Published for SISSA by Springer

Received: *September 5, 2015*
Revised: *November 29, 2015*
Accepted: *December 7, 2015*
Published: *December 15, 2015*

Partially twisted superconformal M5 brane in R-symmetry gauge field backgrounds

Dongsu Bak[a] **and Andreas Gustavsson**[a,b]

[a] *Physics Department, University of Seoul, Seoul 130-743, Korea*

[b] *School of Physics, Korea Institute for Advanced Study, Seoul 130-012, Korea*

E-mail: dsbak@uos.ac.kr, agbrev@gmail.com

Abstract: We obtain the action for a curved superconformal abelian M5 brane with the background R-symmetry gauge field turned on. We then restrict ourselves to superconformal M5 brane on a sphere times flat Minkowski space. We choose R-symmetry SO(1, 4) instead of SO(5), which enables us to partially twist on Minkowski space and replace it by some curved Lorentzian manifold. We obtain M5 brane actions on $M_{1,1} \times S^4$ and $M_{1,2} \times S^3$ where actions and all fields, including the background gauge field, are real. Dimensional reduction along time gives real 5d SYM actions with nonabelian generalizations.

Keywords: Supersymmetric gauge theory, Field Theories in Higher Dimensions, Topological Field Theories, M-Theory

ArXiv ePrint: 1508.04496

doi:10.1007/JHEP12(2015)093

Contents

1 Introduction

A superconformal M5 brane can be put in a generic conformal supergravity background [1]. The corresponding supergravity background fields in the dimensionally reduced 5d SYM theory has been analyzed in [2] following the approach of [3]. Using this result, 5d SYM theories on $\mathbb{R}^3 \times S^2$ [4] and on $\mathbb{R} \times S^4$ [5] have been obtained. However the corresponding Lagrangian of the abelian M5 brane has not been obtained,[1] perhaps due to the belief that no such Lagrangian can be written down because of the selfdual tensor field in 6d. However, by also including the wrong chirality tensor field as a decoupled spectator field, we can write down a superconformal Lagrangian in 6d. But another reason that no 6d Lagrangian has been obtained in the literature might be the following. In the applications to the AGT correspondence [7] and the 3d-3d correspondence [4],[2] we like to put M5 brane on $\mathbb{R}^p \times S^{6-p}$ for $p = 2$ and $p = 3$ respectively, and then perform a partial topological twist with the SO(5) R symmetry that enables us to put the theory on $M_p \times S^{6-p}$ for a general p-manifold M_p. The theory being topological on M_p means that we can scale the size of M_p without affecting any observables in the theory. By taking the size to be small we obtain a dimensionally reduced SYM theory on S^{6-p}. By taking the size to be large we obtain a theory on M_p. These theories will be equivalent thanks to the topological property of the theory on M_p.

However, one obstacle in carrying out such a computation explicitly is that no M5 brane Lagrangian can exist in Euclidean signature with real fermions. If we consider the theory in Lorentzian signature, we should, for $p = 2$ consider the manifold $\mathbb{R}^{1,1} \times S^4$. However, as was mentioned in [9], we cannot twist this theory partially on $\mathbb{R}^{1,1}$ if the R symmetry group is SO(5).

In this paper we propose to solve this problem by instead taking the R symmetry group to be SO$(1, 4)$ [11, 12].[3] This enables us to twist an SO$(1, p-1)$ subgroup with the Lorentz group SO$(1, p-1)$ on $\mathbb{R}^{1,p-1}$. We may then put the theory on a general Lorentzian p-manifold $M_{1,p-1}$ times S^{6-p}.

For $p = 1, 2, 3$ we can find solutions for the background gauge potential, and the full M5 brane Lagrangian becomes real in Lorentzian signature. It is required that the bosonic part of the Lagrangian is real in order to have a unitarity of the theory [13]. What is problematic though, is that with SO$(1, 4)$ R symmetry we have an indefinite kinetic energy for the scalar fields. But this kind of problem might be cured by finding a suitable integration cycle where the path integral is convergent. For more details we refer to section 3.

We will also perform dimensional reduction along time. This will perhaps justify our choice of R symmetry group as SO$(1, 4)$ a bit further. After that we dimensionally reduce

[1] A related question was addressed in [8]. Here the abelian M5 brane Lagrangian was obtained on geometries of the form $\mathbb{R}^{1,1} \times M_4$ where a partial topological twist of Donaldson-Witten type was performed on M_4.

[2] The many original papers that proposed the 3d-3d correspondence can be found in the reference list of [4].

[3] We use a signature convention such that SO$(1, 4)$ refers to the group of transformations that leaves the metric diag$(-1, +1, +1, +1, +1)$ invariant. We then also refer to this space as $\mathbb{R}^{1,4}$ or as a space of signature $(1, 4)$.

JHEP12(2015)093

flat M5 brane with SO(1, 4) R symmetry along time, we find precisely the 5d SYM that has global symmetry $\mathrm{SO}(5)\times\mathrm{SO}(1,4)$ that also can be obtained by dimensionally reducing 10d SYM with SO(1, 9) global symmetry by reduction along time and 4 spatial directions. The latter approach has been used in for example [14] to derive a SYM Lagrangian on a four-sphere from 10d SYM with real fermions.

In this paper we will restrict ourselves to just turning on the supergravity background gauge field that is associated with the SO(1, 4) R symmetry. Thus we will put all the other supergravity fields to zero. Our restriction has the unfortunate limitation that we cannot consider squashed spheres as these require other background fields also being turned on. The AGT-like correspondences of course become much more interesting if one can include an additional squashing parameter in the correspondence. We plan to return to this problems in a future publication.

2 Abelian 6d theory with SO(1, 4) R symmetry group

In the introduction we have motivated why we like to study 6d (2, 0) theory with SO(1, 4) R symmetry group. This can be thought of as embedding a Lorentzian M5 brane into 11 dimensional space with signature (2, 9). Let us now work out the supersymmetry transformations assuming SO(1, 4) R-symmetry group. We start by considering M5 brane on flat $\mathbb{R}^{1,5}$. We use 11d gamma matrices that we split as Γ^M $(M = 0, \ldots, 5)$ and $\hat{\Gamma}^A$ $(A = 0', \ldots, 4')$ and define the 6d chirality matrix $\Gamma = \Gamma^{012345}$. The spinor and the supersymmetry parameter have opposite 6d chiralities. We choose the convention

$$\begin{aligned}\Gamma\psi &= \psi\\ \Gamma\epsilon &= -\epsilon\end{aligned}$$

The 11d Majorana conditions (or, equivalently, the 6d SO(1, 4)-Majorana conditions) for these chiral spinors read

$$\begin{aligned}\bar{\psi} &= \psi^T C\\ \bar{\epsilon} &= \epsilon^T C\end{aligned}$$

where $\bar{\psi} = \psi^\dagger\Gamma^0\hat{\Gamma}^{0'}$. We find that the following supersymmetry variations

$$\begin{aligned}\delta\phi^A &= \bar{\epsilon}\hat{\Gamma}^A\psi\\ \delta B_{MN} &= i\bar{\epsilon}\Gamma_{MN}\psi\\ \delta\psi &= -\frac{i}{12}\Gamma^{MNP}\epsilon H_{MNP} + \Gamma^M\hat{\Gamma}_A\epsilon\partial_M\phi^A\end{aligned}$$

close on-shell,

$$\begin{aligned}[][\delta_\eta, \delta_\epsilon]\phi^A &= -2\bar{\epsilon}\Gamma^P\eta\partial_P\phi^A\\ [\delta_\eta, \delta_\epsilon]B_{MN} &= -2\bar{\epsilon}\Gamma^P\eta H_{PMN}\\ [\delta_\eta, \delta_\epsilon]\psi &= -2\bar{\epsilon}\Gamma^P\eta\partial_P\psi + \frac{3}{4}\bar{\epsilon}\Gamma^P\eta\Gamma_P\Gamma^M\partial_M\psi - \frac{1}{4}\bar{\eta}\Gamma^M\hat{\Gamma}^A\epsilon\Gamma_M\hat{\Gamma}_A\Gamma^N\partial_N\psi\end{aligned}$$

JHEP12(2015)093

To obtain the closure relation for the fermion we have used the Fierz identity that we have collected in appendix D. For closure we must use the fermionic equation of motion

$$\Gamma^M \partial_M \psi = 0$$

Let us notice that

$$(\bar{\epsilon}\Gamma^M \eta)^* = -(-1)^{\frac{q(q+1)}{2}} \bar{\epsilon}\Gamma^M \eta$$

where q counts the number of timelike components in the R symmetry group $\mathrm{SO}(q, 5-q)$. In particular then, while we have that $\bar{\epsilon}\Gamma^M\eta$ is purely imaginary for SO(5) R symmetry, we find that $\bar{\epsilon}\Gamma^M\eta$ becomes real for SO(1,4) R symmetry. This explains why we do not get the usual factor of i in the closure relations, such as $\sim 2i\bar{\epsilon}\Gamma^M\eta\,\partial_M\phi^A$ as we get when the R symmetry is SO(5).

By using the 11d Majorana condition, one can see that $\delta\phi^A$ and δB_{MN} are real, and that the variation $\delta\psi$ again satisfies the 11d Majorana condition. We notice that the factors of i sit at different places compared to the more commonly used supersymmetry transformations for the (2,0) theory that has SO(5) R symmetry group.

As usual, from $\Gamma\epsilon = -\epsilon$, we can find that the gauge field part of the above supersymmetry variations can be also written in the form[4]

$$\begin{aligned}
\delta H^+_{MNP} &= \frac{i}{2}\bar{\epsilon}\Gamma^Q\Gamma_{MNP}\partial_Q\psi \\
\delta H^-_{MNP} &= 0 \\
\delta\psi &= -\frac{i}{12}\Gamma^{MNP}\epsilon H^+_{MNP} + \dots
\end{aligned}$$

where we define

$$H^\pm_{MNP} = \frac{1}{2}\left(H_{MNP} \pm \frac{1}{6}\epsilon_{MNP}{}^{UVW}H_{UVW}\right)$$

This means that H^-_{MNP} is not part of the tensor multiplet, but we include it in order to write down a neat supersymmetric Lagrangian, which is given by

$$\mathcal{L} = -\frac{1}{24}H^{MNP}H_{MNP} + \frac{1}{2}\partial^M\phi^A\partial_M\phi_A - \frac{1}{2}\bar{\psi}\Gamma^M\partial_M\psi$$

First we notice that the whole Lagrangian is real. In particular we have

$$(\bar{\psi}\Gamma^M\partial_M\psi)^\dagger = \bar{\psi}\Gamma^M\partial_M\psi$$

up to a boundary term produced by an integration by parts. Second, we notice that the gauge potential kinetic term and the scalar field kinetic term cannot both have the right sign simultaneously. However, for the kinetic term of the scalar fields, we also need to remember that the signature of the R symmetry group is SO(1,4) which means that it is never possible for all the five scalar fields to have the right sign of the kinetic term. It is therefore the most natural to assign the gauge potential the right sign kinetic term, and then $\phi^{a'}$ for $a' = 1', 2', 3', 4'$ will have the wrong sign kinetic term.

