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Conformal blocks of $\mathcal{W}_N$ minimal models and AGT correspondence

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Abstract: We study the AGT correspondence between four-dimensional supersymmetric gauge field theory and two-dimensional conformal field theories in the context of $\mathcal{W}_N$ minimal models. The origin of the AGT correspondence is in a special integrable structure which appears in the properly extended conformal theory. One of the basic manifestations of this integrability is the special orthogonal basis which arises in the extended theory. We propose modification of the AGT representation for the $\mathcal{W}_N$ conformal blocks in the minimal models. The necessary modification is related to the reduction of the orthogonal basis. This leads to the explicit combinatorial representation for the conformal blocks of minimal models and employs the sum over N-tupels of Young diagrams with additional restrictions.

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1 Introduction

The bootstrap approach to 2d CFTs is based on the requirements of conformal symmetry, associativity of the operator algebra and crossing symmetry for the correlation functions [1]. One of the main ingredients in the conformal bootstrap approach is the conformal block function, which sums up the contributions of conformal descendants of a given primary field. The bootstrap approach allows in principle to define the structure constants of the operator algebra and then to construct arbitrary multi-point correlation functions. Important class of the conformal field theories is the minimal models, where there is a finite number of irreducible representations of the conformal algebra closed with respect to the operator algebra.

The AGT correspondence [2] and its generalizations establish connections between different 2d conformal field theories and instanton moduli spaces in 4d $\mathcal{N}=2$ supersymmetric gauge quiver theories in the Omega background [3]. In the framework of this correspondence the conformal blocks are represented by the instanton partition functions which are known explicitly [3]. The AGT representation for $\mathcal{W}_N$ conformal blocks was considered in refs. [4–6]. In ref. [7] the connection between an integrable structure of the theory with chiral algebra

$$\mathcal{A}_N = \mathcal{H} \otimes \frac{\widehat{\mathfrak{sl}}(N)_1 \otimes \widehat{\mathfrak{sl}}(N)_{n-1}}{\widehat{\mathfrak{sl}}(N)_n}\,, \tag{1.1}$$

and $\mathcal{W}_N$ conformal blocks was found. Here $\mathcal{H}$ denotes the Heisenberg algebra and the second term gives the standard coset realisation of the $\mathcal{W}_N$ algebra with the central charge defined in terms of the parameter n. The theory possesses the following remarkable property. The algebra $\mathcal{A}_N$ is some special limit of the quantum toroidal $\mathbf{gl}_1(\mathrm{q}_1, \mathrm{q}_2)$ algebra with simple action on the cohomologies of equivariant K theories. In particular, in the module of the toroidal algebra there exists the orthogonal basis enumerated by plane partitions. $\mathcal{A}_N$ module inherits the structure of the basis of the module of the toroidal algebra [8]. As a result of the limiting procedure the basis is enumerated by N ordinary Young diagrams. In the $\mathcal{A}_N$ module there exists some special orthogonal basis such that the matrix element of the composite vertex operators in this basis are known explicitly in terms of the simple rational functions of the basic parameters [7, 9]. For this reason, using the orthogonal basis automatically leads to the explicit results for arbitrary correlation functions.

It is interesting to study the consequences of the AGT correspondence for the conformal blocks of $\mathcal{W}_N$ minimal models. The first example of the AGT correspondence for rational CFT models with Virasoro symmetry was considered in ref. [10]. In [11] the properties of the conformal blocks of some special degenerate fields in the conformal field theories with extended $\mathcal{W}_N$ symmetry were studied in the context of the AGT correspondence. However, the problem of constructing the general combinatorial representations for conformal blocks in $\mathcal{W}_N$ minimal models and more generally the problem of constructing the correlation functions remains open. Interesting applications of $\mathcal{W}_N$ minimal models, where the question of constructing conformal blocks is relevant, can be found within the AdS_3/CFT_2 higher spin correspondence, see, e.g., refs. [12, 13].

In this paper we propose AGT-like combinatorial representation for the conformal block functions in $\mathcal{W}_N$ minimal models. We formulate a necessary modification of the AGT combinatorial representation for the Virasoro minimal models. The modification is reduced to additional restrictions on the region of summation over Young diagrams. In particular, we perform some checks for Virasoro conformal blocks comparing with the exact results which follows from the definition of the conformal block. Also, we formulate the conjecture on the form of AGT representation for conformal blocks of $\mathcal{W}_N$ minimal models. Our conjecture relies upon the properties of the orthogonal basis in $\mathcal{A}_N$ modules described in refs. [14, 15].

2 AGT for non-degenerate Virasoro representations

As an example of the AGT representation for non-degenerate representations of Virasoro algebra $\mathcal{W}_2$ we consider a 4-point conformal block on the sphere. The consideration of general k-point correlation functions contains the same ingredients. The general answer including degenerate representations of $\mathcal{W}_N$ algebras will be given in section 5.

The 4-point conformal block is a holomorphic contribution of the conformal family $[\Phi_{\Delta_0}]$ of the primary field Φ_{Δ_0} in the correlation function $\langle \Phi_{\Delta_1}(x)\Phi_{\Delta_2}(0)\Phi_{\Delta_3}(1)\Phi_{\Delta_4}(\infty)\rangle$. A standard Liouville parametrization of the conformal dimensions and central charge is

$$\Delta_i = Q^2/4 - P_i^2, \qquad c = 1 + 6Q^2, \qquad Q = b^{-1} + b\,. \tag{2.1}$$

The AGT correspondence gives the following power series expansion for the 4-point conformal block [2]

$$\mathcal{B}(P_i;x) \equiv \sum_{N=0}^{\infty} x^N \mathcal{B}^{(N)}(P_i) = (1-x)^{-\nu} \sum_{N=0}^{\infty} x^N F^{(N)}\,, \tag{2.2}$$

and

$$F^{(N)} = \sum_{\vec{\lambda},|\vec{\lambda}|=N} \frac{Z_f(\mu_i,a;\vec{\lambda})}{Z_v(a;\vec{\lambda})}\,. \tag{2.3}$$

