in three, four and higher dimensions have attracted considerable interest (e.g., see reviews [1–5] and references therein), while comparatively little attention has been paid to two-dimensional higher spin theories [6–11]. One of the reasons for this is that higher spin gravity in two dimensions does not necessarily share some of characteristic features of its higher dimensional cousins such as $(A)dS$ background geometry or infinitely many propagating massless modes of all spins. So for example conventional $2d$ Fronsdal-type equations of motion both for massless or massive fields of higher spins $s \geq 1$ do not propagate local degrees of freedom. For that matter the two-dimensional case is somewhat analogous to that in three dimensions, where higher spin Chern-Simons theory also describes no local degrees of freedom [12–15].

It follows that in two dimensions the notion of higher spin gauge fields should be clearly defined. We can, at least formally, introduce gauge fields of higher ranks and impose one or another set of gauge invariant equations and/or constraints. Then some of the resulting gauge systems have no local degrees of freedom, while others describe matter modes as particular components of higher rank gauge fields. In the former case the respective gauge fields often result from higher dimensional gauge systems by taking $d = 2$. In particular, both global and gauge transformations remain intact, while local degrees of freedom disappear.

In view of the above we propose to consider a particular $2d$ topological field theory as higher spin gravity with the cosmological constant. The theory is formulated as two-dimensional BF model with $\mathcal{A}$-valued 0-form and 1-form fields, where $\mathcal{A}$ is some finite-dimensional or infinite-dimensional higher spin Lie algebra.[1] In [10] we explicitly considered the finite-dimensional case of $\mathcal{A} = sl(N, \mathbb{R})$ for $N \geq 2$. The point is that the gauge algebra can be represented in the higher spin basis where generators are arranged as subalgebra $sl(2, \mathbb{R})$ rank-s irreps so that the respective connections are identified with two-dimensional spin-$(s + 1)$ fields. The case of $N = 2$ corresponds to the Jackiw-Teitelboim dilaton gravity [20–25], while taking $N \geq 3$ gives rise to particular higher spin extensions. The $N = 3$ theory was also discussed in [11] in the framework of Poisson sigma-models, mainly form the holographic perspective.

It is remarkable that a ground state of the model under consideration is given by the AdS_2 spacetime. It follows that the gauge sector of the $sl(N, \mathbb{R})$ higher spin gravity model comprises gauge fields in AdS_2 space with spins $s = 2, 3, \ldots, N$ and masses $m_s^2 = s(s-1)\Lambda$, where Λ is the cosmological constant. Using their global symmetry properties one finds that

[1] Two-dimensional topological gravity and its higher spin extensions can be defined in a different way as topological field theories of Witten type [16–19]. The higher spin gravity we elaborate here is obviously a topological field theory of Schwarz type.

JHEP10(2014)122

the fields are to be treated as "topological partially-massless" fields of maximal depth [10]. Recall that the system does not have local degrees of freedom. It follows that the AdS_2 higher spin gravity can be interpreted as a consistent theory of topological yet interacting partially-massless higher spin fields given in a closed form. It is worth noting that partially-massless fields in higher dimensions do have local degrees of freedom [26–30], while their interactions at the action level are known only in the cubic approximation [31–35].

In this paper we formulate AdS_2 higher spin gravity with (in)finitely many fields as BF theory for the infinite-dimensional higher spin gauge algebra $\mathcal{A} = \mathrm{hs}[\nu]$ and its finite-dimensional truncations [36, 37]. Note that similar models with an infinite higher spin algebra were partly discussed in [7, 9]. Here we focus on the following issues.

- Local tensor fields in the AdS_2 higher spin gravity: frame-like versus the metric-like formulation. We study in detail the interplay between the BF formulation of the higher spin gravity which is actually the frame-like formulation and its metric-like formulation which extends the original Jackiw-Teitelboim dilaton gravity.

- Global higher spin symmetry algebras:[2] a formulation using the Howe duality $o(2,1)-sp(2)$ between AdS_2 global symmetry algebra and auxiliary symplectic algebra. We explicitly describe previously unknown realization of higher spin algebras $\mathcal{A} = \mathrm{hs}[\nu]$ in terms of $o(2,1) - sp(2)$ vector doublet variables.[3] Gauging algebra $\mathcal{A}$ defines local invariance of the BF theory under consideration.

- BF action for $\mathcal{A}$-valued gauge fields: introducing particular trace operation on the infinite-dimensional gauge algebra $\mathcal{A}$ we define various (in)finite-dimensional truncations directly at the action level. We study a perturbative expansion of the action around the AdS_2 background.

The outline of the paper is as follows.

Section 2: the linearized AdS_2 higher spin gravity is formulated via the BF action functional. The action, the equations of motion, and the gauge symmetry transformations are given explicitly. The BF formulation under consideration is treated as a particular frame-like formulation which is known to be a generalization of the zweibein description of $2d$ gravitational systems. As a by-product, we propose a higher spin generalization of $2d$ Maxwell theory obtained as higher spin BF theory extended by a particular quadratic potential.

Section 3: BF systems are treated in the framework of the unfolded formulation that pursues the cohomological understanding of both lower spin and higher spin systems (see the review [2] for details). The section contains a detailed discussion of various mathematical structures underlying the cohomological interpretation of the dynamics. The main

[2] By global symmetry algebra in topological field theory we understand (generalized) Killing symmetries of a given vacuum solution to the theory. In a theory with local degrees of freedom this notion naturally extends to conventional global symmetry algebras acting on the space of one-particle states.

[3] The present construction of $\mathrm{hs}[\nu]$ uses six independent oscillators which is a minimal number of variables allowing for the Howe duality. Other approaches with less number of oscillators were known in the earlier literature [38, 39].

JHEP10(2014)122

objects here are the so-termed σ_+ and σ_- nilpotent operators acting on the field space of the model. Elements of the space are differential p-forms taking values in any rank $o(2,1)$ finite-dimensional irreps. Using the $\sigma_\pm$-cohomology we perform a cohomological reduction of the initial field space to a certain subspace: a transition from the frame-like formulation of the model to its metric-like form. We compute $\sigma_\pm$-cohomology groups that completely identify the local structure of the (linearized) metric-like theory: gauge symmetry, independent metric-like fields, equations of motion and their Bianchi identities.

Sections 4 and 5: Nilpotent operators σ_+ and σ_- correspond to two different cohomological reductions of the initial field space. So, in the one-form sector of the BF higher spin model we find that the system is equivalent either to massive scalar theory with a mass proportional to the cosmological constant and dependent on the spin, or to higher rank current conservation conditions. The scalar/current equations are invariant with respect to particular type of trivial on-shell symmetries/improvements that eliminate all local degrees of freedom. We suggest that these two forms of a single system are analogous to the well-known classical duality phenomenon occurring in the WZNW theory when second-order equations can be represented as the first-order conservation condition [40, 41]. The same analysis is done in the zero-form sector of the model.

Section 6: it summarizes the metric-like formulation developed in the previous sections. We list the metric-like equations of motions in the zero-form and one-form sectors of the BF higher spin gravity model in both cases of the $\sigma_\pm$ cohomological reductions. Finally, the model is interpreted as the higher spin gauge-dilaton theory extending the Jackiw-Teitelboim dilaton gravity. Also, we consider two metric-like action functionals which give rise to dual metric-like equations of motion. We find out that the BF action is a "parent" action for the two dual metric-like formulations.

Section 7: using manifestly covariant $o(2,1) - sp(2)$ vector notation we elaborate a realization of the one-parametric higher spin algebra hs$[\nu]$ introduced in refs. [36, 37]. Our realization is derived from the general d-dimensional oscillator description of the Eastwood-Vasiliev higher spin algebra for $d \geq 3$ [42, 43]. The approach is based on the Howe dual pair $o(2, d-1) - sp(2)$ realization in the bimodule of formal power series in auxiliary doublet variables [43, 44]. Specifying to $d = 2$ we find out that hs$[\nu]$ is to be identified as quotient algebra obtained by singling out a particular ideal. The Howe duality $o(2,1) - sp(2)$ used to describe quotient higher spin algebras may be useful in many respects, in particular, for considering general non-linear two-dimensional higher spin models not necessarily of BF type. Indeed, the Howe duality is known to be crucial to built a consistent interacting higher spin theory in $d \geq 4$ dimensions [43].

Section 8: it defines the full non-linear BF formulation of the AdS_2 higher spin gravity. Since the gauge algebras are realized as quotient algebras, the corresponding BF actions are formulated using particular projecting technique that allows to factor out elements of ideals directly inside the action. Quadratic higher spin actions studied in section 2 result from a linearization around the AdS_2 background solution.

Section 9: it summarizes our results and discusses future research directions. Details of the $\sigma_\pm$-cohomology computation are given in appendix A. Details of the projecting technique are given in appendix B.

JHEP10(2014)122

2 Quadratic higher spin BF action

Let $\mathcal{G}_s$ be a linear space of differential p-forms on a two-dimensional manifold taking values in finite-dimensional $o(2,1)$ totally symmetric and traceless representations of arbitrary rank[4]

$$F_{(p)}^{A_1\ldots A_{s-1}} = dx^{m_1}\wedge\cdots\wedge dx^{m_p}\, F_{m_1\ldots m_p}^{(A_1\ldots A_{s-1})}\,, \qquad \eta_{BC}F_{(p)}^{BCA_3\ldots A_{s-1}} = 0\,, \tag{2.1}$$

where $p=0,1,2$ is a rank of a differential form (at $p\geq 3$ differential forms are identically zero). Using $o(2,1)$ Levi-Civita tensor one shows that all non-symmetric finite-dimensional $o(2,1)$ irreducible representations either vanish identically, or are described by hook-type traceless tensors

$$F_{(p)}^{A_1\ldots A_m} \sim F_{(p)}^{A_1\ldots A_m,\,B_1}\,. \tag{2.2}$$

Two-dimensional higher spin fields are defined to be elements of $\mathcal{G}_s$. In two spacetime dimensions both massless and massive Wigner groups trivialize and whence it follows that only scalar and spinor modes may propagate. However, by a slight abuse of notation, we identify parameter s as a spin.

When considering gravitational systems parameterized by the negative cosmological constant Λ, it is convenient to represent gravitational fields as $o(2,1)$ connection 1-forms $W^A(x)\,T_A = dx^m\,W^A_m(x)\,T_A$, where T_A are $o(2,1)$ basis elements (see, e.g., [23]). Using antisymmetric basis one represents the connection as $W^{AB}_m = -W^{BA}_m$ which is dual to the original connection via $W_{mAB} = \epsilon_{ABC}\,W^C_m$. Flat connections satisfy the zero-curvature condition, which component form is given by

$$\mathcal{R}^A_{mn} \equiv \partial_m W^A_n - \partial_n W^A_m - \epsilon^{ABC}\,W_{m,\,B}\,W_{n,\,C} = 0\,. \tag{2.3}$$

The frame field and Lorentz spin connection are introduced in a standard fashion using the compensator V^A normalized such that $V^A V_A = -L^2$. In what follows, we use V^A in the form $V^A = (0,0,L)$. The $o(2,1)$ covariant decomposition of W^A_m is given by

$$W^A_m = E^A_m + V^A\omega_m\,, \tag{2.4}$$

where the transversality conditions $V_A E^A_m = 0$ and $\omega_m = \Lambda\,W^A_m V_A$ give rise to $E^A_m = (e^a_m, 0)$ and $W^A_m = \big(e^a_m, -1/\sqrt{-\Lambda}\,\omega_m\big)$.

It is well-known that AdS_2 spacetime solves constraint (2.3). The corresponding connection will be denoted $W_0 = \big(h^a_m, -1/\sqrt{-\Lambda}\,w_m\big)$. The zero-curvature constraint expresses Lorentz spin connection w_m via the frame h^a_m, while the latter defines AdS_2 spacetime metric g_{mn} through the standard identification $g_{mn} = \eta_{ab}h^a_m h^b_n$, where $\eta_{ab} = (+-)$ is the fiber Minkowski metric.

[4]A spacetime $\mathcal{M}^2$ is a general two-dimensional manifold with local coordinates x^m, Lorentz world indices run $m,n=0,1$, Lorentz fiber indices run $a,b=0,1$, $o(2,1)$ fiber indices run $A,B,C=0,1,2$, $o(2,1)$ invariant metric is $\eta^{AB}=(+--)$. The spacetime derivative is denoted as $\partial_m = \partial/\partial x^m$, the de Rham differential is $d = dx^m\partial_m$. The Levi-Civita tensor ϵ_{ABC} is normalized as $\epsilon_{012}=+1$. Two-dimensional anti-de Sitter spacetime AdS_2 has a radius L and a signature $(+-)$, so that the cosmological constant is $\Lambda = -1/L^2$. The Levi-Civita tensor ϵ_{mn} is normalized as $\epsilon_{01}=+1$. Symmetrization of indices has a unit weight and is labelled by parentheses.

Let us consider particular elements of the space $\mathcal{G}_s$ which are 0-form field $\Phi^{A_1\ldots A_{s-1}}$, 1-form field $\Omega^{A_1\ldots A_{s-1}}$ along with 2-form field strength

$$\Phi^{A_1\ldots A_{s-1}}\,, \qquad \Omega^{A_1\ldots\, A_{s-1}} = dx^m\,\Omega_m{}^{A_1\ldots\, A_{s-1}}\,, \qquad R_1^{A_1\ldots\, A_{s-1}} = D_0\Omega^{A_1\ldots\, A_{s-1}}\,, \tag{2.5}$$

where D_0 is $o(2,1)$ covariant background derivative,

$$D_0 F_{(p)}^{A_1\ldots A_k} = dT_{(p)}^{A_1\ldots A_k} + \epsilon^{BC(A_1}\, W_{0B} F_{(p)C}{}^{A_2\ldots A_k)} + \ldots + \epsilon^{BC(A_k} W_{0B} T_{(p)C}{}^{A_1\ldots A_{k-1})}\,. \tag{2.6}$$

From now on, we systematically omit the wedge product symbol $\wedge$. Representing the zero-curvature condition (2.3) evaluated on the background connection W_0 as $\mathcal{R}(W_0) \equiv D_0 D_0 = 0$ one observes that higher spin field strengths are invariant with respect to the following gauge transformations

$$\delta\Omega^{A_1\ldots\, A_{s-1}} = D_0\xi^{A_1\ldots\, A_{s-1}}\,, \tag{2.7}$$

where gauge parameters $\xi^{A_1\ldots\, A_{s-1}}$ are 0-forms taking values in the same finite-dimensional representations. Note that the Bianchi identities $D_0 R$ in two dimensions are trivial since any 3-form vanishes identically. The 0-form fields are assumed to be gauge invariant,

$$\delta\Phi^{A_1\ldots A_{s-1}} = 0\,. \tag{2.8}$$

Fields (2.5) are referred to as frame-like fields as these generalize the gravitational connection W_m^A to any number of fiber indices and any rank of differential form.

Let us consider now the BF action for a single rank-s system,

$$S_0[\Omega,\Phi] = \int_{\mathcal{M}^2} \Phi_{A_1\ldots A_{s-1}} R_1^{A_1\ldots A_s - 1}\,. \tag{2.9}$$

The equations of motion obtained by varying with respect to $\Phi_{A_1\ldots A_{s-1}}$ and $\Omega_{A_1\ldots A_{s-1}}$ are

$$R_1^{A_1\ldots A_{s-1}} = 0\,, \qquad D_0\Phi^{A_1\ldots A_{s-1}} = 0\,. \tag{2.10}$$

Both the action and the equations of motion are invariant with respect to gauge transformations (2.7) and (2.8). In section 8 the action (2.9) will be obtained from a full non-linear BF higher spin action by a linearization around AdS_2 background W_0.

The original BF theory (2.9) can be deformed in various ways. For instance, augmenting its action by a quadratic term

$$S_0[\Omega,\Phi] = \int_{\mathcal{M}^2} \Big(\Phi_{A_1\ldots A_{s-1}} R_1^{A_1\ldots A_s - 1} - \frac{1}{2}\,\Phi^{A_1\ldots A_{s-1}}\Phi_{A_1\ldots A_{s-1}}\mathcal{V}_2\Big)\,, \tag{2.11}$$

where $\mathcal{V}_2 = \epsilon_{ab}\, h^a \wedge h^b$ is the volume 2-form built of AdS_2 background frame fields, one obtains the following equations

$$R_1^{A_1\ldots A_{s-1}} = \mathcal{V}_2\Phi^{A_1\ldots A_{s-1}}\,, \qquad D_0\Phi^{A_1\ldots A_{s-1}} = 0\,. \tag{2.12}$$

Eliminating the auxiliary field $\Phi^{A_1\ldots A_{s-1}}$ by using its own equation of motion one arrives at the action of the form

$$S_0[\Omega] = \int_{\mathcal{M}^2} R^{\star}_1{}_{A_1\ldots A_{s-1}} R_1^{A_1\ldots A_s - 1}\,, \tag{2.13}$$

where $R_1^\star$ is the Hodge dual field strength. Note that now the action explicitly depends on the background AdS_2 metric. The rank-s equations of motion following from (2.13)

$$D_0^m R_{1\,mn}^{A_1\dots A_{s-1}} = 0\,, \tag{2.14}$$

generalize the Maxwell equations and describe no local degrees of freedom (see also our comments in the end of section 4.3). For the simplest case $s = 1$ the action (2.11) is the well-known action for the Maxwell field A_m on the background metric g_{mn} with the auxiliary scalar variable f: $S_0[A, f] = \int d^2x\sqrt{g}\,(f\epsilon_{mn}F^{mn} - \frac{1}{2}f^2)$, where $F_{mn} = \nabla_m A_n - \nabla_n A_m$. Representing the Maxwell action in this form is useful in the analysis of $2d$ Maxwell-dilaton theories of gravity, since the dynamical field enters the action linearly (see, e.g., [45]).

3 Cohomological view of BF equations

In order to analyze the dynamical content of the BF action (2.9) we employ homological tools developed within the unfolded formulation (see the review [2] for details). Indeed, one observes that the BF equations of motion are explicitly formulated as the zero-curvature and the covariant constancy conditions imposed on the frame-like fields which are differential forms taking values in certain $o(2,1)$ irreps, see (2.10). Fortunately, such a geometrical setting naturally fits the unfolded formulation.

Most importantly, using the unfolded machinery helps to obtain the metric-like formulation of the BF theory. For instance, in order to obtain the Jackiw-Teitelboim dilaton gravity theory from $o(2,1)$ BF theory one should carefully identify the metric and scalar fields along with auxiliary fields, use local Lorentz symmetry to set an antisymmetric part of the zweibein equal to zero, split all the equations into dynamical and constraint ones [23–25]. It is remarkable that all these operations can be done in a systematic manner using cohomology groups of certain nilpotent operators called $\sigma_\pm$ acting on the field space $\mathcal{G}_s$ (2.1). In other words, using the $\sigma_\pm$-cohomology provides precise guidelines how to pass from a frame-like (i.e., BF) formulation to a metric-like formulation where the higher spin fields are higher rank Lorentz tensor fields.

In order to make using the cohomological methods more manifest it is convenient to reformulate given BF equations as off-shell system. It means that the right-hand-sides of BF equations are not zero but some arbitrary sources. Sending the sources to zero implies going on-shell. Indeed, put equations (2.10) off-shell as follows

$$D_0\Phi^{A_1\dots A_{s-1}} = B_{(1)}^{A_1\dots A_{s-1}}\,, \tag{3.1}$$

$$D_0\Omega^{A_1\dots A_{s-1}} = C_{(2)}^{A_1\dots A_{s-1}}\,, \tag{3.2}$$

where the right-hand-sides of the equations are now arbitrary differential 1-form and 2-form, respectively, taking values in rank-$(s-1)$ irreducible $o(2,1)$ representation, (2.1). By definition, sources $B_{(1)}$ are $C_{(2)}$ are invariant with respect to gauge symmetry transformations (2.7) and (2.8), and therefore the off-shell system (3.1)–(3.2) retains the same gauge symmetry as the on-shell one (2.10).

3.1 $\sigma_\pm$ operators

Most conveniently, the cohomological analysis of off-shell $o(2,1)$ covariant equation system (3.1)–(3.2) is performed in terms of Lorentz $o(1,1) \subset o(2,1)$ algebra component fields. To this end, we rewrite elements of the field space $\mathcal{G}_s$ (2.1) in Lorentz basis,

$$T_{(p)}{}^{A_1 \ldots A_{s-1}} = \bigoplus_{k=0}^{s-1} T_{(p)}{}^{a_1 \ldots a_k} \,, \tag{3.3}$$

where Lorentz fields are totally symmetric and traceless,

$$T_{(p)}{}^{a_1 \ldots a_k} = T_{(p)}{}^{(a_1 \ldots a_k)} \,, \qquad \eta_{bc}\, T_{(p)}{}^{bca_3 \ldots a_k} = 0 \,. \tag{3.4}$$

Therefore, in Lorentz basis space $\mathcal{G}_s$ is given by a direct sum of subspaces spanned by differential p-forms $T^{a_1\ldots a_k}_{(p)}$ with fixed value of $k = 0, \ldots, s-1$. Such elements will be denoted as $T_{(p)}(k)$. It is worth recalling that a $o(1,1)$ totally symmetric and traceless tensor $T^{a_1\ldots a_k}$ has just two independent components. This is most obvious in the light-cone parametrization $T^{a_1\ldots a_k} \sim (T^{++\cdots+}, T^{--\cdots-})$, where a number of $\pm$ equals k. However, keeping $o(1,1)$ symmetry manifest is convenient when analyzing the dynamical content of the theory.

