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An analytic result for the two-loop seven-point MHV amplitude in $\mathcal{N} = 4$ SYM

John Golden and Marcus Spradlin

Department of Physics, Brown University, Box 1843, Providence, RI 02912-1843, U.S.A.

E-mail: John_Golden@brown.edu, Marcus_Spradlin@brown.edu

Abstract: We describe a general algorithm which builds on several pieces of data available in the literature to construct explicit analytic formulas for two-loop MHV amplitudes in $\mathcal{N} = 4$ super-Yang-Mills theory. The non-classical part of an amplitude is built from A_3 cluster polylogarithm functions; classical polylogarithms with (negative) cluster $\mathcal{X}$-coordinate arguments are added to complete the symbol of the amplitude; beyond-the-symbol terms proportional to π^2 are determined by comparison with the differential of the amplitude; and the overall additive constant is fixed by the collinear limit. We present an explicit formula for the seven-point amplitude $R_7^{(2)}$ as a sample application.

Keywords: Supersymmetric gauge theory, Scattering Amplitudes, Differential and Algebraic Geometry

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Contents

1 Introduction

This note is a natural continuation of the research program that has been pursued in the papers [1–4] and has been heavily guided by earlier mathematical work of Goncharov on both the structure of polylogarithm functions and on cluster algebras (see in particular [5] and [6]). The physics goal of our program is, narrowly, to understand the rich mathematical structure of two-loop amplitudes in $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) theory [7], and more broadly, to develop a toolkit of mathematical techniques useful for unlocking the structure of multi-loop amplitudes in general field theories. An example of the latter is the symbol calculus, which following [1] has become a very useful workhorse for dealing with the kinds of polylogarithm functions which are ubiquitous in multi-loop calculations, while the intimate connection between amplitudes and cluster algebras unearthed in [3] is a prime example of the very special structure exemplified by SYM theory in particular.

In this paper we tie together several threads which have run through the earlier work [1–4] but have not yet been fully wrapped up. Our immediate goal will be to construct an explicit analytic formula for the two-loop seven-point MHV amplitude $R_7^{(2)}$ in SYM theory.[1] While it may be interesting in its own right, we do not view the formula itself as the primary result of this paper. Rather our aim is to first review the various obstacles that arise in the pursuit of writing such analytic formulas, and then to bring together the relevant ideas and results from [1–4, 13] to argue that the problem of constructing analytic formulas for $R_n^{(2)}$ for any desired n may be considered "solved" (modulo the availability of sufficient computer

[1] More precisely $R_n^{(L)}$ stands for the n-particle L-loop *remainder function*, after the infrared singularities of the amplitude have been subtracted in a now standard way following [8, 9]. Dual conformal symmetry requires $R_n^{(L)}$ to vanish for $n<6$ at any loop order [10, 11], but a numerical study [12] established that $R_6^{(2)}$ is nonzero.

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power, of course). By this we mean that we describe an algorithm which, building on the scaffolding provided by Caron-Huot's computation [14] of the symbol of $R_n^{(2)}$, may be used to construct an analytic formula for any desired n. The result for $n = 6$ appeared in [1], and we present a result for $n = 7$ here as a specific application of our algorithm. Numerical studies of $R_n^{(2)}$ have been carried out for $n = 6$ in [12, 15, 16] and for higher n in [17, 18], and explicit formulas are known for the special case when all particles have momenta lying in a common $\mathbb{R}^{1,1}$ subspace of four-dimensional Minkowski space [19–21].

We do not address here the question of how the computational complexity of our algorithm scales with n because we hope that this will ultimately be an irrelevant question. As has happened often before in physics, and especially so in the study of SYM theory, we believe that once suitably packaged and digestible results accumulate for various relatively small values of n, the structure might become clear enough that one can extrapolate an all-n formula, which could subsequently be proven to be correct or at least could be checked to be consistent with all known properties of the true amplitudes.

Amplitudeology is a data-driven enterprise where insights gleaned by analyzing the results of a seemingly difficult calculation have often revealed hidden structure which trivialize the original calculation, and help to make the next set of calculations simpler (or even just possible). We very much anticipate that the formula we obtain for $R_7^{(2)}$ will not be the simplest or "best" one possible, but hope that the algorithm described in this paper will prove useful for generating new data for the amplitude community.

Section 2 contains some brief background material and definitions. Section 3 comments on the difficulties of integrating symbols in general, and on the tools we employ to overcome these difficulties. We also discuss the relation of our work to a complementary approach to similar problems which has been used by Dixon and collaborators to achieve several impressive results on multi-loop six-point amplitudes [22–25]. Section 4 outlines our general algorithm, while section 5 discusses its application to the specific case of $R_7^{(2)}$, culminating in the construction of a complete analytic formula for this amplitude, some properties of which are discussed in section 6.