The summation on the right hand side runs over pairs of Young diagrams $\vec{\lambda} = (\lambda_1, \lambda_2)$ and the norm $|\vec{\lambda}|$ denotes the total number of cells. The explicit form of $Z_f(\mu_i, a; \vec{\lambda})$ and $Z_v(a; \vec{\lambda})$ reads

$$\begin{aligned} Z_f(\mu_i,a;\vec{\lambda}) = &\prod_{s\in\lambda_1} (\phi(a,s)+\mu_1)(\phi(a,s)+\mu_2)(\phi(a,s)+\mu_3)(\phi(a,s)+\mu_4) \\ &\times \prod_{s\in\lambda_2} (\phi(-a,s)+\mu_1)(\phi(-a,s)+\mu_2)(\phi(-a,s)+\mu_3)(\phi(-a,s)+\mu_4)\,, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} Z_v(a;\vec{\lambda}) = &\prod_{s\in\lambda_1} E_{\lambda_1,\lambda_2}\big(2a\big|s\big)\big(Q-E_{\lambda_1,\lambda_2}\big(2a\big|s\big)\big)E_{\lambda_1,\lambda_1}\big(0\big|s\big)\big(Q-E_{\lambda_1,\lambda_1}\big(0\big|s\big)\big) \\ &\times \prod_{s\in\lambda_2} E_{\lambda_2,\lambda_1}\big(-2a\big|s\big)\big(Q-E_{\lambda_2,\lambda_1}\big(-2a\big|s\big)\big)E_{\lambda_2,\lambda_2}\big(0\big|s\big)\big(Q-E_{\lambda_2,\lambda_2}\big(0\big|s\big)\big)\,. \end{aligned} \tag{2.5}$$

Functions $E_{\lambda,\mu}(x|s)$ and $\phi(x,s)$ are defined as

$$\begin{aligned} E_{\lambda,\mu}\big(x\big|s\big) &= x - b\, l_\mu(s) + b^{-1}(a_\lambda(s)+1)\,, \\ \phi(x,s) &= x + b(i-1) + b^{-1}(j-1)\,. \end{aligned} \tag{2.6}$$

To explain our notation we adjust $(\lambda)_i$ to ith row of the Young diagram λ and denote $(\lambda)_j^T$ the length of the jth column, where T stands for a matrix transposition. For a cell $s = (i, j)$ such that i and j label a respective row and a column, the arm-length function $a_\lambda(s)$ and the leg-length function $l_\lambda(s)$ are given by

$$a_\lambda(s) = (\lambda)_i - j\,, \qquad l_\lambda(s) = (\lambda)_j^T - i\,. \tag{2.7}$$

The parameters of Nekrasov partition function are related to the parameters of the conformal block as follows

$$\begin{aligned} \mu_1 &= \frac{Q}{2} - (P_1+P_2), & \mu_2 &= \frac{Q}{2} - (P_1-P_2), \\ \mu_3 &= \frac{Q}{2} - (P_3+P_4), & \mu_4 &= \frac{Q}{2} - (P_3-P_4), \\ a &= P_0, & \nu &= 2\left(\frac{Q}{2}-P_1\right)\left(\frac{Q}{2}-P_3\right). \end{aligned} \tag{2.8}$$

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Following ref. [9] one can find an orthogonal basis $|P, \vec{\lambda}\rangle$ numerated by pairs of Young diagrams in $\mathcal{A}_N$ module that reproduces Nekrasov decomposition (2.2)–(2.3). The Nekrasov sum is obtained simply by inserting the following unity decomposition

$$\sum_{\vec{\lambda}} \frac{|P, \vec{\lambda}\rangle\langle P, \vec{\lambda}|}{\langle P, \vec{\lambda}|P, \vec{\lambda}\rangle} = \mathbb{I} \tag{2.9}$$

in the correlation function $\langle \Phi_{\Delta_1}(x)\Phi_{\Delta_2}(0)\Phi_{\Delta_3}(1)\Phi_{\Delta_4}(\infty)\rangle$ between each two of the primary fields.

3 The AGT-like representation for the $\mathcal{W}_2$ minimal models

Conformal field theories $\mathcal{M}_{p,p'}$ are characterized by the central charge of the Virasoro algebra

$$c = 1 - 6\frac{(p'-p)^2}{pp'}, \qquad b = i\sqrt{\frac{p'}{p}}\,. \tag{3.1}$$

There are $(p-1)\times(p'-1)/2$ primary fields $\Phi_{l,k}$ ($l = 1,\ldots,p-1$ and $k = 1,\ldots,p'-1$) in the model. The conformal dimensions are determined by the Kac formula

$$\Delta_{m,n} = \frac{(p'm - pn)^2 - (p'-p)^2}{4pp'}\,. \tag{3.2}$$

Note that there is a symmetry $\Delta_{m,n} = \Delta_{p-m,p'-n}$ and $\Delta_{m,n} = \Delta_{p+m,p'+n}$. In the Liouville parametrization the values of the parameter P corresponding to the degenerate values are

$$P_{m,n} = P(\Delta_{m,n}) = \frac{mb + nb^{-1}}{2}\,. \tag{3.3}$$

For the Virasoro minimal models the fusion rules which describe conformal families that appear in the operator product expansion of two primary fields are [1]

$$\Phi_{(r,s)} \otimes \Phi_{(m,n)} = \sum_{\substack{k=|m-r|+1,\\ k-m+r-1 \text{ even}}}^{\min(m+r-1,2p'-1-m-r)} \; \sum_{\substack{l=|n-s|+1,\\ l-n+s-1 \text{ even}}}^{\min(n+s-1,2p-1-n-s)} \left[\Phi_{(k,l)}\right]. \tag{3.4}$$

3.1 Reduction of the basis in the minimal models

The AGT representation (2.2) is not directly applicable to minimal models. The reason is that the fields of minimal models are degenerate. Indeed, vectors of invariant submodules possess zero norms so that expression (2.2) which contains these norms in the denominator is singular in this case. One comment about relations between general and minimal models conformal block is in order. The 4-point conformal blocks on the sphere in $\mathcal{M}_{p,p'}$ minimal model can be derived from the expression for the non-degenerate fields by means of the following procedure of the analytic continuation.