[4]The dots represent the scalar field part.

JHEP12(2015)093

2.1 SO(4,1) R symmetry group

For completeness, we also work out the supersymmetry variations with SO(4,1) R symmetry group, which corresponds to (5,6) signature in 11 dimensions. We have the supersymmetry variations

$$\begin{aligned}\delta\phi^A &= i\bar{\epsilon}\hat{\Gamma}^A\psi\\ \delta B_{MN} &= i\bar{\epsilon}\Gamma_{MN}\psi\\ \delta\psi &= \frac{1}{12}\Gamma^{MNP}\epsilon H_{MNP} + \Gamma^M\hat{\Gamma}_A\epsilon\partial_M\phi^A\end{aligned}$$

where the Dirac conjugation $\bar{\chi}$ is now defined by $\chi^\dagger\Gamma^0\hat{\Gamma}^{1'2'3'4'}$ in this SO(4,1) theory. By using the 11d Majorana condition in this signature, one can see that $\delta\phi^A$ and δB_{MN} are real, and that the variation $\delta\psi$ again satisfies the 11d Majorana condition. Closure relations are

$$\begin{aligned}[\delta_\eta,\delta_\epsilon]\phi^A &= -2i\bar{\epsilon}\Gamma^P\eta\partial_P\phi^A\\ [\delta_\eta,\delta_\epsilon]B_{MN} &= -2i\bar{\epsilon}\Gamma^P\eta H_{PMN}\\ [\delta_\eta,\delta_\epsilon]\psi &= -2i\bar{\epsilon}\Gamma^P\eta\partial_P\psi + \frac{3i}{4}\bar{\epsilon}\Gamma^P\eta\Gamma_P\Gamma^M\partial_M\psi - \frac{i}{4}\bar{\eta}\Gamma^M\hat{\Gamma}^A\epsilon\Gamma_M\hat{\Gamma}_A\Gamma^N\partial_N\psi\end{aligned}$$

That is, we have on-shell closure on the fermionic equation of motion

$$\Gamma^M\partial_M\psi = 0$$

The supersymmetric Lagrangian is

$$\mathcal{L} = -\frac{1}{24}H^{MNP}H_{MNP} - \frac{1}{2}\partial^M\phi^A\partial_M\phi_A + \frac{i}{2}\bar{\psi}\Gamma^M\partial_M\psi$$

This theory can be obtained from the above theory in signature (2,9) by the following map,

$$\begin{aligned}\Gamma^M &\to \Gamma^M\\ \hat{\Gamma}^A &\to -i\hat{\Gamma}^A\\ g_{AB} &\to -g_{AB}\\ \psi &\to i\psi\\ \epsilon &\to \epsilon\\ C &\to -iC\end{aligned}$$

together with $\bar{\psi}\to\bar{\psi}$ and $\bar{\epsilon}\to -i\bar{\epsilon}$ which follow from the definitions of the Dirac conjugation and the Gamma matrix transformation rule. Thus the SO(4,1) twisted and the time reduced theories are equivalent to those from the (2,9) theory.

JHEP12(2015)093

2.2 SO(5) R symmetry group

The supersymmetry variations and the Lagrangian for the usual Lorentzian M5 brane with SO(5) R symmetry are in our conventions given by

$$
\begin{aligned}
\delta B_{MN} &= i\bar{\epsilon}\Gamma_{MN}\psi \\
\delta\phi^A &= i\bar{\epsilon}\hat{\Gamma}^A\psi \\
\delta\psi &= \frac{1}{12}\Gamma^{MNP}\epsilon H_{MNP} + \Gamma^M\hat{\Gamma}_A\epsilon\partial_M\phi^A
\end{aligned}
$$

and

$$
\mathcal{L} = -\frac{1}{24}H^{MNP}H_{MNP} - \frac{1}{2}\partial^M\phi^A\partial_M\phi_A + \frac{i}{2}\bar{\psi}\Gamma^M\partial_M\psi
$$

Although these variations and the Lagrangian are on the same form as for the case of SO(4,1) R symmetry above, there is no simple relation between the SO(1,4) or SO(4,1) theories and the usual SO(5) theory since there is no natural map from the Dirac conjugate $\bar{\psi} = \psi^\dagger\Gamma^0$ to the Dirac conjugates of the SO(1,4) or SO(4,1) theories.

3 Unitarity

As we have changed signatures, it is important to check unitarity of the theory. To illustrate unitarity, we follow the arguments in [13]. Let us consider some Lagrangian

$$
L = \frac{1}{2}g_{ij}\dot{q}^i\dot{q}^j + ih_{ij}\psi^{*i}\dot{\psi}^j
$$

where g_{ij} and h_{ij} are invertible matrices with inverses g^{ij} and h^{ij}. This system can be quantized by imposing the commutation relations

$$
[q^i, p_j] = i\hbar\delta^i_j \tag{3.1}
$$

$$
\{\psi^i, \psi^{j\dagger}\} = \hbar h^{ij} \tag{3.2}
$$

where p_i are the conjugate momenta of q^i. These have the unitary representations $p_i = -i\hbar\partial/\partial q^i$ irrespectively of the signature of g_{ij}, in the sense that the translation operators $U = \exp iL^i p_i$ are unitary for any real distances L^i, provided the bosonic part of the Lagrangian is real. However, if g_{ij} is indefinite the energy is unbounded from below. For the fermions the situation is the opposite; we see that h^{ij} has to be positive definite to have a unitarity representation of (3.2). On the other hand we do not encounter negative energy states by filling up the Dirac sea.

Let us now consider our theory. The bosonic part of our Lagrangian is real although we have indefinite g_{ij}. Hence the bosonic part describes a unitary theory. The fermionic part does not however. Here we have

$$
\{\psi, \psi^\dagger\} \sim \hbar\hat{\Gamma}^0
$$

which is indefinite. Hence our 6d theory is non-unitary. This happens for both SO(1, 4) and SO(4, 1) R symmetry. On the other hand, if the R symmetry is SO(5) the 6d theory is unitary since then we have

$$\{\psi, \psi^\dagger\} \sim \hbar \mathbb{I}$$

where $\mathbb{I}$ denotes the 16×16 identity matrix.

The microscopic structure of an 11d theory is of course unclear, but it seems reasonable to think that such a theory would have two time-directions and global symmetry group SO(2, 9), that is broken by an embedding of M5 brane down to SO(1, 5) × SO(1, 4). But if we have two time-directions, then time-evolution will be rather different from what we are used to and a new concept should replace that of unitarity, which is based on time evolution with just one time direction.

Since we are not aware of any formalism with two time directions, let us stick to one time direction. Here we can also find that a unitary theory may appear to be non-unitary if we have one time direction and one space direction, if we interpret the space direction as 'time'. To illustrate this, let us consider an action of a 2-component spinor with σ^3 the third Pauli matrix,

$$S = \int d^2x \psi^\dagger (i\partial_0 + \sigma^3 \partial_1)\psi$$

If we let x^0 play the role of time, we quantize the theory by imposing

$$\{\psi^\dagger, \psi\} = \hbar$$

and we have a unitary representation. But we can also quantize this theory by declaring that x^1 is the direction of time evolution, in which case we shall impose the commutation relation

$$\{\psi^\dagger, \psi\} = \sigma^3$$

which has no unitary representation as the matrix σ^3 is indefinite. One might now speculate that our non-unitary M5 brane theory might appear to be non-unitary for a similar reason that is related to the fact that one time direction of the 11d theory is outside the worldvolume of the M5 brane.

More concrete statements can be made related to unitarity if we reduce our M5 brane theory along the world-volume time direction. This dimensional reduction gives rise to 5d SYM theory with global symmetry SO(5) × SO(1, 4) and can be exactly mapped to the 5d SYM theory that one would also obtain by reducing 10d SYM theory with SO(1, 9) Lorentz symmetry, along time and four space directions. We present the map in full detail in appendix B. As the 10d SYM theory is a unitary theory and the dimensional reduction is a physically consistent procedure, we conclude that there is no problem with our M5 brane theory with SO(1, 4) R symmetry after this theory has been reduced along the time direction down to 5d SYM theory.

Let us finally comment on the issue of convergence of the path integral. If the R symmetry group is $\mathrm{SO}(1,4)$, then we have the wrong sign of the kinetic term in the Lagrangian for one of the scalar fields, say ϕ^0. We may Wick rotate this into $i\phi^0$ to get the right sign kinetic term. We can indeed Wick rotate the R symmetry $\mathrm{SO}(1,4)$ including the fermionic part, into the $\mathrm{SO}(5)$ R symmetry and get the usual M5 brane theory. But we can also carry on with our $\mathrm{SO}(1,4)$ R symmetry, and perform some partial twist of say an $\mathrm{SO}(1,p-1)$ subgroup of the R symmetry where $p = 2,3,\ldots$. In this case, the R symmetry will be reduced by the twist with the Lorentz group to $\mathrm{SO}(5-p)$. Nevertheless, we can Wick rotate ϕ^0 into $i\phi^0$ and get the right sign kinetic term. If we do that after the twist, then we get a different theory that can not be related to the familiar M5 brane theory with $\mathrm{SO}(5)$ R symmetry.

4 Superconformal symmetry

The Lagrangian has not only the usual Poincare supersymmetry, but also a special conformal supersymmetry. We can relax the condition that the supersymmetry parameter is constant, to the condition that it satisfies the superconformal Killing spinor equation [10]

$$D_M\epsilon = \frac{1}{6}\Gamma_M\Gamma^N D_N\epsilon$$

Once we have done that, we can also admit more general curved six-manifolds where this equation has some solution. The Ricci curvature scalar may be defined by the equation

$$\Gamma^{MN}D_M D_N\epsilon = -\frac{1}{4}R\,\epsilon$$

and $D_M = \partial_M + \omega_M$ is the covariant derivative where ω_M is the spin connection.