The space $\mathcal{G}_s$ incorporates all tensor fields of the theory, including zero-forms, one-forms and associated two-forms (2.5), along with their 0-form gauge symmetry parameters (2.7). For a given spin s there are two natural gradings in the space $\mathcal{G}_s$: by a rank of differential forms and by a number of Lorentz indices. On the other hand, there exist two nilpotent algebraic operators acting on $\mathcal{G}_s$ that shift the gradings by one.

Let us define operators $\sigma_\pm$ acting on $\mathcal{G}_s$ as follows $\sigma_\mp :\ T_{(p)}(k \pm 1) \to T_{(p+1)}(k)$. Their component action is given by[5]

$$\begin{aligned}
\sigma_- : \qquad & \alpha_{(k)}\, h_c\, T^{ca_1\ldots a_k}_{(p)} = T^{a_1\ldots a_k}_{(p+1)} \,, \\
\sigma_+ : \qquad & \beta_{(k)} \Big[h^{(a_1}\, T^{a_2\ldots a_k)}_{(p)} + \gamma_{(k)}\, \eta^{(a_1 a_2} h_c\, T^{ca_3\ldots a_k)}_{(p)} \Big] = T^{a_1\ldots a_k}_{(p+1)} \,,
\end{aligned} \tag{3.5}$$

where h^a_m is the AdS_2 background frame, while exact expressions for coefficients $\alpha_{(k)}, \beta_{(k)}$ and $\gamma_{(k)}$ are given below, see (3.14) and (3.12). The operators satisfy

$$\sigma_-^2 = 0 \,, \qquad \sigma_+^2 = 0 \,, \qquad \nabla^2 + \sigma_-\sigma_+ + \sigma_+\sigma_- = 0 \,, \tag{3.6}$$

where covariant derivative $\nabla_m = \partial_m + w_m$ is evaluated with respect to AdS_2 background Lorentz spin connection w_m. It is worth noting that conditions (3.6) can be understood as realization of the zero-curvature condition $D_0^2 = 0$ (2.3) in the Lorentz component basis [46],

$$D_0 = \nabla + \sigma_- + \sigma_+ \,. \tag{3.7}$$

[5] It stands to mention that conventional σ_- operator in $d \geq 4$ dimensions turns to σ_+ in $d = 2$ dimensions. This is because in the case $d \geq 4$ the field space $\mathcal{G}_s$ consists of two-row rectangle $o(2, d-1)$ traceless tensors that are replaced by one-row $o(2,1)$ traceless tensors in the case of $d = 2$. In the spin-2 case this is achieved by using the Levi-Civita tensor what changes the roles of σ_- and σ_+ operators in dualized pictures. Note, however, that this difference is purely notational.

It is convenient to define the Euler operator N counting a number of Lorentz indices, $NT_{(p)}(k) = kT_{(p)}(k)$. Then, $[N, \sigma_\pm] = \pm\sigma_\pm$ and $[N, \nabla] = 0$. Operator N provides the space $\mathcal{G}_s$ with a finite grading,

$$\mathcal{G}_s = \mathcal{G}_s^{(0)} \oplus \cdots \oplus \mathcal{G}_s^{(s-1)} \,, \tag{3.8}$$

where a subspace $\mathcal{G}_s^{(k)}$ is spanned by homogeneous elements of degree k. By definition, operator σ_- decreases a degree by one, operator σ_+ increases a degree by one.

The space $\mathcal{G}_s$ can be endowed with an inner product given by

$$\langle A|B\rangle = \delta_{k,\,l}\delta_{m+n,\,2}\int_{\mathcal{M}^2} A_{(m)}{}^{a_1\ldots a_k}\, B_{(n)a_1\ldots a_l}\,, \qquad A, B \in \mathcal{G}_s\,. \tag{3.9}$$

Modulo an overall coefficient, operators σ_- and σ_+ are mutually conjugated with respect to the above inner product. The following properties are elementary:

$$\langle A|B\rangle = 0\,, \qquad\qquad A \in \mathcal{G}_s^{(k)}\,, \quad B \in \mathcal{G}_s^{(l)}\,, \quad k \neq l\,, \tag{3.10}$$

$$\langle \sigma_\pm A|B\rangle = 0\,, \qquad\qquad \forall A \in \mathcal{G}_s\,, \quad \forall B \in Ker\,\sigma_\mp\,. \tag{3.11}$$

Exact expressions for the coefficients. Coefficients $\gamma_{(k)}$ in (3.5) are fixed by the algebraic symmetry conditions (3.4) as

$$\gamma_{(1)} = 0\,, \qquad \gamma_{(k)} = -\frac{1}{k-1}\,, \qquad k = 2, 3\ldots, s-1\,. \tag{3.12}$$

Coefficients $\alpha_{(k)}$ and $\beta_{(k)}$ are defined by conditions (3.6). Namely, one arrives at the equation system,

$$\rho_{(k)} \equiv \alpha_{(k)}\beta_{(k+1)} \;:\qquad \Lambda + \rho_{(k)}\big[\gamma_{(k+1)} - 1\big] + \rho_{(k-1)} = 0\,, \tag{3.13}$$

for $k = 1, \ldots, s-1$. The explicit solution reads

$$\rho_{(k)} = -\Lambda\frac{(s-k-1)(s+k)}{2(k+1)}\,. \tag{3.14}$$

Using proper field redefinitions one can set either $\beta_{(k)} = 1$ or $\alpha_{(k)} = 1$ for $k = 1, \ldots, s-1$ so that the solution is unique. Here, we choose the former case indicating that the dynamical systems under consideration are extended from Minkowski to AdS space.

3.2 Cohomological analysis

Below we shortly describe the general idea of the cohomological reduction of the off-shell BF system (3.1)–(3.2) using $\sigma_\pm$ nilpotent operators (see ref. [2] for more details).

Let us consider p-form gauge fields $\pi_{(p)}(k) \in \mathcal{G}_s$. Then, using the decomposition (3.7), the unfolded equations (3.1), (3.2) can be represented in the Lorentz component form as follows

$$\nabla\pi_{(p)}(k) + \sigma_-\pi_{(p)}(k+1) + \sigma_+\pi_{(p)}(k-1) = Z_{(p+1)}(k)\,, \tag{3.15}$$

where differential $(p+1)$-forms $Z_{(p+1)}(k)$ are the sources, while $k = 0, \ldots, s-1$, and a rank of differential forms runs $p = 0, 1$ since for $p = 2$ the above expression vanishes identically.

JHEP10(2014)122

The unfolded equations (3.15) are invariant with respect to gauge transformations given by

$$\delta\pi_{(p)}(k) = \nabla\varepsilon_{(p-1)}(k) + \sigma_-\varepsilon_{(p-1)}(k+1) + \sigma_+\varepsilon_{(p-1)}(k-1)\;, \tag{3.16}$$

where $(p-1)$-forms $\varepsilon_{(p-1)}(k)$ are gauge parameters. In fact, the gauge symmetry transformation appears at $p=1$ only. Indeed, in the case $p=0$ the gauge fields have no associated gauge parameters, while in the case $p=2$ the corresponding equations of motion vanish identically.

Quantities $Z_{(p+1)}(k)$ on the right-hand-side of (3.15) are not completely arbitrary and are restricted by the Bianchi identity

$$\nabla Z_{(p+1)}(k) + \sigma_- Z_{(p+1)}(k+1) + \sigma_+ Z_{(p+1)}(k-1) = 0\;, \tag{3.17}$$

which is a differential $(p+2)$-form. It is obtained by using conditions (3.6). For $p=1$ the Bianchi identity is a 3-form that vanishes identically.

Note that the unfolded equations, gauge transformations and identities are decomposed according to the grade degree (3.8). On the other hand, operators $\sigma_\pm$ enter all equations algebraically. It suggests that the gauge system (3.15)–(3.17) can be analyzed recurrently, starting either from the minimal grade degree $k=0$ equations or, from the maximal grade degree $k=s-1$ equations. In both cases, one arrives at the linear systems of the type

$$\sigma_\pm X = Y\;, \tag{3.18}$$

for some $X, Y \in \mathcal{G}_s$ built of the sources, fields, parameters, and their derivatives. It follows that one is inevitably led to compute $Im\,\sigma_\pm$ and $Ker\,\sigma_\pm$, and, moreover, the cohomology group $H(\sigma_\pm) = Ker\,\sigma_\pm / Im\,\sigma_\pm$ as the operators $\sigma_\pm$ are nilpotent.

By way of example, let us identify independent equations of motion contained in the gauge system (3.15)–(3.17). Consider the equations of motion (3.15) parameterized by the sources $Z_{(p+1)}(k)$.[6] Those $o(1,1)$ irreducible components of the sources $Z_{(p+1)}(k)$ that belong to $Im\,\sigma_\pm$ can be shifted to zero by appropriate shift redefinitions of fields in terms $\sigma_\pm\pi_{(p)}(k\mp1)$ in (3.15). Representing now the Bianchi identity as (3.18) one finds that non-vanishing irreducible components of $Z_{(p+1)}(k)$ not belonging to $Ker\,\sigma_\pm$ are auxiliary. That is to say these components are expressed through the derivatives of components belonging to the cohomology $H^{(p+1)}(\sigma_\pm) = Ker\,\sigma_\pm / Im\,\sigma_\pm\big|_{p+1}$, where the slash denotes restriction to $(p+1)$-forms. Note that cohomology elements of $H^{(p+1)}(\sigma_\pm)$ represent independent equations of motion and these nonetheless are not arbitrary being restricted by the residual Bianchi identity.

A number of independent identities between independent equations of motion is equal to a number of independent elements of the next cohomology group $H^{(p+2)}(\sigma_\pm)$. Note that for $p=1$ the Bianchi identities are identically vanishing 3-forms and therefore any 2-from always belongs to $Ker\,\sigma_\pm$. Consequently, there are no differential constraints in this case

[6]Note that a differential form $Z_{(p+1)}(k)$ is a tensor product of two groups of indices: $(p+1)$ antisymmetric world indices and k totally symmetric traceless fiber indices. For $p=1$ world indices form a singlet, and, therefore, the tensor product contains a single $o(1,1)$ irreducible component given by totally symmetric traceless tensor. For $p=0$ the tensor product contains two components given by formula (A.2).

and only field redefinitions associated with $Im\, \sigma_\pm$ are possible. These field redefinitions allow one to shift all non-zero tensors on the right-hand-side of the unfolded equations (3.15) to zero except for the cohomology elements.

Independent fields and gauge parameters can be considered similarly. So, the independent dynamical fields are particular $o(1,1)$ irreducible components of $\pi_{(p)}$ identified with elements of $H^{(p)}(\sigma_\pm)$, while other irreducible $o(1,1)$ components are either auxiliary fields expressed via dynamical ones, or Stueckelberg fields that can be shifted to zero by appropriate gauge transformation. Residual gauge parameters are given by $o(1,1)$ irreducible components identified with elements of $H^{(p-1)}(\sigma_\pm)$.

In this way, for a given $p = 0, 1, 2$ we come to the well-known dynamical interpretation of different cohomology groups [2, 47, 48] specified to two spacetime dimensions:

$$\begin{array}{cc} \text{parameters} \in H^{(p-1)}(\sigma_\pm) & \text{fields} \in H^{(p)}(\sigma_\pm) \\ \\ \text{equations} \in H^{(p+1)}(\sigma_\pm) & \text{identities} \in H^{(p+2)}(\sigma_\pm) \end{array} \tag{3.19}$$

All higher cohomology groups are empty, $H^{(p)}(\sigma_\pm) = \varnothing$ for $p \geq 3$, because in $d = 2$ dimensions differential p-forms with $p \geq 3$ vanish identically. As a corollary, there are no reducible gauge parameters and identities for identities.

It is important to note that the above interpretation of the cohomology elements (3.19) gives rise to different forms of one dynamical system reduced via either σ_+ or σ_- operators. Generally, this happens because the respective cohomology groups are non-isomorphic (see below).

Theorem. *The cohomology groups of operators $\sigma_\pm$ in $\mathcal{G}_s$ are given by*

$$H^{(p)}(\sigma_-) = \begin{cases} p = 0: \; T \\ \\ p = 1, s = 1: \; T^{a_1} \\ p = 1, s > 1: \; T\;, T^{a_1 \ldots a_s} \\ \\ p = 2: \; T^{a_1 \ldots a_{s-1}} \end{cases} \qquad H^{(p)}(\sigma_+) = \begin{cases} p = 0: \; T^{a_1 \ldots a_{s-1}} \\ \\ p = 1, s = 1: \; T^{a_1} \\ p = 1, s > 1: \; T\;, T^{a_1 \ldots a_s} \\ \\ p = 2: \; T \end{cases} \tag{3.20}$$

where $T^{a_1 \ldots a_m}$ are totally symmetric and traceless $o(1,1)$ tensors.

The proof is straightforward and relegated to appendix A.[7] A few comments are in order.

- The cohomology groups establish a cross-duality relation:

$$H^{(p+2)}(\sigma_\pm) \approx H^{(p)}(\sigma_\mp)\;, \qquad p = 0, 1, 2;\; (p+2) \bmod 2\;. \tag{3.21}$$

  It underlines dual interpretations of the BF higher spin theory that we develop in the following sections.

[7]Our results on $H(\sigma_+)$ cohomology (see a comment in footnote 5) can be obtained from d-dimensional consideration of [48] by taking $d = 2$. However, the case of $d = 2$ is strongly degenerate so that making a direct substitution of $d = 2$ should not be taken for granted. Also, $H(\sigma_-)$ has not been discussed before. In particular, an explicit computation of the cohomology has technical features specific to two dimensions that are crucial when analyzing the reduced unfolded equations.

JHEP10(2014)122

- Elements of group $H^{(1)}(\sigma_\pm)$ are not double traceless (Fronsdal) spin-s tenors for $s > 2$.

- Scalar elements of $H^{(1)}(\sigma_\pm)$ are two different scalar components of grade $k = 1$ element of $\mathcal{G}_s$, while tensor components are given by the same maximally symmetric traceless component of maximal grade $k = s - 1$ element of $\mathcal{G}_s$ (see appendix A for more details).

- Each of the second cohomology groups $H^{(2)}(\sigma_\pm)$ contains a single non-vanishing element. It is worth noting that in $d \geq 4$ dimensions $H^{(2)}(\sigma_-)$ contains two non-vanishing elements called the Einstein cohomology elements and the Weyl cohomology elements [46].[8] These cohomology elements are given by differential gauge-invariant combinations of d-dimensional Fronsdal fields and have an elegant interpretation. Indeed, in order to obtain Fronsdal equations of motion one equates the Einstein cohomology element to zero, while the Weyl cohomology element remains arbitrary modulo the Bianchi identities. It follows that the Weyl elements parameterize on-shell nontrivial gauge invariant combinations of dynamical fields, *i.e.*, the physical degrees of freedom. In the $d = 2$ case $H^{(2)}(\sigma_\pm)$ is spanned by a single element.[9] Equating this element to zero inevitably makes the theory topological. We refer elements of $H^{(2)}(\sigma_\pm)$ to as the Weyl tensors/scalars.

4 Off-shell unfolded equations for one-form fields

Component form of fields. Lorentz components of 0-form gauge parameters (2.7), 1-form gauge fields (2.5), and 2-form field strengths (2.5) will be denoted as

$$\xi^{a_1\ldots a_k}\,, \qquad \omega_m^{a_1\ldots a_k}\,, \qquad R_{mn}^{a_1\ldots a_k}\,, \qquad k = 0, \ldots, s-1\,; \tag{4.1}$$

all of them satisfy the irreducibility conditions (3.4).

Using general formulas (3.15), along with (3.5) and (3.12), (3.14), we find that the component form of the field strength is given by [10]

$$\begin{aligned} R_{mn}^{a_1\ldots a_k}(\omega) = \nabla_{[m}\,\omega_{n]}^{a_1\ldots a_k} - \Lambda\frac{(s-k-1)(s+k)}{2(k+1)}h_{[m,\,c}\,\omega_{n]}^{ca_1\ldots a_k} + \\ + \Big[h_{[m}^{(a_1}\;\omega_{n]}^{a_2\ldots a_k)} - \frac{1}{k-1}\,\eta^{(a_1a_2}h_{[m,\,c}\;\omega_{n]}^{ca_3\ldots a_k)}\Big]\,. \end{aligned} \tag{4.2}$$

[8] See footnote 5. In higher spacetime dimensions one considers the σ_- cohomology only because its elements are interpreted as fields, parameters, and equations of the Fronsdal theory of massless fields. A dynamical interpretation of the higher spacetime dimensional σ_+ cohomology has not been elaborated yet.

[9] Along with the second item above this may imply that Fronsdal action in two dimensions at $s > 1$ is a total derivative. *E.g.*, in the $s = 2$ case, the Einstein tensor does vanish identically. On the other hand, the $2d$ Maxwell action is not a total derivative: the respective variational equation is $\partial_m F = 0$, where F stands for dualized Maxwell tensor. Nonetheless, the theory is topological because the general solution reads $F = const$ allowing for linear potentials only.

Analogously, the component form of the gauge symmetry transformations (2.7) is given by

$$\delta\omega_m^{a_1\ldots a_k} = \nabla_m\xi^{a_1\ldots a_k} - \Lambda\frac{(s-k-1)(s+k)}{2(k+1)}h_{m,c}\,\xi^{ca_1\ldots a_k} + \\ + \Big[h_m^{(a_1}\,\xi^{a_2\ldots a_k)} - \frac{1}{k-1}\,\eta^{(a_1a_2}h_{m,c}\,\xi^{ca_3\ldots a_k)}\Big]\,. \tag{4.3}$$

Off-shell equations of motion. Consider now the unfolded equations in the one-form sector (3.2) written in the Lorentz component form as follows

$$R^{a_1\ldots a_k} = \mathcal{V}_2 C_{(s)}^{a_1\ldots a_k}\,, \qquad k=0,\ldots,s-1\,, \tag{4.4}$$

where $o(1,1)$ totally symmetric and traceless tensors $C_{(s)}^{a_1\ldots a_k}$ are the Lorentz components of the 2-form source $C_{(2)}^{A_1\ldots A_{s-1}}$ parameterizing the right-hand-side of (3.2). The expression $\mathcal{V}_2 = \epsilon^{cd}h_c\wedge h_d$ is the volume 2-form (dual to 0-form) built of AdS_2 background frame fields.

In the case $s=1$, the cohomology groups are isomorphic, $H^{(p)}(\sigma_+)\approx H^{(p)}(\sigma_-)$ for $\forall\, p$. Therefore, the only equation of motion in (4.4) says that the Maxwell tensor admits a dual representation, i.e., $R_{mn}\equiv F_{mn} = \epsilon_{mn}C_{(1)}$. Whence it follows that there are no restrictions imposed on F_{mn}, and the theory is off-shell. By some means, going on-shell constrains $C_{(1)}$. For instance, by taking $C_{(1)}=0$ one obtains the BF topological theory; other possible constraints are discussed in section 4.3. In what follows we always assume $s\geq 2$.

For $s\geq 2$ and $p\neq 1$ the cohomology groups $H^{(p)}(\sigma_\pm)$ are not isomorphic. This implies that the cohomological reduction of the equation system (4.4) could be done in two different ways giving rise to two different but dynamically equivalent theories.

Following the general discussion of section 3.2, the unfolded equations (4.4) can be represented in two forms depending on particular operator $\sigma_\pm$:

$$R^{a_1\ldots a_k} = \delta_{k,0}\left(C_{(s)} + \nabla_{b_1}C_{(s)}^{b_1} + \cdots + \nabla_{b_1}\cdots\nabla_{b_{s-1}}C_{(s)}^{b_1\ldots b_{s-1}}\right)\mathcal{V}_2\,, \tag{4.5}$$

within the σ_+ cohomological reduction, and,

$$R^{a_1\ldots a_k} = \delta_{k,s-1}\big(C_{(s)}^{a_1\ldots a_{s-1}} + \nabla^{(a_1}C_{(s)}^{a_2\ldots a_{s-1})} + \cdots + \nabla^{(a_1}\cdots\nabla^{a_{s-1})}C_{(s)} + \cdots\big)\mathcal{V}_2\,, \tag{4.6}$$

within the σ_- cohomological reduction. In (4.6) the ellipsis refers to proper symmetrizations of derivatives and trace terms. The proof is analogous to that of the theorem of section 3.2. The representations (4.5) and (4.6) are convenient in practice because all field redefinitions have been done that remove all right-hand-side tensors $\notin H^{(2)}(\sigma_\pm)$. In both cases, we see that field redefinitions produce derivative transformations setting all the source components to zero except for those corresponding to the cohomology elements.