Let us consider first CFT for general value of the central charge parameter, and let

$$\langle \Phi_{\Delta_1}(x)\Phi_{\Delta_2}(0)\Phi_{\Delta_3}(1)\Phi_{\Delta_4}(\infty)\rangle \tag{3.5}$$

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be the correlation function of four non-degenerate fields. Suppose we have some orthogonal basis in the Verma module $[\Phi_\Delta]$ denoted by $|N\rangle$. For the conformal block we have

$$\mathcal{B}(x)=\sum_{N=0}^{\infty} x^N \frac{\langle\Phi_{\Delta_1}\Phi_{\Delta_2}|N\rangle\langle N|\Phi_{\Delta_3}\Phi_{\Delta_4}\rangle}{\langle N|N\rangle}\,. \tag{3.6}$$

Now, we are interested in the conformal blocks of minimal models. They can be derived from (3.6) in two steps. First, we fix $c=c_{p,p'}$ (3.1) and external dimensions $\Delta_i=\Delta_{m_i,n_i}(p,p')$ (3.2). We note that the set of the dimensions should be admissible for the fusion rules of the minimal models. Second, we take a limit $\Delta\to\Delta_{mn}$. More precisely, we use the parametrization $\Delta=a(Q-a)$ and take $a\to a_{mn}$. Among the descendants of the primary field Φ_Δ will appear singular vectors (and their descendants). Let us denote the singular vector creation operator as D_{mn}. One can derive [16] that the norm of the vector $D_{mn}\Phi_{\Delta(a)}$ has zero of the first order in the limit $a\to a_{mn}$, or, explicitly,

$$\langle D_{mn}\Phi_{\Delta(a)}|D_{mn}\Phi_{\Delta(a)}\rangle\sim(a-a_{mn})\,. \tag{3.7}$$

One can check that each of the three-point functions in the numerator of (3.6) has also zero of the first order in this limit as it was also shown in ref. [16]. Hence we get a second-order zero in the numerator and a first-order zero in the denominator. The same result occurs for all descendants of the basic singular vectors $D_{mn}\Phi_{\Delta(a)}$ and $D_{p-m,p'-n}\Phi_{\Delta(a)}$. So we can just drop out the contribution of the vectors in the decomposition (3.6) which fall in the invariant subspace generated by the singular vector $D_{mn}\Phi_{\Delta(a)}$. In fact, this procedure can be considered as a definition of the conformal blocks of the minimal models.

On the other hand, the above procedure can be effectively used to re-derive conformal blocks in minimal models from the AGT representation for non-degenerate fields.

3.2 Combinatorial representation

Using the idea of the orthogonal basis for minimal models we must drop out all basis elements belonging to invariant submodules. Even though the explicit construction of the vectors is not known we can use (2.2) to find for which elements of the AGT basis the norm vanishes

$$\langle\vec{\lambda}|\vec{\lambda}\rangle=0\,. \tag{3.8}$$

This leads us to some additional restrictions on the form of Young diagrams parametrising basis elements in the irreducible representations of minimal models.

Proposition 1. *Consider a degenerate module parameterized by $P_{n,m}$ (3.3). Function $Z_v(\Delta_{n,m},\lambda_1,\lambda_2)$ (2.5) is not equal to zero provided that Young diagrams are ordered as*

$$(\lambda_1)_i\geq(\lambda_2)_{i+m-1}-n+1\,, \tag{3.9}$$

where $(\lambda_\alpha)_i$ are lengths of i-th rows of Young diagrams λ_α, $\alpha=1,2$.

The proof is relegated to appendix A. Next, we consider a minimal model $\mathcal{M}_{p,p'}$ so that parameter b is given by formula (3.1). It follows that the function Z_v can have additional zeros that restrict the basis. One proves[1]

[1] The analogous theorem has been established in [14] within the representation theory of the toroidal algebra $\mathbf{gl}_1(\mathrm{q}_1,\mathrm{q}_2)$.

Proposition 2. *Consider a degenerate module with dimension $\Delta_{n,m}$ in a minimal model $\mathcal{M}_{p,p'}$. Function $Z_v(p,p'|\Delta_{n,m},\lambda_1,\lambda_2)$ is not equal to zero provided the set of Young diagrams belongs to the region*

$$R^{(p,q)}_{n,m}: \quad (\lambda_\alpha)_i \geq (\lambda_{\alpha+1})_{i+m_\alpha-1} - n_\alpha + 1\,, \qquad \alpha = 1,2\,, \tag{3.10}$$

where $(\lambda_\alpha)_i$ are lengths of i-th rows of Young diagrams λ_α, and $(n_1,m_1) = (n,m)$ and $(n_2,m_2) = (p-n,p'-m)$. We use the identification $\lambda_3 = \lambda_1$.

The proof is relegated to appendix A. From the Proposition 2 we derive the following explicit representation for the 4-point conformal block in the Virasoro minimal models

$$\mathcal{B}(P_{n_i,m_i};P_{n,m};x) = (1-x)^{-\nu}\sum_{N=0}^{\infty} x^N \sum_{\vec{\lambda}\in R^{(p,q)}_{n,m}}^{|\vec{\lambda}|=N} \frac{Z_f(\mu_i,a;\vec{\lambda})}{Z_v(a;\vec{\lambda})}\,, \tag{3.11}$$

where P_{n_i,m_i} denote external conformal dimensions, $P_{n,m}$ denote internal one and the summation region $R^{(p,q)}_{n,m}$ is defined in (3.10). An analogues result on the conformal blocks in the Virasoro minimal models is obtained in ref. [17].

4 Testing AGT for Virasoro minimal models

In this section we consider a 4-point conformal block $\mathcal{B}(\Delta_{1,n_j};x)$ with at least one degenerate field $\Phi_{1,2}$. This function satisfies the null vector equation which turns out to be the Riemann equation [1]

$$\frac{d^2\mathcal{B}}{dx^2} + \left(\frac{1-\alpha-\alpha'}{x} + \frac{1-\gamma-\gamma'}{x-1}\right)\frac{d\mathcal{B}}{dx} + \left(\frac{\alpha\alpha'}{x^2} + \frac{\gamma\gamma'}{(x-1)^2} + \frac{\beta\beta'-\alpha\alpha'-\gamma\gamma'}{x(x-1)}\right)\mathcal{B} = 0\,, \tag{4.1}$$

with the parameters

$$\begin{aligned} \alpha &= \frac{n_1-1}{2}\kappa\,, & \alpha' &= 1 - \frac{n_1+1}{2}\kappa\,, \\ \beta &= \frac{2-n_3}{2}\kappa\,, & \beta' &= \frac{n_3+2}{2}\kappa - 1\,, \\ \gamma &= \frac{n_2-1}{2}\kappa\,, & \gamma' &= 1 - \frac{n_2+1}{2}\kappa\,. \end{aligned} \tag{4.2}$$

Here, the parameter κ is related to the labels (p,p') of the minimal model as $\kappa = p/p'$. The equation (4.1) has two linearly independent solutions with power-law behavior at $x=0$. They correspond to two conformal blocks in the S-channel

$$\begin{aligned} \mathcal{B}_1(x) &= x^{\alpha}(1-x)^{\gamma}{}_2F_1(\alpha+\beta+\gamma,\alpha+\beta'+\gamma,1+\alpha-\alpha';x)\,, \\ \mathcal{B}_2(x) &= x^{\alpha'}(1-x)^{\gamma}{}_2F_1(\alpha'+\beta+\gamma,\alpha'+\beta'+\gamma,1+\alpha'-\alpha;x)\,. \end{aligned} \tag{4.3}$$

In what follows we reproduce the above hypergeometric functions in the form of the diagrammatic decomposition (2.2), (3.11) applied to the Lee-Yang model.

