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Neutrino masses and mixing from flavour antisymmetry

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Abstract: We discuss consequences of assuming (*i*) that the (Majorana) neutrino mass matrix M_ν displays flavour antisymmetry, $S_\nu^T M_\nu S_\nu = -M_\nu$ with respect to some discrete symmetry S_ν contained in SU(3) and (*ii*) S_ν together with a symmetry T_l of the Hermitian combination $M_l M_l^\dagger$ of the charged lepton mass matrix forms a finite discrete subgroup G_f of SU(3) whose breaking generates these symmetries. Assumption (*i*) leads to at least one massless neutrino and allows only four textures for the neutrino mass matrix in a basis with a diagonal S_ν if it is assumed that the other two neutrinos are massive. Two of these textures contain a degenerate pair of neutrinos. Assumption (*ii*) can be used to determine the neutrino mixing patterns. We work out these patterns for two major group series $\Delta(3N^2)$ and $\Delta(6N^2)$ as G_f. It is found that all $\Delta(6N^2)$ and $\Delta(3N^2)$ groups with even N contain some elements which can provide appropriate S_ν. Mixing patterns can be determined analytically for these groups and it is found that only one of the four allowed neutrino mass textures is consistent with the observed values of the mixing angles θ_{13} and θ_{23}. This texture corresponds to one massless and a degenerate pair of neutrinos which can provide the solar pair in the presence of some perturbations. The well-known groups A_4 and S_4 provide examples of the groups in respective series allowing correct θ_{13} and θ_{23}. An explicit example based on A_4 and displaying a massless and two quasi degenerate neutrinos is discussed.

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Contents

1 Introduction

Orderly pattern of neutrino mixing appears to hide some symmetry, discrete or continuous. It is possible to connect a given mixing pattern with some discrete symmetries of the leptonic mass matrices. Such symmetries may however be residual symmetries arising from a bigger symmetry in the underlying theory. One can obtain a possible larger picture by assuming that these symmetries are a part of a bigger group operating at the fundamental level whose breaking leads to the symmetries of the mass matrices. There is an extensive literature on study of possible residual symmetries of the mass matrices and of the groups which harbor them [1–16], see [17–19] for reviews and additional references.

Starting point in these approaches is to assume the existence of some symmetries S_ν (usually a $Z_2 \times Z_2$) and T_l (usually $Z_N, N \geq 3$) of the (Majorana) neutrino and the charged lepton mass matrices

$$T_l^\dagger M_l M_l^\dagger T_l = M_l M_l^\dagger , \tag{1.1}$$

$$S_\nu^T M_\nu S_\nu = M_\nu \, . \tag{1.2}$$

Matrices diagonalizing the 3×3 symmetry matrices S_ν, T_l can be related to the mixing matrices in each sector. The structures of these matrices can also be independently fixed if one assume that S_ν and T_l represent specific elements of some discrete group G_f in a given three dimensional representation. In this way, the leptonic mixing can be directly related to group theoretical structures. This reasoning has been used for the determination of the neutrino mixing angles in case of the three non-degenerate neutrinos [1–15, 20], two or three degenerate neutrinos [21, 22] and one massless and two non-degenerate neutrinos [23, 24].

The residual symmetries may arise from the spontaneous breaking of G_f if the vacuum expectation values of the Higgs fields responsible for generating neutrino (the charged lepton) masses break G_f but respect S_ν (T_l). We wish to study in this paper consequences of an alternative assumption that the spontaneous breaking of G_f leads to an M_ν which displays antisymmetry instead of symmetry, i.e. assume that eq. (1.2) gets replaced by

$$S_\nu^T M_\nu S_\nu = -M_\nu \tag{1.3}$$

but eq. (1.1) remains as it is. These assumptions prove to be quite powerful and are able to simultaneously restrict both the mass patterns and mixing angles when embedding of S_ν, T_l into G_f is considered. In the case of symmetry, the vacuum determining neutrino masses is invariant under S_ν. This invariance implies eq. (1.2). In the case of eq. (1.3), the vacuum is not invariant under S_ν but transforms under it in such a way that the neutrino mass matrix displays a residual antisymmetry. We shall give an explicit example based on A_4 of this phenomena in section 5. Since S_ν in the antisymmetric case does not survive as a residual symmetry of G_f, eq. (1.3) and the requirement that S_ν belongs to G_f are logically independent requirements. But they both can be consistent. Assuming that S_ν belongs to G_f helps in determining the mixing pattern and we shall make this assumption. We shall further assume that G_f is some finite discrete subgroup of SU(3). Then the first consequence of imposing eq. (1.3) is that $\mathrm{Det}\, M_\nu = 0$, i.e. at least one of the neutrinos remains massless. Since cases with two (or three!) massless neutrinos are not phenomenologically interesting, we shall restrict ourselves to cases with only one massless neutrino. Then as a second consequence of eq. (1.3), one can determine all the allowed forms of M_ν in a given basis for all possible S_ν contained in SU(3). There exist only four possible M_ν (and their permutations) consistent with eq. (1.3) in a particular basis with a diagonal S_ν. Two of these give one massless and two non-degenerate neutrinos and the other two give a massless and a degenerate pair of neutrinos which may be identified with the solar pair.

We determine all the allowed textures of the neutrino mass matrix in the next section. Subsequently, we discuss groups $\Delta(3N^2)$ and $\Delta(6N^2)$ and identify those which can give correct description of mixing using flavour antisymmetry. In section 4, we introduce $Z_2 \times Z_2$ as neutrino residual symmetry and present an example in which neutrino mass matrix gets fully determined group theoretically except for an overall scale. We discuss a realization of the basic idea with a simple example based on the A_4 group in section 5. Section 6 contains summary and comparison with earlier relevant works.

2 Allowed textures for neutrino mass matrix

We shall first consider the case of only one S_ν satisfying eq. (1.3) and subsequently generalize it to include two. The unitary matrix S_ν can be diagonalized by another unitary matrix V_{S_ν}:

$$V_{S_\nu}^\dagger S_\nu V_{S_\nu} = \tilde{S}_\nu$$

where $\tilde{S}_\nu$ is a diagonal matrix having the form:

$$\tilde{S}_\nu = \text{diag.}(\lambda_1, \lambda_2, \lambda_3)\,. \tag{2.1}$$

Unitarity of S_ν implies that $\lambda_{1,2,3}$ are some roots of unity. They satisfy $\lambda_1\lambda_2\lambda_3 = 1$ due to the condition $\text{Det}\, S_\nu = +1$. We now go to the basis with a diagonal S_ν. Defining $\tilde{M}_\nu = V_{S_\nu}^T M_\nu V_{S_\nu}$, eq. (1.3) can be rewritten as:

$$(\tilde{M}_\nu)_{ij}(1 + \lambda_i\lambda_j) = 0 \qquad (i, j \text{ not summed})\,. \tag{2.2}$$

It follows that a given element $(\tilde{M}_\nu)_{ij}$ is non-zero only if the factor in bracket multiplying it is zero. This cannot happen for an arbitrary set of λ_i and one needs to impose specific relation among them to obtain a non-trivial $\tilde{M}_\nu$. We now argue that only two possible forms of $\tilde{S}_\nu$ and their permutations lead to neutrino mass matrices with two massive neutrinos. The third mass will always be zero as a consequence of eq. (1.3) and the assumption that S_ν belongs to SU(3). These forms of $\tilde{S}_\nu$ are given by:

$$\begin{aligned}\tilde{S}_{1\nu} &= \text{diag.}(\lambda, -\lambda^*, -1)\,,\\ \tilde{S}_{2\nu} &= \text{diag.}(\pm i, \mp i, 1)\,.\end{aligned} \tag{2.3}$$

λ is an arbitrary root of unity. This can be argued as follows. Assume that at least one off-diagonal element of $\tilde{M}_\nu$ is non-zero which we take as the 12 element for definiteness. In this case, eq. (2.2) immediately implies the first of eq. (2.3) as a necessary condition. One can distinguish three separate cases of this condition[1] (I) $\lambda = 1$ (II) $\lambda = \pm i$ and (III) $\lambda \neq \pm 1, \pm i$. The structures of $\tilde{M}_\nu$ get determined in these cases from condition eq. (2.2) as follows:

$$\text{Texture I:} \quad \tilde{S}_{1\nu} = (1, -1, -1)\,; \qquad \tilde{M}_\nu = m_0 \begin{pmatrix} 0 & c & s e^{i\beta} \\ c & 0 & 0 \\ s e^{i\beta} & 0 & 0 \end{pmatrix}, \tag{2.4}$$

where $c = \cos\theta, s = \sin\theta$. This structure implies one massless and two degenerate neutrinos with a mass $|m_0|$. In case of (II),

$$\text{Texture II:} \quad \tilde{S}_{1\nu} = (\pm i, \pm i, -1)\,; \qquad \tilde{M}_\nu = \begin{pmatrix} x_1 & y & 0 \\ y & x_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.5}$$

This case corresponds to one massless and two non-degenerate neutrinos. In the third case one gets

$$\text{Texture III:} \quad \tilde{S}_{1\nu} = (\lambda, -\lambda^*, -1)\,; \qquad \tilde{M}_\nu = m_0 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad (\lambda \neq \pm 1, \pm i) \tag{2.6}$$

which implies a massless and a pair of degenerate neutrinos.