The Lagrangian is now given by

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$$

where

$$\begin{aligned}
\mathcal{L}_0 &= -\frac{1}{24}H_{MNP}^2 + \frac{1}{2}(D_M\phi_A)^2 - \frac{1}{2}\bar{\psi}\Gamma^M D_M\psi \\
\mathcal{L}_1 &= \frac{R}{10}\phi^A\phi_A
\end{aligned}$$

The superconformal symmetry variations can be expressed as

$$\delta = \delta_0 + \delta_1$$

where

$$\begin{aligned}
\delta_0\phi^A &= \bar{\epsilon}\hat{\Gamma}^A\psi \\
\delta_0 B_{MN} &= i\bar{\epsilon}\Gamma_{MN}\psi \\
\delta_0\psi &= -\frac{i}{12}\Gamma^{MNP}\epsilon H_{MNP} + \Gamma^M\hat{\Gamma}_A\epsilon D_M\phi^A
\end{aligned}$$

and

$$\begin{aligned}
\delta_1\phi^A &= 0\\
\delta_1 B_{MN} &= 0\\
\delta_1\psi &= \frac{2}{3}\Gamma^M\hat{\Gamma}_A(D_M\epsilon)\phi^A
\end{aligned}$$

When we vary the Lagrangian, we find it most convenient to bring the variation into the following form,

$$\delta\mathcal{L} = D_M b^M + \frac{1}{4}D_M H^{MNP}\delta B_{NP} - D_M D^M\phi^A\delta\phi_A + \frac{R}{5}\phi^A\delta\phi_A - \bar{\psi}\Gamma^M D_M\delta\psi$$

where the boundary term

$$\begin{aligned}
b^M &= -\frac{1}{4}H^{MNP}\delta B_{NP} + D^M\phi^A\delta\phi_A + \frac{1}{2}\bar{\psi}\Gamma^M\delta\psi\\
&= -\frac{i}{24}\bar{\epsilon}\Gamma^M\Gamma^{PQR}\psi H_{PQR} + \frac{1}{2}\bar{\epsilon}\hat{\Gamma}_A\Gamma^M\Gamma^P\psi D_P\phi^A - 2(D^M\bar{\epsilon})\hat{\Gamma}_A\psi\phi^A
\end{aligned}$$

is non-vanishing if the M5 brane has a boundary. If there is no boundary, then the variation is vanishing if the supersymmetry parameter ϵ is a superconformal Killing spinor. We then find the following superconformal variations,

$$\begin{aligned}
\delta_0\mathcal{L}_0 &= 4\bar{\psi}\hat{\Gamma}_A(D^N\epsilon)D_N\phi^A\\
\delta_1\mathcal{L}_0 &= -4\bar{\psi}\hat{\Gamma}_A(D^N\epsilon)D_N\phi^A - \frac{R}{5}\phi^A\bar{\epsilon}\hat{\Gamma}_A\psi\\
\delta_0\mathcal{L}_1 &= \frac{R}{5}\phi^A\bar{\epsilon}\hat{\Gamma}_A\psi\\
\delta_1\mathcal{L}_1 &= 0
\end{aligned}$$

where we have used the conformal Killing spinor equation and ignore the total derivative contribution $D_M b^M$. Hence $\delta\mathcal{L} = \delta_0\mathcal{L}_0 + \delta_1\mathcal{L}_0 + \delta_0\mathcal{L}_1 + \delta_1\mathcal{L}_1 = 0$ up to the total derivatives. If we then replace ϵ by $f\epsilon$ where f is a function on spacetime, then we pick up a variation that is proportional to $\partial_M f$, which is again up to total derivatives. From this we can read off the supercurrent. We only need to consider the last term since this is the only term that can produce something $\sim \partial_M f$. We find that

$$\delta\mathcal{L} = j^M\partial_M f$$

where

$$j^M = -\frac{i}{12}\bar{\epsilon}\Gamma^{PQR}\Gamma^M\psi H_{PQR} + \bar{\epsilon}\hat{\Gamma}_A\Gamma^P\Gamma^M\psi D_P\phi^A + 4(D^M\bar{\epsilon})\hat{\Gamma}_A\psi\phi^A$$

For this computation, we may use the variation

$$\delta_1\psi = \frac{2}{3}\Gamma^M\hat{\Gamma}_A f(D_M\epsilon)\phi^A$$

When the equations of motion are satisfied, we will have that the action is stationary under any variation. Hence

$$0 = \int d^6x\sqrt{g}\delta\mathcal{L} = \int j^M D_M f = -\int D_M j^M f$$

and since f is arbitrary, it follows that $D_M j^M = 0$.

4.1 Coupling to background R symmetry gauge potential

We introduce a background gauge potential $A_M{}^A{}_B$ and corresponding covariant derivatives

$$\begin{aligned}
D_M\phi^A &= \nabla_M\phi^A + A_M{}^A{}_B\phi^B \\
D_M\psi &= \nabla_M\psi + \frac{1}{4}A_{MAB}\hat\Gamma^{AB}\psi
\end{aligned}$$

Here ∇_M is the covariant derivative of the background geometry.

We can now find a superconformal Lagrangian by imposing the following Weyl projection

$$\begin{aligned}
\frac{1}{2}\Gamma^{MN}\hat\Gamma_A\epsilon F_{MN}{}^A{}_B &= \hat\Gamma_A\,\epsilon\,P^A{}_B \\
P_{AB} &= P_{BA}
\end{aligned}$$

From this, it follows that

$$\begin{aligned}
\frac{1}{2}\Gamma^{MN}\hat\Gamma^{AB}\,\epsilon F_{MNAB} &= -\epsilon\,P^A{}_A \\
\Gamma^{MN}D_M D_N\epsilon &= -\frac{1}{4}(R+P)\,\epsilon
\end{aligned}$$

Here we define

$$P = P^A{}_A$$

After we gauge the R symmetry, we find new terms in the variation of the Lagrangian

$$\begin{aligned}
\delta_0\mathcal{L}_0 &= \cdots - \frac{1}{2}\bar\psi\Gamma^{MN}\hat\Gamma^A\epsilon F_{MNAB}\phi^B = \cdots - \bar\psi\hat\Gamma^A\epsilon P_{AB}\phi^B \\
\delta_1\mathcal{L}_0 &= \cdots - \frac{4}{5}\bar\psi\hat\Gamma_A(\Gamma^{MN}D_M D_N\epsilon)\phi^A = \cdots + \frac{P}{5}\bar\psi\hat\Gamma_A\epsilon\phi^A
\end{aligned}$$

where $\cdots$ are terms of the same form as we had before. We cancel these terms by adding the following terms

$$\Delta\mathcal{L} = \frac{1}{2}\left(\frac{1}{5}\eta_{AB}P - P_{AB}\right)\phi^A\phi^B$$

to the Lagrangian.

4.2 Dimensional reduction along time to 5d SYM

We assume six-manifold of the form $\mathbb{R} \times M_5$ with time along $\mathbb{R}$, and with a rather generic R symmetry gauge field. The natural split of the 6d conformal Killing spinor equation for this analysis will be to write $6 = 1 + 5$, which means that we will assume the following equations

$$\begin{aligned}\Gamma^0 D_0 \epsilon &= \frac{1}{5}\Gamma^m D_m \epsilon \\ D_m \epsilon &= \frac{1}{5}\Gamma_m \Gamma^n D_n \epsilon\end{aligned}$$

where we also put

$$\partial_0 \epsilon = 0$$

in order to preserve supersymmetry under the dimensional reduction.

By dimensional reduction along time, we get the following Lagrangian

$$\begin{aligned}\mathcal{L}_0 &= \frac{1}{4}F_{mn}^2 + \frac{1}{2}(D_m \phi_A)^2 - \frac{1}{4}[\phi_A, \phi_B]^2 - \frac{1}{2}\bar{\psi}\Gamma^m D_m \psi - \frac{1}{2}\bar{\psi}\Gamma^0 \hat{\Gamma}^A [\phi_A, \psi] \\ \mathcal{L}_1 &= -\frac{1}{2}(D_0 \phi_A)^2 - \frac{1}{2}\bar{\psi}\Gamma^0 D_0 \psi + \frac{1}{2}M_{AB}\phi^A \phi^B \\ \mathcal{L}_2 &= \frac{i}{6}\epsilon^{ABCDE} A_{0AB} \phi_C [\phi_D, \phi_E]\end{aligned}$$

where the mass matrix is given by

$$M_{AB} = \frac{1}{5}\eta_{AB}(R + P) - P_{AB}$$

The action is invariant under

$$\begin{aligned}\delta \phi_A &= \bar{\epsilon}\hat{\Gamma}_A \psi \\ \delta A_m &= i\bar{\epsilon}\Gamma_m \Gamma_0 \psi \\ \delta \psi &= -\frac{i}{2}\Gamma^{mn}\Gamma^0 \epsilon F_{mn} + \Gamma^m \hat{\Gamma}_A \epsilon D_m \phi^A - \frac{1}{2}\hat{\Gamma}^{AB}\Gamma^0 [\phi_A, \phi_B] \\ &\quad + \Gamma^0 \hat{\Gamma}_A \epsilon D_0 \phi^A + 4\Gamma^0 \hat{\Gamma}^A D_0 \epsilon \phi^A\end{aligned}$$

To check supersymmetry, we only need to check this for the nonabelian type of terms that involve the curvature corrections. Collecting all such terms, we find the following contributions

$$\begin{aligned}\delta \mathcal{L}_0 &= -\frac{3}{2}\bar{\psi}\hat{\Gamma}^{AB} D_0 \epsilon [\phi_A, \phi_B] - \bar{\psi}\hat{\Gamma}^{AB}\epsilon [\phi_A, D_0 \phi_B] - \bar{\psi}\epsilon [\phi^A, D_0 \phi_A] \\ \delta \mathcal{L}_1 &= -\frac{1}{2}\bar{\psi}\hat{\Gamma}^{AB} D_0 \epsilon [\phi_A, \phi_B] - \bar{\psi}\hat{\Gamma}^{AB}\epsilon [\phi_A, D_0 \phi_B]\end{aligned}$$

Then we note

$$\begin{aligned}\hat{\Gamma}^{AB}\hat{\Gamma}^{CD} &= -2\eta^{AB,CD} + 4\eta^{BC}\hat{\Gamma}^{AD} + \hat{\Gamma}^{ABCD} \\ D_0 \epsilon &= \frac{1}{4}\hat{\Gamma}^{AB}\epsilon A_{0AB}\end{aligned}$$

and we get

$$\delta(\mathcal{L}_0+\mathcal{L}_1) = -\frac{1}{2}\bar{\psi}\hat{\Gamma}^{ABCD}\epsilon A_{0CD}[\phi_A,\phi_B] = -\delta\mathcal{L}_2$$

5 Summary

The M5 brane Lagrangian is given by

$$\mathcal{L} = \mathcal{L}_0+\mathcal{L}_1$$

where

$$\begin{aligned}\mathcal{L}_0 &= -\frac{1}{24}H^{MNP}H_{MNP}+\frac{1}{2}\nabla^M\phi^A\nabla_M\phi_A-\frac{1}{2}\bar{\psi}\Gamma^M\nabla_M\psi,\\ \mathcal{L}_1 &= A^M_{AB}\phi^B\partial_M\phi^A+\frac{1}{2}M_{AB}\phi^A\phi^B-\frac{1}{8}\bar{\psi}\Gamma^M\hat{\Gamma}^{AB}\psi A_{MAB}\end{aligned}$$

where ∇_M is the covariant derivative of the background geometry and

$$M_{AB} = \frac{1}{5}\eta_{AB}(P+R)-P_{AB}+A_M{}^C{}_AA^M{}_{CB}$$

We have the superconformal transformations

$$\begin{aligned}\delta\phi_A &= \bar{\epsilon}\hat{\Gamma}_A\psi\\ \delta B_{MN} &= i\bar{\epsilon}\Gamma_{MN}\psi\\ \delta\psi &= -\frac{i}{12}\Gamma^{MNP}\epsilon H_{MNP}+\Gamma^M\hat{\Gamma}^A\epsilon\partial_M\phi_A+\frac{1}{p}\Gamma^\mu\hat{\Gamma}_A\hat{\Gamma}^{BC}\epsilon A_{\mu BC}\phi^A+\Gamma^\mu\hat{\Gamma}^A\epsilon A_{\mu AB}\phi^B\end{aligned}$$

where P_{AB} is a symmetric tensor that we deduce from the curvature of the R-symmetry connection through the Weyl projection

$$\frac{1}{2}\Gamma^{MN}\hat{\Gamma}_A\epsilon F_{MN}{}^A{}_B = \hat{\Gamma}_A\epsilon P^A{}_B$$

It would be interesting to see whether one can give P_{AB} a geometric interpretation, perhaps as the Ricci tensor in normal directions to the M5 brane.