4.1 Lee-Yang model

For the Lee-Yang model $\mathcal{M}_{2,5}$ there is only one non-trivial primary field in the Kac table $\Phi_{1,2}(=\Phi_{1,3})$ with conformal weight $-1/5$. Due to the fusion rules (3.4), there are two possible intermediate channels $\Phi_{1,3}$ and $\Phi_{1,1}$, the corresponding 4-point conformal blocks are (we omit four external parameters $\Delta_{1,2}$):

$$\begin{aligned}
\mathcal{B}(\Delta_{1,1};x) &= (1-x)^{1/5}{}_2F_1(2/5,3/5,6/5,x) = 1-\frac{x^2}{55}-\frac{x^3}{55}-\frac{9x^4}{550}-\frac{4x^5}{275}+\dots\,,\\
\mathcal{B}(\Delta_{1,3};x) &= (1-x)^{1/5}{}_2F_1(1/5,2/5,4/5,x) = 1-\frac{x}{10}-\frac{4x^2}{75}-\frac{9x^3}{250}-\frac{962x^4}{35625}+\dots\,.
\end{aligned} \tag{4.4}$$

In this example, eq. (3.11) restricts the sum over Young diagrams to be of a general form $(\lambda,\varnothing)$ and $(\varnothing,\lambda)$ for the expansions coefficients $F^{(N)}(\Phi_{1,1})$ and $F^{(N)}(\Phi_{1,3})$ defined in (2.2).

At level 1, we have $(\square,\varnothing)$ and $(\varnothing,\square)$. The corresponding contributions are

$$\begin{aligned}
F^{(1)}_{\Delta_{1,1}} &= -\frac{(a+\mu_1)(a+\mu_2)(a+\mu_3)(a+\mu_4)}{2a\epsilon_1\epsilon_2(2a+\epsilon_1+\epsilon_2)}\,,\\
F^{(1)}_{\Delta_{1,3}} &= -\frac{(a-\mu_1)(a-\mu_2)(a-\mu_3)(a-\mu_4)}{2a\epsilon_1\epsilon_2(2a-\epsilon_1-\epsilon_2)}\,,
\end{aligned} \tag{4.5}$$

where $\epsilon_1=b$ and $\epsilon_2=b^{-1}$. With (2.8) one can check that ($\nu=-1/5$)

$$F^{(1)}_{\Delta_{1,1}}+\nu=0\,,\qquad F^{(1)}_{\Delta_{1,3}}+\nu=-\frac{1}{10}\,. \tag{4.6}$$

At level 2, we have $(\square\!\square,\varnothing)$ and $(\varnothing,\square\!\square)$. The corresponding contributions can be easily derived, and one can check that

$$F^{(2)}_{\Delta_{1,1}}+\nu F^{(1)}_{\Delta_{1,1}}+\frac{\nu(\nu+1)}{2}=-\frac{1}{55}\,,\qquad F^{(2)}_{\Delta_{1,3}}+\nu F^{(1)}_{\Delta_{1,3}}+\frac{\nu(\nu+1)}{2}=-\frac{4}{75}\,. \tag{4.7}$$

Let us compare these results with the diagrammatic decomposition at the arbitrary level N. Recall that intermediate fields are $\Phi_{1,1}$ and $\Phi_{1,3}$, while all external fields are $\Phi_{1,2}$.

Intermediate field $\Phi_{1,1}$. Following the general consideration of section 3.1 we conclude that diagrams falling out of the Nekrasov decomposition correspond to zeros of functions $\phi(a|s)+\mu_i=0$ or $\phi(-a|s)+\mu_i=0$, cf. (2.4).

One obtains that $\Delta_{1,1}=0$ and respective $P_0=\pm Q/2$. We choose $P_0=Q/2\equiv a$, see (2.8). Also,

$$\mu_1=\mu_3=\frac{Q}{2}-2P_{1,2}=a-2P_{1,2}\,,\qquad \mu_2=\mu_4=\frac{Q}{2}=a\,. \tag{4.8}$$

Then, consider a factor $\phi(-a|s)+\mu_2=\phi(-a|s)+a=0$ in the product over cells of the second Young diagram λ_2, cf. (2.4). It follows that this equation reduces to $b^{-1}(i-1)+b(j-1)=0$, and the zeros are given by $i=j=1$. It follows that $\lambda_2=\varnothing$. On the other hand, consider a factor $\phi(a|s)+\mu_1=2a-2P_{1,2}+b^{-1}(i-1)+b(j-1)=0$ in the product over cells of the first Young diagram λ_1, cf. (2.4). The resulting equation is

$b^{-1}(i-2)+b(j-1)=0$ and the zeros are given by $i=2$ and $j=1$. It follows that λ_1 is an arbitrary length N row, where N is the level.

We conclude here that a decomposition involves pairs of diagrams of the form

$$(\lambda_1 = \text{ a row of length } N, \;\; \lambda_2 = \varnothing). \tag{4.9}$$

Provided these facts the diagrammatic decomposition yields the final formula

$$F^{(N)}_{\Delta_{1,1}} = \frac{1}{N!}\prod_{n=1}^{N} \frac{\big(b(n-1)-b^{-1}\big)\big(bn+b^{-1}\big)}{b\big(b(n+1)+2b^{-1}\big)}\,. \tag{4.10}$$

The right-hand-side is given by N-th expansion coefficient of the hypergeometric function (4.4). One can check that $F^{(1)}_{\Delta_{1,1}} = -1/5$ that agrees with (4.6).

Intermediate field $\mathbf{\Phi_{1,3}}$. Quite analogously, we consider the case of another intermediate dimension. One obtains that $\Delta_{1,2}$ corresponds to $P_0 = a = P_{1,3} = \frac{1}{2}(b+3b^{-1})$. An equivalent form of a reads $a = Q/2 + b^{-1}$ or $Q/2 = a - b^{-1}$. Also,

$$\mu_1 = \mu_3 = a - 2P_{1,2} - b^{-1}\,, \qquad \mu_2 = \mu_4 = \frac{Q}{2} = a - b^{-1}\,. \tag{4.11}$$

Consider then a factor $\phi(-a|s)+\mu_2 = \phi(-a|s)+a-b^{-1} = 0$ in the product over cells of the second Young diagram λ_2, cf. (2.4). It follows that this equation reduces to $b^{-1}(i-2)+b(j-1)=0$, and the zeros are given by $i=2$, $j=1$. It follows that $\lambda_2 =$ an arbitrary length row. On the other hand, consider a factor $\phi(a|s)+\mu_1 = 2a-2P_{1,2}-b^{-1}+b^{-1}(i-1)+b(j-1)=0$ in the product over cells of the first Young diagram Y_1, cf. (2.4). The resulting equation is $b^{-1}(i-1)+b(j-1)=0$ and the zeros are given by $i=j=1$. It follows that $\lambda_1 = \varnothing$ and therefore λ_1 is an arbitrary length N row, where N is the level.