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[1] $\lambda = -1$ case corresponds to permutation of the case with $\lambda = 1$.

The cases (I, III) lead to the same mass spectrum but different mixing patterns. $\tilde{M}_\nu$ in eq. (2.4) is diagonalized as $V_\nu^T \tilde{M}_\nu V_\nu = \text{diag.}(m_0, m_0, 0)$ with

$$V_\nu = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ \frac{c}{\sqrt{2}} & \frac{ic}{\sqrt{2}} & -s \\ \frac{s}{\sqrt{2}}e^{-i\beta} & \frac{is}{\sqrt{2}}e^{-i\beta} & ce^{-i\beta} \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{2.7}$$

The arbitrary rotation by an angle ϕ originates due to degeneracy in masses. The texture II in eq. (2.5) is diagonalized by a unitary rotation in the 12 plane while the one in eq. (2.6) by a similar matrix with the angle $\frac{\pi}{4}$.

The permutations of entries in $\tilde{M}_\nu$ give equivalent structures and are obtained by permuting entries in $\tilde{S}_{1\nu}$. The case which is not equivalent to above textures follows with a starting assumption that one of the diagonal elements of $\tilde{M}_\nu \neq 0$ say, $(\tilde{M}_\nu)_{11} \neq 0$. In this case one requires $\tilde{S}_\nu = \text{diag.}(\pm i, \lambda', \mp i\lambda^{*\prime})$ with $|\lambda'| = 1$. The case with $\lambda' = \pm i$ gives $\tilde{S}_{1\nu}$ which is already covered. $\lambda' = \mp i$ implies the condition $\tilde{S}_{2\nu}$ in (2.3). This leads to a new texture

$$\text{Texture IV:} \quad \tilde{S}_\nu = (i, -i, 1)\,; \qquad \tilde{M}_\nu = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{2.8}$$

For $\lambda' = \pm 1$ one gets permutation of $\tilde{S}_{1\nu}$ or $\tilde{S}_{2\nu}$ and for $\lambda' \neq \pm 1, \pm i$ only 11 element of $\tilde{M}_\nu$ is non zero and two neutrinos remain massless. Thus conditions eq. (2.3) and their permutations exhaust all possible textures of $\tilde{M}_\nu$ consistent with the antisymmetry of M_ν, eq. (1.3) and two massive neutrinos. Any G_f admitting an element with these sets of eigenvalues will give a viable choice for flavour antisymmetry group. Note that texture III (IV) can be obtained from I (II) by putting $s(y)$ to zero. But the residual symmetries in all four cases are different. Because of this, the embedding groups G_f can also be different. We therefore discuss all these cases separately.

The mixing matrix in texture I contains two unknowns θ and β apart from an overall complex scale m_0. This is a reflection of the fact that the corresponding S_ν is a Z_2 symmetry and contains two degenerate eigenvalues -1. These unknowns can be fixed by imposing another residual Z_2 symmetry commuting with S_ν and satisfying eq. (1.2) or (1.3). We shall discuss such choices in section 4.

3 Group theoretical determination of mixing

The physical neutrino mixing matrix $U_{\text{PMNS}} \equiv U$ depends on the structure of M_ν and $M_l M_l^\dagger$. The latter can be determined if the symmetry T_l as in eq. (1.1) is known. We now make an assumption that S_ν satisfying eq. (1.3) and T_l as in eq. (1.1) are elements of some discrete subgroup (DSG) of SU(3) denoted by G_f. The DSG of SU(3) have been classified in [25–27]. They are further studied in [28–38]. These can be written in terms of few 3×3 presentation matrices whose multiple products generate various DSG. Two main groups series called C and D [37] constitute bulk of the DSG of SU(3). Of these, we shall explicitly study two infinite groups series $\Delta(3N^2)$ and $\Delta(6N^2)$ which are examples of the type C and D respectively. See [39–41] for earlier studies of neutrino mixing using the groups $\Delta(3N^2)$ and $\Delta(6N^2)$ and neutrino symmetry rather than antisymmetry.

Eq. (1.1) implies that T_l commutes with $M_l M_l^\dagger$. Thus, the matrix U_l diagonalizing the former also diagonalizes $M_l M_l^\dagger$ and corresponds to the mixing matrix among the left handed charged leptons. Similarly, the matrix U_ν diagonalizing M_ν gets related to the structure of S_ν. In this way, the knowledge of S_ν and T_l can be used to determine the mixing matrix

$$U \equiv U_{\text{PMNS}} = U_l^\dagger U_\nu \,. \tag{3.1}$$

This is the strategy followed in the general approach and we shall also use this to determine all possible mixing pattern for a given G_f consistent with eqs. (1.1) and (1.3).

Not all the groups G_f can admit an S_ν which will provide a legitimate antisymmetry operator S_ν, i.e. an element with eigenvalues specified by eq. (2.3). Our strategy would be to determine a class of groups which will have one or more allowed S_ν and then look for all viable T_l within these groups. There would be different mixing patterns associated with each choice of S_ν, T_l and it is possible to determine all of them analytically for $\Delta(3N^2)$ and $\Delta(6N^2)$ groups.

3.1 $\Delta(3N^2)$

The $\Delta(3N^2)$ groups are isomorphic to $(Z_N \times Z_N) \rtimes Z_3$, where $\rtimes$ denotes the semi-direct product. The group theoretical details for $\Delta(3N^2)$ are discussed in [29, 42]. For our purpose, it is sufficient to note that all the elements of the group are generated from the multiple product of two basic generators defined as:

$$F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & \eta^* \end{pmatrix} , \qquad E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{3.2}$$

with $\eta = e^{\frac{2\pi i}{N}}$. Here F generates one of the Z_N groups and E generates Z_3 in the semi-direct product $(Z_N \times Z_N) \rtimes Z_3$. The other Z_N group is generated by EFE^{-1}. The above explicit matrices provide a faithful three dimensional irreducible representation of the group and multiple products of these matrices therefore generate the entire group whose elements can be labeled as:

$$W \equiv W(N,p,q) = \begin{pmatrix} \eta^p & 0 & 0 \\ 0 & \eta^q & 0 \\ 0 & 0 & \eta^{-p-q} \end{pmatrix} , \qquad R \equiv R(N,p,q) = \begin{pmatrix} 0 & 0 & \eta^p \\ \eta^q & 0 & 0 \\ 0 & \eta^{-p-q} & 0 \end{pmatrix} ,$$
$$V \equiv V(N,p,q) = \begin{pmatrix} 0 & \eta^p & 0 \\ 0 & 0 & \eta^q \\ \eta^{-p-q} & 0 & 0 \end{pmatrix} . \tag{3.3}$$

All elements of $\Delta(3N^2)$ are obtained by varying p, q over the allowed range $p, q = 0, 1, 2, \ldots,$ $N-1$ in the above equation. Thus each matrices W, R, V have N^2 elements giving in total $3N^2$ elements corresponding to the order of $\Delta(3N^2)$. The eigenvalue equation for the $2N^2$

non-diagonal elements R and V is simply given by $\lambda^3 = 1$. These elements therefore have eigenvalues $(1, \omega, \omega^2)$ with $\omega = e^{\frac{2\pi i}{3}}$. These are not in the form of eq. (2.3) required to get the neutrino antisymmetry operator S_ν. Thus S_ν has to come from the N^2 diagonal elements. This requires that N, p, q should be such that $W(N, p, q) = \text{diag.}(\eta^p, \eta^q, \eta^{-p-q})$ matches the required eigenvalues $\tilde{S}_\nu$ of S_ν given by eq. (2.3) or their permutations. This cannot happen for all the values of variables and one can easily identify the viable cases. It is found that

- W can match any of $\tilde{S}_\nu$ only for even N. Thus only $\Delta(12k^2)$ groups with $k = 1, 2, \ldots$ contain neutrino antisymmetry operator S_ν.

- The eigenvalue set $\tilde{S}_\nu = (1, -1, -1)$ is always contained as a diagonal generator for all $\Delta(12k^2)$ groups and can be chosen as $S_\nu = W(2k, 0, k)$. Hence the texture I with two degenerate and one massless neutrino can follow in any $\Delta(12k^2)$. The smallest such group is $\Delta(12) = A_4$ which is one of the most studied flavour symmetry from other points of view [43–52].