2 Background

This section is a brief review of some of the more advanced mathematics that will appear throughout the rest of the paper, namely the coproduct δ and cluster algebras. For a more thorough introduction to these topics, see [2, 4].

The space of polylogarithm functions modulo products is a Lie coalgebra with coproduct[2] δ. The coproduct maps a polylogarithm function of weight 4 (the case of relevance to two-loop amplitudes) into two component spaces, $\Lambda^2 B_2$ and $B_3 \otimes \mathbb{C}^*$. Here, B_k refers to the Bloch group, which roughly speaking represents the space of classical weight k polylogarithm functions modulo functional relationships amongst Li_k and modulo products of functions of lower weight. Elements of B_k are linear combinations of objects denoted by $\{x\}_k$, which stands for the equivalence class containing the function $-\mathrm{Li}_k(-x)$. The $\Lambda^2 B_2$

[2]Throughout this paper, we use the word "coproduct" to denote δ, which satisfies $\delta^2 = 0$, rather than Δ which operates by simply deconcatenating the symbol. We refer the reader to [2, 26] for additional details.

component of the coproduct captures the obstruction to writing a function in terms of the classical polylogarithm functions Li_k [27, 28]. The $B_3 \otimes \mathbb{C}^*$ component of the coproduct encapsulates all of the intrinsically weight 4 terms in a function.

Cluster algebras are generated by a preferred set of variables ("cluster coordinates") grouped in disjoint sets called clusters related to each other by a transformation called mutation. The cluster algebra relevant for two-loop MHV scattering amplitudes in SYM theory is the $\mathrm{Gr}(4, n)$ Grassmannian cluster algebra, which is related to the kinematic configuration space for n particles, $\mathrm{Conf}_n(\mathbb{P}^3)$. These coordinates come in two flavors, $\mathcal{A}$- and $\mathcal{X}$- coordinates. An example of $\mathcal{A}$-coordinates are the standard Plücker coordinates $\langle ijkl \rangle = \det(Z_i Z_j Z_k Z_l)$ (in terms of momentum-twistor variables [29]). Slightly more complicated examples that will appear later in this paper are of the type

$$\langle a(bc)(de)(fg) \rangle \equiv \langle abde \rangle \langle acfg \rangle - \langle abfg \rangle \langle acde \rangle, \tag{2.1}$$

$$\langle ab(cde) \cap (fgh) \rangle \equiv \langle acde \rangle \langle bfgh \rangle - \langle bcde \rangle \langle afgh \rangle. \tag{2.2}$$

Cluster $\mathcal{X}$-coordinates are a special class of cross-ratios built from $\mathcal{A}$-coordinates.

These two topics, polylogarithms and cluster algebras, merge beautifuly in the arena of SYM theory. Firstly, only cluster $\mathcal{A}$-coordinates for $\mathrm{Gr}(4, n)$ appear in the symbol for $R_n^{(2)}$. Moreover, the coproduct of $R_7^{(2)}$ was calculated in [2] and it was noted that the elements $\{x\}_2$ and $\{x\}_3$ appearing in the coproduct were cluster $\mathcal{X}$-coordinates of the $\mathrm{Gr}(4, 7)$ Grassmannian cluster algebra. Furthermore, it was noted that the function for $R_6^{(2)}$ obtained in [1] can be written purely in terms of classical polylogarithms Li_k with (negative) $\mathcal{X}$-coordinates as arguments. In this paper we extend these connections to a general algorithm for constructing the function $R_n^{(2)}$.

Let us note that the $\mathrm{Gr}(4, n)$ cluster algebra has infinitely many $\mathcal{A}$- and $\mathcal{X}$-coordinates when $n > 7$, but we believe that this presents no obstruction to our algorithm since it is evident from the result of [14] that only finitely many (in fact, precisely $\frac{3}{2}n(n-5)^2$) of the $\mathcal{A}$-coordinates actually appear in the two-loop MHV amplitude $R_n^{(2)}$, and our experience has shown that the "most complicated part" of these amplitudes (see [4] for details) can be expressed in terms of the $\mathcal{X}$-coordinates belonging to finitely many A_3 subalgebras of $\mathrm{Gr}(4, n)$. For the special cases $n = 6, 7$, we expect that the two-loop symbol alphabet (which contains already all available $\mathcal{A}$-coordinates) will be sufficient to express all amplitudes (whether MHV or not) to all loop order, but for $n > 7$ we know of no reason to exclude the possibility that the symbol alphabet could grow larger at higher loops (indeed we expect it to become infinite for ten-point N^3MHV amplitudes starting already at only two loops).