The existence of two operators $\sigma_\pm$ used for the corresponding cohomological reductions implies two dual descriptions of the same system (4.4).[10] We show that the σ_+ cohomological reduction yields the massive scalar Klein-Gordon equation on the hyperboloid with non-vanishing right-hand-side given by scalar Weyl tensor. The σ_- cohomological reduction yields the current conservation condition with non-vanishing right-hand-side given by the higher rank Weyl tensor. In both cases, we impose partial gauge conditions setting a part of dynamical fields to zero.

Recall that the Bianchi identity (3.17) for the equation system (4.4) is trivial thereby implying that the cohomology elements are arbitrary. Imposing algebraic and/or differential constraint on Weyl scalars/tensors is discussed in section 4.3. For instance, equating all the cohomology elements to zero one obtains the BF higher spin theory with the action (2.9).

4.1 Explicit σ_+ - reduction: one-form sector

For convenience, we use the representation (4.5) with $C_{(s)}^{b_1\ldots b_k} = 0$, where $k = 1, 2, \ldots, s-1$. It follows that the unfolded equations take the form

$$R = \mathcal{V}_2 C^{(s)}\,, \quad R^{a_1\cdots a_k} = 0\,, \qquad k = 1, \ldots, s-1\,, \tag{4.7}$$

where Weyl scalar $C^{(s)} \in H^{(2)}(\sigma_+)$ is arbitrary function of spacetime variables, and the field strengths $R^{a_1\cdots a_k}(\omega)$ are given by (4.2).

The cohomological approach says that the field space $\mathcal{G}_s$ in the sector of 1-form fields $\omega_m^{a_1\cdots a_k}$ decomposes into Stueckelberg fields, auxiliary fields, and dynamical fields given by the cohomology $H^{(1)}(\sigma_+)$. The above three types of fields appear as particular irreducible Lorentz components of $\omega_m^{a_1\cdots a_k}$, cf. (A.2).

In the case $s > 1$, the vanishing higher rank field strengths at $k \neq 0$ are constraints allowing to express auxiliary fields via derivatives of independent dynamical fields given by a scalar and a rank-s traceless tensor $\varphi, \varphi^{a_1\ldots a_s} \in H^{(1)}(\sigma_+)$ (3.20). Other Lorentz components of $\omega_m^{a_1\cdots a_k}$ are Stueckelberg ones shifted to zero by algebraic parts of the gauge transformations (4.3).

The minimal grade degree equation $\epsilon^{mn}R_{mn} = C^{(s)}$ is the only off-shell equation of motion for dynamical fields. Gauge fixing all Stueckelberg fields to zero and expressing all auxiliary fields via the dynamical fields, one shows that the minimal grade equation is reduced to the following order-s differential equation

$$\kappa_s\left(\epsilon_{a_1 b}\nabla^b\nabla_{a_2}\ldots\nabla_{a_s}\varphi^{a_1 a_2\ldots a_s}\right) + \rho_s(\Box_{AdS_2} + m_s^2)\varphi = C^{(s)}\,, \tag{4.8}$$

with

$$m_s^2 = 2\rho_0 \equiv -s(s-1)\Lambda\,, \quad s \geq 2\,, \tag{4.9}$$

where $\Box_{AdS_2} = \nabla^a\nabla_a$ is the wave operator on the AdS_2 background, and coefficient ρ_0 is given by (3.14). Non-vanishing spin-dependent coefficients κ_s, ρ_s are fixed by gauge

[10] Recall that in the flat space limit $\Lambda = 0$ the operator σ_- disappears (see formula (4.2)) so that the duality phenomena described below are peculiar to $(A)dS_2$ space only. The cohomological analysis based on the remaining operator σ_+ remains valid in Minkowski space as well.

JHEP10(2014)122

symmetry transformations

$$\delta\varphi^{a_1\ldots a_s} = \nabla^{(a_1}\xi^{a_2\ldots a_s)} - \frac{1}{s-1}\,\eta^{(a_1a_2}\nabla_c\,\xi^{a_3\ldots a_{s-1})c}\,, \tag{4.10}$$

$$\delta\varphi = \epsilon_{ba_1}\nabla^b\nabla_{a_2}\cdots\nabla_{a_{s-1}}\xi^{a_1a_2\cdots a_{s-1}}\,, \tag{4.11}$$

with an independent gauge parameter $\xi^{a_1\ldots a_{s-1}} \in H^{(0)}(\sigma_+)$, see (3.20). Lower grade degree $k = 0, 1, \ldots, s-2$ gauge parameters $\xi^{a_1\ldots a_k}$ are Stueckelberg ones used to shift some Lorentz components in $\omega_m^{a_1\cdots a_k}$ to zero.

The dynamical equation (4.8) can be simplified. To this end, a field $\varphi^{a_1\ldots a_s}$ is completely gauged away by imposing the higher-spin gauge

$$\varphi^{a_1\ldots a_s} = 0\,. \tag{4.12}$$

Indeed, a traceless rank-k tensor in $d = 2$ dimensions has two independent components for any $k \geq 2$. It follows that a number of independent components of a rank-$(s-1)$ gauge parameter equals a number of equations in (4.12). The higher spin gauge can be viewed as an extension of the standard conformal gauge in $2d$ gravity which makes the metric proportional to Minkowski tensor. Then, the only dynamical field is given by a scalar component of the cohomology group, $\varphi \in H^{(1)}(\sigma_+)$.

Imposing the higher spin gauge (4.12) and solving the constraints in (4.7) one finds that the leftover equation reduces to the massive scalar equation with particular value of the mass-like term [10]

$$\Box_{AdS_2}\varphi - s(s-1)\Lambda\varphi = C^{(s)}\,, \tag{4.13}$$

where we redefined the right-hand-side as $\rho_s^{-1}C^{(s)} \to C^{(s)}$.

The massive scalar equation (4.13) is invariant with respect to residual gauge transformations (4.11) provided that the gauge parameter $\xi^{a_1\ldots a_{s-1}} \in H^{(0)}(\sigma_+)$, satisfies the generalized Killing equation on the hyperboloid,

$$\nabla^{(a_1}\xi^{a_2\ldots a_s)} - \frac{1}{s-1}\,\eta^{(a_1a_2}\nabla_c\,\xi^{a_3\ldots a_{s-1})c} = 0\,, \tag{4.14}$$

The above constraint is clearly explained as the stability transformation of the higher spin gauge condition (4.12) for transformations (4.10).

A few comments are in order.

- In the spin-2 case the above equation reproduces the gauge-fixed linearized equation of motion of the Jackiw-Teitelboim model in the one-form sector [10, 23–25, 49]. We see that the higher spin extension is described by the scalar field as well, but with a different spin-dependent mass term (4.9) and higher derivative leftover gauge symmetry (4.14).

- Mass m_s^2 (4.9) differs from the conformal value of mass $m^2_{\rm conf} = -\Lambda\, d(d-2)/4 = 0$ in $d = 2$ dimensions.

- Mass m_s^2 coincides with the value of the Casimir operator of $o(2,1)$ global symmetry algebra of AdS_2 space realized on tensor fields .

- Since the theory propagates no local degrees of freedom, the scalar field equation (4.13) at $C^{(s)} \neq 0$ becomes a constraint equation for auxiliary field φ that can be solved by defining the respective Green's function: $\varphi(x) = (\Box_{AdS_2} + m_s^2)^{-1} C^{(s)}(x)$.

4.2 Explicit σ_- - reduction: one-form sector

Using the representation (4.6) with $C_{(s)}^{b_1 \ldots b_k} = 0$, $k = 0, 1, \ldots, s-2$, one arrives at the following unfolded equations

$$R^{a_1 \ldots a_{s-1}} = \mathcal{V}_2 C^{a_1 \ldots a_{s-1}} \,, \quad R^{a_1 \ldots a_k} = 0 \,, \qquad k = 0, \ldots, s-2 \,, \tag{4.15}$$

where Weyl tensor $C_{(s)}^{a_1 \ldots a_{s-1}} \in H^{(2)}(\sigma_-)$ is arbitrary function of spacetime variables, and the field strengths $R^{a_1 \cdots a_k}(\omega)$ are given by (4.2).

In the case $s > 1$, the vanishing higher rank field strengths at $k = 0, \ldots, s-2$ are constraints allowing to express auxiliary fields via derivatives of independent dynamical fields given by a scalar and a rank-s traceless tensor $\phi, \phi^{a_1 \ldots a_s} \in H^{(1)}(\sigma_+)$ (3.20). Other Lorentz components of $\omega_m^{a_1 \cdots a_k}$ are Stueckelberg ones shifted to zero by algebraic parts of the gauge transformations (4.3).

Solving the constraints (4.15) yields the following expression

$$\omega_{m|a_1 \ldots a_{s-1}} = \phi_{m a_1 \ldots a_{s-1}} + \tau_s \big(\eta_{m a_1} \nabla_{a_2} \ldots \nabla_{a_{s-1}} \phi + \ldots \big) \,, \tag{4.16}$$

where τ_s is some non-vanishing spin-dependent coefficient, the parenthesis contain terms that depend on field ϕ only, while the ellipsis refers to appropriate symmetrizations of derivatives and trace terms. Independent gauge transformations are given by

$$\delta\phi = (\Box_{AdS_2} + m_s^2)\xi \,, \tag{4.17}$$

$$\delta\phi_{a_1 \ldots a_s} = \frac{1}{\Lambda} \nabla_{a_1} \cdots \nabla_{a_s} \xi + \ldots \,, \tag{4.18}$$

where the ellipsis refers to proper symmetrizations and trace terms, while a scalar gauge parameter $\xi \in H^{(0)}(\sigma_-)$ (3.20). The mass coefficient m_s^2 is given by (4.9).

The maximal grade degree equation $R^{a_1 \ldots a_{s-1}} = \mathcal{V}_2 C^{a_1 \ldots a_{s-1}}$ is the only off-shell equation of motion for dynamical fields. Gauge fixing all Stueckelberg fields to zero and expressing all auxiliary fields via the dynamical fields using (4.16), one shows that the maximal grade equation is reduced to the following order-$(s-1)$ differential equation

$$\nabla^m \phi_{m a_1 \ldots a_{s-1}} - \tau_s \nabla_{a_1} \ldots \nabla_{a_{s-1}} \phi + \ldots = C^{(s)}_{a_1 \ldots a_{s-1}} \,, \tag{4.19}$$

where the ellipsis refers to proper symmetrizations and trace terms.

Higher order equation (4.19) can be simplified by imposing a gauge condition. Indeed, using the scalar field transformations (4.17) one introduces the scalar gauge condition along with the residual gauge parameter equation

$$\phi = 0 \,, \qquad \Box_{AdS_2} \xi - m_s^2 \xi = 0 \,, \tag{4.20}$$

which are dual cousins of higher spin gauge condition (4.12) and generalized Killing equations (4.14). It follows that dynamical equation (4.19) takes the form

$$\nabla^n \phi_{n a_1 \ldots a_{s-1}} = C^{(s)}_{a_1 \ldots a_{s-1}} \,. \tag{4.21}$$

JHEP10(2014)122

For equation (4.21) with the vanishing right-hand-side $C^{(s)}_{a_1\dots a_{s-1}} = 0$ one identifies $\phi_{a_1\dots a_s}$ with spin-s conserved current on the hyperboloid.[11] Higher order derivative transformations (4.18) with the scalar gauge parameter satisfying (4.20) are treated now as "improvement" transformations for conserved currents. Indeed, "improvements" are higher order derivative transformations with an antisymmetric tensor parameter which in $d = 2$ dimensions is dualized to a scalar via the Levi-Civita tensor.

Our analysis of the σ_- cohomological reduction applied to the unfolded equations in the one-form sector yields the following interpretation of the cohomology groups $H^{(p)}(\sigma_-)$, which conforms the general scheme (3.19). Namely, elements $C^{(s)}_{a_1\dots a_{s-1}} \in H^{(2)}(\sigma_-)$ are conservation conditions. Element $\phi \in H^{(1)}(\sigma_-)$ can be chosen a pure gauge, so that another cohomology element $\phi_{a_1\dots a_s} \in H^{(1)}(\sigma_-)$ can be identified with a conserved current. Element $\xi \in H^{(0)}(\sigma_-)$ plays the role of an "improvement" transformation parameter.

4.3 Off-shell field spaces

In the framework of the unfolded formulation one may introduce the so-called Weyl module as a linear space which elements parameterize all possible gauge-invariant differential combinations of dynamical fields $\in H^{(1)}(\sigma_\pm)$ that remain arbitrary on-shell. In $d \geq 4$ dimensions, the Weyl module is derived by solving the Bianchi identities: one "unfolds" the original higher spin Weyl tensor, i.e. introduces new variables (infinite of them) that parameterize independent combinations of derivatives of the Weyl tensor [2].

In $d = 2$ dimensions the Bianchi identities in the one-form sector trivialize due to $H^{(3)}(\sigma_\pm) = \varnothing$, see (3.17) and (3.19). Whence, the Weyl tensor $\in H^{(2)}(\sigma_\pm)$ remains completely arbitrary function of spacetime variables. However, it does not yield local degrees of freedom in the theory. Indeed, recall that contrary to the higher-dimensional case, the cohomology $H^{(2)}(\sigma_\pm)$ contains the only element, cf. (3.20). In other words, the Einstein cohomology (higher spin equations of motion) and the Weyl cohomology (higher spin Weyl tensors) coincide in two dimensions. It follows that keeping the Weyl element arbitrary implies the theory is off-shell. On the other hand, choosing the Weyl element to be a particular function can be treated as "going on-shell". *E.g.*, setting all Weyl tensors to zero results in the zero-curvature equations of motion (2.10). There are various ways of how to put our topological system on-shell. We discuss some of them in section 4.3.2.

4.3.1 Unfolding Weyl tensors

Despite the lack of 2d Bianchi identities, one can still associate to Weyl tensors infinite sets of components which comprise their all possible derivative combinations. Namely, by off-shell field space for the Weyl scalar $C^{(s)} \in H^{(2)}(\sigma_+)$ we call the following set of components

$$\mathcal{W}_0 = \{W^{(s)}_{b_1\dots b_k}\,, \quad k = 0, 1, 2, \dots\}\,, \tag{4.22}$$

[11] For particular models, switching on non-vanishing tensors on the right-hand-side may be visualized as a sort of covariantization characteristic to non-Abelian interaction theories, which therefore is not conservation violation but rather a map $\nabla_m \to D_m$, where D_m is some new field-dependent covariant derivative.

where elements are totally symmetric and traceful, $\eta^{mn}\,W^{(s)}_{mn\,b_1\dots b_{k-2}} \neq 0$ for $k = 2,3,\dots$, so that one identifies an index-free component with the original Weyl scalar, $W^{(s)} \equiv C^{(s)}$. Elements of $\mathcal{W}_0$ are equated with all possible derivatives of original scalar $C^{(s)}$, i.e.,

$$W^{(s)}_{b_1\dots b_k} - \mathcal{P}_{b_1\dots b_k} C^{(s)} = 0\;, \qquad \mathcal{P}_{b_1\dots b_k} = \nabla_{b_1}\cdots\nabla_{b_k} + \cdots\;, \tag{4.23}$$

where the ellipses in (4.23) refers to proper symmetrizations and all possible trace terms. For a given k the projector $\mathcal{P}_{b_1\dots b_k}$ contains a finitely many arbitrary coefficients not fixed by the above definition of $\mathcal{W}_0$. Note that in $d = 2$ dimensions only symmetric combinations of covariant derivatives are possible because any non-symmetric $\nabla^{a_1}\dots\nabla^{a_k}C$ can be reduced to a collection of symmetrized combinations by using the Levi-Civita tensor and commutator $[\nabla,\nabla] \sim \Lambda$.

Quite analogously, by off-shell field space for the Weyl tensor $C^{(s)}_{a_1\dots a_{s-1}} \in H^{(2)}(\sigma_-)$ we call the following set of components

$$\mathcal{W}_{s-1} = \big\{W^{(s)}_{a_1\dots a_{s-1}|b_1\dots b_k}\;, \quad k = 0,1,2,\dots\big\}\;, \tag{4.24}$$

where elements are totally symmetric in each group of indices, and traceless with respect to the first group of indices, $\eta^{mn}\,W^{(s)}_{mna_1\dots a_{s-3}|b_1\dots b_k} = 0$, and traceful with respect to the second group of indices, $\eta^{mn}\,W^{(s)}_{a_1\dots a_{s-1}|b_1\dots b_{k-2}mn} \neq 0$. The $k = 0$ element is identified with the original Weyl tensor, $W^{(s)}_{a_1\dots a_{s-1}} \equiv C^{(s)}_{a_1\dots a_{s-1}}$. Elements of $\mathcal{W}_{s-1}$ are equated with all possible derivatives of original tensor $C^{(s)}_{a_1\dots a_{s-1}}$, i.e.,

$$W^{(s)}_{a_1\dots a_s|b_1\dots b_k} - \mathcal{P}_{b_1\dots b_k} C^{(s)}_{a_1\dots a_{s-1}} = 0\;, \qquad \mathcal{P}_{b_1\dots b_k} = \nabla_{b_1}\cdots\nabla_{b_k} + \cdots\;. \tag{4.25}$$

Generally, off-shell field space elements are not related to each other. A natural option suggested in [47] is to consider particular constraints for elements of the off-shell field space relating components with different values of k as

$$W^{(s)}_{b_1\dots b_{k+1}} \sim \nabla_{b_1} W^{(s)}_{b_2\dots b_{k+1}}\;, \tag{4.26}$$

while element $W^{(s)}$ remains arbitrary. It follows that the form of relations (4.23) is not changed, while arbitrary coefficients in projectors $\mathcal{P}_{b_1\dots b_k}$ are uniquely fixed modulo a single free coefficient to be identified with the mass parameter. We refer the off-shell field space $\mathcal{W}_0$ supplemented with constraints (4.26) to as the off-shell Weyl module $\widetilde{\mathcal{W}_0}$. The same consideration can be applied to off-shell module $\mathcal{W}_{s-1}$.

4.3.2 Going on-shell

Recall now that dynamical fields propagated by the unfolded equations (4.4) are considered as auxiliary, see our comments in the end of section 4.1. Indeed, these are completely expressed via the Weyl tensors which parameterize the right-hand-sides of the dynamical equations. Such a phenomenon is characteristic of topological field theories coupled to external dynamical systems with or without local degrees of freedom (see a recent discussion in [50]). In particular, this is the way one couples matter fields to $3d$ topological Chern-Simons theory. In this case, Chern-Simons strength tensor turns out to be proportional to a matter current so that respective gauge fields are auxiliary carrying no physical degrees of freedom. However, added topological modes may have a profound impact on dynamics of the matter system, giving rise, for instance, to anyonic statistics.

In our case, the problem of coupling a field theory with an (in)finite number of degrees of freedom to the topological unfolded theory given by equations (4.4) reduces to the equivalent problem of specifying Weyl tensors via imposing appropriate constraints on elements of the off-shell field spaces. Note that choosing particular Weyl tensors actually puts the topological system (4.4) on-shell. Other way round, going on-shell in the topological theory (4.4) is nicely interpreted as coupling to external field theory.

By way of example, specify the off-shell field space $\mathcal{W}_0$ to the off-shell Weyl module $\widetilde{\mathcal{W}}_0$ given by (4.26), and impose the tracelessness condition

$$\eta^{mn}\, W^{(s)}_{mn\, b_3\dots\, b_k} = 0\,. \tag{4.27}$$

The above constraint yields the massive Klein-Gordon equation of motion on AdS_2 space-time imposed on the Weyl scalar $C^{(s)}$ [8, 47]. It follows that an external field theory is identified here as the scalar field theory coupled to (linearized) topological spin-s BF theory. The dynamical field φ in equation (4.13) is auxiliary and expresses now in terms of the Klein-Gordon field $C^{(s)}$.

As another possible option let us mention a truncation of the off-shell Weyl $\widetilde{\mathcal{W}}_0$ by imposing the following constraint

$$W^{(s)}_{b_1\dots b_k} = 0 \quad \text{for} \quad k = m, m+1, \dots, \infty\,, \tag{4.28}$$

at some fixed m. The above truncation is most easily analyzed in the spin $s = 1$ case. Here, there are two standard choices of $m = 1$ and $m = 0$. Truncating $\mathcal{W}_0$ by imposing $W^{(1)}_b = 0$ is equivalent to constraint $\nabla_b F = 0$ which is the dualized Maxwell equation. Recall here that dualized Maxwell tensor $F_{mn} = \epsilon_{mn} F$ is identified with scalar $C^{(1)}$ and two off-shell field spaces considered above coincide, being actually a single space $\mathcal{W}_0$. Also, one may truncate all elements of $\mathcal{W}_0$ by imposing constraint $W^{(1)} \equiv F = 0$ that appears as the equation of motion in the Abelian BF theory.