We conclude here that a decomposition involves pairs of diagrams of the form

$$(\lambda_1 = \varnothing, \;\; \lambda_2 = \text{ a row of length } N). \tag{4.12}$$

Provided these facts the diagrammatic decomposition yields the final formula

$$F^{(N)}_{\Delta_{1,3}} = \frac{1}{N!}\prod_{n=1}^{N} \frac{\big(b(n-1)-b^{-1}\big)\big(b(n-2)-3b^{-1}\big)}{b\big(b(n-1)-2b^{-1}\big)}\,. \tag{4.13}$$

These ratios are again expansion coefficients of the hypergeometric function in (4.4).

5 Generalization for $\mathcal{W}_N$ minimal models

In this section we are interested in unitary $\mathcal{W}_N$ minimal models $\mathcal{M}_{p,p+1}(N)$ with the central charge

$$c = (N-1)\Big(1 - \frac{N(N+1)}{p(p+1)}\Big)\,. \tag{5.1}$$

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The primary fields are labelled by two $\widehat{\mathfrak{sl}}_N$ weights $\Lambda_+ = \sum_{s=1}^{N-1}(m_s-1)\omega_s$ and $\Lambda_- = \sum_{s=1}^{N-1}(n_s-1)\omega_s$, $(m_s, n_s \in \mathbb{Z}_{>0})$, where ω_s are the fundamental weights of the Lie algebra $\mathfrak{sl}_N$, and

$$\sum_{s=1}^{N} m_s = p, \qquad \sum_{s=1}^{N} n_s = p+1\,, \tag{5.2}$$

where m_N and n_N are defined by the above formulas. In the Liouville-like parametrization we write Φ_P, where the vector $P = (P^{(1)}, \ldots, P^{(N-1)})$, and

$$P = Q\rho - \mathbf{a}, \qquad a_{m,n} = -mb - nb^{-1}, \qquad Q = b + b^{-1} \quad \text{and} \quad b^2 = -\frac{p}{p+1}\,, \tag{5.3}$$

where ρ is the Weyl vector (a half-sum of positive roots). Unlike the Virasoro case, in a $\mathcal{W}_N$ theory the conformal blocks are not fixed by conformal and $\mathcal{W}_N$ invariance [18]. The bootstrap program for k-point correlation functions can be performed only if the charges of $k-2$ fields are proportional to the first fundamental weight ω_1 of the Lie algebra $\mathfrak{sl}_N$ [19]. We consider the correlation functions of this kind

$$\langle \Phi_P(z_1)\Phi_{a_2}(z_2)\Phi_{a_3}(z_3)\ldots\Phi_{a_{k-1}}(z_{k-1})\Phi_{\hat{P}}(z_k)\rangle\,, \tag{5.4}$$

where parameters in points $z_2, \ldots, z_{k-1}$ correspond to degenerate representations of minimal models described above, while fields in z_1 and z_k are general primary fields of the minimal models. It is convenient to change the variables

$$z_{i+1} = q_i q_{i+1}\cdots q_{k-3} \quad \text{for} \quad i = 1, \ldots, k-3\,. \tag{5.5}$$

The holomorphic dependence of the correlation functions is encoded in the conformal block functions

$$\mathcal{B}(q_1, \ldots, q_{k-3}|P, P_1, \ldots, P_{k-3}, \hat{P}|a_2, \ldots, a_{k-1})\,, \tag{5.6}$$

where the momenta $P_1, \ldots, P_{k-3}$ correspond to the fields in the intermediate channels of the conformal block decomposition. Recall that in the minimal models, the weights of all external and intermediate fields are related by fusion rules.

In [14, 15] there was proposed the orthogonal basis for modules of the toroidal algebra in some special limits. This basis is labelled by the special sets of N-tuples of Young diagrams. It was shown that the characters of these diagrams coincide with the characters of $\mathcal{W}_N$ minimal models up to a contribution related to the presence of extra Heisenberg algebra. It follows that the found basis should define AGT basis in the highest weight representations of the $\mathcal{H}\otimes\mathcal{W}_N$ algebra [7] (see also [20]) restricted for the minimal models thereby giving rise to the AGT representation for conformal blocks in $\mathcal{W}_N$ minimal models. The basis vectors are enumerated by N-tuples of Young diagrams with some additional restrictions formulated below.

We conjecture the following explicit form of the k-point conformal block in $\mathcal{W}_N$ minimal models $\mathcal{M}_{p,p+1}$ $(p \geq N-1)$. In this case the conformal block (5.6) for non-degenerate parameters is related to the instanton part of the Nekrasov partition function for the quiver gauge theory and can be written explicitly [4, 7]

$$\mathcal{B} = \prod_{j=1}^{k-3}\prod_{l=j}^{k-3}(1 - q_j\cdots q_l)^{-a_{j+1}(Q-a_{l+2}/N)}\;\mathcal{F}\,, \tag{5.7}$$

and

$$\mathcal{F} = 1 + \sum_{\vec{j}} q_1^{j_1} q_2^{j_2} \dots q_{k-3}^{j_{k-3}} \, \mathbb{Z}_{\vec{j}} \,, \tag{5.8}$$

where $\vec{j} = (j_1, \dots, j_{k-3})$, with the coefficients

$$\mathbb{Z}_{\vec{j}} = \sum_{\vec{\lambda}_i \in \mathcal{R}_{\mathbf{m}_i, \mathbf{n}_i}}^{|\vec{\lambda}|=r_i} \mathbb{N}^{-1}(\vec{\lambda}_1, P_1) \dots \mathbb{N}^{-1}(\vec{\lambda}_{k-3}, P_{k-3}) \times \tag{5.9}$$
$$\times \mathbb{F}_{\varnothing}^{\vec{\lambda}_1}(a_2, P_1, P) \mathbb{F}_{\vec{\lambda}_1}^{\vec{\lambda}_2}(a_3, P_2, P_1) \dots \mathbb{F}_{\vec{\lambda}_{k-4}}^{\vec{\lambda}_{k-3}}(a_{k-2}, P_{k-3}, P_{k-4}) \mathbb{F}_{\vec{\lambda}_{k-3}}^{\varnothing}(a_{k-1}, \hat{P}, P_{k-3}) \,.$$