A salient feature of cluster $\mathcal{X}$-coordinates is that they are positive when evaluated inside the positive Grasmmannian, defined as the subset of the Euclidean domain where $\langle ijkl \rangle > 0$ whenever $i < j < k < l$. This is incredibly important because it allows us to impose analyticity inside the positive domain with relative ease (since $\mathrm{Li}_k(x)$ is smooth for $x < 0$), in particular without having to worry about branch cuts. It would be interesting to check the extension of our final formula to more general Euclidean kinematics, for which it would be necessary to specify where to take the branch cut of each $\mathrm{Li}_k(x)$ (as was done for example in [1] for $n = 6$). It would also be interesting to explore the analytic continuation to other regions outside the Euclidean domain, for example to make contact with work on the seven-point amplitude in the multi-Regge regime [30–32].

Before we describe our algorithm we would first like to clarify the difficulties that our cluster algebraic approach allows us to overcome.

3 The problem of integrating symbols

The problem of finding an explicit polylogarithm function whose symbol matches a given random (but integrable) symbol is hopeless; no algorithm exists in general. Fortunately, amplitudes in SYM theory do not have random symbols, nor do we expect them to be expressed in terms of completely random functions.

In such happier cases the problem can be tractable if the desired function may be expressed in terms of some class of generalized polylogarithm functions whose arguments are all drawn from some particular finite collection of well-behaved variables. Then the problem of integrating the symbol becomes simply one of linear algebra: one writes a general linear combination of the functions in the ansatz, and chooses the coefficients to match the desired symbol. Ideally, the ansatz should be just big enough to contain the answer, and not too big. If the ansatz is too overcomplete[3] there can be considerable ambiguity in choosing a functional representative for the integrated symbol.

If one were merely interested in being able to obtain numerical values for SYM amplitudes, then such ambiguity would be of little concern. If the goal however is to unlock their mathematical structure, then it is desirable to have functional representations which manifest, to the extent possible, all of their known properties. From this point of view, any ambiguity in how to write an amplitude is seen as an inefficiency, a wasted opportunity.

In a series of papers [22–25], Dixon and collaborators have pursued one approach to this problem by studying "hexagon functions", defined as polylogarithm functions whose symbol can be expressed in terms of a certain 9-letter alphabet (in our terminology, the alphabet of $\mathcal{A}$-coordinates for the $\mathrm{Gr}(4,6)$ Grassmannian cluster algebra) and which have the appropriate analytic structure for scattering amplitudes (specifically, that they must be analytic everywhere inside the Euclidean domain, with branch points on the boundary of the Euclidean domain when $\langle i\, i{+}1\, j\, j{+}1\rangle = 0$ for some i, j). By systematically classifying such hexagon functions through weight eight, and by using physical input about the near-collinear limit derived from the Wilson loop OPE approach [34–38] and from the multi-Regge limit [23, 39–46], they have determined analytic expressions for the six-point NMHV amplitude at two loops, and the six-point MHV amplitude at three and four loops.

It would be extremely interesting to pursue a similar approach for $n > 6$, by exploring for example the space of "heptagon functions". Our trepidation to take this route stems from the fact that the required symbol alphabet grows rapidly with n: as mentioned above,

[3]Some overcompleteness is inevitable in our approach due to Li_k identities involving configurations of points in projective space (see for example [27, 28, 33]), but such identities are rare when the arguments are restricted to be (negative) cluster $\mathcal{X}$-coordinates. The only currently known non-trivial identities of this type are the 5-term A_2 identity (Abel's identity) for Li_2 and the 40-term D_4 identity for Li_3 [2].

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the symbol alphabet for $R_n^{(2)}$ has $\frac{3}{2}n(n-5)^2$ entries [14], so the space of weight-four symbols has dimension[4] $\mathcal{O}(n^{12})$.

We have pursued instead the somewhat orthogonal approach of organizing our calculations not from left-to-right in the symbol, but rather in order of decreasing mathematical complexity of the functional constituents. At weight four, this means that we first focus our attention on the "non-classical" part of the amplitude: the $\Lambda^2 B_2$ component of its coproduct. The remaining purely classical pieces of an amplitude can be systematically computed in order from most to least complicated by following the procedure outlined in [1]. This approach has the disadvantage of leaving the analytic properties of amplitudes obscure, while it has the advantage of making some remarkable mathematical properties — the relation to the cluster structure on the kinematic domain — manifest.