Another example of a theory with no local degrees of freedom identified with an external field theory is given by equations (3.1)–(3.2) with the right-hand-sides given by (2.12). In this case, the right-hand-side of unfolded equation (4.4) is parameterized by 0-form field subjected to another unfolded equation which describes no local degrees of freedom as well (see the next section).

5 Off-shell unfolded equations for zero-form fields

Consider now the unfolded equations in the zero-form sector (3.15). By analogy with (3.3) $o(2,1)$ covariant 0-form fields can be decomposed into Lorentz algebra $o(1,1) \subset o(2,1)$ components as

$$\Phi^{A_1\dots\, A_{s-1}} = \bigoplus_{k=0}^{s-1} \phi^{a_1\dots\, a_k}\,, \tag{5.1}$$

where Lorentz components satisfy irreducibility conditions (3.4). Using general formulas (3.15), along with (3.5) and (3.12), (3.14), one finds that Lorentz component form of equations (3.15) reads as

$$D^{a_1\ldots a_k|m} = B^{a_1\ldots a_k|m}\;, \qquad k = 0,\ldots,s-1\;, \tag{5.2}$$

where

$$\begin{aligned} D^{a_1\ldots a_k}_m = \nabla_m\phi^{a_1\ldots a_k} - \Lambda\frac{(s-k-1)(s+k)}{2(k+1)}h_{m,c}\,\phi^{ca_1\ldots a_k} + \\ +\Big[h_m^{(a_1}\,\phi^{a_2\ldots a_k)} - \frac{1}{k-1}\,\eta^{(a_1a_2}h_{m,c}\,\phi^{ca_3\ldots a_k)}\Big]\;, \end{aligned} \tag{5.3}$$

where $D^{a_1\ldots a_k|m} = h^{m,n}D^{a_1\ldots a_k}_n$ and the slash says that two groups of fiber indices are not related by permutations, tensors $B^{a_1\ldots a_k|m}$ are $o(1,1)$ components of differential 1-form $B^{A_1\ldots A_{s-1}}_{(1)}$ (3.1).

The 0-form fields have no associated gauge symmetry (2.8). However, the equations of motion (5.2) satisfy the Bianchi identities taking the following component form, cf. (3.17),

$$\begin{aligned} \nabla_{[m}\,D^{a_1\ldots a_k}_{n]} - \Lambda\frac{(s-k-1)(s+k)}{2(k+1)}h_{[m,\,c}\,D^{ca_1\ldots a_k}_{n]} + \\ +\Big[h^{(a_1}_{[m}\;D^{a_2\ldots a_k)}_{n]} - \frac{1}{k-1}\,\eta^{(a_1a_2}h_{[m,\,c}\;D^{ca_3\ldots a_k)}_{n]}\Big] \equiv 0\;. \end{aligned} \tag{5.4}$$

According to the general consideration of section 3.2, the system (5.2) can be algebraically reduced using one or another type of nilpotent operators $\sigma_\pm$. In both cases, the cohomological theorem (3.20) guarantees that the true dynamical fields in the system are either $\phi \in H^{(0)}(\sigma_-)$, or $\phi^{a_1\ldots a_{s-1}} \in H^{(0)}(\sigma_+)$. Cohomology elements $B^{\pm(s)}$, $B^{\pm(s)}_{a_1\ldots a_s} \in H^{(1)}(\sigma_\pm)$ represent independent equations of motion. A number of independent identities between equations of motion corresponds to a number of independent elements of the second cohomology group, *i.e.*, $I^{(s)}_{a_1\ldots a_{s-1}} \in H^{(2)}(\sigma_-)$ and $I^{(s)} \in H^{(2)}(\sigma_+)$.

Note that the right-hand side of the equation system (5.2) cannot be set to $\delta_{k,1}\,(\epsilon_{ma_1}B^{+(s)} + \eta_{ma_1}B^{-(s)}) + \delta_{k,s-1}(B^{+(s)}_{a_1\ldots a_{s-1}m} + B^{-(s)}_{a_1\ldots a_{s-1}m})$ as in the case of the unfolded equations in the one-form sector (4.4). Not only the cohomology elements, but also other components $B^{a_1\ldots a_k|m}$ are generally non-vanishing. While the cohomology represents the independent equations of motion, the other components are auxiliary, i.e., are expressed through the independent ones by virtue of the Bianchi identities, see section 3.2.

It is worth noting that the right-hand-sides of the independent equations of motion obtained through the cohomological reduction are parameterized by two independent elements of $H^{(1)}(\sigma_\pm)$. In this respect, the situation is different from that in the gauge sector, where the reduced equations of motion are parameterized by a single Weyl scalar/tensor. It is similar to the higher dimensional picture, where the right-hand-sides of the equations also contain two independent cohomology elements, the Einstein part and the Weyl part, see the discussion in the end of section 3.2.[12]

[12]It would be instructive to explicitly build Weyl-like linear spaces that parameterize solutions to the Bianchi identities. See our discussion of the off-shell field spaces in the gauge sector in section 4.3.

5.1 Explicit σ_+ - reduction: zero-form sector

The σ_+ cohomological reduction of the unfolded equations (5.2) gives rise to the following independent equations of motion

$$
\begin{aligned}
&\epsilon_{mn_1}\nabla^m\nabla_{n_2}\cdots\nabla_{n_{s-1}}\varphi^{n_1\cdots n_{s-1}} = B^{+(s)}\;,\\
&\nabla_{(a_1}\varphi_{a_2\ldots a_s)} - \frac{1}{s-1}\,\eta_{(a_1a_2}\nabla^c\varphi_{a_3\ldots a_{s-1})c} = B^{+(s)}_{a_1\ldots a_s}\;,
\end{aligned}
\tag{5.5}
$$

where $\varphi_{a_1\cdots a_{s-1}}\in H^{(0)}(\sigma_+)$ and $B^{+(s)}, B^{+(s)}_{a_1\ldots a_s}\in H^{(1)}(\sigma_+)$, and indices are symmetrized with a unit weight. The tensors on the right-hand-sides of (5.5) are not arbitrary and are subjected to the Bianchi identities (5.4). Following (3.19) and (3.20), we find that there is a single identity between independent equations (5.5) corresponding to a scalar element $I^{(s)}\in H^{(2)}(\sigma_+)$,

$$
\kappa_s\left(\epsilon^{a_1b}\nabla_b\nabla^{a_2}\ldots\nabla^{a_s}B^{+(s)}_{a_1a_2\ldots a_s}\right) + \rho_s\big(\Box_{AdS_2}+m_s^2\big)B^{+(s)} = 0\,,
\tag{5.6}
$$

where κ_s,ρ_s are some non-vanishing spin-dependent coefficients, cf. (4.8), while mass parameter m_s^2 is given by (4.9).

By way of example, consider the spin-2 case. Here, the unfolded equations of motion (5.2) read

$$
\nabla_m\varphi - \Lambda h^c_m\varphi_c = B_m\;,\qquad \nabla_m\varphi^a + h^a_m\varphi = B^a_m\;,
\tag{5.7}
$$

where Λ is the cosmological constant, and B and B^a are subjected to the Bianchi identities (5.4)

$$
\nabla_m B - \Lambda h^c_m B_c = 0\;,\qquad \nabla_m B^a + h^a_m B = 0\;.
\tag{5.8}
$$

Using the σ_+ cohomological reduction and field redefinitions one finds from the second equation in (5.7) that $\varphi = -\frac{1}{2}\nabla^a\varphi_a$. Considering the Bianchi identities (5.8) one shows that the first equation in (5.7) is a differential consequence of the second equation. The resulting equations that follow from the second equation in (5.7) for the independent field $\varphi^a\in H^{(0)}(\sigma_+)$ read

$$
\nabla_a\varphi_b + \nabla_b\varphi_a - \eta_{ab}\nabla^c\varphi_c = B^+_{ab}\;,\qquad \epsilon^{ab}\nabla_a\varphi_b = B_+\;,
\tag{5.9}
$$

where $B^+, B^+_{(ab)}\in H^{(1)}(\sigma_+)$; cf. equations (5.5). Note that redefining fields by a dualization via ϵ^{ab}-tensor yields the following system $\nabla_a\varphi_b+\nabla_b\varphi_a = B_{ab}$, where $B_{ab} = B^+_{ab}+\epsilon_{ab}B^+$,[13]. This form is useful when analyzing Killing symmetries of the gauge dynamical field, see section 5.3. The Bianchi identities (5.6) take the form

$$
\epsilon^{ac}\nabla_c\nabla^b B^+_{ab} + \big(\Box_{AdS_2} - 2\Lambda\big)B^+ = 0\,,
\tag{5.10}
$$

or, equivalently, $\epsilon_{ab}\big(\nabla^a\nabla^c B^b{}_c + \Lambda B^{ab}\big) = 0$. We see that there is a single identity corresponding to a single element of the second cohomology $I\in H^{(2)}(\sigma_+)$.

[13] Here we used formula (A.3). The trace component is set to zero by a shift field redefinition because it belongs to $Im\,\sigma_+$.

5.2 Explicit σ_- - reduction: zero-form sector

The σ_- cohomological reduction of the unfolded equations (5.2) gives rise to the following independent equations of motion

$$\begin{aligned} (\Box_{AdS_2} + m_s^2)\phi = B^{-(s)} \;, \\ (\nabla_{a_1}\cdots\nabla_{a_s}\phi + \dots) = B^{-(s)}_{a_1\dots a_s} \;, \end{aligned} \tag{5.11}$$

where $\phi \in H^{(0)}(\sigma_-)$ and $B^{-(s)}, B^{-(s)}_{a_1\dots,a_s} \in H^{(1)}(\sigma_-)$, coefficient m_s^2 is given by (4.9); the ellipses refers to proper symmetrizations and trace terms. The right-hand-sides of equations (5.11) are not arbitrary and are subjected to the Bianchi identities (5.4). Following (3.19) and (3.20), we find that there is a tensor identity between independent equations (5.11) corresponding to a tensor element $I^{(s)}_{a_1\dots a_{s-1}} \in H^{(2)}(\sigma_-)$,

$$\nabla^n B^{-(s)}_{na_1\dots a_{s-1}} - \tau_s(\nabla_{a_1}\dots\nabla_{a_{s-1}}B^{-(s)} + \dots) = 0 \;, \tag{5.12}$$

where τ_s is some non-vanishing spin-dependent coefficients, cf. (4.8); the ellipses refers to proper symmetrizations and trace terms.

By way of example, consider the spin-2 case. Here, the equations of motion and the Bianchi identities are the same as in the previous section, see (5.7) and (5.8). The cohomological analysis goes along the same lines. So, using the σ_- cohomological reduction one finds from the first equation in (5.7) that $\phi^a = -\nabla^a\phi$. It follows that the resulting equation for the independent field $\phi \in H^{(0)}(\sigma_-)$ reads $\nabla_a\nabla_b\phi - \eta_{ab}\Lambda\phi = B_{ab}$, where tensor $B_{ab} = B^-_{ab} + \eta_{ab}B^-$, while $B^-, B^-_{ab} \in H^{(1)}(\sigma_-)$. The trace and traceless parts of the above equation are

$$\Box_{AdS_2}\phi - 2\Lambda\phi = B^- \;, \qquad \nabla_a\nabla_b\phi - \frac{1}{2}\eta_{ab}\Box_{AdS_2}\phi = B^-_{ab} \;, \tag{5.13}$$

cf. equations (5.11). Equations (5.13) reproduce the Jackiw-Teitelboim linearized equations in the zero-form sector [23–25]. The Bianchi identities (5.12) take the form

$$\nabla^b B^-_{ab} - \nabla_a B^- = 0 \;, \tag{5.14}$$

or, equivalently, $\epsilon^{bc}\nabla_b B_c{}^a = 0$. We see that there is an $o(1,1)$ vector identity corresponding to independent elements of the second cohomology $I^a \in H^{(2)}(\sigma_-)$.

5.3 Background symmetries

The unfolded equations in the zero-form sector (3.1) can be considered from a different perspective. Provided the right-hand-side is vanishing, the equations (3.1) are interpreted as stability transformations for a particular 1-form background gauge field Ω_0. From (2.7) it follows that the stability transformation equation reads

$$D_0\xi^{A_1\dots A_{s-1}} = 0 \;, \tag{5.15}$$

JHEP10(2014)122

while its $o(1,1)$ component form read off from (4.3) is given by

$$\nabla_m \xi^{a_1\ldots a_k} - \Lambda \frac{(s-k-1)(s+k)}{2(k+1)} h_{m,c}\, \xi^{ca_1\ldots a_k} + \\ + \Big[h_m^{(a_1}\, \xi^{a_2\ldots a_k)} - \frac{1}{k-1}\, \eta^{(a_1 a_2} h_{m,c}\, \xi^{ca_3\ldots a_k)} \Big] = 0\,. \tag{5.16}$$

Taking into account the analysis of the unfolded equations in the zero-form sector, the system (5.16) can be treated in two different ways, using either σ_- or σ_+ cohomological reduction. Whence it follows that there are two possible interpretations of the stability transformations.

Using the σ_+ cohomological reduction one finds out that (5.16) reduces to equations (4.10)–(4.11) or (5.5) on tensor parameters $\xi^{a_1\cdots a_{s-1}}$ at $s = 1, 2, \ldots, \infty$ subjected to the Bianchi identity (5.6). For a given s, the solution to the stability equations depends on a finitely many integration constants interpreted as constant $o(1,1)$ tensors parameterizing higher spin global symmetry transformations of the AdS_2 background spacetime.[14] For instance, in the spin-2 case stability transformation equations can be rewritten in the form $\nabla^a \xi^b + \nabla^b \xi^a = 0$ (see our comments below (5.9)) and their explicit solution reproduces the well-known $o(2,1)$ Killing vector parameterized by three integration constants representing three $o(2,1)$ generators.

On the other hand, using the σ_- cohomological reduction one finds out that (5.16) reduces to equations (4.17)–(4.18) or (5.11) on a scalar parameter $\xi^{(s)}$ at $s = 1, 2, \ldots, \infty$ subjected to the Bianchi identities (5.12). In this case the stability transformations describe trivial "improvement" transformations of the respective spin-s conserved currents. Contrary to the general "improvement" transformations that are invariance transformations of the conservation condition, the trivial "improvements" do not change the conserved current itself. It seems that there is no any "background conserved current" similar to the background spacetime, so that an interpretation of trivial "improvements" remains unclear.

6 Summary of the metric-like formulation

6.1 Metric-like equations of motion

Below we list the metric-like equations following from the $\sigma_\pm$ cohomological reductions of the original spin $s > 1$ unfolded equation system (3.1) and (3.2) analyzed in sections 4 and 5.

- **σ_+ - reduction**

$$\text{1-form sector:} \qquad \big(\Box_{AdS_2} - s(s-1)\Lambda\big)\varphi = C\,, \qquad \varphi^{a_1\ldots a_s} = 0 \tag{6.1}$$

[14] Detailed discussion of global higher spin symmetries in higher dimensions and their representations can be found, e.g., in [1, 39, 51–53].

$$\text{0-form sector:} \qquad \epsilon_{mn_1}\nabla^m\nabla_{n_2}\cdots\nabla_{n_{s-1}}\varphi^{n_1\cdots n_{s-1}} = B_+$$
$$\nabla_{(a_1}\varphi_{a_2\ldots a_s)} - \frac{1}{s-1}\,\eta_{(a_1a_2}\nabla^c\,\varphi_{a_3\ldots a_{s-1})c} = B^+_{a_1\ldots a_s} \tag{6.2}$$

- **σ_- - reduction**

$$\text{1-form sector:} \qquad \nabla^n\phi_{na_1\ldots a_{s-1}} = C_{a_1\ldots a_{s-1}}\;, \qquad \phi = 0 \tag{6.3}$$

$$\text{0-form sector:} \qquad (\Box_{AdS_2} - s(s-1)\Lambda)\phi = B^-$$
$$\big(\nabla_{a_1}\cdots\nabla_{a_s}\phi + \ldots\big) = B^-_{a_1\ldots a_s} \tag{6.4}$$

Recall that the metric-like equations in the one-form sector have been obtained using the higher spin gauge (4.12) in the σ_+ case, and the scalar gauge (4.20) in the σ_- case. In particular, the above equations are supplemented with the leftover gauge transformations and the Bianchi identities in the one-form and the zero-form sectors, respectively. Note also that the metric-like equations of motion are of order $1, 2, s-1, s$ in derivatives.

6.2 Dual metric-like higher spin actions

Let us consider linearized frame-like action (2.9) in the metric-like form. To this end, we represent the action in Lorentz basis

$$S_0[\phi,\omega] = \sum_{k=0}^{s-1}\int_{\mathcal{M}^2}\phi_{a_1\ldots a_k}R^{a_1\ldots a_k}(\omega)\;, \tag{6.5}$$

where 0-form fields $\phi_{a_1\ldots a_k}$ are given by (5.1) and 2-form field strength $R^{a_1\ldots a_k}(\omega)$ is expressed via 1-form gauge fields $\omega^{a_1\cdots a_k}$ (4.2). The corresponding equations of motion are given by (4.4) and (5.2) with vanishing right-hand-sides.

The idea is to fix Stueckelberg (shift) gauge symmetries and eliminate auxiliary fields using their own equations of motion substituting then the independent metric-like fields and the field strengths back to the frame-like action (6.5). In particular, this is the way one shows the equivalence of the frame-like $o(2,1)$ BF theory with the original metric-like Jackiw-Teitelboim model [23–25].

As we have already seen, a reduction to the independent dynamical sector can be done in two equivalent ways associated either to σ_+ or σ_- cohomology. Moreover, when considering both one-form and zero-form sectors simultaneously one has four equivalent reductions which we denote as $(\sigma_\pm,\sigma_\pm)$ reduction, where the first and second sigmas refer to corresponding reduction in the one-form and zero-form sector, respectively. However, at the action level one finds out that there are only two possible ways to perform a reduction to the metric-like form. Equations obtained via (σ_-,σ_-) or (σ_+,σ_+) reductions cannot be derived as variational equations since the number of the independent fields do not coincide with the number of the equations of motion.

JHEP10(2014)122

Equations obtained via the $(\sigma_+,\ \sigma_-)$ reduction can be derived as the Euler-Lagrange equations of motion following from the action

$$S_0^{+-}[\varphi, \varphi_{a_1\ldots a_s}|\phi] = \int_{\mathcal{M}^2} \phi\, R(\varphi, \varphi_{a_1\ldots a_s})\ , \tag{6.6}$$

where $R(\varphi, \varphi_{a_1\ldots a_s})$ is the 2-from field strength of grade degree $k = 0$ (4.7) expressed in terms of the dynamical fields. The equations of motion of the theory (6.6) take the form (6.1) and (6.4) (using the higher spin gauge). In particular, the linearized action and equations of the Jackiw-Teitelboim model follow from (6.6) at $s = 2$.

Analogously, equations obtained via the (σ_-, σ_+) reduction follow from the other action

$$S_0^{-+}[\phi, \phi_{a_1\ldots a_s}|\varphi_{a_1\ldots a_{s-1}}] = \int_{\mathcal{M}^2} \varphi_{a_1\ldots a_{s-1}}\, R^{a_1\ldots a_{s-1}}(\phi, \phi_{a_1\ldots a_s})\ , \tag{6.7}$$

where $R^{a_1\cdots a_{s-1}}(\phi, \phi_{a_1\ldots a_s})$ is the 2-form field strength of grade degree $k = s-1$ (4.15) expressed in terms of the dynamical fields. The equations of motion of the theory (6.7) take the form (6.2) and (6.3) (using the scalar gauge).

The form of actions (6.6) and (6.7) can be explained by resorting to the cross-duality (3.21) exhibited by the cohomology groups $H^{(m)}(\sigma_+)$ and $H^{(n)}(\sigma_-)$ that gives, in particular, $H^{(2)}(\sigma_\pm) \approx H^{(0)}(\sigma_\mp)$. To this end, one employs inner product (3.9) on the space $\mathcal{G}_s$ and reformulates action (6.5) as $S_0[\phi, \omega] = \int_{\mathcal{M}^2} \langle\phi|R\rangle$, where $\phi, \omega, R \in \mathcal{G}_s$. Then, eliminating the auxiliary fields via their own equations of motion one finds that fields of the metric-like formulation are elements of the cohomology, 0-forms $\langle\phi| \in H^{(0)}(\sigma_\pm)$ and reduced 2-forms $|R\rangle \in H^{(2)}(\sigma_\mp)$. After that, using the properties (3.10), (3.11) along with the above cross-duality relation one arrives at the two actions considered above.

On the other hand, both types of the cohomological reductions describe the same dynamical system. It suggests there exists a duality mapping between two linear theories given by (6.6) and (6.7). It would be interesting to provide an exact definition of such a mapping originated from the cohomology cross-duality and to study its properties and implications beyond the linear approximation.

6.3 The model interpretation

The equations of motion in the one-form sector have been previously interpreted as describing topological maximal depth partially-massless higher spin fields on the AdS_2 background [10]. It should be noted that such an interpretation follows from (σ_+, σ_-) - reduction described by action (6.6).