By dimensional reduction along time, we can also find a nonabelian generalization

$$\begin{aligned}\mathcal{L}_0 &= \mathrm{tr}\bigg(\frac{1}{4}F^{mn}F_{mn}+\frac{1}{2}\nabla^m\phi^A\nabla_m\phi_A-\frac{1}{4}[\phi^A,\phi^B][\phi_A,\phi_B]\\ &\qquad -\frac{1}{2}\bar{\psi}\Gamma^m\nabla_m\psi-\frac{1}{2}\bar{\psi}\Gamma^0\hat{\Gamma}^A\psi[\phi_A,\psi]\bigg)\\ \mathcal{L}_1 &= \mathrm{tr}\bigg(A^M_{AB}\phi^B\partial_M\phi^A+\frac{1}{2}M_{AB}\phi^A\phi^B-\frac{1}{8}\bar{\psi}\Gamma^M\hat{\Gamma}^{AB}\psi A_{MAB}\\ &\qquad +\frac{i}{2}\epsilon^{ABCDE}A_{0AB}\phi_C[\phi_D,\phi_E]\bigg)\end{aligned}$$

JHEP12(2015)093

6 Six-manifolds on the form $\mathbb{R}^{1,p-1} \times S^{6-p}$

We will now restrict ourselves to six-manifolds on the form $\mathbb{R}^{1,p-1} \times S^{6-p}$ where p can take any of the values $p = 1,2,3,4,5,6$. We will subsequently perform a partial topological twist along $\mathbb{R}^{1,p-1}$, although for $p = 1$ this twist cannot be done since the Lorentz group on $\mathbb{R}$ is rather trivial. For our M5 brane theory on $\mathbb{R}^{1,5}$ we have deliberately chosen the global symmetry group $SO(1,5) \times SO(1,4)$. If we break this symmetry down to $SO(1,p-1) \times SO(6-p) \times SO(1,p-1) \times SO(5-p)$, we can perform a partial twist and identify the two $SO(1,p-1)$ subgroups and declare that the diagonal subgroup of these, times $SO(6-p)$, is the new twisted Lorentz group. Thus after the twist, we have the global symmetry $SO(1,p-1)' \times SO(6-p) \times SO(5-p)_R$. We then first need how the M5 brane spinor in the representation $(4';4)$ of $SO(1,5) \times SO(1,4)$ transforms under the subgroups for the various values of p. Here we denote by a prime as in $4'$ the anti-Weyl representation. The supersymmetry parameter is subject to the anti-Weyl projection $\Gamma\epsilon = -\epsilon$. After the split we find the following representations

$$\begin{aligned}
p &= 1 \qquad (4;4)\\
p &= 2 \qquad \left(2_{-\frac{i}{2}} \oplus 2'_{+\frac{i}{2}}; 2_{\frac{i}{2}} \oplus 2_{-\frac{i}{2}}\right)\\
p &= 3 \qquad \left(2,2;2_{+\frac{1}{2}}\right) \oplus \left(2,2;2_{-\frac{1}{2}}\right)\\
p &= 4 \qquad \left(2_{-\frac{1}{2}} \oplus 2'_{+\frac{1}{2}}, 2 \oplus 2'\right)\\
p &= 5 \qquad (4;4)
\end{aligned}$$

where subscripts denote either $SO(1,1)$ or $SO(2)$ charges respectively. Our convention for these charges are $Q^{MN} = -\frac{i}{2}\Gamma^{MN}$ so that for instance $Q^{01} = \pm\frac{i}{2}$ and $Q^{45} = \pm\frac{1}{2}$. After the identification of the $SO(1,p-1)$ groups, these representations become

$$\begin{aligned}
p &= 1 \qquad (4;4)\\
p &= 2 \qquad (2,2)_0 \oplus (2',2)_0 \oplus (2,2)_{-i} \oplus (2',2)_{+i}\\
p &= 3 \qquad (1,2)_{+\frac{1}{2}} \oplus (3,2)_{+\frac{1}{2}} \oplus (1,2)_{-\frac{1}{2}} \oplus (3,2)_{\frac{1}{2}}\\
p &= 4 \qquad 1_{-\frac{1}{2}} \oplus 3^+_{-\frac{1}{2}} \oplus 4_{-\frac{1}{2}} \oplus 4_{+\frac{1}{2}} \oplus 3^-_{+\frac{1}{2}} \oplus 1_{+\frac{1}{2}}\\
p &= 5 \qquad 1 \oplus 5 \oplus 10
\end{aligned}$$

Here 3^+ refers to a selfdual two-form of $SO(1,3)$. Let us turn to the Weyl projections for the singlet supercharges. First we have the 6d Weyl projection

$$\Gamma^{01}\Gamma^{23}\Gamma^{45}\epsilon = -\epsilon$$

For $p = 2$ we have the singlet representations $(2,2)_0 \oplus (2',2)_0$ i.e. neutral under $SO(1,1)$. For these representations we have

$$\Gamma^{01}\hat{\Gamma}_{0'1'}\epsilon = \epsilon$$

For $p = 3$ we have the singlet representations $(1,2)_{+\frac{1}{2}} \oplus (1,2)_{-\frac{1}{2}}$ i.e. singlets under $\mathrm{SO}(1,2)$. For these representations we have

$$\Gamma^{01}\hat{\Gamma}_{0'1'}\epsilon = \epsilon$$
$$\Gamma^{12}\hat{\Gamma}_{1'2}\epsilon = \epsilon$$

These two projections project onto the singlet state in the tensor product representation of two spin-1/2 representations of $\mathrm{SO}(1,2)$. With the gamma matrix representation as below, these two projections amount to

$$\begin{aligned}(\sigma^3)^{s_0}{}_{s_0'}(\sigma^3)^{t_0}{}_{t_0'}\eta^{s_0't_0'} &= -\eta^{s_0t_0}\\ (\sigma^2)^{s_0}{}_{s_0'}(\sigma^2)^{t_0}{}_{t_0'}\eta^{s_0't_0'} &= -\eta^{s_0t_0}\end{aligned}$$

The first projection picks states with spins $s_0 + t_0 = 0$, that is either $|+-\rangle$ or $|-+\rangle$. Then the second projection projects out the even linear combination $|+-\rangle + |-+\rangle$ leaving us with the singlet state $|+-\rangle - |-+\rangle$ of $\mathrm{SO}(1,2)$. In other words, $\eta^{s_0t_0} = \epsilon^{s_0t_0}\eta$ where $\epsilon^{s_0t_0}$ is the antisymmetric tensor with $\epsilon^{+-} = 1$. This is why we chose the notation η for the supersymmetry parameter, in order to not confuse it with the antisymmetric tensor.

After having performed the partial topological twist, we may put the theory on $M_{1,p-1} \times S^{6-p}$ where $M_{1,p-1}$ can be any Lorentzian p-dimensional manifold, while preserving a certain amount of supersymmetry. For $p = 2$ this will then have applications to the AGT correspondence relating SYM theory on S^4 to Toda theory on $M_{1,1}$. For $p = 3$ we should expect to find the 3d-3d correspondence with a complex Chern-Simons theory living on $M_{1,2}$. For $p = 5$ we have a trivial circle reduction from 6d down to 5d SYM and $p = 6$ is flat M5 brane on $\mathbb{R}^{1,5}$. The case $p = 1$ has been considered in [6] and in many subsequent papers.

Let us now begin the detailed computations. We split the 6d vector index $M = (\mu, i)$ where μ lives on $\mathbb{R}^{1,p-1}$ (and more generally on $M_{1,p-1}$ after the twist) and i lives on S^{6-p}. We assume that the background gauge field has no components along S^{6-p},

$$A_i = 0$$

and we require the 6d conformal Killing spinor equation holds along with the conditions that the supersymmetry parameter is constant on $\mathbb{R}^{1,p-1}$,

$$\partial_\mu\epsilon = 0$$

This implies that

$$\begin{aligned}\Gamma^\mu D_\mu\epsilon &= \frac{p}{6-p}\Gamma^i D_i\epsilon\\ D_i\epsilon &= \frac{1}{6-p}\Gamma_i\Gamma^j D_j\epsilon\\ D_\mu\epsilon &= \frac{1}{p}\Gamma_\mu\Gamma^\nu D_\nu\epsilon\end{aligned}$$

and, for $p = 2, 3, 4$,

$$P = -\frac{p(p-1)}{(6-p)(5-p)} R$$

where we have

$$\begin{aligned} \Gamma^{\mu\nu} D_\mu D_\nu \epsilon &= \frac{1}{8} \Gamma^{\mu\nu} \hat{\Gamma}^{AB} \epsilon F_{\mu\nu AB} = -\frac{1}{4} P \epsilon \\ \Gamma^{ij} D_i D_j \epsilon &= -\frac{1}{4} R \epsilon \end{aligned} \tag{6.1}$$

Let us comment that once we put $\partial_\mu \epsilon = 0$ we descend to an ordinary Killing spinor equation on M_{6-p}

$$D_i \epsilon = \frac{1}{4p} \Gamma_i \Gamma^\mu \hat{\Gamma}^{AB} \epsilon A_{\mu AB}$$

For $p = 1$ we may instead use the relation

$$D_0 D^0 \epsilon = +\frac{1}{80} R \epsilon$$

to determine $A_{0,AB}$

We have the curvature condition

$$\frac{1}{2} \Gamma^{\mu\nu} \hat{\Gamma}^{AB} \epsilon F_{\mu\nu AB} = -P\epsilon$$

Assuming that $p = 2, 3, 4$ we can solve this equation as

$$\begin{aligned} F_{\mu\nu}^{\mu'\nu'} &= -\frac{2P}{p(p-1)} \delta_{\mu\nu}^{\mu'\nu'} \\ F_{\mu\nu}^{ab} &= 0 \\ F_{\mu\nu}^{\mu' a} &= 0 \end{aligned}$$

if we imposing the Weyl projection

$$\frac{1}{p(p-1)} \Gamma^{\mu\nu} \hat{\Gamma}_{\mu'\nu'} \epsilon = \epsilon \tag{6.2}$$

We find that if we make the assumptions we make, then the curvature R must be constant, and it leads us to consider manifolds on the form $\mathbb{R}^{1,p-1} \times S^{6-p}$. If r denotes the radius of S^{6-p}, then we have

$$\begin{aligned} R &= \frac{(6-p)(5-p)}{r^2} \\ P &= -\frac{p(p-1)}{r^2} \end{aligned}$$

We further find that

$$P^{\mu'}{}_{\nu'} = -\frac{p-1}{r^2} \delta^{\mu'}_{\nu'}$$

We now proceed to solve the conformal Killing spinor equation on $\mathbb{R}^{1,p-1}$ with respect to the background gauge field. To this end, it is convenient to introduce the notations

$$X_\mu = \frac{1}{4}\hat{\Gamma}^{AB}A_{\mu AB}$$
$$Y_\mu = X_\mu\epsilon$$

The equation we have to solve then reads

$$Y_\mu = \frac{1}{p}\Gamma_\mu\Gamma^\nu Y_\nu$$

For $p \neq 1$, we can rewrite this in the form

$$Y_\mu = \frac{1}{p-1}\Gamma_\mu{}^\nu Y_\nu \tag{6.3}$$

We solve this iteratively in p. If we know the solution for p, then we can construct the solution for $p+1$. For $p+1$, we have the equations

$$Y_\mu = \frac{1}{p}\Gamma_\mu{}^\nu Y_\nu + \frac{1}{p}\Gamma_\mu{}^p Y_p \tag{6.4}$$
$$Y_p = \frac{1}{p}\Gamma_p{}^\mu Y_\mu \tag{6.5}$$

Inserting (6.5) into (6.4), we find the equation (6.3). Let us now take $p = 2$ which is the lowest value of p for which the conformal Killing spinor on $\mathbb{R}^{1,p-1}$ is nontrivial. For $p = 2$ we get

$$Y_\mu = \Gamma_\mu{}^\nu Y_\nu$$

By induction we then find that the most general solution for general p can be expressed as

$$Y_\mu = \Gamma_\mu{}^p Y_p \tag{6.6}$$

for $\mu = 0, \cdots, p-1$.