Here $\vec{\lambda}_i = (\lambda_i^{(1)}, \dots, \lambda_i^{(N)})$ are N-tuples of Young diagrams, and index $i = 1, \dots, k-3$ enumerates intermediate channels. Each component of the vectors $\lambda_i^{(s)}$ is a finite integer partition. We define the norms as a total number of boxes in the Young diagram representation $|\vec{\lambda}_i| = \sum_{s=1}^{N} |\lambda_i^{(s)}|$. The norms are $\mathbb{N}(\vec{\lambda}, P) = \mathbb{F}_{\vec{\lambda}}^{\vec{\lambda}}(0, -P, P)$, and the general matrix element is given by [7]

$$\mathbb{F}_{\vec{\lambda}'}^{\vec{\lambda}}(a, P, P') = \prod_{i,j=1}^{N} \prod_{t' \in \lambda'_i} (Q - E_{\lambda'_i, \lambda_j}(x_j - x'_i | t') - a/N) \prod_{t \in \lambda_j} (E_{\lambda_j, \lambda'_i}(x'_i - x_j | t) - a/N) \,, \tag{5.10}$$

where $x_j = (h_j, P)$ (vectors h_i are the weights of the first fundamental representation of $\mathfrak{sl}_N$ with the the highest weight ω_1, i.e. $h_i = \omega_1 - e_1 - \dots - e_{i-1}$, where e_k are simple roots, and $h_i h_j = 1 - \frac{1}{N}\delta_{ij}$), while the function $E_{\lambda,\mu}(x|t)$ is defined in (2.6).[2]

Extending the Proposition 2 we conjecture that the summation in formula (5.9) is further restricted by the following region

$$\mathcal{R}_{\mathbf{m},\mathbf{n}} = \left\{ \vec{\lambda} \;\middle|\; \left(\lambda^{(s)}\right)_j \geq \left(\lambda^{(s+1)}\right)_{j+m_s-1} - n_s + 1, \text{ where } s = 1, \dots, N, \;\; j \in \mathbb{Z}_{>0} \right\} \,, \tag{5.11}$$

where $\lambda^{(N+1)} \equiv \lambda^{(1)}$, while m_s and n_s are components of extended $\mathfrak{sl}_N$ weight vectors $\mathbf{m} = (m_1, \dots, m_N)$ and $\mathbf{n} = (n_1, \dots, n_N)$, see (5.2).

6 Conclusions

In this paper we studied application of the AGT correspondence to minimal models of Virasoro and $\mathcal{W}_N$ algebras. We used the conjecture (supported by the comparison of the characters of the corresponding representations) that the chiral $\mathcal{W}_N$ algebra appears in the conformal limit from the toroidal algebra $\mathbf{gl}_1(\mathrm{q}_1, \mathrm{q}_2)$ [14, 15]. This connection reveals a nice integrable structure of degenerate modules of the algebra $\mathcal{A}_N = \mathcal{H} \otimes \mathcal{W}_N$. It appears for the $\mathcal{W}_N$ central charge corresponding to the minimal models $\mathcal{M}_{p,p'}$ arising from $\mathbf{gl}_1(\mathrm{q}_1, \mathrm{q}_2)$ once the parameters are constrained by the following wheel condition $\mathrm{q}_1^p \mathrm{q}_2^{p'} = 1$. In particular, $\mathcal{A}_N$ module inherits some (reduced) orthogonal basis defined naturally in $\mathbf{gl}_1(\mathrm{q}_1, \mathrm{q}_2)$ modules. Hence, it is natural to use this basis for the evaluation of the $\mathcal{W}_N$

[2]Strictly speaking, formula (5.10) is valid for the case $N \geq 3$ only. For $N = 2$ one should use a different normalization.

conformal block functions. We have checked that using this basis for Virasoro minimal models we get the AGT-like representation for the conformal blocks of minimal models once the consequences of emergence of invariant subspaces in the degenerate representations is taken into account.

Our main result is the explicit expression for the conformal blocks in the Virasoro minimal models (3.11) and the conjecture on the form of AGT representation for $\mathcal{W}_N$ conformal blocks (5.9)–(5.11). The difference from the original AGT expression for non-degenerate representations is encoded in additional restrictions on the summation region over Young diagrams.

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A Proof of the propositions

In order to analyse zeros of functions Z_f and Z_v we use simple method described below. It turns out that for particular values of external/internal dimensions and/or the central charge these functions take the form of a vector on the (b^{-1}, b) plane with integer-valued coordinates

$$F(s) \equiv (j-K)b \pm (i-L)b^{-1} = 0\,, \tag{A.1}$$

where K, L are some integers and (i, j) are coordinates of a cell $s \in \lambda_1$ or $s \in \lambda_2$. For general value of the Liouville coupling b there is a unique solution $i = L$ and $j = K$ which defines a cell where function $F(s) = 0$. Obviously, only positive numbers K, L make sense so one can define a set of "admissible" diagrams (λ_1, λ_2) which do not contain cells with coordinates (L, K). In what follows $(\bar{x}, \bar{y})$ and $(\bar{\bar{x}}, \bar{\bar{y}})$ denote coordinates of excluded cells (those that give zeros of the functions) in Young diagrams λ_1 and λ_2, respectively.

For the minimal models $\mathcal{M}_{p,p'}$ the Liouville coupling takes particular value (3.1) and therefore equation (A.1) allows for more (infinitely many) solutions, namely, $i = L \pm \alpha p'$ and $j = K + \alpha p$. Here arbitrary parameter $\alpha \in \mathbb{Z}$ because p and p' are coprimes. Note that the case $\alpha = 0$ reproduces zeros described above for general Liouville coupling b. It follows that for the minimal models more zeros appear but sometimes new zeros are "weaker" than those for $\forall\, b$. It is worth noting that actually there are infinitely many new zeros but generally values their coordinates are restricted from below by minimal values that define the form of admissible diagrams.

Proof of the Proposition 1. First of all we note that the terms with $E_{\lambda_\alpha,\lambda_\alpha}(x|s)$ are not equal to zero. The only source of zeros is provided by terms containing $E_{\lambda_\alpha,\lambda_\beta}(x|s)$ with the pair of different Young diagrams.