In this case, the equations of motion in both zero-form and one-form sectors (in the gauge fixed form) are given by the same Klein-Gordon equation $\big(\Box_{AdS_2} - s(s-1)\Lambda\big)\varphi = 0$ and $\big(\Box_{AdS_2} - s(s-1)\Lambda\big)\phi = 0$ for two scalars φ and ϕ. The general solution depends on two arbitrary functions of spacetime coordinates so that it can be interpreted as left and right waves. However, there are gauge symmetry in the one-form sector and additional tensor constraint along with the Bianchi identities in the zero-form sector that eventually eliminate the functional freedom leaving no local modes (only a finitely many integration constants). The absence of propagating degrees of freedom leaves enough room for interpretation of the equations of motion under consideration. We set that fields in the one-form sector are gauge fields, while those in the zero-form sector are dilaton fields, both topological.

The spectrum of the model can be interpreted as follows. The BF higher spin theory given by action (6.6) describes: (one-form sector) topological $s = 1$ massless Maxwell field and $s = 2$ graviton field along with increasing spin $s = 3, 4, ...$ partially-massless gauge fields of the maximal depth; (zero-form sector) topological dilaton fields with increasing masses $m_s^2 = -s(s-1)\Lambda$. In this form action (6.6) can be treated as a higher spin gauge-dilaton extension of the original (linearized) Jackiw-Teitelboim dilaton gravity model.

7 The higher spin algebras in two dimensions

To formulate a non-linear BF higher spin theory the fields should be represented as connections of some (in)finite Lie algebra. In the case of finitely many fields a higher spin algebra can be identified with $sl(N, \mathbb{R})$ algebra provided that its basis elements are represented as

$$T_{A_1} \oplus T_{A_1 A_2} \oplus \cdots \oplus T_{A_1 \ldots A_{N-1}} , \tag{7.1}$$

where generators $T_{A_1 \ldots A_k}$ are rank-k totally symmetric and traceless $sl(2, \mathbb{R})$ algebra tensors [14, 15, 54]. Gauging algebra (7.1) yields a finite collection of 0-form and 1-form fields of the type (2.5). A natural infinite-dimensional generalization of (7.1) should have the following structure

$$\bigoplus_{s=1}^{\infty} \bigoplus_{l_s} T^{(l_s)}_{A_1 \ldots A_{s-1}} , \tag{7.2}$$

where the numbers l_s are multiplicities of spin-s basis elements. Note that (7.2) contains also infinitely many copies of $gl(1, \mathbb{R})$ generator T corresponding to the spin-1 Abelian connection.

A convenient way to realize higher spin algebras with generators $T_{A_1 \ldots A_{s-1}}$ (7.2) is to represent them as homogeneous polynomials of degree-$(s-1)$ in auxiliary vector variables. It is remarkable that such a vector realization can be obtained using d-dimensional oscillator approach based on the $o(2, d-1) - sp(2)$ Howe duality proposed by Vasiliev [43, 44]. In what follows, we use the $o(2, 1) - sp(2)$ Howe duality to describe the one-parametric family of $2d$ higher spin algebras hs[ν] originally introduced by Feigin as quotients of the universal enveloping algebra $\mathcal{U}(sl(2))$ [36], and by Vasiliev as the enveloping algebra of the Wigner deformed oscillator algebra [37].

7.1 Oscillator approach

Following the original papers [43, 44], we consider auxiliary doublet variables Y^A_α, with $sp(2)$ vector index α and $o(2, M)$ vector index A,[15] and consider polynomials expanded in the auxiliary variables as follows

$$F(Y) = \sum_{k=0}^{\infty} F^{\alpha_1 \ldots \alpha_k}_{A_1 \ldots A_k} Y^{A_1}_{\alpha_1} \ldots Y^{A_k}_{\alpha_k} = \sum_{m,n=0}^{\infty} F_{A_1 \ldots A_m | B_1 \ldots B_n} Y^{A_1}_1 \cdots Y^{A_m}_1 Y^{B_1}_2 \cdots Y^{B_n}_2 , \tag{7.3}$$

where expansion coefficients are totally symmetric in both groups of indices.

[15] In this section symplectic indices $\alpha, \beta, \gamma, ... = 1, 2$, vector indices $A, B, C ... = 0, ..., M + 1$, the $o(2, M)$ invariant metric is $\eta_{AB} = (+ - .... - +)$, symplectic indices are raised and lowered with the $sp(2)$ invariant metric $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$.

Define now the Weyl star-product

$$(F*G)(Y)=\frac{1}{\pi^{2M}}\int dSdT\,F(Y+S)\,G(Y+T)\,\exp(-2S^A_\alpha T^\alpha_A)\,. \tag{7.4}$$

It follows that the auxiliary variables satisfy the following commutation relations $\big[Y^A_\alpha,Y^B_\beta\big]_*=\epsilon_{\alpha\beta}\eta^{AB}$. A space of polynomials (7.3) endowed with the star-product (7.4) is the Weyl algebra $\mathcal{A}_{M+2}$.

The algebra $\mathcal{A}_{M+2}$ is a bi-module over $o(2,M)$ and $sp(2)$ algebras. Their basis elements are realized as bilinear combinations of the auxiliary variables

$$T^{AB}=\frac{1}{2}\epsilon^{\alpha\beta}\big\{Y^A_\alpha,Y^B_\beta\big\}_*\,,\qquad t_{\alpha\beta}=\frac{1}{2}\eta_{AB}\big\{Y^A_\alpha,Y^B_\beta\big\}_*\,. \tag{7.5}$$

Bilinears T^{AB} and $t_{\alpha\beta}$ commute, $[T^{AB},t_{\alpha\beta}]_*=0$. Moreover, the two algebras form a Howe dual pair $o(2,M)-sp(2)$ [55]. It follows that $sp(2)$ highest weight conditions imposed on elements of $\mathcal{A}_{M+2}$ single out particular finite-dimensional $o(2,M)$ irreducible representations (see section 7.2 for more details).

Using (7.5) one finds that quadratic Casimir operators $C_2=\frac{1}{2}T_{AB}*T^{AB}$ of $o(2,M)$ algebra and $c_2=\frac{1}{2}t_{\alpha\beta}*t^{\alpha\beta}$ of $sp(2)$ algebra are related as

$$C_2=\frac{1}{4}(M^2-4)+c_2\,. \tag{7.6}$$

Higher spin algebras considered below are various quotients of the $*$-product algebra $\mathcal{S}_{M+2}\subset\mathcal{A}_{M+2}$ of all polynomials spanned by $sp(2)$ invariant elements

$$\big[t_{\alpha\beta},F(Y)\big]_*=0\,. \tag{7.7}$$

Endowing the associative algebra $\mathcal{S}_{M+2}$ with the commutator $[F,G]_*$, where $F,G\in\mathcal{S}_{M+2}$ one obtains the Lie algebra denoted as $hc(1|2{:}[M,2])$ [44].[16]

In general, associative algebra $\mathcal{S}_{M+2}$ (as well as Lie algebra $hc(1|2{:}[M,2])$) contains various two-sided ideals $\mathcal{I}$. For instance, there exists the maximal ideal spanned by elements

$$\mathcal{I}_1=\big\{\,g(Y)=t_{\alpha\beta}*g^{\alpha\beta}(Y)\,\big\}\,,\qquad \big[t_{\alpha\beta},g^{\gamma\rho}\big]_*=\delta^\gamma_\beta\,g_\alpha{}^\rho+3\text{ terms}\,, \tag{7.8}$$

where $g^{\alpha\beta}(Y)$ is an arbitrary polynomial transforming as an $sp(2)$ symmetric tensor. Using ideals $\mathcal{I}$ one defines quotient algebras $\mathcal{H}=\mathcal{S}_{M+2}/\mathcal{I}$. So, factoring out the maximal ideal (7.8) gives rise to associative algebra $\mathcal{S}_{M+2}/\mathcal{I}_1$. A particular real form of the respective Lie algebra $hc(1|2{:}[M,2])/\mathcal{I}_1$ is denoted as $hu(1|2{:}[M,2])$ [44]. It is singled out by reality conditions

$$\big(F(Y)\big)^\dagger=-F(Y)\,, \tag{7.9}$$

where the involution $\dagger$ of the complex algebra $\mathcal{S}_{M+2}$ is defined as $(Y^A_\alpha)^\dagger=Y^A_\alpha$ and $\big(aF(Y)\big)^\dagger=\bar a\big(F(Y)\big)^\dagger$, where $a\in\mathbb{C}$, and the bar stands for the complex conjugation. Gauging $hu(1|2{:}[M,2])$ yields totally symmetric massless (Fronsdal) fields of increasing spins $s=1,2,...,\infty$.

[16] In what follows, by a slight abuse of notation, we denote associative algebras and Lie algebras obtained by taking the commutators with respect to the associative product by the same symbols.

In what follows, we explicitly consider the case of $M = 1$ and study quotient higher spin algebras corresponding to different ideals, including the maximal one. We show that $hc(1|2{:}[1,2])/\mathcal{I}_1$ is a finite-dimensional algebra. Therefore, in order to produce an infinite-dimensional higher spin algebra one should use non-maximal ideals. We identify two infinite families of ideals that yield both finite- and infinite-dimensional quotient higher spin algebras. Our analysis also applies to the case of $M = 2$, where the AdS_3 global symmetry algebra $o(2,2) \approx o(2,1) \oplus o(2,1)$, and each factor can be considered by analogy with the case of $M = 1$.

7.2 Howe dual realization of $\mathcal{U}(o(2,1))$

Howe dual algebras $sp(2)$ and $o(M,2)$ act on $\mathcal{A}_{M+2}$ so that expansion coefficients of $F(Y)$ in the auxiliary variables (7.3) are both $sp(2)$ and $o(2,M)$ tensors. On the other hand, the $sp(2)$ invariance condition (7.7) says that these tensors are of particular index symmetry type. It follows that the resulting expansion coefficients of (7.3) are $o(2,M)$ traceful tensors with index symmetry described by rectangular two-row Young diagrams

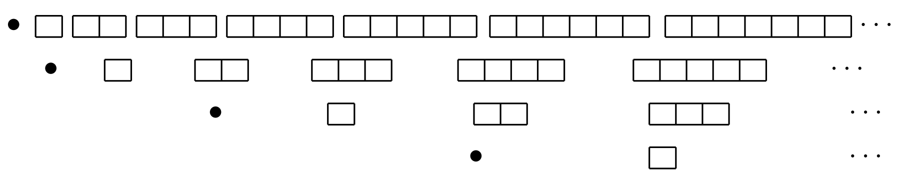
$$F_{A_1\dots A_m,\, B_1\dots B_m} \quad : \qquad F_{(A_1\dots A_m,\, B_1)B_2\dots B_m} \equiv 0\;. \tag{7.10}$$

In the $M = 1$ case, any $o(2,1)$ traceful two-row rectangular tensor (7.10) can be decomposed into one-row tensors because any traceless $o(2,1)$ tensor with indices described by two-row Young diagram with more than one cell in the second row vanishes identically, while those with a single cell in the second row are dualized using the Levi-Civita tensor, see (2.2).

It follows that a linear space of the algebra $\mathcal{S}_3$ spanned by $sp(2)$ singlets (7.7) can be represented as an infinite collection of one-row traceless Young diagrams. Indeed, let T_m denote a spin-m $o(2,1)$ irrep given by a totally symmetric traceless $o(2,1)$ tensor. Then, one can show that a linear space of $\mathcal{S}_3$ as o(2,1) module is decomposed in a direct sum

$$\mathcal{S}_3 = \bigoplus_{m=0}^{\infty} \bigoplus_{l=1}^{\infty} T_m^{(l)}\;, \tag{7.11}$$

where a superscript l stands for multiplicity, cf. (7.2). Elements of linear space (7.11) can be depicted on the following plot:



(7.12)

Here, irreps T_k are depicted as length-k Young diagrams, dots $\bullet$ correspond to scalar components T_0. Irreps T_k resulted from decomposing a traceful two-row rectangle of length $m-1$ are disposed vertically, $k = 0, \dots, m$. Note that an each line on the plot successively depicts all basis elements of $gl(N)$ algebra, where $N = 1, 2, 3, \dots$.

The other way around, traceless symmetric tensors can be rearranged as traces of a given totally symmetric traceful tensor. It suggests that the linear space can be described

JHEP10(2014)122

by traceful symmetric tensors of all ranks from zero to infinity, each in a single copy. It can be equivalently seen by dualizing traceful rectangular $o(2,1)$ diagrams (7.10). It follows that the linear space of $\mathcal{S}_3$ can be represented as

$$\mathcal{S}_3 = \bigoplus_{k=0}^{\infty} G_k \ , \tag{7.13}$$

where G_k denotes a rank-k symmetric traceful $o(2,1)$ tensor; it follows that $G_k = T_k \oplus T_{k-2} \oplus \cdots$. On the plot (7.12) a tensor G_k corresponds to the k-th vertical column.

Let us now notice that when indices $A, B, \ldots$ run just three values it is possible to introduce new variables

$$T_A = \epsilon_{ABC}\epsilon^{\alpha\beta}Y^B_\alpha Y^C_\beta \ , \tag{7.14}$$

which are in fact Hodge dualized $o(2,1)$ basis elements (7.5), and hence satisfy the commutation relations $\big[T_A, T_B\big]_* = \epsilon_{ABC}T^C$. One can show that any $sp(2)$ singlet $F(Y)$ can be equivalently rewritten as an arbitrary polynomial $F(T)$. Indeed, the $sp(2)$ invariance condition (7.19) says that expansion coefficients of any $F \in \mathcal{S}_3$ (7.3) have even numbers of $sp(2)$ and $o(2,1)$ vector indices, and can be represented as

$$F^{\alpha_1\ldots\alpha_{2m}}_{A_1\ldots A_{2m}} = \epsilon^{\alpha_1\alpha_2}\ldots\epsilon^{\alpha_{2m-1}\alpha_{2m}}F_{A_1A_2|\cdots|A_{2m-1}A_{2m}} \tag{7.15}$$

where each group of two vector indices $|A_iA_{i+1}|$ is antisymmetric (see [56] for more details). Using the definition (7.14) along with (7.15) one finds that (7.3) can be completely rewritten as polynomials of $o(2,1)$ bilinears T^A with totally symmetric expansion coefficients. Note that T^A are $sp(2)$ singlets. It follows that the space $\mathcal{S}_3$ of $sp(2)$ singlets is now naturally realized as functions of $sp(2)$ invariant variables. The action of Howe dual algebra $sp(2)$ becomes implicit.

In this way, we establish that the associative algebra $\mathcal{S}_3$ of $sp(2)$ singlets and the universal enveloping algebra $\mathcal{U}(o(2,1))$ are isomorphic,

$$\mathcal{S}_3 \approx \mathcal{U}(o(2,1)) \ . \tag{7.16}$$

Note that the above consideration applies to $\mathcal{S}_{M+2}$ for any M. However, its basis elements are parameterized by $o(2,M)$ two-row rectangle $o(2,M)$ diagrams (7.10) so that $\mathcal{S}_{M+2}$ cannot be interpreted as the universal enveloping algebra $\mathcal{U}(o(2,M))$. In the case of $M=1$ two-row rectangle diagrams become arbitrary one-row diagrams making isomorphism (7.16) possible.

Trace decomposition. Subtracting $o(2,1)$ traces can be done systematically if one employs $sp(2)$ Howe dual algebra. To this end, consider first $o(2,M)$ trace decompositions. From the definition of $sp(2)$ basis elements $t_{\alpha\beta}$ (7.5) it follows that all three possible traces of a tensor with indices described by $o(2,M)$ two-row Young diagram can be collectively represented as three independent $sp(2)$ generators. In particular, any multiple trace of $F \in \mathcal{S}_3$ is to be proportional to the following combination [56]

$$t_{\alpha\beta}\cdots t_{\gamma\rho}\, c_2 \cdots c_2 \ . \tag{7.17}$$

Here, $sp(2)$ indices are assumed to be symmetrized. Totally antisymmetric combinations of $t_{\alpha\beta}$ produces powers of the $sp(2)$ Casimir element c_2.

By way of example consider particular polynomial $F(Y) = F_{AB|CD}Y_1^A Y_1^B Y_2^C Y_2^D$ subjected to the $sp(2)$ invariance condition (7.7). It follows that an expansion coefficient $F_{AB,CD}$ is described by a "window" Young diagram $\boxplus$. On the other hand, the expansion coefficient is traceful so that a decomposition into traceless parts yields a linear combination

$$F_{AB,CD} = F^0_{AB,CD} + \eta_{AB}F^1_{CD} + \eta_{AB}\eta_{CD}F^2 + \dots \,, \tag{7.18}$$

where the ellipsis denote proper symmetrization of indices, while $F^0_{AB,CD}$, F^1_{AB}, and F^2 are traceless components. Substituting the above decomposition into $F(Y)$ one finds that the second term is proportional to $t_{\alpha\beta}$, while the third term is proportional to c_2, i.e., $F(Y) = F_0(Y) + t_{\alpha\beta}F_1^{\alpha\beta}(Y) + c_2 F_2$. For the case of $M = 1$ the first term in decomposition (7.18) identically vanishes, $F^0_{AB,CD} = 0$. The second and the third terms correspond to T_2 and T_0 elements depicted in the third vertical column on the plot (7.12).

It follows that a trace decomposition of any $F(Y) \in \mathcal{S}_3$ reads [56]

$$F(Y) = F_0 + F_1(Y) + \sum_{k,m=0}^{\infty} F_{(m)}^{\alpha_1\dots\alpha_{2k}}(Y)\, t_{\alpha_1\alpha_2}\cdots t_{\alpha_{2k-1}\alpha_{2k}}\,(c_2)^m \,, \tag{7.19}$$

where F_0 and $F_1(Y)$ denote the scalar and the vector components, while $F_{(m)}^{\alpha_1\dots\alpha_{2k}}(Y)$ are totally symmetric $sp(2)$ rank-$2k$ tensors, a subscript m stands for a multiplicity. Using the symmetry property $F^{\dots\alpha\beta\dots}_{\dots AB\dots} = F^{\dots\beta\alpha\dots}_{\dots BA\dots}$ one concludes that expansion coefficients in (7.19) are given by totally symmetric $o(2,1)$ traceless tensors. It is worth noting that analogous decomposition for elements of $\mathcal{S}_{M+2}$ algebra is 3-parametric, while taking $M = 1$ leaves only 2 parameters. The absent branch corresponds to traceless two-row rectangular $o(2,M)$ Young diagrams. In the case $M = 1$ this branch reduces to the two first terms.

One concludes that the first line in (7.12) contains T_k for $k \geq 2$ that appear as coefficients in front of symmetrized combinations $t_{(\alpha_1\alpha_2} * \dots * t_{\alpha_{2k-1}\alpha_{2k})}$, while subsequent lines necessarily contain powers of c_2. Any tensor on the plot (7.12) is proportional to particular combination (7.17) except for the first two scalar T_0 and vector T_1 representations.

7.3 Quotient higher spin algebras

Algebra $\mathcal{S}_3$ is not simple. In what follows, we consider two types of ideals $\mathcal{I} \subset \mathcal{S}_3$ along with respective quotient algebras $\mathcal{S}_3/\mathcal{I}$ which we call vertical and horizontal ones according to their graphical interpretation (7.12) and trace decomposition (7.19).

For instance, factoring out the maximal ideal $\mathcal{I}_1$ spanned by elements (7.8) yields the quotient $\mathcal{H}_1 = \mathcal{S}_3/\mathcal{I}_1$ spanned by a finitely many basis elements

$$\mathcal{H}_1 = T_0 \oplus T_1 \,, \tag{7.20}$$

corresponding to $gl(2,\mathbb{R}) \approx gl(1,\mathbb{R}) \oplus sl(2,\mathbb{R})$ algebra. Indeed, using the trace decomposition (7.19) one notes that all elements in (7.12) save for T_0 and T_1 are proportional to $sp(2)$ generators $t_{\alpha\beta}$. It follows that all such elements belong to the ideal $\mathcal{I}_1$ and therefore are to be factored out.