The very first step in this approach is the one most fraught with peril, as we now explain. The $\Lambda^2 B_2$ component of the coproduct of $R_n^{(2)}$ can be expressed [2, 13] as a linear combination of various $\{x_i\}_2 \wedge \{x_j\}_2$ where the x's are drawn from the $\mathcal{X}$-coordinates of the $\mathrm{Gr}(4,n)$ cluster algebra. Moreover, the x's always appear together in pairs satisfying $\{x_i, x_j\} = 0$ with respect to the natural Poisson structure on the kinematic domain $\mathrm{Conf}_n(\mathbb{P}^3)$; this implies that each pair of variables generates an $A_1 \times A_1$ subalgebra of the $\mathrm{Gr}(4,n)$ cluster algebra.

For several years a guiding aim of this research program, strongly advocated by Goncharov, has been that it should be possible to write each amplitude under consideration as a linear combination of special functions associated with smaller building blocks ("atoms"). For example, it is well-known that the function[5]

$$L_{2,2}(x,y) = \frac{1}{2}\int_0^1 \frac{dt}{t}\,\mathrm{Li}_2(-tx)\,\mathrm{Li}_1(-ty) - (x \leftrightarrow y) \tag{3.1}$$

has the simple $\Lambda^2 B_2$ coproduct component $\{x\}_2 \wedge \{y\}_2$. Therefore one might be tempted to construct the non-classical part of a desired $R_n^{(2)}$ by writing down an appropriate linear combination of $L_{2,2}(x_i, x_j)$ functions; the difference between this object and $R_n^{(2)}$ must then be expressible in terms of the classical functions Li_k only.

The fatal flaw in this approach is that while $L_{2,2}(x_i, x_j)$ indeed has a simple coproduct, it is poorly adapted to applications where one wants to manifest cluster structure because its symbol has some entries of the form $x_i - x_j$, which is never expressible as a product of cluster $\mathcal{A}$-coordinates (and thus can never be an $\mathcal{X}$-coordinate). Therefore one would have to considerably enlarge the symbol alphabet under consideration in order to fit all of the classical pieces of the amplitude left over by subtracting a linear combination of $L_{2,2}$'s. Just as bad, one would almost inevitably generate Li_k functions whose arguments range

[4]This is rather too pessimistic; the analyticity condition cuts this down by one power of n and the integrability condition no doubt cuts down by some more powers of n.

[5]We caution the reader that several variants of this function exist in the literature, beginning with [27], all of which differ from each other by the addition of terms proportional to Li_4, or products of lower-weight Li_k's. In fact even in this short paper we will use a second variant $K_{2,2}$ momentarily. All of these variants have the same $\Lambda^2 B_2$ coproduct component. The particular $L_{2,2}(x,y)$ used here may also be expressed as $L_{2,2}(x,y) = \frac{1}{2}\,\mathrm{Li}_{2,2}(x/y, -y) - (x \leftrightarrow y)$ in terms of the $\mathrm{Li}_{2,2}$ function.

over the entire real line, greatly complicating the problem of arranging all of the branch cuts of the individual terms to conspire to cancel out everywhere in the positive domain.

So if we want to maintain a connection to the cluster structure (and, more practically, to avoid enormously complicating the calculation by being forced to clean up unwanted mess in the symbol), we should abandon the idea that each individual term $\{x_i\}_2 \wedge \{x_j\}_2$ may be thought of as an atom.[6] The problem of identifying the smallest building block manifesting all of the known cluster properties of $R_n^{(2)}$ was solved (at least, for a few of the simplest cluster algebras, and more generally conjectured) in [4]. The solution is a function associated to the A_3 cluster algebra which we can write in the form

$$f_{A_3}(x_1, x_2, x_3) = \sum_{i=1}^{3} K_{2,2}(x_{i,1}, x_{i,2}), \tag{3.2}$$

where

$$\begin{aligned} x_{1,1} &= x_1, & x_{1,2} &= 1/x_3, \\ x_{2,1} &= (x_1 x_2 + x_2 + 1)\, x_3, & x_{2,2} &= \frac{x_1 x_2 + x_2 + 1}{x_1}, \\ x_{3,1} &= \frac{x_2 x_3 + x_3 + 1}{x_2}, & x_{3,2} &= \frac{x_2 x_3 + x_3 + 1}{x_1 x_2 x_3} \end{aligned} \tag{3.3}$$

and

$$K_{2,2}(x,y) = L_{2,2}(x,y) - \left[\mathrm{Li}_4(x/y) - \frac{1}{3}\,\mathrm{Li}_3(x/y)\log(x/y) - (x \leftrightarrow y)\right] - \frac{1}{2}\,\mathrm{Li}_2(-x)\,\mathrm{Li}_2(-y). \tag{3.4}$$

The expression for $K_{2,2}$ given here differs from the one presented in [4] by the addition of terms proportional to products of logarithms as well as the final $\mathrm{Li}_2\,\mathrm{Li}_2$ term, none of which affect the coproduct of $K_{2,2}$.