- The set $\tilde{S}_\nu = (\pm i, \pm i, -1)$ arises only for N multiple of 4, i.e. in case of groups $\Delta(48l^2)$, $l = 1, 2 \ldots$. These groups also contain a $\tilde{S}_\nu$ satisfying the second of eq. (2.3). Thus textures $\mathrm{I}, \mathrm{II}, \mathrm{IV}$ are possible for all $\Delta(48l^2)$ groups.

- The set $\tilde{S}_\nu = (\lambda, -\lambda^*, -1)$ with $\lambda \neq \pm 1, \pm i$ and the associated texture III is viable in $\Delta(12k^2)$ with $k \geq 3$.

Let us now turn to the mixing pattern allowed within the $\Delta(12k^2)$ groups. S_ν has to be a diagonal operator identified above. Then T_l can be any other diagonal operator $W(2k, p, q)$ or any of $R(2k, p, q)$ or $V(2k, p, q)$. In the former case, $U_l = \mathbb{1}$, where $\mathbb{1}$ denotes a 3×3 identity matrix. The neutrino mixing in this case coincides with V_ν diagonalizing any of the four textures of $\tilde{M}_\nu$ giving $U_{\text{PMNS}} = V_\nu$. None of the allowed V_ν are suitable to give the correct mixing pattern with a non-zero θ_{13}. Thus, T_l needs to be any of the non-diagonal element R, V. The matrices $V_{R,V}$ diagonalizing R, V are given by

$$\begin{aligned} V_R(N, p, q) &= \text{diag.}(1, \eta^q, \eta^{-p}) U_\omega\,, \\ V_V(N, p, q) &= \text{diag.}(1, \eta^{-p}, \eta^{-p-q}) U_\omega^*\,, \end{aligned} \tag{3.4}$$

where,

$$U_\omega = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}. \tag{3.5}$$

The final mixing matrix depends upon the choice of specific texture for $\tilde{M}_\nu$. Consider the texture I which arises within all the $\Delta(12k^2)$ groups. $U_\nu = V_\nu$ in this case is given by eq. (2.7) and $U_{\text{PMNS}} = V_{R,V}^\dagger V_\nu$. Since a neutrino pair is degenerate, the solar mixing angle θ_{12} remains undetermined in the symmetry limit. This is reflected by the presence of an unknown angle ϕ in eq. (2.7). In this case, the neutrino mass hierarchy is inverted and the third column of $U_{\rm PMNS} \equiv U$ needs to be identified with the massless state. It is independent of the angle ϕ. We get for $T_l = R(N,p,q)$,

$$U_{i3} = \frac{1}{\sqrt{3}}(ce^{-i\beta}\eta^{p+q} - s, c\omega e^{-i\beta}\eta^{q+p} - s\omega^2, c\omega^2 e^{-i\beta}\eta^{q+p} - s\omega)^T \tag{3.6}$$

with $\eta = e^{\frac{\pi i}{k}}$ for the group $\Delta(12k^2)$. p, q take discrete values $0 \ldots 2k-1$ in above equation while β and θ are unknown quantities appearing in the neutrino mixing matrix eq. (2.7). The entries in U_{i3} can be permuted by reordering the eigenvalues of T_l. We will identify the minimum of $|U_{i3}|^2$ with s_{13}^2. If the minimum of the remaining two is identified with $c_{13}^2 s_{23}^2$ then one will get a solution with the atmospheric mixing angle $\theta_{23} \leq 45^\circ$. In the converse case, one will get a solution $\geq 45^\circ$. The experimental values of the leptonic angles are determined through fits to neutrino oscillation data [53–55]. Throughout, we shall specifically use the fits presented in [53] for definiteness. The texture I corresponds to the inverted hierarchy and the best fit values and 3σ ranges appropriate for this case are given [53] by:

$$\begin{aligned}
\sin^2\theta_{12} &= 0.308 \ (0.259\text{–}0.359)\,, \\
\sin^2\theta_{23} &= 0.455 \ (0.380\text{–}0.641)\,, \\
\sin^2\theta_{13} &= 0.0240 \ (0.0178\text{–}0.0298)\,.
\end{aligned} \tag{3.7}$$

Let us mention salient features of results following from eq. (3.6).

- It is always possible to obtain correct θ_{13}, θ_{23} by choosing unknown quantities θ and β of $\tilde{M}_\nu$. This should be contrasted with situation found in [22] which used neutrino symmetry instead of antisymmetry to obtain a degenerate pair of neutrinos. As discussed there, none of the $\Delta(3N^2)$ groups could simultaneously account for the values of θ_{13}, θ_{23} within 3σ.

- It is possible to obtain more definite predictions by choosing specific values of θ and or β. In contrast to θ and β which are unknown, the choice of p, q is dictated by the choice of T_l and it is possible to consider any specific choice of p, q in the range $0, \ldots N-1$. Consider a very specific choice of real $\tilde{M}_\nu$, i.e. $\beta = 0$ and a residual symmetry $T_l = E^2$ corresponding to putting $p = q = 0$ in eq. (3.6). This equation in this case gives a prediction $|U_{23}| = |U_{33}|$ which holds for all values of θ. This relation is equivalent to a maximal θ_{23} which lies within the 1σ range of the global fits [53]. θ then can be chosen to get the correct θ_{13}. Since the specific choice $p = q = 0$ is allowed within all the $\Delta(12k^2)$ groups, all of them can predict the maximal θ_{23} and can accommodate correct θ_{13}.

- The relation $|U_{23}| = |U_{33}|$ does not hold for a complex η^{p+q} even if $\beta = 0$. Such choices of T_l give departures from maximality in θ_{23}. It is then possible to reproduce both the angles correctly by choosing θ. This is non-trivial since a single unknown θ determines both θ_{13} and θ_{23} for a specific choice of group (i.e. N) and a residual symmetry T_l (i.e. p and q). The resulting prediction can be worked out numerically

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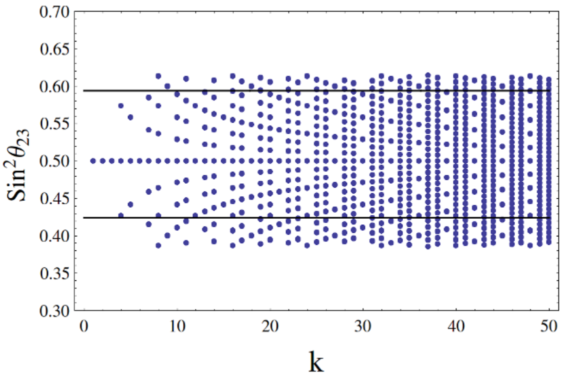

Figure 1. Predictions for $\sin^2\theta_{23}$ for the groups $\Delta(12k^2)$ as a function of k when $\sin^2\theta_{13}$ is allowed to vary within the 1σ range as obtained through global fits in [53]. Horizontal lines show 1σ limits on $\sin^2\theta_{23}$.

by varying p, q, N over the allowed integer values and θ over continuous range from 0 to 2π. Values of s^2_{23} obtained this way are depicted in figure 1. This is obtained by requiring that s^2_{13} lies within the allowed 1σ range. The phase β is put to zero. It is seen from the figure 1 that all the $\Delta(12k^2)$ groups always allow maximal θ_{23} as already discussed. But solutions away from maximal are also possible for $k \geq 4$. The minimal group capable of doing this is $\Delta(192)$. The next group $\Delta(300)$ can lead to near to the best fit values of the parameters. Specifically the choice $T_l = R(10, 0, 7)$, $S_\nu = W(10, 0, 5)$ within the group and $\theta \sim 54.3^\circ$ gives $s^2_{13} \sim 0.024$ and $s^2_{23} \sim 0.442$ to be compared with the best fit values 0.024 and 0.455 in [53].

- p, q can only be zero or 1 and η is real for the smallest group $\Delta(12) = A_4$. In this case, one immediately gets the prediction $\theta_{23} = \frac{\pi}{4}$ for $\beta = 0$. μ-τ symmetry is often used to predict the maximal θ_{23}. This is not even contained in A_4 which has only even permutations of four objects. Still the use of antisymmetry rather than symmetry allows one to get the maximal θ_{23} and it also accommodates a non-zero θ_{13} within A_4. This should be contrasted with the situation obtained in case of the use of symmetry condition eq. (1.2) instead of (1.3). It is known that in this case A_4 group gives democratic value $\frac{1}{3}$ for s^2_{13}, see for example [6].