7.3.1 Horizontal factorization

The maximal ideal is the first element in a family of two-sided ideals

$$\mathcal{I}_k = \left\{ T_{\alpha_1\ldots\alpha_{2k}} * g^{\alpha_1\ldots\alpha_{2k}}(Y) \right\} , \qquad k\in\mathbb{N} , \tag{7.21}$$

where

$$T_{\alpha_1\alpha_2\ldots\alpha_{2k}} = t_{(\alpha_1\alpha_2} * \ldots * t_{\alpha_{2k-1}\alpha_{2k})} , \tag{7.22}$$

and $g^{\alpha_1\ldots\alpha_{2k}}(Y)$ is a rank-$2k$ symmetric $sp(2)$ tensor: $\left[t_{\gamma\rho}, g^{\alpha_1\alpha_2\cdots}\right]_* = \delta^{\alpha_1}_{\rho}\, g_{\gamma}{}^{\alpha_2} + \ldots$, where the ellipses denotes all possible symmetrizations. Using the associativity of the $*$-product, the $sp(2)$-invariance condition (7.7), and the following elementary properties

$$\begin{gathered}
\left[t_{\gamma\rho}\, ,\, g^{\gamma\rho\alpha_3\ldots\alpha_{2k}}(Y)\right]_* = 0 ,\\
\left[T_{\alpha_1\ldots\alpha_{2k}}\, ,\, g^{\alpha_1\ldots\alpha_{2k}}(Y)\right]_* = 0 ,\\
\left[F(Y)\, ,\, T_{\alpha_1\alpha_2\ldots\alpha_{2k}}\right]_* = 0 ,
\end{gathered} \tag{7.23}$$

where $F(Y)\in\mathcal{S}_3$, one shows that $\mathcal{I}_k\subset\mathcal{S}_3$ is a two-sided ideal. Note that ideals (7.21) form an infinite flag sequence

$$\mathcal{I}_1 \supset \mathcal{I}_2 \supset \cdots \supset \mathcal{I}_k \supset \cdots . \tag{7.24}$$

A quotient algebra $\mathcal{H}_k = \mathcal{S}_3/\mathcal{I}_k$ is given by

$$\mathcal{H}_k = \bigoplus_{m=0}^{2k-1} G_m . \tag{7.25}$$

cf. (7.13). It is finite-dimensional and isomorphic to a direct sum of general linear algebras

$$\mathcal{H}_k \approx gl(2,\mathbb{R})\oplus \ldots \oplus gl(2k-2,\mathbb{R})\oplus gl(2k,\mathbb{R}) . \tag{7.26}$$

To prove (7.26) one notes that factoring out elements proportional to (7.22) for a given k is equivalent to truncating the plot (7.12) starting from $(2k+1)$-th column. The remaining elements form (7.25).

7.3.2 Vertical factorization

Another type of ideals is given by a family

$$\mathcal{I}^t = \left\{ I_t(c_2) * g(Y) \;,\;\; \forall g\in\mathcal{S}_3 \right\} , \tag{7.27}$$

where $I_t(c_2)$ is a t-th order $*$-product polynomial in the $sp(2)$ Casimir element c_2. Using the $sp(2)$ invariance condition (7.7) one shows that $\mathcal{I}^t\subset\mathcal{S}_3$ are two-sided ideals. From (7.19) and (7.12) it follows that the resulting quotient algebra $\mathcal{H}^t = \mathcal{S}_3/\mathcal{I}^t$ is given by

$$\mathcal{H}^t = \bigoplus_{m=0}^{\infty}\bigoplus_{l=1}^{t} T^{(l)}_m . \tag{7.28}$$

Any polynomial $I_t(c_2)$ can be decomposed into elementary monomials, so that an ideal corresponding to $I_1 = c_2 + \nu$, where ν is a constant parameter,

$$\mathcal{I}^1_\nu = \left\{ \, (c_2 + \nu) * g(Y) \, , \ \ \forall g \in \mathcal{S}_3 \right\} \, , \tag{7.29}$$

is special. Taking $t = 1$ in (7.28) one arrives at the quotient algebra $\mathcal{H}^1_\nu = \mathcal{S}_3/\mathcal{I}^1_\nu$ given by

$$\mathcal{H}^1_\nu = \bigoplus_{m=0}^{\infty} T_m \, . \tag{7.30}$$

Recalling that $\mathcal{S}_3 \approx \mathcal{U}(o(2,1))$ (7.16) and using the relation $c_2 = C_2 + \frac{3}{4}$ obtained by taking $M = 1$ in formula (7.6), one finds that the above factorization is equivalent to factoring out elements proportional to $C_2 + \frac{3}{4}$ from the universal enveloping algebra $\mathcal{U}(o(2,1))$. In this way, we obtain that $\mathcal{H}^1_\nu = \mathcal{U}(o(2,1))/\mathcal{I}_{C_2+\frac{3}{4}+\nu}$, and, therefore, $\mathcal{H}^1_\nu$ is isomorphic to the higher spin algebra hs$[\nu]$ [36, 37, 57]. On the other hand, the algebra hs$[\nu]$ is spanned by polynomials of two spinor variables q_α and an idempotent element K with commutation relations $[q_\alpha, q_\beta] = 2i\epsilon_{\alpha\beta}(1 + \nu K)$, $\{q_\alpha, K\} = 0$ [37].

Note that the two types of factorizations can be visualized on the plot (7.12). The horizontal factorization corresponds to truncating the plot horizontally starting from $(2k+1)$-th column. The vertical factorization corresponds to truncating the plot vertically starting from t-th row.

7.3.3 Double factorizations

For particular integer ν algebra $\mathcal{H}^1_\nu$ (7.30) contains an additional (infinite-dimensional) ideal. The corresponding quotient is a finite-dimensional general linear algebra [36, 37]. Using the $o(2,1) - sp(2)$ Howe duality this can be seen as follows.

For a given ν, all other ideals $\mathcal{I}^1_\mu$ for $\mu \neq \nu$ and ideals $\mathcal{I}_k$ (7.21) for any k in the quotient $\mathcal{S}_3/\mathcal{I}^1_\nu$ become the trivial ideal which is the entire quotient itself.

Indeed, factoring out $\mathcal{I}^1_\nu$ one obtains that in the quotient algebra $\mathcal{H}^1_\nu$ the $sp(2)$ Casimir element takes a particular value $c_2 = -\nu$. Consider now ideal $\mathcal{I}_\mu \subset \mathcal{S}_3$ with parameter $\mu \neq \nu$. Using definition (7.29) one shows that elements of $\mathcal{I}_\mu$ restricted to quotient $\mathcal{H}^1_\nu$ are of the form $(\mu - \nu)g$, where $g \in \mathcal{H}^1_\nu$. As a result, $\mathcal{I}^1_\mu \approx \mathcal{H}^1_\nu$ for $\mu \neq \nu$, and $\mathcal{I}^1_\mu \approx \varnothing$ for $\mu = \nu$, so that the ideal becomes trivial.

The same reasoning applies to another type of ideals $\mathcal{I}_k$ restricted to the quotient algebra $\mathcal{H}^1_\nu$. To this end, taking in (7.21) elements $g^{\alpha_1 \ldots \alpha_{2k}}(Y) = T^{\alpha_1 \ldots \alpha_{2k}}(Y) * g(Y)$, where $\forall g(Y) \in \mathcal{S}_3$, and using the formula

$$T_{\alpha_1\alpha_2\ldots\alpha_{2k}} * T^{\alpha_1\alpha_2\ldots\alpha_{2k}} = \tau_k \prod_{m=0}^{k-1} * \, (c_2 + \alpha_m) \, , \qquad \alpha_m = m(2m+1) \, , \tag{7.31}$$

where τ_k is some non-vanishing normalization coefficient, one shows that $\mathcal{I}_k$ contains elements $g(Y) * \prod_{m=0}^{k-1} * \, (c_2 + \alpha_m)$, where $\alpha_m = m(2m+1)$. Substituting the quotient value $c_2 = -\nu$ one finds that $\mathcal{I}_k$ contains elements of the form $g(Y) \prod_{m=0}^{k-1} (\alpha_m - \nu_0)$, where $g(Y) \in \mathcal{H}^1_{\nu_0}$. For general values ν the appearance of these elements implies that the ideal $\mathcal{I}_k$ is trivial, i.e., $\mathcal{I}_k \approx \mathcal{H}_\nu$.

However, for particular integer values

$$\nu_0 = (k-1)(2k-1) \,, \qquad k \in \mathbb{N} \,, \tag{7.32}$$

one finds that the ideal $\mathcal{I}_k$ restricted to $\mathcal{H}^1_{\nu_0}$ is non-trivial, and, therefore, can be factored out. Indeed, ideal $\mathcal{I}_k$ restricted to $\mathcal{H}^1_{\nu_0}$ does not contain any powers of the $sp(2)$ Casimir element since $c_2 = -\nu_0$. On the other hand, it contains combinations $T_{\alpha_1\alpha_2\ldots\alpha_{2l}}$ for $l \geq k$ only, cf. (7.22) and (7.24). Since the horizontal factorization yields a finite-dimensional quotient, we conclude that the result of such a double factorization is finite-dimensional as well: examining the plot (7.12) one finds out that basis elements of the double factorization span a general linear algebra,

$$\mathcal{H}^1_{\nu_0}/\mathcal{I}_k \approx gl(2k, \mathbb{R}) \,. \tag{7.33}$$

Note that the rank of the algebra (7.33) is even. In Conclusions 9 we discuss how to take account of odd values.

Finally, one can use a combination of the two types of ideals in a single factorization. For instance, consider a composite two-sided ideal $\mathcal{I}^p_1 = \big\{\, t_{\alpha\beta} * I_p(c_2) * g^{\alpha\beta}(Y) \,\big\}$ provided that a $sp(2)$ symmetric tensor $g^{\alpha\beta}$ is not proportional to $t^{\alpha\beta}$, and $I(c_2)$ is some p-th order polynomial in c_2. The resulting quotient algebra is given by

$$\mathcal{H}^p_1 = \Big[T_0 \oplus T_1\Big] \oplus \Big[\bigoplus_{m=0}^{\infty} \bigoplus_{l=1}^{p} T^{(l)}_m\Big] \,. \tag{7.34}$$

7.4 Factorization via (quasi-)projectors

To describe quotients of algebra $\mathcal{S}_3$ explicitly one employs the projecting technique elaborated in [44, 46].[17] Given a quotient $\mathcal{H}$ of algebra $\mathcal{S}_3$ with respect to some ideal $\mathcal{I}$ one introduces a quasi-projector Δ satisfying the basic property

$$\Delta * h = h * \Delta = 0 \,, \qquad \forall\, h \in \mathcal{I} \,. \tag{7.35}$$

Then, it follows that elements of quotient $\mathcal{H} = \mathcal{S}_3/\mathcal{I}$ can be parameterized as follows

$$\mathcal{H} = \Big\{g \in \mathcal{H} : \;\; g = \Delta * F \,,\; \forall\, F \in \mathcal{S}_3\Big\} \,. \tag{7.36}$$

An educated guess is to consider the following ansatz

$$\Delta = \Delta(z) \,, \qquad z = Y_{\alpha A} Y^A_\beta Y^\alpha_B Y^{\beta B} \,. \tag{7.37}$$

Note that $z = 2c_2 - 9/2$, where c_2 is $sp(2)$ Casimir operator. Variable z is invariant with respect to both $sp(2) - o(2,1)$ Howe dual algebras, $[t_{\alpha\beta}, z]_* = 0$ and $[T^A, z]_* = 0$. In particular,

$$\forall\, F \in \mathcal{S}_3 : \qquad \Delta * F = F * \Delta \,. \tag{7.38}$$

In appendix B we explicitly analyze the projecting conditions (7.35) imposed on $\Delta(z)$ (7.37). We show that the horizontal projecting condition is given by an ordinary

[17] The projecting technique was also discussed in refs. [58–62].

$2k$-th order differential equation for function $\Delta_k(z)$. The vertical projecting condition is an ordinary 4-th order differential equation for function $\Delta_\nu(z)$. In both cases the searched-for solutions have the form of the series $\Delta(z) = \kappa_0 z^\alpha + \kappa_1 z^{\alpha+1} + \kappa_2 z^{\alpha+2} + \cdots$, for some degree $\alpha \geq 0$ and fixed coefficients κ_i depending on either k or ν. Also, we analyze solutions with parameter ν taking particular values (7.32).

8 Non-linear higher spin BF action

As a starting point, we formulate a non-linear higher spin theory in two dimensions as BF theory with gauge fields taking values in the adjoint representation of the infinite-dimensional Lie algebra $hc(1|2{:}[1,2])$ explicitly discussed in section 7.2. After that, using the factorization procedure of section 7.4 we describe reduced theories with fields taking values in the quotient higher spin Lie algebras.

The fields of the theory are 0-forms and 1-forms taking values in $hc(1|2{:}[1,2])$ algebra

$$\Psi(Y|x)\;, \qquad W(Y|x) = dx^m W_m(Y|x)\;. \tag{8.1}$$

From (7.11) it follows that the expansion coefficients in the auxiliary variables of (8.1) are 0-form and 1-form fields taking values in totally symmetric traceless $o(2,1)$ representations of any rank. Each independent field enters in infinitely many copies, cf. (7.2). We assume that fields (8.1) satisfy the reality conditions

$$\Psi^\dagger(Y) = -\Psi(Y)\;, \qquad W^\dagger(Y) = -W(Y)\;, \tag{8.2}$$

where the conjugation $\dagger$ is defined by (7.9).

The higher spin curvature associated to 1-form gauge fields (8.1) is defined as

$$\mathcal{R}(Y|x) = dx^m dx^n \mathcal{R}_{mn}(Y|x) = dW(Y|x) + W(Y|x) * W(Y|x)\;, \tag{8.3}$$

while the infinitesimal gauge transformations are

$$\delta_\varepsilon W = D\varepsilon\;, \qquad \delta_\varepsilon \Psi = [\Psi, \varepsilon]_*\;, \qquad \delta_\varepsilon \mathcal{R} = [\mathcal{R}, \varepsilon]_*\;, \tag{8.4}$$

where $\varepsilon = \varepsilon(Y|x)$ is 0-form gauge parameter taking values in the algebra $hc(1|2{:}[1,2])$, and

$$DF = dF + [W, F]_*\;, \qquad d = dx^m \frac{\partial}{\partial x^m}\;, \tag{8.5}$$

is the gauge covariant derivative.

Consider now an invariant bilinear form on the higher spin algebra needed to build a BF action. To this end, define a trace of any element $F(Y) \in hc(1|2{:}[1,2])$ as follows [63]

$$\mathrm{Tr}(F(Y)) = F(0)\;. \tag{8.6}$$

The trace satisfies the cyclic property

$$\mathrm{Tr}\big(F * G - G * F\big) = 0\;, \qquad \forall F, G \in hc(1|2{:}[1,2])\;, \tag{8.7}$$

that can be directly shown using the definition (7.4) and the property that F is even function, $F(Y) = F(-Y)$. It follows that the algebra $hc(1|2\!:\![1,2])$ can be endowed with the following invariant bilinear form

$$\langle F, G \rangle = \mathrm{Tr}(F * G) \,, \tag{8.8}$$

which is symmetric $\langle F, G \rangle = \langle G, F \rangle$ and invariant $\big\langle [F, G]_*, H \big\rangle = \big\langle G, [H, F]_* \big\rangle$. From (7.4) it follows that the invariant form has an integral representation useful in practice.

Using the invariant bilinear form (8.8) one defines the higher spin BF action as

$$S[\Psi, W] = g \int_{\mathcal{M}^2} \mathrm{Tr}\,(\Psi * \mathcal{R}) \tag{8.9}$$

where g is a dimensionless coupling constant. The above action can be invariantly extended by adding potentials which are linear combinations of Casimir polynomials $\kappa_i I_i(\Psi)$ on the algebra, where κ_i are coupling constants.

The equations of motion obtained by varying with respect to $W_m(Y|x)$ and $\Psi(Y|x)$ are

$$\mathcal{R}_{mn}(Y|x) = 0 \,, \tag{8.10}$$

and

$$D_m \Psi(Y|x) = 0 \,, \tag{8.11}$$

where the gauge covariant derivative D_m is given by (8.5). The equation (8.10) is the covariance constancy condition involving both fields Ψ and W_m, while equation the (8.11) is the zero-curvature condition involving fields W_m only. It follows that the gauge sector of the theory can be analyzed independently. Adding invariant potentials to the action results in that the curvature acquires non-vanishing right-hand-side. For instance, additional terms proportional to the second-order invariant operator $I_2 = \mathrm{Tr}(\Psi * \Psi)$ yields the deformation (2.12) discussed earlier within the linearized theory.

By construction, the higher spin BF action is invariant under the gauge symmetry transformations (8.4). On the other hand, the theory is manifestly diffeomorphism invariant as it is formulated via differential forms, while containing no metric tensor. The diffeomorphism transformations of fields (8.1) are given by the respective Lie derivatives

$$\delta_\xi \Psi = \xi^m \partial_m \Psi \,, \qquad \delta_\xi W_n = \xi^m \partial_m W_n + \partial_n \xi^m W_m \,, \tag{8.12}$$

that can be represented as follows

$$\delta_\xi \Psi = \big[\Psi, \xi^m W_m\big]_* + \xi^m D_m \Psi \,, \qquad \delta_\xi W_n = D_m\big(\xi^n W_n\big) + \xi^n \mathcal{R}_{nm} \,. \tag{8.13}$$

The terms proportional to the field equations represent the trivial invariance transformations vanishing on the mass-shell. Indeed, given any action $S[\phi_i]$ depending on fields ϕ_i, $i = 1, 2, 3, ...$ one has a trivial invariance transformation $\delta\phi_i = M_{ij}\, \delta S/\delta\phi_j$, where the parameter matrix is antisymmetric $M_{ij} = -M_{ij}$. Symmetries which differ by these trivial terms are equivalent. In our case, 0-form Ψ and 1-form W are identified with ϕ_1 and ϕ_2.

It follows that modulo the trivial transformations the diffeomorphisms are just a particular gauge transformation with a field-dependent gauge parameter, and, therefore, can be disregarded as independent symmetries.[18]

8.1 Linearization around AdS_2 background

The higher spin theory (8.9) contains the gravitational subsector since the higher spin algebras under consideration always contain $o(2,1)$ subalgebra. Moreover, the ground state of the model is identified with the AdS_2 spacetime. It seems natural to have AdS_2 spacetime as the background, because in this way higher dimensional higher spin gauge theories extend to the $2d$ case while keeping their main characteristic features intact: higher spin gauge fields and the AdS background geometry. One should note, however, that contrary to $d \geq 4$ higher spin theories the AdS_2 background is not necessarily required to have a consistent interacting theory.[19] Recall that switching on the cosmological constant $\Lambda \neq 0$ is indispensable to guarantee consistent gravitational interactions of gauge massless higher spin fields. In two and three dimensions it seems that taking $\Lambda = 0$ does not prevent having a consistent theory with higher spin symmetries because higher spin fields carry no local degrees of freedom.

Fixing the background connection W_0 we treat dynamical fields Ω as fluctuations,

$$W(Y|x) = W_0(Y|x) + \Omega(Y|x)\;, \tag{8.14}$$

where W_0 satisfies the $o(2,1)$ zero-curvature condition (2.3) and describes AdS_2 spacetime. A background value of Ψ is discussed below, while perturbations over Ψ_0 are defined as

$$\Psi(Y|x) = \Psi_0(Y|x) + \Phi(Y|x)\;, \tag{8.15}$$

where Φ are dynamical fields. Up to the second order in the fields the non-linear curvature (8.3) decomposes as

$$\mathcal{R}(Y|x) = \mathcal{R}_0(Y|x) + R(Y|x) + \ldots\;, \tag{8.16}$$

where

$$\mathcal{R}_0 = dW_0 + W_0 * W_0\;, \qquad R = d\Omega + W_0 * \Omega + \Omega * W_0\;. \tag{8.17}$$

Substituting the perturbative expansions (8.14), (8.15) into the equations of motion (8.10), (8.11) one finds that the background fields satisfy the following equations

$$dW_0 + W_0 * W_0 = 0\;, \qquad d\Psi_0 + [W_0, \Psi_0]_* = 0\,. \tag{8.18}$$

The first equation above is the zero curvature-condition (2.3), while the background field Ψ_0 remains unknown. Next, the first-order equations are given by

$$d\Omega + [W_0, \Omega]_\star = 0\;, \qquad d\Phi + [W_0, \Phi]_* + [\Omega, \Psi_0]_* = 0\,. \tag{8.19}$$

[18] In particular, for the spin $s=1$ two components of the diffeomorphism parameter $\xi^n(x)$ combine into a single scalar gauge parameter $\varepsilon(x)$. For the spin $s=2$ case one shows that the gauge transformation of the frame with $o(1,1)$ vector parameter $\varepsilon^a(x)$ and the diffeomorphism with parameter $\xi^n(x)$ are identified [64]. For the higher spins $s>2$ diffeomorphism parameters form a subspace in the gauge parameter space.

[19] See, e.g., refs. [65, 66], where $3d$ flat higher spin theory was discussed.

Suppose now that Ψ_0 is x-independent, that is $d\Psi_0 = 0$. Then, the second equation in (8.18) says that

$$[W_0, \Psi_0]_* = 0\,. \tag{8.20}$$

It follows that $o(2,1)$-invariant non-vanishing vacuum value of the 0-form field is a function of the $sp(2)$ basis elements only

$$\Psi_0(Y) = a_{(0)} + a_{(0)}^{\alpha\beta} t_{\alpha\beta} + a_{(1)} c_2 + \ldots\; = \sum_{k,\,l=0}^{\infty} a_{(l)}^{\alpha_1\alpha_2\ldots\alpha_{2k}}\; T_{\alpha_1\alpha_2\ldots\alpha_{2k}} * (c_2 *)^l\,, \tag{8.21}$$

where $a_{(l)}^{\alpha_1\alpha_2\ldots\alpha_{2k}}$ are some (Y,x)-independent (constant) $sp(2)$ symmetric tensor parameters, $T_{\alpha_1\alpha_2\ldots\alpha_{2k}}$ is given by (7.22) and c_2 is $sp(2)$ Casimir operator.[20] Recall that these properties guarantee the $sp(2)$ invariance of Ψ_0, cf. (7.19). The fluctuation field Ω is also $sp(2)$ invariant, and therefore it commutes with any combination of $t_{\alpha\beta}$. As a result, $[\Omega, \Psi_0]_* = 0$.