We also have to satisfy the condition that comes from the curvature by commuting two covariant derivatives as in equation (6.1) that amounts to the condition

$$\Gamma^{\mu\nu}[X_\mu, X_\nu]\epsilon = -\frac{1}{2}P\,\epsilon \tag{6.7}$$

We will now proceed to solve the equations (6.6) and (6.7) while imposing the Weyl projection in (6.2) for various values on p.

6.1 M5 brane on $\mathbb{R}^{1,0} \times S^5$

For $p = 1$ we find the solution

$$A_{0,ab} = \left(\frac{1}{2r} - \lambda\right)\epsilon_{ab}$$
$$A_{0,a'b'} = \left(\frac{1}{2r} + \lambda\right)\epsilon_{a'b'}$$

where $a = 1', 2'$ and $a' = 3', 4'$. These solutions are valid only if we impose the projection

$$\hat{\Gamma}^{1'2'3'4'}\epsilon = -\epsilon$$

unless $\lambda = \pm\frac{1}{2r}$ when this projection is not necessary. The Lagrangian is

$$\begin{aligned}\mathcal{L} = \mathcal{L}_0 &+ \left(\frac{1}{2r} - \lambda\right)\epsilon_{ab}\phi^a\partial_0\phi^b + \left(\frac{1}{2r} + \lambda\right)\epsilon_{a'b'}\phi^{a'}\partial_0\phi^{b'} + \left(\frac{15}{8r^2} - \frac{\lambda^2}{2}\right)\left(\phi_a\phi^a + \phi_{a'}\phi^{a'}\right)\\ &+\frac{\lambda}{2r}\left(\phi_a\phi^a - \phi_{a'}\phi^{a'}\right) + \frac{2}{r^2}\phi_{0'}\phi^{0'} - \frac{1}{4r}\bar{\psi}^-\Gamma^0\hat{\Gamma}^{1'2'}\psi^- + \frac{\lambda}{2}\bar{\psi}^+\Gamma^0\hat{\Gamma}^{1'2'}\psi^+\end{aligned}$$

where $\psi^\pm = \frac{1}{2}\left(1 \pm \hat{\Gamma}^{1'2'3'4'}\right)\psi$.

6.2 M5 brane on $\mathbb{R}^{1,1} \times S^4$

For $p = 2$ we find the solution

$$A_{\mu,\nu'4'} = \frac{1}{r}\epsilon_{\mu\nu'}$$

where $\epsilon_{01'} = 1$ and antisymmetric, in the sense that $\epsilon_{10'} = -1$. The Weyl projection is

$$\Gamma^{01}\hat{\Gamma}_{0'1'}\epsilon = \epsilon$$

The M5 brane Lagrangian is

$$\mathcal{L} = \mathcal{L}_0 + \frac{2}{r}\epsilon^{\mu\nu'}\phi^{4'}\partial_\mu\phi_{\nu'} + \frac{1}{r^2}\left(-\phi_{0'}^2 + \phi_{1'}^2 + \phi_{2'}^2 + \phi_{3'}^2\right) - \frac{1}{4r}\bar{\psi}\Gamma^\mu\hat{\Gamma}^{\nu'4'}\psi\epsilon_{\mu\nu'}$$

6.3 M5 brane on $\mathbb{R}^{1,2} \times S^3$

For $p = 3$ we find the solution

$$A_{\mu,\nu'\lambda'} = \frac{1}{r}\epsilon_{\mu\nu'\lambda'}$$

where $\epsilon_{01'2'} = 1$ and totally antisymmetric. We have the Weyl projections

$$\begin{aligned}\Gamma^{01}\hat{\Gamma}_{0'1'}\epsilon &= \epsilon\\ \Gamma^{12}\hat{\Gamma}_{1'2'}\epsilon &= \epsilon\end{aligned}$$

The M5 brane Lagrangian is

$$\mathcal{L} = \mathcal{L}_0 + \frac{1}{r}\epsilon^{\mu\nu'\lambda'}\phi_{\lambda'}\partial_\mu\phi_{\nu'} - \frac{1}{8r^2}\epsilon_{\mu\nu'\lambda'}\bar{\psi}\Gamma^\mu\hat{\Gamma}^{\nu'\lambda'}\psi$$

6.4 M5 brane on $\mathbb{R}^{1,3} \times S^2$

For $p = 4$ we find the solution

$$A_{\mu,\nu'4'} = \frac{i}{r}\eta_{\mu\nu'}$$

and Weyl projections

$$\Gamma^{01}\hat{\Gamma}_{0'1'}\epsilon = \epsilon$$
$$\Gamma^{12}\hat{\Gamma}_{1'2'}\epsilon = \epsilon$$
$$\Gamma^{23}\hat{\Gamma}_{2'3'}\epsilon = \epsilon$$

The M5 brane Lagrangian is

$$\mathcal{L} = \mathcal{L}_0 + \frac{2i}{r}\phi^{4'}\partial^\mu\phi_{\mu'} - \frac{3}{r^2}\phi^{4'}\phi_{4'} - \frac{i}{4r}\bar{\psi}\Gamma_\mu\hat{\Gamma}^{\mu'4'}\psi$$

Here we could not find a real solution for the background gauge potential. The 5d SYM action can be real for R symmetry group SO$(2,3)$ if the signature is $(2,3)$ (section 9.2 in [12]). We find that the bosonic part of the action is real once we Wick rotate $\phi^{4'}$ which suggests R symmetry is Wick rotated from SO$(1,4)$ into SO$(2,3)$. If we do that Wick rotation of R symmetry then $\hat{\Gamma}^{4'}$ shall also be Wick rotated and the full action becomes real on $R^3 \times S^2$ if the signature is $(2,3)$ with the S^2 part timelike.

7 Partially twisted theory on $\mathbb{R}^{1,1} \times \mathbb{R}^4$

For our gamma matrix conventions for this twist, we refer to appendix C.1. On $\mathbb{R}^{1,1}$ we have the flat metric

$$ds^2 = -e^0e^0 + e^1e^1 = -2e^+e^- - 2e^-e^+$$

where $e^0 = dx^0$ and $e^1 = dx^1$ and we define

$$e^\pm = \frac{1}{2}\left(e^0 \pm e^1\right)$$

and $\pm$ denote flat lightcone indices. We define

$$\phi^\pm = \frac{1}{2}\left(\phi^0 \pm \phi^1\right)$$

and

$$\gamma^\pm = \frac{1}{2}\left(\gamma^0 \pm \gamma^1\right)$$

whose nonvanishing components are $(\gamma^+)^+{}_- = 1$ and $(\gamma^-)^-{}_+ = -1$ respectively. We then have

$$\gamma^{+-} = -\frac{1}{2}\gamma$$

We have the following anti-hermitian SO$(1,1)$ charge generator

$$Q = \frac{i}{2}\gamma^{01}$$
$$Q = 2i(\delta^{01})_\mu{}^\nu$$

in the spinor and vector representations. It acts on the vector infinitesimally as

$$\delta\phi_\mu = -\frac{i}{2}\epsilon_{\kappa\tau}(Q^{\kappa\tau})_\mu{}^\nu\phi_\nu$$

which yields

$$\delta\phi_\pm = \pm\epsilon_{01}\phi_\pm =: -i\epsilon_{01}Q\phi_\pm$$

which shows that $\phi_\pm$ carry SO(1,1) charge $Q = \pm i$.

We define twisted spinor components as

$$\begin{aligned}\psi_0^{(\pm)\alpha t} &= \psi^{\pm\alpha\mp t}\\ \chi_\pm^{(\pm)\alpha t} &= \psi^{\pm\alpha\pm t}\end{aligned}$$

Here, on the left hand side, stands the twisted spinor fields, and $\pm, 0$ without round brackets refers to the twisted SO(1,1) charge. The $(\pm)$ refers to the SO(4) Weyl projection on the Dirac spinor index α. On the right hand side stands the untwisted spinor fields, and the $\pm$ there refers to SO(1,1) and SO$(1,1)_R$ charges respectively. Hence the total charge of $\psi_0^{(\pm)\alpha t}$ is zero, while $\chi_\pm^{(\pm)\alpha t}$ carry SO(1,1) charges $\pm i$ respectively, just like $\phi_\pm$ do. In the sequel we will use the following shorthand notations,

$$\begin{aligned}\psi^{(\pm)\alpha t} &:= \psi_0^{(\pm)\alpha t}\\ \chi_\pm^{\alpha t} &:= \chi_\pm^{(\pm)\alpha t}\end{aligned}$$