We now argue that the other three textures though possible within $\Delta(12k^2)$ groups do not give the the correct mixing pattern. Texture II has one massless and in general two non-degenerate neutrinos. This texture can give both the normal and the inverted hierarchy. The mixing matrix V_ν is block-diagonal with a 2×2 matrix giving mixing among two massive states. Given this form for V_ν and a general U_l as given in eq. (3.4), one finds that the case with inverted hierarchy leads to the prediction $\sin^2\theta_{13} = \frac{1}{3}$ while the normal hierarchy gives instead $\cos^2\theta_{13}\cos^2\theta_{12} = \frac{1}{3}$. Neither of them come close to their experimental values.

The texture III having degenerate pair corresponds to the inverted hierarchy. V_ν in this case is block diagonal with an unknown solar angle. Given the most general form, eq. (3.4) for V_l one obtains once again the wrong prediction $\sin^2\theta_{13} = \frac{1}{3}$ ruling out this texture as well. Likewise, texture IV also gets ruled out. This corresponds to a diagonal $\tilde{M}_\nu$ with $V_\nu = \mathbb{1}$ and $|U_{\rm PMNS}| = |U_l|$ has the universal structure $|U| = \frac{1}{3}\mathbb{1}$.

To sum up, all the groups $\Delta(12k^2)$ contain a neutrino antisymmetry operator S_ν and allow a neutrino mass spectrum with two degenerate and one massless neutrino and can reproduce correctly two of the mixing angles θ_{13}, θ_{23}. The values for the solar angle and the solar scale have to be generated by small perturbations within these group. We shall study an example based on the minimal group $A_4 = \Delta(12)$ in this category in section 5.

3.2 $\Delta(6N^2)$

$\Delta(6N^2)$ groups are isomorphic to $(Z_N \times Z_N) \rtimes S_3$ with $N = 1, 2, 3\ldots$. The S_3 group in the semi-direct product is generated by E in eq. (3.2) and a matrix

$$G = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{3.8}$$

The matrices E, F, G provide a faithful irreducible representation of $\Delta(6N^2)$ [30] and generate the entire group with $6N^2$ elements. $3N^2$ elements generated by E, F give the $\Delta(3N^2)$ subgroup. The additional $3N^2$ elements are generated from the multiple products of G with elements of $\Delta(3N^2)$. These new elements can be parameterized by:

$$S \equiv S(N,m,n) = -\begin{pmatrix} \eta^m & 0 & 0 \\ 0 & 0 & \eta^n \\ 0 & \eta^{-m-n} & 0 \end{pmatrix}, \qquad T \equiv T(n,m,n) = -\begin{pmatrix} 0 & 0 & \eta^n \\ 0 & \eta^m & 0 \\ \eta^{-m-n} & 0 & 0 \end{pmatrix},$$
$$U \equiv U(n,m,n) = -\begin{pmatrix} 0 & \eta^n & 0 \\ \eta^{-m-n} & 0 & 0 \\ 0 & 0 & \eta^m \end{pmatrix}. \tag{3.9}$$

Here $0 \leq (m,n) < N-1$. Since $\Delta(3N^2)$ is a subgroup of $\Delta(6N^2)$, the neutrino mass and mixing patterns derived in the earlier section can also be obtained here. But the new elements S, T, U allow more possibilities now. In particular, they allow more elements which can be used as neutrino antisymmetry S_ν. To see this, note that the eigenvalues of S, T, U are given by $(\eta^{-m/2}, -\eta^{-m/2}, -\eta^m)$. This can have the required form, eq. (2.3) when $m = 0$ or $m = N/2$. The eigenvalues in respective cases are $(1,-1,-1)$ or $(-i, i, 1)$ and one gets the textures I or IV by using any of S, T, U as neutrino antisymmetry with $m = 0$ and $m = N/2$ respectively. Similarly, possible choices of the charged lepton symmetry T_l also increases. It can be any of the six types of elements: W,R,V as before or S, T, U. Important difference compared to $\Delta(3N^2)$ is that the texture I can now be obtained for both odd and even values of N by choosing any of the S, T, U with $m = 0$ as neutrino antisymmetry. Texture IV still requires $m = N/2$ and hence even N for its realization. We determine mixing matrix U for each of these textures and discuss them in turn.

Case	S_ν	T_l	U_l	U_ν	U_{PMNS}
1A	$W(2k,0,k)$	$W(N,p,q)$	$\mathbb{1}$	V_ν	V_ν
1B	$W(2k,0,k)$	$P(N,p,q)$	$V_P(N,p,q)$	V_ν	$V_P^\dagger(N,p,q)V_\nu$
1C	$W(2k,0,k)$	$Q(N,p,q)$	$V_Q(N,p,q)$	V_ν	$V_Q^\dagger(N,p,q)V_\nu$
2A	$P(N,0,n)$	$W(N,p,q)$	$\mathbb{1}$	$V_P(N,0,n)V_\nu$	$V_P(N,0,n)V_\nu$
2B	$P(N,0,n)$	$P'(N,p,q)$	$V_{P'}(N,p,q)$	$V_P(N,0,n)V_\nu$	$V_{P'}^\dagger(N,p,q)V_P(N,0,n)V_\nu$
2C	$P(N,0,n)$	$Q(N,p,q)$	$V_Q(N,p,q)$	$V_P(N,0,n)V_\nu$	$V_Q^\dagger(N,p,q)V_P(N,0,n)V_\nu$

Table 1. All possible choices of the residual symmetries S_ν and T_l within $\Delta(6N^2)$ groups and the corresponding PMNS mixing matrices. P, P' collectively denote any of S, T, U defined in the text. Q denotes R and V. The mixing matrices V_P, V_Q and V_ν appearing above are given in eq. (3.10), eq. (3.4) and eq. (2.7) respectively.

3.2.1 Texture I

The residual anti symmetries which lead to texture I can be either (1) $S_\nu = W(2k,0,k)$ or (2) $S_\nu = P(N,0,n)$ where $P = S,T,U$. The residual symmetry T_l of $M_l M_l^\dagger$ can be any elements in the group which we divide in three classes: (A) $T_l = W(N,p,q)$, (B) $T_l = P(N,p,q)$ and (C) $T_l = Q(N,p,q)$. Here and in the following, we use symbols P and Q to collectively denote $P = S,T,U$ and $Q = R,V$. We use the basis as specified in eqs. (3.9), (3.3) for S_ν, T_l. Then the neutrino mixing matrix is given by $U_\nu = V_\nu$ in case (1) while it is given by $U_\nu = V_P(N,0,n)V_\nu$ in case (2). This follows by noting that the texture $\tilde{M}_\nu$ given in eq. (2.4) holds in a basis with diagonal S_ν but S_ν in the chosen group basis of eq. (3.9) is non-diagonal in case (2). The neutrino mass matrix in this basis is thus given by $M_\nu = V_P^* \tilde{M}_\nu V_P^\dagger$ where V_P diagonalizes $P(N,0,n)$. The matrix U_ν which diagonalizes M_ν is then given by $U_\nu = V_P(N,0,n)V_\nu$ where V_ν diagonalizes $\tilde{M}_\nu$. Explicitly, $V_P^\dagger(N,p,q)P(N,p,q)V_P(N,p,q) = \text{diag.}(\eta^{-p/2}, -\eta^{-p/2}, -\eta^p)$ with

$$V_S(N,p,q) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & \sqrt{2} \\ 1 & \eta^{q+p/2} & 0 \\ -\eta^{-q-p/2} & 1 & 0 \end{pmatrix}; \quad V_U(N,p,q) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \eta^{q+p/2} & 0 \\ -\eta^{-q-p/2} & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix},$$

$$V_T(N,p,q) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \eta^{q+p/2} & 0 \\ 0 & 0 & \sqrt{2} \\ -\eta^{-q-p/2} & 1 & 0 \end{pmatrix}. \tag{3.10}$$

We have chosen the ordering of columns of V_P in such a way that the first column always corresponds to the eigenvalue $\eta^{-p/2}$. With this ordering one gets the texture I given in eq. (2.4) when $P(N,0,n)$ is used as neutrino antisymmetry.

The matrices U_l diagonalizing T_l in three cases above are given in the same basis by $U_l = \mathbb{1}, V_P(N,p,q), V_Q(N,p,q)$ in cases (A), (B), (C) respectively where V_Q are given in eq. (3.4). Thus we have six (four) different choices for U_l (U_ν) giving in all 24 leptonic mixing matrices U_{PMNS}. We list these choices and the corresponding U_{PMNS} matrices in table 1.

Not all of 24 mixing matrices listed in table 1 give independent predictions for the third column of U_{PMNS} which determines s_{13} and s_{23}. We discuss the independent ones below.

The choice (1A) giving $U_{\rm PMNS}=V_\nu$ has one of the entries zero and thus cannot lead to correct θ_{13} or θ_{23}. The choice (1C) involves only elements belonging to the $\Delta(3N^2)$ subgroup and its predictions are already discussed in the previous section. The remaining choices give new predictions.