It follows that the linearized equations of motion (8.19) take the form

$$d\Omega + [W_0, \Omega]_* = 0\,, \qquad d\Phi + [W_0, \Phi]_* = 0\,. \tag{8.22}$$

The Abelian part of the gauge transformation (8.4) for fluctuations has the form

$$\delta_\varepsilon \Omega = D_0 \varepsilon \equiv d\varepsilon + [W_0, \varepsilon]\,, \qquad \delta_\varepsilon \Phi = 0\,, \qquad \delta_\varepsilon R = 0\,, \tag{8.23}$$

where the linearized derivative D_0 reproduces the definition (2.6), while the above transformations themselves reproduce (2.7) and (2.8).

Now, the trace decomposition (7.19) that brings the higher spin algebra $hc(1|2{:}[1,2])$ into the basis where all basis elements are given by traceless $o(2,1)$ tensors (7.12) is expressed via the $sp(2)$ generators. It follows that field Ω_m decomposes into irreducible components as

$$\Omega_m := \bigoplus_{s=1}^{\infty} \bigoplus_{k=0}^{\infty} \Omega_m^{(s,k)}\,, \tag{8.24}$$

where components $\Omega_m^{(s,k)}$ are 1-form spin-s gauge fields $\Omega^{(k)}{}_m^{A_1\ldots A_{s-1}}$ with $s-1$ totally symmetric traceless $o(2,1)$ indices, while the label k stands for a multiplicity, cf. (7.2).

On the other hand, field equations (8.22) can be represented via the background covariant derivative as $D_0\Omega = 0$ and $D_0\Phi = 0$, cf. (3.1), (3.2). Therefore, using $D_0 t_{\alpha\beta} = 0$ one finds out that the field equations (8.22) can be decomposed into $o(2,1)$ irreducible components as well. In each irreducible spin-s sector equations of motion take the form (2.10); each pair of equations (2.10) comes in infinitely many copies. Whence, the spectrum of the model contains infinitely many copies of all integer spin-s subsystems,

$$1_{[\infty]},\;\; 2_{[\infty]},\;\; 3_{[\infty]},\;\; \ldots\;\; , \infty_{[\infty]}\,, \tag{8.25}$$

where $1, 2, 3, ..$ denote spins, while a subscript $[\cdot]$ denotes a multiplicity, which in the present case is infinite, cf. (8.24).

[20] Choosing $\Psi_0 = t_{\alpha\beta} a^{\alpha\beta}$ in (8.21) is similar to non-vanishing vacuum value of the 0-form in the BF higher spin model considered in ref. [8].

JHEP10(2014)122

8.2 Reduced BF higher spin models

The spectrum of the AdS_2 higher spin gravity model (8.9) is infinite and degenerate. It can be truncated in two possible ways.

- Horizontally reduced model: finitely many fields with spins bounded from above, each field appears in several copies.
- Vertically reduced model: infinitely many fields of all spins from zero to infinity, each field appears in a single copy.

It is clear that such reduced models are governed by respectively horizontal and vertical quotient higher spin algebras of section 7.3.

We propose to describe reduced models with fields taking values in the quotient higher spin algebras by the BF action (8.9) modified by the projecting operator Δ in the following manner[21]

$$S_\Delta[\Psi, W] = g \int_{\mathcal{M}^2} \mathrm{Tr}\Big[\Delta * \Psi * \mathcal{R}\Big] \,, \tag{8.26}$$

where, according to particular factorization, one chooses either the horizontal projector Δ_k or the vertical projector Δ_u of section 7.4. By inserting Δ we reduce the original spectrum of fields to a smaller subset of fields identified with representatives of the quotient algebra. Indeed, Δ is defined to send all elements of the corresponding ideals in $hc(1|2\!:\![1,2])$ to zero (7.35).

Action (8.26) can be understood by introducing a new invariant form. Indeed, we replace the invariant form (8.8) on the algebra $hc(1|2\!:\![1,2])$ by the following form

$$\langle F, G\rangle_\Delta = \mathrm{Tr}(\Delta * F * G) \,, \qquad F, G \in hc(1|2\!:\![1,2]) \,. \tag{8.27}$$

The invariance and symmetry properties are not spoiled by Δ as it commutes with F and G, (7.35). However, the invariant form (8.27) is degenerate since $\langle F, G\rangle_\Delta = 0$ for $\forall\, F \in hc(1|2\!:\![1,2])$ and $\forall\, G \in \mathcal{I}$.

Reduced action (8.26) is invariant with respect to the gauge transformations (8.4). Additionally, it acquires a new type of invariance due to a degeneracy of the form (8.27),

$$\begin{aligned} \delta\Psi(Y|x) &= A(Y|x) \,, \qquad A \in \mathcal{I} \,, \\ \delta W(Y|x) &= B(Y|x) \,, \qquad B \in \mathcal{I} \,. \end{aligned} \tag{8.28}$$

If the factorization with respect to the ideal $\mathcal{I}$ gives a quotient algebra which is not simple, then there happens a symmetry enhancement governed by an additional ideal. This is the case of the double factorization described in sections 7.3.3 and 7.4.

The equations of motion of the reduced theory (8.26) are

$$\Delta * \mathcal{R}_{mn}(Y|x) = 0 \,, \tag{8.29}$$

[21] Action functionals of this type were previously considered within AdS_5 higher spin interacting theories [46, 60, 62]

and

$$\Delta * D_m \Psi(Y|x) = 0 \,, \tag{8.30}$$

where the covariant derivative D_m is given by (8.5). The equations are invariant with respect to the standard gauge transformations, while the shift transformations (8.28) yield additional algebraic Bianchi identities.

Let us consider a perturbative expansion of the reduced model (8.26). Both zeroth-order and first-order equations are again equations (8.18) and (8.19) but now multiplied by Δ. A natural choice for the background is to take the AdS_2 connection W_0 as the vacuum 1-form field because it solves the equation of motion (8.29). As the background 0-form field we take an x-independent $\Psi_0(Y)$. From (8.30) it follows that $\Delta * [W_0, \Psi_0]_* = 0$ which means that Ψ_0 can be chosen to be an element of the ideal, $\Psi_0 \in \mathcal{I}$. However, using the shift symmetry (8.28) one observes that it can be equivalently set to zero. Therefore, from the very outset one can choose $W = W_0$ and $\Psi_0 = 0$ as representatives of the zeroth equivalence class in the quotient higher spin algebra.

On the other hand, the projector is $o(2,1)$-invariant since $D_0\Delta(Y) = 0$, where D_0 is the background $o(2,1)$ covariant derivative (2.6). Introducing the quotient algebra representatives $\bar\Omega = \Delta * \Omega$ and $\bar\Phi = \Delta * \Phi$ one rewrites the linearized equations of motion as $D_0\bar\Omega(x|Y) = 0$ and $D_0\bar\Psi(x|Y) = 0$. It follows that the linearized equations factorize into independent spin-s subsystems described by previously studied equations (2.10).

In the case of the horizontal factorization, the respective quotient higher spin algebra is given by a direct sum of general linear algebras (7.26). It follows that for a given parameter of the horizontal factorization $k = 1, 2, ...$, a spectrum of the reduced model is degenerate. It contains independent subsystems of spins:

$$2k_{[1]}, \;\; (2k-1)_{[1]}, \;\; (2k-2)_{[2]}, \;\; (2k-3)_{[2]}, \;\; (2k-4)_{[3]}, \;\; (2k-5)_{[3]}, \;\; \ldots \tag{8.31}$$

where $2k - i$ denotes spin, while a subscript $[j]$ denotes a multiplicity. Spin-1 and spin-2 subsystems have a maximal multiplicity $[k]$. For instance, the maximal horizontal factorization ($k = 1$) gives spin $s = (2_{[1]}, 1_{[1]})$ system that obviously reproduces the original Jackiw-Teitelboim model plus the Maxwell BF theory. A spectrum of the next-to-maximal horizontal factorization ($k = 2$) reads $4_{[1]}, 3_{[1]}, 2_{[2]}, 1_{[2]}$.

In the case of the vertical factorization, the resulting higher spin algebra $\mathrm{hs}[\nu]$ is infinite-dimensional and parameterized by continuous parameter ν. A spectrum of the reduced model is non-degenerate. It contains independent subsystems of spins:

$$\nu \neq \nu_0 \;: \qquad 1_{[1]}, \;\; 2_{[1]}, \;\; 3_{[1]}, \; \ldots \,, \infty_{[1]} \,. \tag{8.32}$$

Generally, the spectrum does not depend on ν, but for the special values (7.32) it is truncated to a finite subset of subsystems with spins:

$$\nu_0 = (k-1)(2k-1) \;: \qquad 1_{[1]}, \;\; 2_{[1]}, \;\; 3_{[1]}, \;\; \ldots \,, (2k-1)_{[1]}, \;\; (2k)_{[1]} \,, \tag{8.33}$$

that immediately follows from that the reduced higher spin algebra is $gl(2k, \mathbb{R})$ (7.33).[22]

[22]One can also discuss reduced models based on double factorizations of the form (7.34).

9 Conclusions and outlooks

In this paper, we proposed a new class of two-dimensional higher spin models interpreted as the AdS_2 higher spin gravity and explored some of its global and local properties. The model is formulated by virtue of topological BF action for fields taking values in particular higher spin symmetry algebra containing $o(2,1) \approx sl(2,\mathbb{R})$ subalgebra. Our analysis follows methods used within the unfolded approach to higher spin dynamics. In particular, we developed a two-dimensional version of the unfolded formulation resulting in a cohomological understanding of the BF dynamics. Using two different nilpotent operators acting on the field space of BF model we elaborate two metric-like formulations of the model. Our analysis of the linearized BF equations of motion both for 0-forms and 1-forms accomplishes the analysis of the 1-form sector performed earlier in [10]. We also discuss a new type of duality between two metric-like formulations obtained from a single BF frame-like theory.

We suggested a particular formulation of two-dimensional higher spin algebra hs$[\nu]$ employing the $o(2,1) - sp(2)$ Howe duality. In this way we extend the Vasiliev oscillator construction of $d \geq 4$ higher spin Eastwood-Vasiliev algebras to the $d = 2$ case. Infinite-dimensional higher spin algebras and their finite-dimensional truncations are realized as particular quotient algebras for which reason we classified relevant cases of ideals and corresponding factorizations. We explicitly described the projecting technique used to define the BF actions for fields taking values in the quotient algebras.

The $d = 2$ classification of ideals and factorizations extends to any d case. Obviously, using the ideals generated by the $sp(2)$ Casimir operator and its powers one arrives at some quotient algebra with connections identified with higher spin partially-massless fields of any depth (e.g., see discussion in [56]). It should be realized as the symmetry algebra of higher order singleton representations of $o(2,d)$ algebra [67].

It is important to note that a given BF theory with a finite-dimensional algebra is necessarily topological one. The situation is more intricate in the case of an infinite-dimensional algebra. For instance, the BF action for higher spin algebras considered in this paper is topological. On the other hand, a particular BF theory proposed in ref. [8] describes self-interactions of matter fields via higher spin currents built of these matter fields. Nonetheless, the model is not topological because BF fields take values in a peculiar infinite-dimensional algebra containing hs$[\nu]$ as a subalgebra. The rationale behind this observation is that a BF action formulated on an infinite-dimensional field space may leave a room for local degrees of freedom.

In particular, it follows that BF actions may contain current interactions of matters fields, and, therefore, it is tempting to speculate that higher spin BF action has to do somehow both with currents and matter fields on equal footing. This idea conforms with the duality between the metric-like formulations described in this paper. Indeed, we find out that BF equations of motion can be simultaneously treated as matter field equations and conservation conditions.

Below we list some interesting issues left beyond the scope of the paper.

- The form and properties of the mapping between two metric-like descriptions of the free field higher spin theory discussed in section 6.2. The original linearized BF higher spin action functional can be treated as a parent action for the two dual formulations.

- One may consider the supersymmetric Howe dual pair $o(2, M) - osp(1, 2)$ underlying the construction of the higher spin algebra $hc(1|(1, 2):[M, 2])$ which describes hook-type mixed-symmetry higher spin fields in AdS_{M+1} [44]. For $M = 1$ all mixed-symmetry fields are dual to totally symmetric ones (2.2). One can classify ideals of $hc(1|(1, 2) : [1, 2])$ as in section 7.3, and study respective quotient algebras. In particular, it should result in odd values of the rank of general linear algebras obtained via the double factorization (7.33).

- It is interesting to realize the universal enveloping algebra $\mathcal{U}(o(2, 1))$ in terms of extended $o(2, 1) - osp(n, 2)$ Howe dual pairs with arbitrary $n \geq 2$.

- The role of parameter ν in the vertical reduced model is to be clarified. We have seen that the linearized equations of motion are independent on ν. It appears that ν comes out in the next orders.[23]

- The flat space limit $\Lambda \to 0$ in the BF higher spin models. The resulting theory should be a higher spin extension of the two-dimensional Poincare gravity suggested in [69] and further discussed in [49, 70, 71]. It should be governed by a non-semisimple higher spin algebra extending the $(1 + 1)$ Poincare algebra.

Among other things, the AdS_2 higher spin gravity is interesting because the respective action functional is given in a closed form that makes possible to analyze many conventional questions like higher spin black hole solutions, supersymmetric higher spin extensions, quantization, etc. In particular, it is interesting to consider matter fermions interacting via higher spin fields and, therefore, to formulate a higher spin extension of the Schwinger model in AdS_2 spacetime.[24] Further, topological field theories are known to induce local degrees of freedom at the boundary. This is also the case for two-dimensional higher spin theories of the type considered in the present paper. The problem has been already partly discussed in the literature [9, 11].

Acknowledgments

I am grateful to M.A. Grigoriev, S.E. Konstein, R.R. Metsaev, and E.D. Skvortsov for many useful discussions and comments.

This research was supported by Russian Science Foundation grant 14-42-00047.

[23]See recent paper [68] on $3d$ Chern-Simons higher spin theories, where the parameter has been related to a spin of infinite-dimensional anyon representations in AdS_3.

[24]*E.g.,* see a discussion of a particle moving in lineal gravitational fields [70, 71].

A Computation of the cohomology groups

In what follows, we compute the cohomology of the nilpotent $\sigma_\pm$ operators acting on the space $\mathcal{G}_s$. To this end, one recalls some relevant group-theoretical facts on $o(1,1)$ Lorentz algebra representations and their tensor products.

Introducing a collective notation for symmetrized indices $(a_1 ... a_k) \equiv a(k)$, one finds that a frame-like tensor $\mathrm{T}_m{}^{a(k)}$ being a tensor product of totally symmetric and traceless tensor with a vector decomposes into two $o(1,1)$ irreps of spins $k-1$ and $k+1$. Recalling that a dimension of any integer spin $o(1,1)$ (non-scalar) irrep equals 2, the above statement can be simply understood as $2^2 = 2+2$. On the other hand, any totally symmetric and traceful frame-like tensor $\mathrm{A}_m{}^{a(k)}$ decomposes into $\bigoplus_{n=0}^{k} \mathrm{T}_m{}^{a(n)}$, where $\mathrm{T}_m{}^{a(n)}$ are traceless with respect to fiber $o(1,1)$ tensors. The decompositions clarify the formula $\dim \mathrm{A}_m{}^{a(k)} = 2(2k+1)$.

To summarize, the following decompositions are useful in practice

$$\mathrm{A}_m{}^{a(k)} = \mathrm{A}^{a(k+1)} \oplus \mathrm{A}^{a(k-1)} \,, \tag{A.1}$$

$$\mathrm{T}_m{}^{a(k)} = \mathrm{T}^{a(k+1)} \oplus \mathrm{T}^{a(k-1)} \,, \tag{A.2}$$

both for traceful $\mathrm{A}^{a(k)}$ and traceless $\mathrm{T}^{a(k)}$ totally symmetric tensors. Decomposition (A.2) for traceless tensors is easily explained in components: a trace part is proportional to antisymmetric dualized part of hook component. The case $k=1$ is special: decomposing $\mathrm{A}_m{}^a \equiv \mathrm{T}_m{}^a$ into $sl(2)$ irreps and then into $o(1,1)$ irreps yields

$$\mathrm{A}_m{}^a \equiv \mathrm{T}_m{}^a := \mathrm{A}^{a(2)} \oplus \mathrm{A} = \mathrm{T}^{a(2)} \oplus \mathrm{T} \oplus \mathrm{A} \,, \tag{A.3}$$

where A and T are two different scalar components. Their appearance is due to the relation $A^{a|b} = \frac{1}{2}\mathrm{A}^{(a|b)} + \frac{1}{2}\mathrm{A}^{[a|b]} = \frac{1}{2}\mathrm{A}^{(a|b)} + \frac{1}{2}\epsilon^{ab}\mathrm{A} = \frac{1}{2}\mathrm{T}^{(ab)} + \frac{1}{4}\eta^{ab}\mathrm{T} + \frac{1}{2}\epsilon^{ab}\mathrm{A}$, where $\eta_{mn}\mathrm{T}^{(mn)} = 0$ and ϵ^{ab} is $2d$ Levi-Civita tensor. Vertical slash denotes independent groups of indices.

Consider operators $\sigma_\pm$ given by (3.5) that act on the module $\mathcal{G}_s$ of differential p-forms which take values in $o(1,1)$ finite-dimensional irreps, $T_{(p)}^{a_1 \ldots a_k}$, where $p = 0,1,2$ and $k = 0,1,...,s-1$, see section 3.1. For the case $s=1$ the cohomology computation is trivial so we give detailed consideration of the spin $s \geq 2$ case only.

σ_-- cohomology. Let us compute cohomology group $H^{(0)}(\sigma_-)$. Since exact forms are absent in this case the cohomology is defined by the closure condition only

$$h_c\, T_{(0)}^{a(k-1)c} = 0 \,, \qquad 0 \leq k \leq s-1 \,. \tag{A.4}$$

Using the background 1-form frame $h_{m,\,c}$ the world index is converted into fiber one so that equation (A.4) is cast into the form $T^{a(k-1)c} = 0$ for $k = 1,2,...,s-1$. The case $k=0$ is exceptional: equation (A.4) does not impose any restrictions on T. Thus, the cohomology group contains a single scalar component T, i.e. we find $H^{(0)}(\sigma_-) = \{T\}$, see (3.20).

Consider now cohomology group $H^{(1)}(\sigma_-)$ which is defined by both closer and exactness conditions

$$h_c \wedge T_{(1)}^{a(k-1)c} = 0 \,, \qquad \delta T_{(1)}^{a(k)} = h_c\, T_{(0)}^{a(k)c} \,, \tag{A.5}$$

JHEP10(2014)122

where $T^{a(k)}_{(1)}$ and $T^{a(k+1)}_{(0)}$, $0 \leq k \leq s-1$, are 1-forms and 0-forms, respectively. Consider the first equation in (A.5). Converting all world indices into fiber ones the equation can equivalently be rewritten as $T^{a(k-1)[c|d]} = 0$. Contracting with ϵ_{cd} and using decomposition (A.2) one finds that rank-$(k-1)$ totally symmetric and traceless component of $T^{a(k)}_{(1)}$ vanishes except for the cases $k = 0$ and $k = s-1$. Then, one considers the exactness condition in (A.5) and shows that rank-$(k+1)$ totally symmetric and traceless component of $T^{a(k)}_{(1)}$ also vanish since it is exact, except for the case $k = s-1$.

Equation (A.5) at $k = 1$ should be analyzed separately because in this case decomposition into irreducible components is different, see (A.3). It follows that the closer condition sets to zero the antisymmetric part, while symmetric one is arbitrary. For $s > 2$ symmetric and traceless component cancels due to the exactness condition, while for $s = 2$ it remains intact. One concludes that cohomology is given by rank-s totally symmetric component and a scalar component T which comes as a trace part of $T^{a}_{(1)}$. Therefore, $H^{(1)}(\sigma_-) = \{T, T^{a_1\ldots a_s}\}$, see (3.20).

Then, consider cohomology group $H^{(2)}(\sigma_-)$ defined by the following chain of conditions

$$h_c \wedge T^{a(k-1)c}_{(2)} \equiv 0\,, \qquad \delta T^{a(k)}_{(2)} = h_c \wedge T^{a(k)c}_{(1)}\,, \qquad \delta T^{a(k)}_{(1)} = h_c\, T^{a(k)c}_{(0)}\,, \tag{A.6}$$

where $T^{a(k)}_{(2)}$, $T^{a(k+1)}_{(1)}$, and $T^{a(k+2)}_{(0)}$, $0 \leq k \leq s-1$, are respectively 2-forms, 1-forms, and 0-forms. Being a 3-from the first equation in (A.6) is identically satisfied. On the other hand, analysis of the exactness conditions in (A.6) is similar to previously done computation of $H^{(0)}(\sigma_-)$ and $H^{(1)}(\sigma_-)$. Repeating the reasoning we find that $H^{(2)}(\sigma_-) = \{T^{a_1\ldots a_{s-1}}\}$, see (3.20).