We define

$$D_\pm = e_\pm^\mu D_\mu$$

We have

$$g^{\mu\nu}D_\mu D_\nu = -\frac{1}{2}\{D_+, D_-\}$$

Using the zweibein to convert μ into flat space indices $\pm$, we find the following twisted Lagrangian

$$\begin{aligned}\mathcal{L}_{\text{tensor}} &= \frac{1}{16}H_{+-}{}^iH_{+-i} + \frac{1}{16}H_-{}^{ij}H_{+ij} + \frac{1}{16}H_+{}^{ij}H_{-ij} - \frac{1}{24}H^{ijk}H_{ijk}\\ \mathcal{L}_{\text{scalars}} &= -\frac{1}{2}g^{\mu\nu}D_\mu\phi_+D_\nu\phi_- - \frac{1}{2}g^{ij}\partial_i\phi_+\partial_j\phi_- + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^a + \frac{1}{2}g^{ij}\partial_i\phi^a\partial_j\phi^a\\ \mathcal{L}_{\text{fermions}} &= \bar\chi_-D_+\psi^- + \bar\chi_+D_-\psi^+ + \bar\chi_-\gamma^iD_i\chi_+ + \bar\psi^-\gamma^iD_i\psi^+\end{aligned}$$

where we define the new Dirac conjugation by $\bar{\psi} = \psi^\dagger$ with the reality condition $\bar{\psi}_{\alpha t} = (\psi^{\alpha t})^* = \psi^{\alpha' t'} C_{\alpha' \alpha} \epsilon_{t' t}$. The action is invariant under the supersymmetry variations

$$
\begin{aligned}
\delta B_{+-} &= -2i\bar{\epsilon}^- \psi^- - 2i\bar{\epsilon}^+ \psi^+ \\
\delta B_{\pm i} &= \pm 2i\bar{\epsilon}^\mp \gamma_i \chi_\pm \\
\delta B_{ij} &= i\bar{\epsilon}^- \gamma_{ij} \psi^- - i\bar{\epsilon}^+ \gamma_{ij} \psi^+ \\
\\
\delta\phi_+ &= -2\bar{\epsilon}^+ \chi_+ \\
\delta\phi_- &= -2\bar{\epsilon}^- \chi_- \\
\delta\phi^a &= \bar{\epsilon}^+ \sigma^a \psi^+ + \bar{\epsilon}^- \sigma^a \psi^- \\
\\
\delta\psi^\pm &= \epsilon^\pm D_\pm \phi_\mp + \gamma^i \sigma^a \epsilon^\mp D_i \phi^a + \frac{i}{4} \gamma^i \epsilon^\mp H_{+-i} \mp \frac{i}{12} \gamma^{ijk} \epsilon^\mp H_{ijk} \\
\delta\chi_\pm &= -\sigma^a \epsilon^\pm D_\pm \phi^a - \gamma^i \epsilon^\mp D_i \phi_\pm \mp \frac{i}{4} \gamma^{ij} \epsilon^\pm H_{\pm ij}
\end{aligned}
$$

8 Partially twisted theory on $M_{1,1} \times \mathbb{R}^4$

We introduce the Grassmannian two-space vector field by

$$\chi_\mu = e^+_\mu \chi_+ + e^-_\mu \chi_-$$

and a scalar

$$\psi = \psi^+ + \psi^-$$

where all the Grassmannian fields are realized in the 8d (α,t) space. The supersymmetry parameter is a Grassmannian scalar given by

$$\epsilon = \epsilon^+ + \epsilon^-$$

For notational convenience let us introduce 6D Weyl projection on χ_μ as

$$\chi^W_\mu = \frac{1}{2} \left(\chi_\mu - \gamma_{(4)} \epsilon_{\mu\nu} \chi^\nu \right)$$

Then χ_μ is subject to the Weyl projection condition

$$\chi_\mu = \chi^W_\mu$$

which leads to the relation

$$\chi_\mu = -\gamma_{(4)} \epsilon_{\mu\nu} \chi^\nu$$

Using this notation, we find the following twisted Lagrangian

$$
\begin{aligned}
\mathcal{L}_{\text{tensor}} &= -\frac{1}{8} H^{\mu\nu i} H_{\mu\nu i} - \frac{1}{8} H^{\mu ij} H_{\mu ij} - \frac{1}{24} H^{ijk} H_{ijk} \\
\mathcal{L}_{\text{scalars}} &= \frac{1}{4} \phi_{\mu\nu} \phi^{\mu\nu} + \frac{1}{2} (\nabla_\mu \phi^\mu)^2 + \frac{1}{2} g^{ij} \partial_i \phi_\mu \partial_j \phi^\mu + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi^a \partial_\nu \phi^a + \frac{1}{2} g^{ij} \partial_i \phi^a \partial_j \phi^a \\
\mathcal{L}_{\text{fermions}} &= 2\partial_\mu \bar{\psi} \chi^\mu - \bar{\chi}^\mu \gamma^i \partial_i \chi_\mu + \frac{1}{2} \bar{\psi} \gamma^i \partial_i \psi
\end{aligned}
$$

The action is invariant under the supersymmetry variations

$$\begin{aligned}
\delta B_{\mu\nu} &= -i\epsilon_{\mu\nu}\,\bar{\epsilon}\psi = i\epsilon_{\mu\nu}\,\bar{\psi}\epsilon \\
\delta B_{\mu i} &= 2i\bar{\epsilon}\gamma_i\gamma_{(4)}\chi_\mu \\
\delta B_{ij} &= -i\bar{\epsilon}\gamma_{ij}\gamma_{(4)}\psi
\end{aligned}$$

$$\begin{aligned}
\delta\phi_\mu &= -2\bar{\epsilon}\chi_\mu \\
\delta\phi^a &= \bar{\epsilon}\sigma^a\psi
\end{aligned}$$

$$\begin{aligned}
\delta\psi &= -\epsilon\,\nabla_\mu\phi^\mu - \gamma_{(4)}\epsilon\,\epsilon^{\mu\nu}\partial_\mu\phi_\nu + \gamma^i\sigma^a\epsilon\,\partial_i\phi^a - \frac{i}{4}\gamma^i\epsilon\,\epsilon^{\mu\nu}H_{\mu\nu i} + \frac{i}{12}\gamma^{ijk}\gamma_{(4)}\epsilon\,H_{ijk} \\
\delta\chi_\mu &= \frac{1}{2}\left(q_\mu - \gamma_{(4)}\epsilon_{\mu\nu}q^\nu\right) \equiv q^W_\mu
\end{aligned}$$

where

$$q_\mu = -\sigma^a\epsilon\,\partial_\mu\phi^a - \gamma^i\epsilon\,\partial_i\phi_\mu - \frac{i}{4}\gamma^{ij}\gamma_{(4)}\epsilon H_{\mu ij}$$

9 Partially twisted theory on $M_{1,1} \times S^4$

Using the notation of the previous section, we find the following twisted Lagrangian

$$\begin{aligned}
\mathcal{L}_{\text{tensor}} &= -\frac{1}{8}H^{\mu\nu i}H_{\mu\nu i} - \frac{1}{8}H^{\mu ij}H_{\mu ij} - \frac{1}{24}H^{ijk}H_{ijk} \\
\mathcal{L}_{\text{scalars}} &= \frac{1}{4}\phi_{\mu\nu}\phi^{\mu\nu} + \frac{1}{2}(\nabla_\mu\phi^\mu)^2 + \frac{1}{2}g^{ij}\partial_i\phi_\mu\partial_j\phi^\mu + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^a + \frac{1}{2}g^{ij}\partial_i\phi^a\partial_j\phi^a \\
&\quad -\frac{2}{r}\phi^4\epsilon^{\mu\nu}\partial_\mu\phi_\nu + \frac{1}{r^2}\left(\phi^\mu\phi_\mu + \phi^{a'}\phi^{a'}\right) \\
\mathcal{L}_{\text{fermions}} &= 2\partial_\mu\bar{\psi}\chi^\mu - \bar{\chi}^\mu\gamma^i D_i\chi_\mu + \frac{1}{2}\bar{\psi}\gamma^i D_i\psi - \frac{1}{2r}\bar{\psi}\gamma_{(4)}\sigma^3\psi
\end{aligned}$$

Here, in $\mathcal{L}_{scalars}$, we assume indices range as $a = (a', 4)$ for $a' = 2, 3$.

The action is invariant under the supersymmetry variations

$$\begin{aligned}
\delta B_{\mu\nu} &= -i\epsilon_{\mu\nu}\,\bar{\epsilon}\psi = i\epsilon_{\mu\nu}\,\bar{\psi}\epsilon \\
\delta B_{\mu i} &= 2i\bar{\epsilon}\gamma_i\gamma_{(4)}\chi_\mu \\
\delta B_{ij} &= -i\bar{\epsilon}\gamma_{ij}\gamma_{(4)}\psi
\end{aligned}$$

and

$$\begin{aligned}
\delta\phi_\mu &= -2\bar{\epsilon}\chi_\mu \\
\delta\phi^a &= \bar{\epsilon}\sigma^a\psi
\end{aligned}$$

where ϕ^2, ϕ^3, and ϕ^4 are respectively matched with σ^1, σ^2 and σ^3 with a little abuse of notation. The fermionic variation beomes

$$\begin{aligned}
\delta\psi &= -\epsilon\,\nabla_\mu\phi^\mu - \gamma_{(4)}\epsilon\,\epsilon^{\mu\nu}\partial_\mu\phi_\nu + \gamma^i\sigma^a\epsilon\,\partial_i\phi^a - \frac{i}{4}\gamma^i\epsilon\,\epsilon^{\mu\nu}H_{\mu\nu i} \\
&\quad + \frac{i}{12}\gamma^{ijk}\gamma_{(4)}\epsilon\,H_{ijk} + \frac{2i}{r}(\gamma_{(4)}\sigma_1\epsilon\phi^3 - \gamma_{(4)}\sigma_2\epsilon\phi^2) \\
\delta\chi_\mu &= \frac{1}{2}\left(q_\mu - \frac{1}{2}\gamma_{(4)}\epsilon_{\mu\nu}q^\nu\right) = q^W_\mu
\end{aligned}$$

where

$$q_\mu = -\sigma^a \epsilon \, \partial_\mu \phi^a - \gamma^i \epsilon \, \partial_i \phi_\mu - \frac{i}{4} \gamma^{ij} \gamma_{(4)} \epsilon H_{\mu ij} - \frac{1}{r} \gamma_{(4)} \sigma^3 \epsilon \, \phi_\mu$$

The Killing spinor equation reads

$$D_i \epsilon = \frac{1}{2r} \gamma_i \gamma_{(4)} \sigma^3 \epsilon$$

whose justification follows from the relation

$$-(\bar{\psi} \Gamma^i D_i \epsilon)|_{\chi_\pm = 0} = -4 \bar{\psi} M \epsilon|_{\chi_\pm = 0} = \bar{\psi} \gamma^i D_i \epsilon$$

where

$$M = \frac{1}{2r} \Gamma^0 \hat{\Gamma}^{14}$$

10 Partially twisted theory on $M_{1,2} \times S^3$

For our gamma matrix conventions for this twist, we refer to appendix C.2. We introduce a Grassmannian vector field ψ_μ and scalar field ψ where all the Grassmannian fields are realized in the 4d (s_1, t_1) space and $\mu = 0, 1, 2$. The supersymmetry parameter is a Grassmannian scalar on $M_{1,2}$ which we denote by η which is related to the original supersymmetry parameter by

$$\epsilon^{s_0 s_1 s_2 | t_0 t_1} = \epsilon^{s_0 t_0} \eta^{s_1 t_1}$$

In the twisted theory, the reality condition on any Grassmanian fields χ becomes

$$\bar{\chi}_{s_1 t_1} = (\chi^{s_1 t_1})^* = i \chi^{s_1' t_1'} \epsilon_{s_1' s_1} \epsilon_{t_1' t_1}$$

which basically defines the induced charge conjugation matrix for our twisted theory. In addition, we introduce (for more details we refer to appendix C.2)

$$\gamma^i = \gamma^i \otimes 1$$

and

$$(\sigma^3, \ \kappa^a) = (1 \otimes \sigma^3, \ 1 \otimes \kappa^a)$$

With these preliminaries, we find the following twisted Lagrangian

$$\begin{aligned}
\mathcal{L}_{\text{tensor}} &= -\frac{1}{24} H^{\mu\nu\lambda} H_{\mu\nu\lambda} - \frac{1}{8} H^{\mu\nu i} H_{\mu\nu i} - \frac{1}{8} H^{\mu ij} H_{\mu ij} - \frac{1}{24} H^{ijk} H_{ijk} \\
\mathcal{L}_{\text{scalars}} &= \frac{1}{4} \phi_{\mu\nu} \phi^{\mu\nu} + \frac{1}{2} (\nabla_\mu \phi^\mu)^2 + \frac{1}{2} g^{ij} \partial_i \phi_\mu \partial_j \phi^\mu + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi^a \partial_\nu \phi^a + \frac{1}{2} g^{ij} \partial_i \phi^a \partial_j \phi^a \\
&\quad + \frac{1}{r} \epsilon^{\mu\nu\lambda} \phi_\mu \partial_\nu \phi_\lambda
\end{aligned}$$

where $a = 3, 4$ and

$$\begin{aligned}\mathcal{L}_{\text{fermions}} &= -2\bar{\psi}^\mu\sigma^3\nabla_\mu\psi - \epsilon^{\mu\nu\lambda}\bar{\psi}_\mu\sigma^3\partial_\nu\psi_\lambda - i\bar{\psi}\gamma^i\sigma^3 D_i\psi + i\bar{\psi}^\mu\gamma^i\sigma^3 D_i\psi_\mu \\ &\quad + \frac{3}{2r}\bar{\psi}\sigma^3\psi + \frac{1}{2r}\bar{\psi}^\mu\sigma^3\psi_\mu\end{aligned}$$