The case (1B) leads to three different $U_{\rm PMNS}$. One obtained with $T_l=S(N,p,q)$ contains a zero entry in the third column and can be used only as a zeroeth order choice. One gets the following result in (1B) if $T_l=T(N,p,q)$

$$|U_{23}|^2=\frac{c^2}{2}\,,\qquad |U_{33}|^2=\frac{c^2}{2}\,,\qquad |U_{13}|^2=s^2. \tag{3.11}$$

The ordering of the entries $|U_{i3}|^2$ can be changed by rearranging the eigenvectors of of T_l appearing in U_l. We have chosen here and below an ordering which is consistent with the values of the parameters s_{13}^2, s_{23}^2 when U is equated with the standard form of the mixing matrix. The result in the third case with $T_l=U(N,p,q)$ can be obtained from above by the replacement $s\leftrightarrow c$. All the three entries above follow for all the choices of p,q and the phase β. The case (1B) in this way gives a universal prediction. Two of the $|U_{i3}|^2$ are equal within this choice and they correspond to $c_{13}^2c_{23}^2$ and $c_{13}^2s_{23}^2$. Equality of the two then implies a θ independent prediction $\theta_{23}=\frac{\pi}{4}$. s_{13}^2 in the above case is then given by s^2 and can match the experimental value with appropriate choice of the unknown θ. Since the choice of S_ν within (1B) is possible only for even N it follows that all the groups $\Delta(24k^2)$ lead to a prediction of the maximal atmospheric mixing angle and can accommodate the correct θ_{13}.

The choice (2A) also gives the same result for $|U_{i3}|^2$ as (1B) with an important difference. The neutrino residual symmetry used in this choice is allowed for all N and not necessarily $N=2k$. Thus one gets a universal prediction of the maximal θ_{23} for all p,q,θ,β within all $\Delta(6N^2)$ groups. The smallest group in this category is the permutation group $S_4=\Delta(24)$ which contain symmetries appropriate for both the cases (1B) and (2A).

There are two independent structures within nine possible choices contained in case (2B). The example of the first one is provided by the choice $S_\nu=S(N,0,n)$ and $T_l=S(N,p,q)$. The elements in the third column of mixing matrix are given in this case by

$$\begin{aligned}
|U_{23}|^2 &= \frac{1}{4}s^2|\eta^n-\eta^{q+p/2}|^2,\\
|U_{33}|^2 &= \frac{1}{4}s^2|\eta^{-n}+\eta^{-q-p/2}|^2,\\
|U_{13}|^2 &= c^2.
\end{aligned} \tag{3.12}$$

While this choice does not give universal prediction as in the case (1B) discussed above it still leads to a prediction for θ_{23} which is independent of the unknown angle θ and phase β:

$$\tan^2\theta_{23}\quad\text{or}\quad\cot^2\theta_{23}=\frac{|\eta^n-\eta^{q+p/2}|^2}{|\eta^{-n}+\eta^{-q-p/2}|^2}$$

This follows from eq. (3.12) when $|U_{13}|^2$ is identified with s_{13}^2. The predicted θ_{23} now depends only on the group theoretical factors N,p,q,n.

Group	T_l	S_ν	Predictions
$\Delta(24k^2)$	$T(2k,p,q)$	$W(2k,0,k)$	Maximal θ_{23} for all β,p,q,n $\sin^2\theta_{13}=\sin^2\theta$
$\Delta(6N^2)$	$W(N,p,q)$	$P(N,0,n)$	Maximal θ_{23} for all β,p,q,n $\sin^2\theta_{13}=\cos^2\theta$
$\Delta(6N^2)$	$S(N,p,q)$	$S(N,0,n)$	Maximal θ_{23} for all β, $\frac{\lvert n-q-p/2\rvert}{N}=\frac{(2l+1)}{4}$ $\sin^2\theta_{13}=\cos^2\theta$
$\Delta(294)$	$S(7,0,2)$	$S(7,0,0)$	$s_{23}^2=0.39$ or 0.61 for all θ,β $\cos^2\theta=s_{13}^2$
$\Delta(486)$	$S(9,0,2)$	$S(9,0,0)$	$s_{23}^2=0.41$ or 0.59 for all θ,β $\cos^2\theta=s_{13}^2$
$\Delta(6N^2)$	$S(N,0,0)$	$T(N,0,n)$	$\sin^2\theta=2s_{13}^2$ $s_{23}^2=0.345$ for $\beta=0$ and the best fit s_{13}^2
$\Delta(150)$	$S(5,2,3)$	$T(5,0,0)$	$\sin^2\theta=2s_{13}^2$ $s_{23}^2=0.452$ for the best fit θ_{13}
$\Delta(6N^2)$	$R(N,0,0)$	$S(N,0,0)$	$s_{23}^2=1/2$ for $\beta=0$ and all θ $s_{13}^2=0.028$ for $\beta=0$, maximal θ
$\Delta(150)$	$R(5,3,1)$	$S(5,0,0)$	$s_{23}^2=0.484$, $s_{13}^2=0.022$ for $\beta=0$ and $\theta\sim 80°$

Table 2. Some illustrative predictions of the mixing angles $\sin^2\theta_{13}$ and $\sin^2\theta_{23}$ using $\Delta(6N^2)$ groups as flavour symmetry. $\sin^2\theta_{12}$ remains undetermined due to degeneracy in two of the masses in all these cases.

Unlike (1B), both the maximal and non-maximal values are allowed for θ_{23} in this case. The former occurs whenever $\cos\frac{2\pi(n-q-p/2)}{N}=0$. The latter occurs for other choices. It is possible to find values of parameters which lead to a non-maximal θ_{23} within the experimental limits. The minimal such choice occurs for $N=7$, i.e. the group $\Delta(294)$ which leads as shown in table 2 to a $\sin^2\theta_{23}$ within the 2σ range as given in [53]. The next example of the group $\Delta(486)$ fairs slightly better.

The other prediction of the case (2B) is obtained with $S_\nu=S(N,0,n)$ and $T_l=T(N,p,q)$. One obtains in this case

$$\begin{aligned}
|U_{23}|^2&=\frac{1}{4}|\sqrt{2}ce^{-i\beta}+s\eta^{q+p/2}|^2,\\
|U_{33}|^2&=\frac{1}{4}|\sqrt{2}ce^{-i\beta}-s\eta^{-q-p/2}|^2,\\
|U_{13}|^2&=\frac{1}{2}s^2.
\end{aligned}\tag{3.13}$$

In this case, θ_{23} is necessary non-maximal if θ_{13} is to be small but non-zero. We may identify, $|U_{13}|^2$ with s_{13}^2 and fix $s^2=2s_{13}^2$. This determines the other two entries of $|U_{i3}|^2$ for a given p,q,β. For $p=q=\beta=0$ one obtains $\sin^2\theta_{23}$ either 0.345 or 0.655. Thus all the $\Delta(6N^2)$ groups with this specific choice give results close to the 3σ range in the global fits. This prediction can be improved by turning on β or choosing different T_l. An example based on the group $\Delta(150)$ giving $\sin^2\theta_{23}$ close to the best fit value [53] is shown in the table 2.

Predictions of the case (2C) can also be similarly worked out. Six different $U_{\rm PMNS}$ are associated with this choice but not all give different predictions for the third column. One of the independent structures corresponds to choosing $T_l = R(N,p,q)$ and $S_\nu = S(N,0,n)$. The $U_{\rm PMNS} = V_R^\dagger V_S V_\nu$ gives

$$\begin{aligned}|U_{13}|^2 &= \frac{1}{6}|-s(\eta^p+\eta^{n-q})+\sqrt{2}ce^{-i\beta}|^2,\\ |U_{23}|^2 &= \frac{1}{6}|-s(\eta^p\omega+\eta^{n-q}\omega^2)+\sqrt{2}ce^{-i\beta}|^2,\\ |U_{33}|^2 &= \frac{1}{6}|-s(\eta^p\omega^2+\eta^{n-q}\omega)+\sqrt{2}ce^{-i\beta}|^2.\end{aligned} \tag{3.14}$$

Rest of the choices within (2C) differ from the above only in the powers of η. Their predictions can be obtained from the above by choosing different values of p,q,n.

Eq. (3.14) gives s_{13}, s_{23} withing 3σ range for a suitable choice of N,p,q,θ,β. In particular, one predicts a maximal θ_{23} if $p=q=\beta=0$ as in the earlier cases. But now the maximal value of θ also becomes a viable choice for all the groups $\Delta(6N^2)$. This makes the choice in this class particularly interesting since such value of θ can be forced by some additional symmetry. With the choice $T_l = E^2$ corresponding to $p=q=0$, eq. (3.14) gives for $n=\beta=0$,

$$\theta_{23} = \frac{\pi}{4}, \qquad s_{13}^2 = \frac{1}{3}|c-\sqrt{2}s|^2.$$

The maximal value of θ then leads to $s_{13}^2 \sim 0.029$ which is close to 2σ range as obtained in [53]. One can obtain a better solution with a different choice for p and q and θ. One particular solution based on the group $\Delta(150)$ is shown in the table 2.