σ_+- cohomology. Computation of σ_+ cohomology is analogous. The only essential difference is the origin of the scalar component in $H^{(1)}(\sigma_\pm)$: for the case of σ_+ this is an antisymmetric component of $A^{m|n}$, while for the case of σ_- the scalar component is identified with the trace of $A^{m|n}$, cf. (A.3). The resulting cohomology groups $H^{(p)}(\sigma_+)$ are given in (3.20).

B Horizontal and vertical (quasi-)projectors

Horizontal projection. Substituting (7.21) into (7.35) one gets a function $\Delta_k(z)$ satisfying the horizontal projecting equation

$$\Delta_k * T_{\alpha_1\ldots\alpha_{2k}} = \big[\, D^{(k)}\Delta\, \big] T_{\alpha_1\ldots\alpha_{2k}} = 0\,, \tag{B.1}$$

where $D^{(k)}$ stands for k-th degree of the second-order differential operator

$$D = 2z\frac{d^2}{dz^2} + 2\frac{d}{dz} + 1\,. \tag{B.2}$$

The ordinary differential equation $D^{(k)}\Delta_k = 0$ has $2k$ independent solutions. Among them we single out only those that have the form of the series $\Delta = \kappa_0 z^\alpha + \kappa_1 z^{\alpha+1} + \kappa_2 z^{\alpha+2} + \cdots$, for some $\alpha \geq 0$. It turns out that $\alpha = 0$ and there are k independent solutions of this type,

Δ_i, $i = 1, ..., k$. Since equation $D^{(k)}\Delta = 0$ comes as differential consequences of equation $D^{(k-1)}\Delta = 0$, one concludes that $k-1$ solutions Δ_i, where $i = 1, ..., (k-1)$ solve equation of lower rank and therefore can be found by induction, while the highest rank solution Δ_k does describe factorization (B.1). From the algebraic perspective, a set of analytical solutions to the horizontal projecting equation is clearly explained by the flag sequence of ideals (7.24).

An explicit form of solutions can be found straightforwardly provided that differential operator (B.2) is represented as $D = 2(N_z + 1)\dfrac{d}{dz} + 1$, where $N_z = z\dfrac{d}{dz}$ is the Euler operator, so that searching for a solution in the form of power series yields a recurrent equation system.

Solutions to equation (B.1) can be expressed via the Bessel functions and their multiple integrals. For instance, in the case $k = 1$ equation (B.1) is in fact the Bessel equation of zeroth order solved by[25]

$$\Delta_{k=1}(z) = I_0(\sqrt{2z})\ . \tag{B.3}$$

In the case $k \geq 2$ equation (B.1) can be expressed via auxiliary combinations $F_m(z) = D^{(k-m-1)}\Delta(z)$ as inhomogeneous Bessel equation $DF_m(z) = F_{m-1}(z)$, where $m = 0, ..., k-1$ and $F_{k-1} \equiv \Delta$.

It is worth noting that using the horizontal factorization via projector (B.1) yields finite-dimensional quotient algebras (7.26) with basis elements realized as infinite formal power series of auxiliary variables Y^A_α, and not as bilinear combinations as one might expect from (7.5).

Vertical projection. Substituting (7.29) into (7.35) one gets a function $\Delta_\nu(z)$ satisfying the vertical projecting condition expressed as the 4-th order differential equation

$$\Delta_\nu * (c_2 + \nu) = z^2 F'' + 4zF' + \frac{1}{2}zF + \frac{9}{4}F + \nu\Delta_\nu = 0\ , \qquad F = D\Delta_\nu\ , \tag{B.4}$$

where differential operator D is given by (B.2). Solutions analytical in $z = 0$ have the form $\Delta_\nu(z) = \gamma_0 + \gamma_1 z + \gamma_2 z^2 + \cdots$, where the coefficients satisfy the following recurrent equation system

$$9\gamma_1 + \big(2\nu + \frac{9}{2}\big)\gamma_0 = 0\,, \quad \gamma_{k-2} + A_k\gamma_k + B_k\gamma_{k-1} = 0\ , \tag{B.5}$$

where A_k and B_k are given by

$$A_k = k^2(2k+1)^2\ , \qquad B_k = 2(k-1)(2k+1) + 2\nu + \frac{9}{2}\ . \tag{B.6}$$

A few first coefficients for $\gamma_0 = 1$ are found to be

$$\Delta_\nu(z) = 1 - \frac{u_\nu}{3^2}\,z + \frac{u_\nu(10+u_\nu) - 9}{(30)^2}\,z^2 + \cdots\ , \quad \text{where} \quad u_\nu = 2\nu + 9/2\ . \tag{B.7}$$

[25] In d dimensions the $k = 1$ equation describes the maximal factorization; the solution is given in the particular integral form [44].

Following the discussion of the double factorization in section 7.3.3, one observes that given a particular value (7.32) quotient $\mathcal{H}_{\nu_0}$ defined by projecting condition (B.4) possesses an additional ideal formed by elements proportional to (7.22). Indeed, using relation (7.31) one shows that operator Δ_{ν_0} satisfying the projecting condition $\Delta_{\nu_0} * (c_2 + \nu_0) = 0$ can be represented in the form

$$\Delta_{\nu_0} = \Delta_k * \prod_{m=0}^{k-2} * (c_2 + \alpha_m) \,, \tag{B.8}$$

where Δ_k fulfills the horizontal projecting condition (B.1). It follows that elements of the quotient $\mathcal{H}_{\nu_0}$ proportional to (7.22) are sent to zero by virtue of the projecting property of the prefactor Δ_k.

For instance, taking $k = 1$ corresponding to $\nu_0 = 0$ (7.32) one finds from (B.8) that the vertical and horizontal projectors coincide, $\Delta_{\nu_0=0} = \Delta_{k=1}$. In particular, substituting $\nu_0 = 0$ into (B.5)–(B.6) one finds the solution (B.7) in a closed form $\Delta_{\nu_0=0}(z) = \sum_{k=0}^{\infty} \frac{(-)^k}{2^k (k!)^2} z^k$ recognized as the Bessel function, $\Delta_{\nu_0=0}(z) = I_0(\sqrt{2z})$ (B.3). On the other hand, we know that the $k = 1$ horizontal projection yields the quotient $\mathcal{H}_k \approx gl(2,\mathbb{R})$ (7.26), while the double factorization in the case $\nu_0 = 0$ yields $\mathcal{H}^1_{\nu_0}/\mathcal{I}_1 \approx gl(2,\mathbb{R})$ (7.33). The resulting quotients obviously coincide. Note, however, that for $k > 1$ the horizontal quotient algebra $\mathcal{H}_k$ and the double quotient algebra $\mathcal{H}^1_{\nu_0}/\mathcal{I}_k$ are not isomorphic anymore, while the respective projectors do not coincide as well, see (B.8).

References

[1] M.A. Vasiliev, *Higher spin gauge theories: Star product and AdS space*, hep-th/9910096 [INSPIRE].

[2] X. Bekaert, S. Cnockaert, C. Iazeolla and M.A. Vasiliev, *Nonlinear higher spin theories in various dimensions*, hep-th/0503128 [INSPIRE].

[3] S. Giombi and X. Yin, *The Higher Spin/Vector Model Duality*, J. Phys. **A 46** (2013) 214003 [arXiv:1208.4036] [INSPIRE].

[4] M.R. Gaberdiel and R. Gopakumar, *Minimal Model Holography*, J. Phys. **A 46** (2013) 214002 [arXiv:1207.6697] [INSPIRE].

[5] V.E. Didenko and E.D. Skvortsov, *Elements of Vasiliev theory*, arXiv:1401.2975 [INSPIRE].

[6] A.K.H. Bengtsson and I. Bengtsson, *Higher 'Spins' in One and Two Space-time Dimensions*, Phys. Lett. **B 174** (1986) 294 [INSPIRE].

[7] E.S. Fradkin and V.Y. Linetsky, *Higher Spin Symmetry in One-dimension and Two-dimensions. 1.*, Mod. Phys. Lett. **A 4** (1989) 2635 [INSPIRE].

[8] M.A. Vasiliev, *Higher spin gauge interactions for matter fields in two-dimensions*, Phys. Lett. **B 363** (1995) 51 [hep-th/9511063] [INSPIRE].

[9] S.-J. Rey, *News from Higher Spins: W_∞, Black Holes and Entropy*, talk given at the workshop *Higher spins and holography*, Simons center, March 2011.

[10] K.B. Alkalaev, *On higher spin extension of the Jackiw-Teitelboim gravity model*, *J. Phys.* **A 47** (2014) 365401 [arXiv:1311.5119] [INSPIRE].

[11] D. Grumiller, M. Leston and D. Vassilevich, *Anti-de Sitter holography for gravity and higher spin theories in two dimensions*, *Phys. Rev.* **D 89** (2014) 044001 [arXiv:1311.7413] [INSPIRE].

[12] M.P. Blencowe, *A Consistent Interacting Massless Higher Spin Field Theory in D = (2+1)*, *Class. Quant. Grav.* **6** (1989) 443 [INSPIRE].

[13] E. Bergshoeff, M. Blencowe, and K. Stelle, *Area Preserving Diffeomorphisms and Higher Spin Algebra*, *Commun. Math. Phys.* **128** (1990) 213.

[14] M. Henneaux and S.-J. Rey, *Nonlinear $W_{infinity}$ as Asymptotic Symmetry of Three-Dimensional Higher Spin Anti-de Sitter Gravity*, *JHEP* **12** (2010) 007 [arXiv:1008.4579] [INSPIRE].

[15] A. Campoleoni, S. Fredenhagen, S. Pfenninger and S. Theisen, *Asymptotic symmetries of three-dimensional gravity coupled to higher-spin fields*, *JHEP* **11** (2010) 007 [arXiv:1008.4744] [INSPIRE].

[16] J.M.F. Labastida, M. Pernici and E. Witten, *Topological Gravity in Two-Dimensions*, *Nucl. Phys.* **B 310** (1988) 611 [INSPIRE].

[17] D. Montano and J. Sonnenschein, *The Topology of Moduli Space and Quantum Field Theory*, *Nucl. Phys.* **B 324** (1989) 348 [INSPIRE].

[18] E.P. Verlinde and H.L. Verlinde, *A Solution of Two-dimensional Topological Quantum Gravity*, *Nucl. Phys.* **B 348** (1991) 457 [INSPIRE].

[19] K. Li, *Construction of topological W(3) gravity*, *Phys. Lett.* **B 251** (1990) 54 [INSPIRE].

[20] B.M. Barbashov, V.V. Nesterenko and A.M. Chervyakov, *The solitons in some geometrical field theories*, *Theor. Math. Phys.* **40** (1979) 572 [INSPIRE].

[21] C. Teitelboim, *Gravitation and Hamiltonian Structure in Two Space-Time Dimensions*, *Phys. Lett.* **B 126** (1983) 41 [INSPIRE].

[22] R. Jackiw, in *Quantum theory of Gravity*, S. Christansen ed., Hilger, Bristol, 1984.

[23] T. Fukuyama and K. Kamimura, *Gauge Theory of Two-dimensional Gravity*, *Phys. Lett.* **B 160** (1985) 259 [INSPIRE].

[24] K. Isler and C.A. Trugenberger, *A Gauge Theory of Two-dimensional Quantum Gravity*, *Phys. Rev. Lett.* **63** (1989) 834 [INSPIRE].

[25] A.H. Chamseddine and D. Wyler, *Topological Gravity in (1+1)-dimensions*, *Nucl. Phys.* **B 340** (1990) 595 [INSPIRE].

[26] S. Deser and R.I. Nepomechie, *Anomalous Propagation of Gauge Fields in Conformally Flat Spaces*, *Phys. Lett.* **B 132** (1983) 321 [INSPIRE].

[27] S. Deser and R.I. Nepomechie, *Gauge Invariance Versus Masslessness in de Sitter Space*, *Annals Phys.* **154** (1984) 396 [INSPIRE].

[28] S. Deser and A. Waldron, *Partial masslessness of higher spins in (A)dS*, *Nucl. Phys.* **B 607** (2001) 577 [hep-th/0103198] [INSPIRE].

[29] Y. Zinoviev, *On massive high spin particles in AdS*, hep-th/0108192 [INSPIRE].

[30] E.D. Skvortsov and M.A. Vasiliev, *Geometric formulation for partially massless fields*, Nucl. Phys. **B 756** (2006) 117 [hep-th/0601095] [INSPIRE].

[31] Y. Zinoviev, *On massive spin 2 interactions*, Nucl. Phys. **B 770** (2007) 83 [hep-th/0609170] [INSPIRE].

[32] R.R. Metsaev, *Gravitational and higher-derivative interactions of massive spin* 5/2 *field in (A)dS space*, Phys. Rev. **D 77** (2008) 025032 [hep-th/0612279] [INSPIRE].

[33] I.L. Buchbinder, T.V. Snegirev, Y. Zinoviev and Y. Zinoviev, *Gauge invariant Lagrangian formulation of massive higher spin fields in* $(A)dS_3$ *space*, Phys. Lett. **B 716** (2012) 243 [arXiv:1207.1215] [INSPIRE].

[34] E. Joung, L. Lopez and M. Taronna, *On the cubic interactions of massive and partially-massless higher spins in (A)dS*, JHEP **07** (2012) 041 [arXiv:1203.6578] [INSPIRE].

[35] N. Boulanger, D. Ponomarev and E.D. Skvortsov, *Non-abelian cubic vertices for higher-spin fields in anti-de Sitter space*, JHEP **05** (2013) 008 [arXiv:1211.6979] [INSPIRE].

[36] B.L. Feigin, *Lie algebras* $gl(\lambda)$ *and cohomologies of Lie algebras of differential operators*, *Russian Math. Surv.* **43** (1988) 169.

[37] M.A. Vasiliev, *Quantization on sphere and high spin superalgebras*, *JETP Lett.* **50** (1989) 374 [INSPIRE].

[38] C.N. Pope, L.J. Romans and X. Shen, *W(infinity) and the Racah-wigner Algebra*, Nucl. Phys. **B 339** (1990) 191 [INSPIRE].

[39] E. Joung and K. Mkrtchyan, *Notes on higher-spin algebras: minimal representations and structure constants*, JHEP **05** (2014) 103 [arXiv:1401.7977] [INSPIRE].

[40] C.R. Nappi, *Some Properties of an Analog of the Nonlinear* σ *Model*, Phys. Rev. **D 21** (1980) 418 [INSPIRE].

[41] E.S. Fradkin and A.A. Tseytlin, *Quantum Equivalence of Dual Field Theories*, Annals Phys. **162** (1985) 31 [INSPIRE].

[42] M.G. Eastwood, *Higher symmetries of the Laplacian*, Annals Math. **161** (2005) 1645 [hep-th/0206233] [INSPIRE].

[43] M.A. Vasiliev, *Nonlinear equations for symmetric massless higher spin fields in (A)dS(d)*, Phys. Lett. **B 567** (2003) 139 [hep-th/0304049] [INSPIRE].

[44] M.A. Vasiliev, *Higher spin superalgebras in any dimension and their representations*, JHEP **12** (2004) 046 [hep-th/0404124] [INSPIRE].

[45] T. Hartman and A. Strominger, *Central Charge for* AdS_2 *Quantum Gravity*, JHEP **04** (2009) 026 [arXiv:0803.3621] [INSPIRE].

[46] M.A. Vasiliev, *Cubic interactions of bosonic higher spin gauge fields in* AdS_5, Nucl. Phys. **B 616** (2001) 106 [*Erratum ibid.* **B 652** (2003) 407] [hep-th/0106200] [INSPIRE].

[47] O.V. Shaynkman and M.A. Vasiliev, *Scalar field in any dimension from the higher spin gauge theory perspective*, Theor. Math. Phys. **123** (2000) 683 [hep-th/0003123] [INSPIRE].

[48] E.D. Skvortsov, *Gauge fields in (A)dS(d) within the unfolded approach: algebraic aspects*, JHEP **01** (2010) 106 [arXiv:0910.3334] [INSPIRE].

JHEP10(2014)122

[49] R. Jackiw, *Gauge theories for gravity on a line*, Theor. Math. Phys. **92** (1992) 979 [hep-th/9206093] [INSPIRE].

[50] A. Kapustin and N. Seiberg, *Coupling a QFT to a TQFT and Duality*, JHEP **04** (2014) 001 [arXiv:1401.0740] [INSPIRE].

[51] S.E. Konstein and M.A. Vasiliev, *Extended Higher Spin Superalgebras and Their Massless Representations*, Nucl. Phys. **B 331** (1990) 475 [INSPIRE].

[52] X. Bekaert, *Higher spin algebras as higher symmetries*, arXiv:0704.0898 [INSPIRE].

[53] N. Boulanger, D. Ponomarev, E.D. Skvortsov and M. Taronna, *On the uniqueness of higher-spin symmetries in AdS and CFT*, Int. J. Mod. Phys. **A 28** (2013) 1350162 [arXiv:1305.5180] [INSPIRE].

[54] F.A. Bais, T. Tjin and P. van Driel, *Covariantly coupled chiral algebras*, Nucl. Phys. **B 357** (1991) 632 [INSPIRE].

[55] R. Howe, *Transcending classical invariant theory*, *J. Amer. Math. Soc.* **3** (1989) 2.

[56] K.B. Alkalaev, *On manifestly sp(2) invariant formulation of quadratic higher spin Lagrangians*, JHEP **06** (2008) 081 [arXiv:0711.3639] [INSPIRE].

[57] E. Bergshoeff, B. de Wit and M.A. Vasiliev, *The Structure of the super-W(infinity) (lambda) algebra*, Nucl. Phys. **B 366** (1991) 315 [INSPIRE].

[58] E. Sezgin and P. Sundell, *Doubletons and* $5-D$ *higher spin gauge theory*, JHEP **09** (2001) 036 [hep-th/0105001] [INSPIRE].

[59] E. Sezgin and P. Sundell, $7-D$ *bosonic higher spin theory: Symmetry algebra and linearized constraints*, Nucl. Phys. **B 634** (2002) 120 [hep-th/0112100] [INSPIRE].

[60] K.B. Alkalaev and M.A. Vasiliev, *N=1 supersymmetric theory of higher spin gauge fields in* AdS_5 *at the cubic level*, Nucl. Phys. **B 655** (2003) 57 [hep-th/0206068] [INSPIRE].

[61] A. Sagnotti, E. Sezgin and P. Sundell, *On higher spins with a strong Sp(2,R) condition*, hep-th/0501156 [INSPIRE].

[62] K. Alkalaev, *FV-type action for* AdS_5 *mixed-symmetry fields*, JHEP **03** (2011) 031 [arXiv:1011.6109] [INSPIRE].

[63] M.A. Vasiliev, *Extended Higher Spin Superalgebras and Their Realizations in Terms of Quantum Operators*, *Fortsch. Phys.* **36** (1988) 33 [INSPIRE].

[64] R. Jackiw, *Gauge covariant conformal transformations*, Phys. Rev. Lett. **41** (1978) 1635 [INSPIRE].

[65] H. Afshar, A. Bagchi, R. Fareghbal, D. Grumiller and J. Rosseel, *Spin-3 Gravity in Three-Dimensional Flat Space*, Phys. Rev. Lett. **111** (2013) 121603 [arXiv:1307.4768] [INSPIRE].

[66] H.A. Gonzalez, J. Matulich, M. Pino and R. Troncoso, *Asymptotically flat spacetimes in three-dimensional higher spin gravity*, JHEP **09** (2013) 016 [arXiv:1307.5651] [INSPIRE].

[67] X. Bekaert and M. Grigoriev, *Higher order singletons, partially massless fields and their boundary values in the ambient approach*, Nucl. Phys. **B 876** (2013) 667 [arXiv:1305.0162] [INSPIRE].

[68] N. Boulanger, P. Sundell and M. Valenzuela, *Three-dimensional fractional-spin gravity*, JHEP **02** (2014) 052 [arXiv:1312.5700] [INSPIRE].

JHEP10(2014)122

[69] C.G. Callan Jr., S.B. Giddings, J.A. Harvey and A. Strominger, *Evanescent black holes*, *Phys. Rev.* **D 45** (1992) 1005 [hep-th/9111056] [INSPIRE].

[70] D. Cangemi and R. Jackiw, *Gauge invariant formulations of lineal gravity*, *Phys. Rev. Lett.* **69** (1992) 233 [hep-th/9203056] [INSPIRE].

[71] D. Cangemi and R. Jackiw, *Geometric gravitational forces on particles moving in a line*, *Phys. Lett.* **B 299** (1993) 24 [hep-th/9210036] [INSPIRE].