We start with empty diagram λ_2 and subsequent increase a number of its rows. Another trick is to consider cases $\Delta_{1,m}$ and $\Delta_{n,1}$ separately so that the general case of $\Delta_{n,m}$ is a straightforward combination of the previous ones. On the plane (b^{-1}, b) we have

$$E_{\lambda_1,\lambda_2}(2a|s) = 0 \;: \qquad b^{-1}(\bar{h}_j - \bar{i} + m + 1) - b(\bar{\bar{k}}_i - \bar{j} - n) = 0 \tag{A.2}$$

or

$$\bar{i} = \bar{h}_j + m + 1\,, \qquad \bar{j} = \bar{\bar{k}}_i - n\,. \tag{A.3}$$

and

$$E_{\lambda_2,\lambda_1}(-2a|s) = 0 \;: \qquad b^{-1}(\bar{\bar{h}}_j - \bar{\bar{i}} - m + 1) - b(\bar{k}_i - \bar{\bar{j}} + n) = 0 \tag{A.4}$$

or

$$\bar{\bar{i}} = \bar{\bar{h}}_j - m + 1\,, \qquad \bar{\bar{j}} = \bar{k}_i + n\,. \tag{A.5}$$

Consider diagrams $(\lambda_1, \varnothing)$ with any λ_1, and show that these do not produce zeros of Z_v. Indeed, in this case $\bar{\bar{k}}_i = \bar{\bar{h}}_j = 0$ so that from (A.3) one derives $\bar{j} = -n < 0$. Equation (A.4) is absent in this case. Therefore, we conclude that pairs $(\lambda_1, \varnothing)$ are admissible.

Dimension $\boldsymbol{\Delta_{1,m}}$. Consider pairs of diagrams (λ_1, λ_2), where $\lambda_2 = (N_1, 0, 0, \ldots)$ is a row of arbitrary length N_1. Coordinates of a cell in a row are $(1, \bar{\bar{j}})$, where $\bar{\bar{j}} = 1, \ldots, N_1$. Our aim is to show that absence of zeros imposes constraints on the form of diagram λ_1.

From equation (A.5) one obtains coordinates of a cell that produces a zero,

$$\bar{\bar{i}} = \bar{\bar{h}}_j - m + 1\,, \qquad \bar{\bar{j}} = \bar{k}_i + 1\,. \tag{A.6}$$

In the case of $\lambda_2 =$ row one derives from the first equation above that $\bar{\bar{h}}_j = m$, and it follows that $m = 1$ to have a solution. Another way around, it implies that zeros appear for those λ_2 that have at least m rows, i.e., $\exists\ \bar{\bar{j}}$ such that $\bar{\bar{h}}_j \geq\ m$.

For $m = 1$ the second equation above says that in order to have a zero a first row of λ_1 is to be of length $\bar{k}_1 \leq\ N_1 - 1$. One concludes that for $m \neq 1$ zeros are absent for any λ_1, while for $m = 1$ zeros are absent for pairs of diagrams with lengths subject to

$$\bar{k}_1 \geq \bar{\bar{k}}_1\ . \tag{A.7}$$

Consider then pairs of diagrams (λ_1, λ_2), where λ_2 is an arbitrary diagram with m rows of ordered lengths $N_1 \geq N_2 \geq \ldots \geq N_m$. Substituting $\bar{\bar{i}} = 1$ and $\bar{\bar{j}} = 1, \ldots, N_m$ to the first equation in (A.6) gives solution, $1 = m - m + 1$. The second equation in (A.6) takes the form $\bar{\bar{j}} = \bar{\bar{k}}_1 + 1$ which defines $\bar{\bar{k}}_1 = 0, 1, \ldots, N_m - 1$, while lengths $\bar{\bar{k}}_\alpha$, where $\alpha \geq 2$ are arbitrary but no bigger than $\bar{\bar{k}}_1$. One concludes that zeros are absent for pairs of diagrams with lengths subject to the following inequality

$$\bar{k}_1 \geq \bar{\bar{k}}_m\ , \tag{A.8}$$

which is obviously generalizes (A.7) to arbitrary value of m.

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As the next step one considers an arbitrary diagram λ_2 with $m+l$ rows, where $l = 1, 2, \ldots$. The diagram λ_2 naturally splits in two subdiagrams $\lambda_2 = \lambda_2' \oplus \lambda_2''$, where the first factor is a diagram composed of first $m-1$ rows of λ_2, while the second one is a diagram composed of the remaining rows of λ_2. Considering equations (A.6) one shows that zeros are absent when $\lambda_2'' \subseteq \lambda_1$. Equivalently,

$$\bar{k}_i \geq \bar{\bar{k}}_{m+i-1} \ . \tag{A.9}$$

This inequality completely describes admissible pairs of diagrams (λ_1, λ_2) in the case of dimension $\Delta_{1,m}$.

Dimension $\boldsymbol{\Delta_{n,1}}$. Consider pairs of diagrams (λ_1, λ_2), where $\lambda_2 = (N_1, 0, 0, \ldots)$ is a row of arbitrary length N_1. From equation (A.5) one obtains coordinates of a cell that produces a zero,

$$\bar{\bar{i}} = \bar{\bar{h}}_j \ , \qquad \bar{\bar{j}} = \bar{k}_i + n \ . \tag{A.10}$$

In the case of $\lambda_2 =$ the first equation is automatically satisfied for $\bar{\bar{j}} = 1, \ldots, N_1$. The second equation says that zeros appear when the first row of λ_1 is of length less than $N_1 - n$. This is to say that zeros are absent for pairs of diagrams with lengths subject to

$$\bar{k}_1 \geq \bar{\bar{k}}_1 - n + 1 \ . \tag{A.11}$$

This inequality naturally generalizes to the case of λ_2 with any number of rows. Namely,

$$\bar{k}_i \geq \bar{\bar{k}}_i - n + 1 \ . \tag{A.12}$$

Dimension $\boldsymbol{\Delta_{n,m}}$. To find admissible diagrams in the case of arbitrary dimension $\Delta_{n,m}$ one simply combines previously considered cases of $\Delta_{1,m}$ and $\Delta_{n,1}$ to obtain formula (3.9)

$$\bar{k}_i \geq \bar{\bar{k}}_{i+m-1} - n + 1 \ . \tag{A.13}$$

In particular, this relation implies that in order to produce a zero the second diagram Y_2 should include a rectangle of length n and height m.

To conclude the proof one notes that zeros are also contained in $E_{\lambda_2,\lambda_1}(-2a|s) - Q = 0$. Equations that define coordinates of a zero are those in (A.5) but with $m \to m+1$ and $n \to n+1$. Admissible diagrams are defined by inequality $\bar{k}_i \geq \bar{\bar{k}}_{i+m} - n$ which is weaker than (A.13) though. Indeed, using a definition of a Young diagram one observes that $\bar{\bar{k}}_{i+m-1} \geq \bar{\bar{k}}_{i+m}$ which takes (A.13) to the form $\bar{k}_i \geq \bar{\bar{k}}_{i+m} - n + 1 > \bar{\bar{k}}_{i+m} - n$.