3.2.2 Texture IV

The diagonal texture IV given in eq. (2.8) can be realized in $\Delta(6N^2)$ for even N with the choice $S_\nu = P(2k,k,n)$. This texture has two non-degenerate and one massless neutrino. Thus both the normal and the inverted hierarchies are possible. The massless state has to be identified with the third (first) column of the mixing matrix for the inverted (normal) hierarchy. The neutrino mixing matrix U_ν in this case is given by the matrix which diagonalizes $P(2k,k,n)$. This is given for the inverted hierarchy by $V_P(2k,k,n)$ as defined in eq. (3.10). For the normal hierarchy, one instead gets $U_\nu = V_P(2k,k,n)Z_{13}$ where Z_{13} exchanges the first and the third column of of the mixing matrix obtained in case of the inverted hierarchy. Possible choice of T_l can be any of the six types of generators and corresponding mixing matrices U_l are the same as given in table 1 with the choice (2A), (2B), (2C). It is then straightforward to work out the final mixing matrices $U_{\rm PMNS}$. As the massless state in the basis with diagonal S_ν is given by $(1,0,0)^T$ and its cyclic permutation for $S_\nu = S,T,U$, the third column of the $U_{\rm PMNS}$ is given by $(U_{\rm PMNS})_{i3} = (U_l^\dagger)_{i1}, (U_l^\dagger)_{i2}, (U_l^\dagger)_{i3}$ when $S_\nu = S,T,U$. It follows from the structure of U_l that the third column has either one or two zero entries or all elements have equal magnitudes. The same applies to the first column of $U_{\rm PMNS}$ in case of the normal hierarchy. In either case, the texture IV cannot give phenomenologically consistent result at the zeroeth order.

4 More predictive scenario: $Z_2 \times Z_2$ symmetry

The neutrino mass matrix M_ν has been assumed so far to be antisymmetric with respect to only one S_ν. This fails in determining $\tilde{M}_\nu$ completely in case of the texture I which still has two unknown quantities θ and β. The situation changes if M_ν is assumed to be symmetric or antisymmetric with respect to one more generator. We give here an example in which an additional residual symmetry of $\tilde{M}_\nu$ determines it completely apart from an overall complex mass scale. We use a generator S'_ν commuting with S_ν for this purpose. It should be such that S_ν, S'_ν and T_l together are contained in some G_f. M_ν may be antisymmetric with respect to transformation by S'_ν also. In this case, it will be symmetric with respect to the product $S_\nu S'_\nu$. Instead we assume that S'_ν is a symmetry of M_ν, i.e.

$$S'^T_\nu M_\nu S'_\nu = M_\nu \,. \tag{4.1}$$

We can transform above equation to the basis with a diagonal S_ν by defining $\tilde{S}'_\nu \equiv V^\dagger_{S_\nu} S'_\nu V_{S_\nu}$. In this basis, we get

$$\tilde{S_\nu}'^T \tilde{M}_\nu \tilde{S_\nu}' = \tilde{M}_\nu \,. \tag{4.2}$$

As before, we demand $\tilde{S}'_\nu$ to be contained in SU(3). If it is diagonal, then $\tilde{S}'_\nu = \text{diag.}(\lambda_1, \lambda_2, \lambda_1^* \lambda_2^*)$ with $\lambda_{1,2}$ being roots of unity. Then eq. (4.2) when applied to $\tilde{M}_\nu$ in eq. (2.7) implies that either $\tilde{S_\nu}'$ is proportional to identity or $s = 0$ or $c = 0$. A non-trivial prediction can be obtained if $\tilde{S_\nu}'$ is non-diagonal. Since $\tilde{S_\nu} = \text{diag.}(1, -1 - 1)$, a general $\tilde{S_\nu}'$ commuting with $\tilde{S_\nu}$ should have a block diagonal structure with the lower 2×2 block non-trivial. This block gets further restricted from the requirement that S_ν, S'_ν, T_l are elements of some discrete group G_f. These requirements can be met within the already considered groups $\Delta(6N^2)$.

Consider the group $\Delta(12k^2)$. The choice $S_\nu = \tilde{S_\nu} = W(2k, 0, k) = \text{diag.}(1, -1, -1)$ within it leads to texture I as already discussed. This commutes with all the discrete symmetries having a general form $S(M, m, n)$ as in eq. (3.9). Thus a viable choice for S'_ν is provided by $S'_\nu = S(M, m, n)$. Note that since S_ν is already diagonal, $\tilde{S_\nu}' = S'_\nu = S(M, m, n)$. Then eq. (4.2) and the form of $\tilde{M}_\nu$ implies a restriction:

$$m = 0 \,, \qquad \theta = \pm \frac{\pi}{4} \,, \qquad \beta = \frac{2\pi n}{M} \tag{4.3}$$

which fixes the unknown angle θ and phase β. Mixing pattern can be determined by choosing appropriate T_l and let us choose $T_l = R(N, p, q)$. Since both T_l and S_ν are contained in $\Delta(12k^2)$ mixing pattern is determined by the corresponding eq. (3.6) but now with θ and β satisfying eq. (4.3) which follows from the inclusion of S'_ν as a residual symmetry. We can vary p, q, M, N, n in eq. (3.6) and look for a viable choice. Consider $M = N$ in which case T_l, S_ν, S'_ν are contained in $\Delta(6N^2)$. By varying p, q, N one finds that the minimum group giving acceptable θ_{13}, θ_{23} is $\Delta(600)$ corresponding to $N = 10$. One possible set of residual symmetries within $\Delta(600)$ is given by

$$S_\nu = W(10, 0, 5) \,, \qquad T_l = R(10, 4, 0) \,, \qquad S'_\nu = S(10, 0, 0) \,.$$

With this choice, the S'_ν coincides with the μ-τ symmetry and eqs. (3.6), (4.3) give a prediction

$$s_{13}^2 \approx 0.029\,, \qquad s_{23}^2 \approx 0.38 \;\text{ or }\; 0.62$$

to be compared with the 3σ region given in eq. (3.7).

5 An A_4 model with flavour antisymmetry

Our discussion so far has been at the group theoretical level. We now present an explicit realization of flavour antisymmetric neutrino mass matrix using A_4 as an example. A_4 has been extensively used for several different purposes, for obtaining degenerate neutrinos [43, 44], to realize tri-bimaximal mixing [45, 48] for obtaining maximal CP phase δ [46, 47, 49, 51, 52] or to obtain texture zeros [50] in the leptonic mass matrices. As we discuss here, it also provides a viable alternative to get a massless and two quasi degenerate neutrinos with correct mixing pattern. In the following, we discuss the required symmetry, Higgs content and obtain the vacuum needed to obtain antisymmetry. We also discuss possible perturbations which can split the degenerate pair and lead to the solar scale and mixing angle.

The group theory of A_4 is discussed extensively in many papers. We follow the basis choice as given for example in [48]. In this basis, all the 12 elements of $A_4 = \Delta(12)$ can be generated from the two elements E and F defined in eq. (3.2) with $\eta = -1$. We will use the following product rule between two three dimensional representations $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$:

$$x \times y = 1 + 1' + 1'' + \begin{pmatrix} x_2y_3 + x_3y_2 \\ x_3y_1 + x_1y_3 \\ x_1y_2 + x_2y_1 \end{pmatrix}_{3S} + \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}_{3A} \tag{5.1}$$

with $1 = x_1y_1 + x_2y_2 + x_3y_3$, $1' = x_1y_1 + \omega x_2y_2 + \omega^2 x_3y_3$, $1'' = x_1y_1 + \omega^2 x_2y_2 + \omega x_3y_3$.

The explicit model presented below is based on the flavour symmetry $A_4 \times Z_3 \times Z_5$. The added symmetry $Z_3 \times Z_5$ plays an important role in restricting the structure of the model in a way that leads to correct vacuum alignment and the required antisymmetric M_ν. The A_4 symmetry determines the mixing angle structure obtained group theoretically in the earlier section.

The model uses the following ingredients (1) supersymmetry with unbroken R symmetry (2) flavon fields with zero R-charge which break the A_4 symmetry and (3) the driving fields with $R = 2$ which appear linearly in suprepotential and lead to correct vacuum alignment among flavons. R symmetry is assumed to be unbroken and driving fields have zero vacuum expectation values. These ingredients have been used in several papers to solve the difficult vacuum alignment problem in case of flavour symmetry, see for a review and references [18]. We use the same mechanism to get antisymmetry. The quantum numbers of the required flavonic superfields under $A_4 \times Z_3 \times Z_5$ symmetry are listed in table 3 and that of the driving fields in table 4. The MSSM Higgs fields H_u, H_d and triplet Δ are invariant under $A_4 \times Z_3 \times Z_5$ symmetry. The model also needs and additional triplet superfield $\bar{\Delta}$ for consistency.