This Lagrangian is invariant under the supersymmetry transformation

$$\begin{aligned}\delta B_{\mu\nu} &= -2\epsilon_{\mu\nu\lambda}\,\bar{\eta}\,\sigma^3\psi^\lambda \\ \delta B_{\mu i} &= 2i\bar{\eta}\,\gamma_i\sigma^3\psi_\mu \\ \delta B_{ij} &= -2\bar{\eta}\,\gamma_{ij}\sigma^3\psi \\ \\ \delta\phi_\mu &= -2i\bar{\eta}\,\psi_\mu \\ \delta\phi^a &= 2i\bar{\eta}\,\sigma^3\kappa^a\psi\end{aligned}$$

and

$$\begin{aligned}\delta\psi &= -\frac{1}{12}\eta\left(\epsilon^{\mu\nu\lambda}H_{\mu\nu\lambda} - \epsilon^{ijk}H_{ijk}\right) - i\sigma^3\eta\nabla_\mu\phi^\mu - \gamma^i\kappa^a\eta\,\nabla_i\phi^a + \frac{2i}{r}\kappa^a\eta\,\phi^a \\ \delta\psi_\mu &= -\frac{1}{4}\gamma^{ij}\eta H_{\mu ij} - \frac{i}{4}\gamma_i\,\eta\epsilon_{\mu\nu\lambda}H^{\nu\lambda i} + i\kappa^a\eta\,\nabla_\mu\phi^a + \gamma^i\sigma^3\eta\,\nabla_i\phi_\mu - i\sigma^3\eta\epsilon_\mu{}^{\nu\lambda}\partial_\nu\phi_\lambda\end{aligned}$$

To verify the supersymmetry of the action, we note that the 6d conformal Killing spinor equation reduces to the usual Killing spinor equation on S^3,

$$D_i\eta = -\frac{i}{2r}\gamma_i\,\eta$$

The main application of this twist is to the 3d-3d correspondence. This will be analyzed elsewhere.

Acknowledgments

DB was supported in part by was supported in part by NRF Grant 2014R1A1A2053737.

A Classification of R symmetry groups for 6d $(2, 0)$ theories

We assume Lorentz group SO$(1, 5)$ and R symmetry group SO$(q, 5 - q)$ and attempt to impose the 11d Majorana condition

$$\bar{\psi} = \psi^T C$$

where we shall define

$$\bar{\psi} = \psi^\dagger\Gamma^0\hat{\Gamma}^{1\cdots q}$$

Let us assume that we can impose this Majorana condition. We can then pick the Majorana representation for the gamma matrices where the charge conjugation matrix is given by

$$C = \Gamma^0$$

Since we also have that

$$\begin{aligned}
(\Gamma^M)^T &= -C\Gamma^M C^{-1} \\
(\hat{\Gamma}^A)^T &= -C\hat{\Gamma}^A C^{-1} \\
\Gamma^{M\dagger} &= \Gamma^0\Gamma^M\Gamma^0 \\
\Gamma^{a\dagger} &= -\Gamma^a \qquad \text{for } a = 1, \cdots, q \\
\Gamma^{a'\dagger} &= \Gamma^{a'} \qquad \text{for } a' = q+1, \cdots, 5
\end{aligned}$$

we see that

$$\begin{aligned}
\Gamma^{M*} &= \Gamma^M \\
\Gamma^{a*} &= -\Gamma^a \\
\Gamma^{a'*} &= \Gamma^{a'}
\end{aligned}$$

The Majorana condition becomes

$$\psi^\dagger\Gamma^{1\cdots q} = (-1)^q\psi^T$$

Applying transpose on both sides, we get

$$C\Gamma^{q\cdots 1}C^{-1}\psi^* = \psi$$

Using $C = \Gamma^0$ we get

$$(-1)^{q+1}\Gamma^{q\cdots 1}\psi^* = \psi$$

Applying $\Gamma^{1\cdots q}$ on both sides, we get

$$\psi^* = -\Gamma^{1\cdots q}\psi$$

If we complex conjugate again, we get

$$\psi^{**} = -(-1)^q\Gamma^{1\ldots q}\psi^* = (-1)^q(\Gamma^{1\cdots q})^2\psi$$

Now we use that

$$(\Gamma^{1\cdots q})^2 = (-1)^{\frac{q(q+1)}{2}}$$

We then get

$$\psi^{**} = (-1)^{\frac{q(q-1)}{2}}\psi$$

This is consistent for

$$q(q-1) \in 4\mathbb{Z}$$

Solutions are $q = 0, 1, 4, 5$ and correspond to SO(5), SO(1, 4), SO(4, 1) and SO(5, 0).

JHEP12(2015)093

B A map from 6d to 10d Weyl projections

To find the non-Abelian generalization, we first put $r = \infty$. We wish to relate the theory with the dimensional reduction of SYM on $\mathbb{R}^{1,9}$, dimensionally reduced down to R^5. For this SYM we have the Weyl projections

$$\begin{aligned} -i\Gamma^0\zeta &= \zeta \\ -i\Gamma^0\omega &= \omega \end{aligned}$$

for the spinor field and the supersymmetry parameter respectively. These will be related by a unitary transformation to our original variables as

$$\begin{aligned} \psi &= U\zeta \\ \epsilon &= U^\dagger\omega \end{aligned}$$

where

$$U = \frac{1}{\sqrt{2}}\left(1 + i\Gamma^0\Gamma\right)$$

which has the properties

$$\begin{aligned} UU^\dagger &= 1 \\ U^2 &= i\Gamma^0\Gamma \\ U\Gamma^0 &= \Gamma^0 U^\dagger \\ U\Gamma^m &= \Gamma^m U \\ U\Gamma^A &= \Gamma^A U^\dagger \end{aligned}$$

We define

$$\begin{aligned} \bar\epsilon &= \epsilon^\dagger\Gamma^0\Gamma^{0'} \\ \bar\omega &= \omega^\dagger\hat\Gamma^{0'} \end{aligned}$$

and so we also have the relations

$$\begin{aligned} \bar\epsilon &= \bar\omega\Gamma_0 U \\ \bar\psi &= \bar\zeta\Gamma_0 U^\dagger \end{aligned}$$

In terms of these new spinor variables, we get

$$\begin{aligned} \delta\phi_A &= i\bar\omega\hat\Gamma_A\zeta \\ \delta A_m &= i\bar\omega\Gamma_m\zeta \\ \delta\zeta &= \frac{1}{2}\Gamma^{mn}\omega F_{mn} + \Gamma^m\hat\Gamma^A\omega\partial_m\phi_A \end{aligned}$$

If we now also flip the sign of the matter fields ϕ_A, we find the standard supersymmetry variations of (1+9)d SYM reduced to 5d, for which we have the non-Abelian generalization that is obtained by substituting ordinary derivative with gauge covariant derivative $D_m = \partial_m - i[A_m, \bullet]$ in the adjoint representation, and by adding one commutator term

$$\delta'\zeta = -\frac{i}{2}\hat{\Gamma}^{AB}\omega[\phi_A, \phi_B]$$

We can then transform this term back into our original, M5 brane adapted, variables and get

$$\delta'\psi = -\frac{1}{2}\hat{\Gamma}^{AB}\Gamma^0\epsilon[\phi_A, \phi_B]$$

Likewise the non-Abelian Lagrangian is in the new variables given by the standard SYM Lagrangian

$$\mathcal{L} = \frac{1}{4}F^{mn}F_{mn} + \frac{1}{2}D^m\phi^A D_m\phi_A - \frac{1}{4}[\phi^A, \phi^B][\phi_A, \phi_B] - \frac{i}{2}\bar{\zeta}\Gamma^m D_m\zeta - \frac{1}{2}\bar{\zeta}\hat{\Gamma}^A[\phi_A, \zeta]$$

that in the M5 brane adapted variables translates into

$$\mathcal{L}_0 = \frac{1}{4}F^{mn}F_{mn} + \frac{1}{2}D^m\phi^A D_m\phi_A - \frac{1}{4}[\phi^A, \phi^B][\phi_A, \phi_B] - \frac{1}{2}\bar{\psi}\Gamma^m D_m\psi - \frac{1}{2}\bar{\psi}\Gamma^0\hat{\Gamma}^A[\phi_A, \psi]$$

C Gamma matrix conventions for partial topological twists

When we perform the partial topological twisting we find it convenient to choose gamma matrices according to the dimension of the manifold over which we obtain the scalar supercharges after the twist.

C.1 Gamma matrices for the 2d-4d split

We choose the SO(1, 1) gamma matrices γ^μ as

$$\begin{aligned}\gamma^0 &= i\sigma^2\\ \gamma^1 &= \sigma^1\end{aligned}$$

and we define the SO(1, 1) chirality matrix as

$$\gamma_{(2)} = \gamma^{01} = \sigma^3$$

We have

$$\begin{aligned}(\gamma^\mu)^T &= -\epsilon\gamma^\mu\epsilon^{-1}\\ \gamma_{(2)}^T &= -\epsilon\gamma_{(2)}\epsilon^{-1}\end{aligned}$$

where $\epsilon = i\sigma^2$.