Proof of the Proposition 2. The proof is similar to that one of Proposition 1. The difference is that more zeros appear. Indeed, reconsider condition $E_{\lambda_2,\lambda_1}(-2a|s) = 0$ from (A.4). The coordinates of zeros are given by

$$\bar{\bar{i}} = \bar{\bar{h}}_j - m + 1 - \alpha p' \ , \qquad \bar{\bar{j}} = \bar{k}_i + n + \alpha p \ , \tag{A.14}$$

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for any $\alpha \in \mathbb{Z}$ because coordinates are integers, while p and p' are coprimes. These equations coincide with those in (A.5) but $n \to n + \alpha p$ and $m \to m + \alpha p'$. For $\alpha \in \mathbb{Z}_+$ one obtains that resulting restrictions of diagrams are weaker than (3.9). To consider the case of $\alpha \in \mathbb{Z}_-$ one recalls that by definition of minimal models $n < p$ and $m < p'$. Then one notices that $n + \alpha p < 0$ and $m + \alpha p' < 0$ which is to say that zeros are absent. We conclude that condition (A.4) does not produce new zeros.

New zeros appear due to condition (A.2). In this case coordinates of zeros are

$$\bar{i} = \bar{h}_j + m + 1 - \alpha p' , \qquad \bar{j} = \bar{\bar{k}}_i - n + \alpha p , \tag{A.15}$$

where $\alpha \in \mathbb{Z}$. Zeros are possible for $\alpha \in \mathbb{N}$ only. The resulting equations coincide with those in (A.14) provided $\lambda_1 \leftrightarrow \lambda_2$ and $n \leftrightarrow (\alpha+1)p - n$ and $m \leftrightarrow (\alpha+1)p' - m$, where now $\alpha \in \mathbb{Z}_+$. Repeating arguments below formula (A.14) one concludes that equations (A.15) impose the following restrictions

$$\bar{\bar{k}}_i \geq \bar{k}_{i+(p'-m)-1} - (p-n) + 1 . \tag{A.16}$$

Introducing parameters $(n_1, m_1) = (n, m)$ and $(n_2, m_2) = (p - n, p' - m)$ one obtains formula (3.10) of the proposition.

References

[1] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, Nucl. Phys. **B 241** (1984) 333 [INSPIRE].

[2] L.F. Alday, D. Gaiotto and Y. Tachikawa, *Liouville correlation functions from four-dimensional gauge theories*, Lett. Math. Phys. **91** (2010) 167 [arXiv:0906.3219] [INSPIRE].

[3] N.A. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, Adv. Theor. Math. Phys. **7** (2004) 831 [hep-th/0206161] [INSPIRE].

[4] N. Wyllard, *$A(N-1)$ conformal Toda field theory correlation functions from conformal $N = 2$ SU(N) quiver gauge theories*, JHEP **11** (2009) 002 [arXiv:0907.2189] [INSPIRE].

[5] A. Mironov and A. Morozov, *The power of Nekrasov functions*, Phys. Lett. **B 680** (2009) 188 [arXiv:0908.2190] [INSPIRE].

[6] A. Mironov and A. Morozov, *On AGT relation in the case of* U(3), Nucl. Phys. **B 825** (2010) 1 [arXiv:0908.2569] [INSPIRE].

[7] V.A. Fateev and A.V. Litvinov, *Integrable structure, W-symmetry and AGT relation*, JHEP **01** (2012) 051 [arXiv:1109.4042] [INSPIRE].

[8] V. Belavin and B. Feigin, *Super Liouville conformal blocks from $N = 2$ SU(2) quiver gauge theories*, JHEP **07** (2011) 079 [arXiv:1105.5800] [INSPIRE].

[9] V.A. Alba, V.A. Fateev, A.V. Litvinov and G.M. Tarnopolskiy, *On combinatorial expansion of the conformal blocks arising from AGT conjecture*, *Lett. Math. Phys.* **98** (2011) 33 [arXiv:1012.1312] [INSPIRE].

[10] R. Santachiara and A. Tanzini, *Moore-Read fractional quantum Hall wavefunctions and* SU(2) *quiver gauge theories*, *Phys. Rev.* **D 82** (2010) 126006 [arXiv:1002.5017] [INSPIRE].

[11] B. Estienne, V. Pasquier, R. Santachiara and D. Serban, *Conformal blocks in Virasoro and W theories: duality and the Calogero-Sutherland model*, *Nucl. Phys.* **B 860** (2012) 377 [arXiv:1110.1101] [INSPIRE].

[12] K. Papadodimas and S. Raju, *Correlation functions in holographic minimal models*, *Nucl. Phys.* **B 856** (2012) 607 [arXiv:1108.3077] [INSPIRE].

[13] C.-M. Chang and X. Yin, *Correlators in* W_N *minimal model revisited*, *JHEP* **10** (2012) 050 [arXiv:1112.5459] [INSPIRE].

[14] B. Feigin, E. Feigin, M. Jimbo, T. Miwa and E. Mukhin, *Quantum continuous* gl_∞*: tensor products of Fock modules and* $\mathcal{W}_N$ *characters*, arXiv:1002.3113.

[15] B. Feigin, E. Feigin, M. Jimbo, T. Miwa and E. Mukhin, *Quantum continuous* gl_∞*: Semi-infinite construction of representations*, arXiv:1002.3100.

[16] A. Zamolodchikov, *Higher equations of motion in Liouville field theory*, *Int. J. Mod. Phys.* **A 19S2** (2004) 510 [hep-th/0312279] [INSPIRE].

[17] M. Bershtein and O. Foda, *AGT, Burge pairs and minimal models*, arXiv:1404.7075 [INSPIRE].

[18] P. Bowcock and G.M.T. Watts, *Null vectors of the* $W(3)$ *algebra*, *Phys. Lett.* **B 297** (1992) 282 [hep-th/9209105] [INSPIRE].

[19] V.A. Fateev and A.V. Litvinov, *Correlation functions in conformal Toda field theory. I*, *JHEP* **11** (2007) 002 [arXiv:0709.3806] [INSPIRE].

[20] S. Mironov, A. Morozov and Y. Zenkevich, *Generalized Jack polynomials and the AGT relations for the* SU(3) *group*, *JETP Lett.* **99** (2014) 109 [arXiv:1312.5732] [INSPIRE].

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