Fields	l_L	$(e^c, , \mu^c, \tau^c)$	χ_e	χ_1	χ_2	ξ_1	ξ_2
A_4	3	$(1, 1', 1'')$	3	3	3	1	1
Z_3	ω	ω^2	1	ω	ω	1	1
Z_5	1	β^3	β^2	β	β^4	β^4	β

Table 3. Transformation properties of leptons and the required flavon fields under the symmetry group $A_4 \times Z_3 \times Z_5$. ω and β satisfy $\omega^3 = \beta^5 = 1$.

Driving Fields	$(\sigma_e^0, \sigma_\mu^0, \sigma_\tau^0)$	χ_1^0	χ_2^0	σ_ν^0	σ_ξ^0	$\sigma_1'^0$	$\sigma_2'^0$
A_4	$(1, 1'', 1')$	3	3	1	1	1	1
Z_3	1	ω	ω	ω	1	ω	ω
Z_5	β	β^3	β^2	1	1	β^3	β^2

Table 4. Transformation properties of the required driving fields under the symmetry group $A_4 \times Z_3 \times Z_5$. ω and β satisfy $\omega^3 = \beta^5 = 1$.

The complete superpotential consists of several parts. We discuss significance of each of them below.

$$W_l = \frac{H_d}{M}\big(h_e (l_L \chi_e)_1\, e_R + h_\mu (l_L \chi_e)_{1''}\, \mu_R + h_\tau (l_L \chi_e)_{1'}\, \tau_R\big)\,, \tag{5.2}$$

$$W_\nu = \frac{1}{2M^2}(l_L^T C \Delta l_L)_{3_S}(h_1 \chi_1 \xi_1 + h_2 \chi_2 \xi_2)\,, \tag{5.3}$$

$$W_{ld} = \beta_\mu (\chi_e \chi_e)_{1'} \sigma_\mu^0 + \beta_\tau (\chi_e \chi_e)_{1''} \sigma_\tau^0\,, \tag{5.4}$$

$$W_{\nu d} = \delta_1 (\chi_1 \chi_1 \chi_1^0)_1 + \delta_2 (\chi_2 \chi_2 \chi_2^0)_1 + \delta_3 (\chi_1 \chi_2)_1 \sigma_\nu^0 + \delta_4 (\xi_1 \xi_2 - \mu_\xi^2) \sigma_\xi^0\,, \tag{5.5}$$

$$W_d' = \beta_e \big((\chi_e \chi_e)_1 - \mu_e \xi_1\big) \sigma_e^0 + \beta_1' \big((\chi_1 \chi_1)_1 - \mu_1^2\big) \sigma_1'^0 + \beta_2' \big((\chi_2 \chi_2)_1 - \mu_2^2\big) \sigma_2'^0\,. \tag{5.6}$$

The subscript a in $(\ldots)_a$ in the above equations labels the A_4 representation according to which the quantity $(\ldots)$ transforms. C is the charge conjugation matrix. The cut-off scale M and the flavon vacuum expectation values generate the effective Yukawa couplings in the model. We have kept only the leading order terms in the above superpotential.

W_l and W_ν respectively determine M_l and M_ν. Note that the assignments given in table 3 forbid terms having only one flavon field in W_ν. Thus neutrino masses contain an extra flavon and suppression factor M compared to the charged lepton masses. The residual symmetry properties of the leptonic mass matrices are determined from the above superpotential by the flavon vacuum expectation values (vev) at the minimum. We shall show that these vev lead to the required symmetries in accordance with the discussion given in section 2. The symmetry T_l of $M_l M_l^\dagger$ is obtained as E or E^2. Only possible choice within A_4 for S_ν leading to flavour antisymmetry is given by F or its cyclic permutations and this can come from the minimization of $W_{\nu d}$.

The minimum of the potential is obtained in the supersymmetric limit by setting F terms corresponding to each superfield to zero. W_{ld} and $W_{\nu d}$ are responsible for the vacuum alignment in the model. Consider derivatives of $W_{\nu d}$ with respect to each component $i = 1, 2, 3$ of triplets $\chi_{1,2}^0$ and σ_ν^0

$$F_{\chi_{1i}^0} = F_{\chi_{2i}^0} = F_{\sigma_\nu^0} = 0\,. \tag{5.7}$$

Using the product rules as given in eq. (5.1), one finds that the first two conditions imply that at most one component of the each of the flavon fields $\chi_{1,2}$ can have non-zero vev.[2] The additional constraint $F_{\sigma_\nu^0}$ then implies that $\langle\chi_1\rangle$ and $\langle\chi_2\rangle$ form an orthogonal pair of vectors. Thus only possible non-zero vev for these fields are given by

$$\langle\chi_1\rangle = v\begin{pmatrix}0\\0\\1\end{pmatrix}, \qquad \langle\chi_2\rangle = u\begin{pmatrix}0\\1\\0\end{pmatrix}, \tag{5.8}$$

or

$$\langle\chi_1\rangle = v\begin{pmatrix}0\\0\\1\end{pmatrix}, \qquad \langle\chi_2\rangle = u\begin{pmatrix}1\\0\\0\end{pmatrix} \tag{5.9}$$

and their cyclic permutations which are related to the above by A_4 symmetry. The last term of $W_{\nu d}$ assures through $F_{\sigma_\xi^0} = 0$ that the fields $\xi_{1,2}$ assume non-zero vev. Flavon vev given in eq. (5.8) displays antisymmetry

$$F\langle\chi_{1,2}\rangle = -\langle\chi_{1,2}\rangle\,.$$

These vevs when substituted in eq. (5.3) lead to a neutrino mass matrix given by texture I, eq. (2.4) with $\tan\theta = \frac{|h_2\langle\xi_2\rangle u|}{|h_1\langle\xi_1\rangle v|}$, $\beta = \mathrm{Arg}\left(h_2\langle\xi_2\rangle u h_1^*\langle\xi_1^*\rangle v^*\right)$ and appropriately defined m_0. Similarly, the vev given in eq. (5.9) display similar antisymmetry with respect to $S_\nu = \mathrm{diag.}(-1,1,-1)$ which is also an element of A_4 and leads to a permutation of the texture I. The supersymmetric solution corresponding to F-terms $F_{\sigma_\mu^0} = F_{\sigma_\tau^0} = 0$ lead from eq. (5.4)

$$(\chi_e\chi_e)_{1'} = (\chi_e\chi_e)_{1''} = 0\,.$$

A solution of these two equations is given by

$$\langle\chi_{e1}\rangle^2 = \langle\chi_{e2}\rangle^2 = \langle\chi_{e3}\rangle^2. \tag{5.10}$$

Equality of vev of all three components is one of the solutions of the above equation. This leads to an $M_l M_l^\dagger$ having a residual symmetry $T_l = E, E^2$ as has been discussed in several papers on A_4. Other solutions of eq. (5.10) corresponds to equal magnitudes but relative minus sign between one or two components. Such solutions are invariant under A_4 elements $f_i E f_i$, with $f_1 = F$, $f_3 = EFE^2$, $f_2 = E^2FE$. The texture I obtained here is diagonalized by $U_\nu = V_\nu$. The resulting mixing pattern is a special case of eq. (3.6) obtained for $\Delta(12k^2)$ with $T_l = R(2k,p,q)$ and $S_\nu = W(2k,0,k)$. The choices $T_l = E, f_iEf_i$ are obtained from above by a suitable allowed value of p,q. Consider the case with $T_l = E^2$ corresponding to $R(2,0,0)$ and $S_\nu = F = W(2,0,1)$. Thus U_{i3} can be obtained by putting $p = q = 0$ and $\eta = -1$ in eq. (3.6). As already discussed, this leads to a prediction $\theta_{23} = \frac{\pi}{4}$ for $\beta = 0$ independent of the choice of θ. The latter can be chosen to give the correct θ_{13} while the solar angle and scale remain unpredicted at this stage due to degeneracy in mass.

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[2] All components of $\chi_{1,2} = 0$ is also a solution. Such solutions do not contribute to the leptonic masses and can be avoided in the presence of suitable supersymmetry breaking mass terms. We shall assume throughout that vev of none of the flavon fields are identically zero.