We then choose the 11d gamma matrices as

$$\begin{aligned}\Gamma^\mu &= \gamma^\mu \otimes 1 \otimes 1 \otimes 1\\ \Gamma^i &= \gamma_{(2)} \otimes \gamma^i \otimes 1 \otimes 1\\ \hat{\Gamma}^{\mu'} &= \gamma_{(2)} \otimes \gamma_{(4)} \otimes \gamma^{\mu'} \otimes 1\\ \hat{\Gamma}^a &= \gamma_{(2)} \otimes \gamma_{(4)} \otimes \gamma_{(2)} \otimes \sigma^a\end{aligned}$$

We let indices range as $\mu = \mu' = 0, 1,\ i = 1, 2, 3, 4$ and $a = 1, 2, 3$. We then find that the 6d chirality matrix becomes

$$\Gamma = \gamma_{(2)} \otimes \gamma_{(4)} \otimes 1 \otimes 1$$

where we define the SO(4) hermitian chirality matrix as

$$\gamma_{(4)} = \gamma^{1234}$$

The 6d Weyl condition amounts to

$$\left(\gamma_{(2)} \otimes \gamma_{(4)} \otimes 1 \otimes 1\right) \psi = \psi$$

The 11d charge conjugation matrix is

$$C_{11d} = \epsilon \otimes C \otimes \sigma^1 \otimes \epsilon$$

which is such that

$$\begin{aligned} C_{11d}^T &= -C_{11d} \\ (\Gamma^M)^T &= -C_{11d}\Gamma^M C_{11d}^{-1} \\ (\hat{\Gamma}^A)^T &= -C_{11d}\Gamma^A C_{11d}^{-1} \end{aligned}$$

We then have $C^T = -C$ and $\epsilon^T = -\epsilon$. An explicit realization of SO(4) gamma matrices is

$$\begin{aligned} \gamma^{1,2,3} &= \sigma^{1,2,3} \otimes \sigma^2 \\ \gamma^4 &= 1 \otimes \sigma^1 \end{aligned}$$

and

$$C = \epsilon \otimes 1$$

Then

$$(\gamma^i)^T = C\gamma^i C^{-1}$$

Also, if we define

$$\gamma_{(4)} = \gamma^{1234} = 1 \otimes \sigma^3$$

then

$$\gamma_{(4)}^T = C\gamma_{(4)}C^{-1}$$

We will use spinor indices as follows,

$$\psi^{s_0 \alpha t_0 t_1}$$

Thus if we write out all spinor indices, we have for instance

$$C_{11d} = \epsilon_{s_0 s_0'} C_{\alpha\beta} \sigma^1_{t_0 t_0'} \epsilon_{tt'}$$

We have that

$$\begin{aligned} C_{\alpha\beta} &= -C_{\beta\alpha} \\ \gamma^i_{\alpha\beta} &= -\gamma^i_{\beta\alpha} \\ \gamma^{ij}_{\alpha\beta} &= \gamma^{ij}_{\beta\alpha} \\ \gamma^{ijk}_{\alpha\beta} &= \gamma^{ijk}_{\beta\alpha} \end{aligned}$$

where we define $\gamma^i_{\alpha\beta} := C_{\alpha\gamma}(\gamma^i)^\gamma{}_\beta$.

We define

$$(\gamma_{(2)})^s{}_t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$(\gamma_{(2)})_{st} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

We denote the twisted SO(1,1) neutral spinor components as

$$\psi^{\alpha t_1}$$

In addition to these, we have the twisted SO(1,1) charged spinor components

$$\chi^{\alpha t_1}$$

which carry the SO(1,1) charge according to their SO(4) chirality.

In total we have 8 neutral (denoted as ψ) and 8 charged (denoted as χ) spinor components. The supersymmetry parameters are neutral under SO(1,1). We denote these as

$$\epsilon^{\alpha t_1}$$

which has $4 \times 2 = 8$ real components. In other words, we have 8 real supercharges.

C.2 Gamma matrices for the 3d-3d split

We choose 11d gamma matrices as ($\mu = 0,1,2,\ i = 3,4,5,\ A = 0',1',2',3',4'$)

$$\begin{aligned} \Gamma^\mu &= \gamma^\mu \otimes 1 \otimes \sigma^2 \otimes 1 \\ \Gamma^i &= 1 \otimes \gamma^i \otimes \sigma^1 \otimes 1 \\ \hat{\Gamma}^A &= 1 \otimes 1 \otimes \sigma^3 \otimes \gamma^A \end{aligned}$$

where $\gamma^\mu = (i\sigma^2, \sigma^1, \sigma^3)$ and $\gamma^i = (\sigma^3, \sigma^1, \sigma^2)$ and where we choose γ^A as follows

$$\begin{aligned}
\gamma^0 &= i\sigma^2 \otimes \sigma^3 \\
\gamma^1 &= \sigma^1 \otimes \sigma^3 \\
\gamma^2 &= \sigma^3 \otimes \sigma^3 \\
\gamma^3 &= 1 \otimes \sigma^2 \\
\gamma^4 &= 1 \otimes \sigma^1
\end{aligned}$$

and we may use the notation

$$\begin{aligned}
\gamma^{\mu'} &= \gamma^{\mu'} \otimes \sigma^3 \\
\gamma^a &= 1 \otimes \kappa^a
\end{aligned}$$

for $\mu' = 0, 1, 2$ and $a = 3, 4$. We have

$$\begin{aligned}
(\gamma^A)^T &= C\gamma^A C^{-1} \\
C^T &= -C
\end{aligned}$$

where

$$C = \epsilon \otimes \sigma^1$$

The 11d charge conjugation matrix is

$$C_{11d} = \epsilon \otimes \epsilon \otimes \sigma^1 \otimes C$$

which is antisymmetric

$$C_{11d}^T = -C_{11d}$$

We expand the spinor as

$$\psi^{s_0 s_1 s_2 t_0 t_1} = \epsilon^{s_0 t_0} \psi^{s_1 t_1} + (\gamma^\mu)^{s_0 t_0} \psi_\mu^{s_1 t_1}$$

Here $\psi^{s_1 \pm}$ transform in the representation $(1, 2)_\pm$ and $\psi_\mu^{s_1 \pm}$ in the representation $(3, 2)_\pm$ of $\mathrm{SO}(1, 2) \times \mathrm{SO}(3) \times \mathrm{SO}(2)_R$. Note that s_2 is determined by the 6d Weyl projection. We have

$$\begin{aligned}
\Gamma^{01} &= (\sigma^3)^{s_0}{}_{s_0'} \\
\Gamma^{23} &= -i(\sigma^3)^{s_0}{}_{s_0'}(\sigma^3)^{s_1}{}_{s_1'}(\sigma^3)^{s_2}{}_{s_2'} \\
\Gamma^{45} &= i(\sigma^3)^{s_1}{}_{s_1'}
\end{aligned}$$

Then

$$\Gamma = \Gamma^{01}\Gamma^{23}\Gamma^{45} = (\sigma^3)^{s_2}{}_{s_2'}$$

We conclude that s_2 gives the 6d chirality of the spinor so that this number is fixed by the spinor. For ψ we have $s_2 = +$ and for the supersymmetry parameter η we have $s_2 = -$.

D Untwisted Fierz identity

We use 11d gamma matrices that we split them into two groups, Γ^M and $\hat{\Gamma}^A$ where $M = 0,1,2,3,4,5$ is for spacetime and $A = 0',1',2',3',4'$ is for SO(1,4) R symmetry. We thus assume that $\{\Gamma^M, \hat{\Gamma}^A\} = 0$ as part of the 11d Clifford algebra. We define the 6d chirality matrix

$$\Gamma = \Gamma^{012345}$$

For two negative chirality spinors $\Gamma\epsilon = -\epsilon$ and $\Gamma\eta = -\eta$, we have the following Fierz identity,

$$\epsilon\bar{\eta} - \eta\bar{\epsilon} = \frac{1}{8}\left[-(\bar{\eta}\Gamma^M\epsilon)\Gamma_M + (\bar{\eta}\Gamma^M\hat{\Gamma}^A\epsilon)\Gamma_M\hat{\Gamma}_A\right]\frac{1}{2}(1+\Gamma) - \frac{1}{192}(\bar{\eta}\Gamma^{MNP}\hat{\Gamma}^{AB}\epsilon)\Gamma_{MNP}\hat{\Gamma}_{AB}$$

We have the following gamma matrix identities,

$$\begin{aligned}
\Gamma^{MNP}\Gamma_Q\Gamma_{NP} &= -20\delta^M_Q - 4\Gamma^M{}_Q \\
\Gamma^{PMN}\Gamma_{QRS}\Gamma_{MN} &= 4\Gamma^P{}_{QRS} + 12\delta^P_{[Q}\Gamma_{RS]} \\
\hat{\Gamma}_A\hat{\Gamma}^B\hat{\Gamma}^A &= -3\hat{\Gamma}^B \\
\hat{\Gamma}_A\hat{\Gamma}^{BC}\hat{\Gamma}^A &= \hat{\Gamma}^{BC}
\end{aligned}$$

References

[1] E. Bergshoeff, E. Sezgin and A. Van Proeyen, $(2,0)$ *tensor multiplets and conformal supergravity in* $D = 6$, *Class. Quant. Grav.* **16** (1999) 3193 [hep-th/9904085] [INSPIRE].

[2] C. Cordova and D.L. Jafferis, *Five-dimensional maximally supersymmetric Yang-Mills in supergravity backgrounds*, arXiv:1305.2886 [INSPIRE].

[3] G. Festuccia and N. Seiberg, *Rigid supersymmetric theories in curved superspace*, *JHEP* **06** (2011) 114 [arXiv:1105.0689] [INSPIRE].

[4] C. Cordova and D.L. Jafferis, *Complex Chern-Simons from M5-branes on the squashed three-sphere*, arXiv:1305.2891 [INSPIRE].

[5] J. Kim, S. Kim, K. Lee and J. Park, *Super-Yang-Mills theories on* $S^4 \times \mathbb{R}$, *JHEP* **08** (2014) 167 [arXiv:1405.2488] [INSPIRE].

[6] H.-C. Kim and S. Kim, *M5-branes from gauge theories on the 5-sphere*, *JHEP* **05** (2013) 144 [arXiv:1206.6339] [INSPIRE].

[7] L.F. Alday, D. Gaiotto and Y. Tachikawa, *Liouville correlation functions from four-dimensional gauge theories*, *Lett. Math. Phys.* **91** (2010) 167 [arXiv:0906.3219] [INSPIRE].

[8] U. Gran, H. Linander and B.E.W. Nilsson, *Off-shell structure of twisted* $(2,0)$ *theory*, *JHEP* **11** (2014) 032 [arXiv:1406.4499] [INSPIRE].

JHEP12(2015)093

[9] L. Anderson and H. Linander, *The trouble with twisting* (2, 0) *theory*, JHEP **03** (2014) 062 [arXiv:1311.3300] [INSPIRE].

[10] H. Linander and F. Ohlsson, (2, 0) *theory on circle fibrations*, JHEP **01** (2012) 159 [arXiv:1111.6045] [INSPIRE].

[11] C.M. Hull and R.R. Khuri, *World volume theories, holography, duality and time*, Nucl. Phys. **B 575** (2000) 231 [hep-th/9911082] [INSPIRE].

[12] C.M. Hull and N. Lambert, *Emergent time and the M5-brane*, JHEP **06** (2014) 016 [arXiv:1403.4532] [INSPIRE].

[13] E. Witten, *Quantization of Chern-Simons gauge theory with complex gauge group*, Commun. Math. Phys. **137** (1991) 29 [INSPIRE].

[14] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, Commun. Math. Phys. **313** (2012) 71 [arXiv:0712.2824] [INSPIRE].

JHEP12(2015)093