The flavon vev v, u remain undetermined at this stage. The last piece W_d' of the superpotential given in eq. (5.6) is introduced to fix these vev. The price for this to be paid in the model is introduction of soft $Z_3 \times Z_5$ breaking mass parameters μ_1 and μ_2. Apart from introduction of these mass parameters, the superpotential given by eqs. (5.2)–(5.6) is the most general superpotntial invariant under the $A_4 \times Z_3 \times Z_5$ symmetry. The soft breaking masses may result from the neglected higher order terms, e.g. μ_1^2 may come from $\frac{\xi_1}{M}\chi_2^2\sigma_1'^0$ and μ_2^2 may come from $\frac{\xi_2}{M}\chi_1^2\sigma_2'^0$ allowed in the superpotential by the $A_4 \times Z_3 \times Z_5$ symmetry. Rather than considering full higher order corrections, we discuss model only at the leading order and regard for the present the masses $\mu_{1,2}^2$ as effective soft $Z_3 \times Z_5$ breaking parameters. This is a technically natural assumption.

Perturbations are needed to split the degenerate pair and stabilize the solar angle. We now discuss possible perturbations within the model. One source of perturbation arises from the shift in the vev of $\chi_{1,2}$ which may arise from higher order effects. Consider for example,

$$\langle \chi_1 \rangle = v \begin{pmatrix} \delta v_1 \\ \delta v_2 \\ 1 \end{pmatrix} ,$$

where, $|\delta v_{1,2}| \ll 1$. The δv_1 generates a non-zero 23 elements in M_ν, eq. (2.4). The δv_2 corrects the already existing entries s, c and can be absorbed in their redefinition. Zero entries in the diagonal part of M_ν can be generated through an A_4 singlet contribution $(l_L^T C \Delta l_L)_1$ which can arise for example from from the higher order $A_4 \times Z_3 \times Z_5$ invariant terms, e.g., $\frac{1}{M^3}(l^T C \Delta l)_1 (\chi_e \chi_2)_1 \xi_1$. Motivated by this, we consider the following perturbed neutrino mass matrix.

$$\tilde{M}_\nu = m_0 \begin{pmatrix} \epsilon_1 & c & s e^{i\beta} \\ c & \epsilon_1 & \epsilon_2 \\ s e^{i\beta} & \epsilon_2 & \epsilon_1 \end{pmatrix} . \tag{5.11}$$

Inclusion all possible non-leading effects may result in more complex M_ν and would also correct M_l. All these corrections will add more parameters to the model and we assume their contribution to be small. Here we show that two parameters $\epsilon_{1,2}$ are sufficient to reproduce the neutrino mixing and scales correctly. They split the degeneracy and generate the solar scale and angle correctly. For example,

$$(\theta, \beta, \epsilon_1, \epsilon_2) = (0.5904, -0.1818 - 0.0579, 0.1186) \tag{5.12}$$

give the following values of the observables

$$\sin^2\theta_{13} \sim 0.024\,, \quad \sin^2\theta_{23} \sim 0.455\,, \quad \sin^2\theta_{12} \sim 0.307\,, \quad \frac{\Delta_\odot}{|\Delta_{\rm atm}|} \sim 0.0317 \tag{5.13}$$

which corresponds to (nearly) best fit values obtained for example with a global fits in [53].

6 Summary

The bottom up approach of finding discrete symmetry groups starting with possible symmetries of the residual mass matrices has been successfully used in last several years to predict leptonic mixing angles. The residual symmetry assumed in these works leaves the neutrino mass matrix M_ν invariant. We have proposed here a different possibility in which M_ν displays antisymmetry as defined in eq. (1.3) under a residual symmetry. The use of antisymmetry is found to be more predictive than symmetry. It is able to restrict both neutrino masses and mixing angles unlike all the previous works in this category which [1–15] could predict only mixing angles. Moreover, the antisymmetry condition by itself is sufficient for determining all possible discrete residual antisymmetry operators S_ν residing in SU(3). This in turn leads to very specific textures of the neutrino mass matrix satisfying antisymmetry condition. These are given by eqs. (2.4), (2.5), (2.6), (2.8).

Just like symmetry, the antisymmetry of M_ν can also come from the spontaneous breaking of some discrete group G_f. We have demonstrated it through a supersymmetric model based on the group $A_4 \times Z_3 \times Z_5$. Just like its counterparts in the case of symmetry [18, 56–58], the present model leading to antisymmetry also needs an elaborate set of flavons and driving fields.

We studied the mixing angle predictions in the specific context of the groups $\Delta(3N^2)$ and $\Delta(6N^2)$. The main results obtained are:

- Only the groups $\Delta(12k^2)$ with $k = 1, 2 \ldots$ and all $\Delta(6N^2)$ groups contain the residual antisymmetry operator.
- Of the four possible neutrino mass textures allowed by antisymmetry, only texture I having one massless and two degenerate neutrinos can lead to correct mixing pattern. This case provides a very good zeroeth order approximation to reality if the neutrino mass hierarchy is inverted.
- There always exists within these groups residual symmetries of $M_l M_l^\dagger$ and M_ν such that the atmospheric neutrino mixing angle is maximal. Correct value of θ_{13} can be accommodated by choosing the unknown angle in eq. (2.4) appropriately. There also exists other choices of residual symmetries which for some groups allow non-maximal values of the atmospheric neutrino mixing angle as well. The results of various cases are summarized in figure 1 and table 2.
- The successful texture I still has two free parameters apart from an overall mass scale. But as we have shown here, predicted atmospheric mixing angle in many cases is independent of these unknowns. The reactor angle θ_{13} depends on it but it is possible to determine these unknown also by enlarging the residual symmetry and we have given an example of a $Z_2 \times Z_2$ residual symmetry which can determine the complete neutrino mass matrix up to an overall scale in terms of group theoretical parameters alone and have identified $\Delta(600)$ as a possible group which can give correct θ_{13} and θ_{23} with this symmetry.

We end this section with a comparison of the present work with some earlier relevant works.

- The texture I, eq. (2.4) has been extensively studied since long in the context of $L_e - L_\mu - L_\tau$ global symmetry which implies it, see for example [59] and references therein. Imposition of this symmetry on the charged lepton mass matrix M_l makes it diagonal after redefinition of θ appearing in (2.4). Thus the matrix V_ν as given in eq. (2.7) corresponds to the final mixing matrix which is now not allowed by the present experimental constraints. This is not the case here since the $M_l M_l^\dagger$ is non-trivial with the imposed discrete symmetry.

- Neutrino mass matrix displaying a specific flavour antisymmetry namely, μ-τ antisymmetry was studied in [60]. This antisymmetry was assumed there to hold in the neutrino flavour basis. In our terminology, this would correspond to study of a specific example within the choice (2A) discussed in section 3.2.1. The structure of the neutrino mass matrix and the mixing angle predictions obtained here for this choice agrees with ref. [60] after suitable basis change. The study presented here is not limited to the μ-τ antisymmetry but encompasses all possible antisymmetry operators within SU(3) and leads to many new phenomenological predictions.

- The antisymmetry condition, eq. (1.3), can be converted to the usually assumed symmetry condition by redefining the operator $S_\nu \to iS_\nu$. The new operator does not however have unit determinant and would belong to a U(3) group. The occurrence of massless state within such group with condition, eq. (1.2) was discussed in [23, 24]. The residual symmetry operators used there had eigenvalues $(\eta, 1, -1)$ (or its permutations) with $\eta \neq \pm 1$. This coincides with eigenvalues of iS_ν for texture IV when $\eta = i$. Only texture IV was considered in [23, 24] and it was shown there that a large class of DSG of U(3) imply $\sin^2\theta_{13}$ to be either 0 or $\frac{1}{3}$ with condition (2). The same conclusion is found to be true here with eq. (1.3) and texture IV in case of the group series $\Delta(3N^2)$ and $\Delta(6N^2)$.

- It is possible to obtain a degenerate pair of neutrinos using symmetry condition, eq. (1.2) and DSG of SU(3). This was studied for the finite von-Dyck groups in [21] and for all DSG of SU(3) having three dimensional IR in [22]. Here, the third state is not implied to be massless. The case of one massless and two degenerate neutrinos can follow from the symmetry condition if DSG of U(3) are used. This was also discussed in [22]. The successful examples found in these two works are different from here because of the difference in the assumed residual symmetries. The cases studied in the context of DSG of SU(3) and U(3) [22] have texture similar to the texture II in the present terminology. It was found there that this texture can give non-trivial values of s_{13}^2, s_{23}^2 in several $\Delta(6N^2)$ groups when symmetry condition (1.2) is used. This does not happen with the antisymmetry condition in case of texture II as argued here. On the other hand, one can obtain correct values for θ_{13} and θ_{23} in all the $\Delta(3N^2)$ groups with texture I when antisymmetry condition is employed. Thus symmetry and antisymmetry conditions appear complementary to each other and allow more possibilities for flavour symmetries G_f.

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