As long as the three x_i generate an A_3 algebra $x_1 \to x_2 \to x_3$ (which could be a subalgebra of a larger algebra), the A_3 function accomplishes a remarkable feat:

- the $\Lambda^2 B_2$ component of its coproduct, $\sum_{i=1}^{3}\{x_{i,1}\}_2 \wedge \{x_{i,2}\}_2$, involves only pairs of Poisson commuting $\mathcal{X}$-coordinates;
- the $B_3 \otimes \mathbb{C}^*$ component of its coproduct can be written in terms of $\mathcal{X}$-coordinates (the Li_4 term in $K_{2,2}$ is crucial here);
- its symbol can be written entirely in terms of $\mathcal{A}$-coordinates (here the Li_3 log term is crucial);
- and it is smooth and real-valued everywhere inside the positive domain (i.e., as long as $x_1, x_2, x_3 > 0$), thanks to the terms which were added compared to [4].

[6] Instead they are perhaps quarks: never allowed to appear alone, but always bound safely together in A_2 functions or perhaps other, not yet discovered, more exotic baryons.

The $\mathrm{Li}_2 \mathrm{Li}_2$ term in (3.2) is completely innocuous and was chosen for inclusion because it was observed to nicely package together most of the $\mathrm{Li}_2 \mathrm{Li}_2$ terms in the amplitude $R_7^{(2)}$. It would be very interesting to see if a more optimal packaging of subleading terms could be obtained, whether for $n = 7$ or even for all n.

Working with A_3 functions, rather than the underlying individual $L_{2,2}$'s, therefore allows us to avoid having to enlarge the symbol alphabet beyond the set of cluster $\mathcal{A}$-coordinates. Moreover, when expressing the classical contributions to an amplitude we are able to restrict our attention to the functions $\mathrm{Li}_k(-x)$, which are smooth and real-valued throughout the positive domain as long as the arguments x are always taken from the set of cluster $\mathcal{X}$-coordinates.

4 The algorithm for $R_n^{(2)}$

The algorithm is naturally broken into four steps. (1) As discussed in the previous section, we start by writing down a linear combination of A_3 cluster functions with the same $\Lambda^2 B_2$ content as the desired $R_n^{(2)}$. After subtracting this linear combination from the amplitude we are left with a function which (2) we express in terms of the classical polylogarithms Li_k following the algorithm described in [1]. One minor difference with respect to [1] is that we prioritize the Li_4 terms over those which can be written as products of lower-weight Li_k's, since only the former contribute to the $B_3 \otimes \mathbb{C}^*$ component of the coproduct. So, to be explicit, we proceed in the following order: f_{A_3}, Li_4, $\mathrm{Li}_2 \mathrm{Li}_2$, $\mathrm{Li}_2 \log\log$, $\mathrm{Li}_3 \log$, $\log\log\log\log$.

At this stage we have a function with the same symbol as the amplitude, so the difference is expected to be equal to π^2 times polylogarithm functions of weight two. We ought not find any terms proportional to $i\pi$ times a function of weight three since at each step we work with functions that are manifestly free of branch cuts in the positive domain. (3) The $\mathcal{O}(\pi^2)$ terms can be found by comparison to the known [3, 14] all-n formula for the differential $dR_n^{(2)}$ of the amplitude. (4) Finally, the overall additive constant in the amplitude can be determined by enforcing smoothness of the collinear limit $R_n^{(2)} \to R_{n-1}^{(2)}$, a property which is built into the definition of the remainder function [9].

5 The construction of $R_7^{(2)}$

We present here some details about the expression for $R_7^{(2)}$ generated by our algorithm. Some of the contributions, in particular the terms of the form $\mathrm{Li}_2 \log\log$ or $\log\log\log\log$, are too numerous to reasonably display in the text, so we refer the reader to the Mathematica file associated to this note for the full symbolic result.[7]

We begin by recalling the representation of the non-classical pieces of $R_7^{(2)}$ in terms of A_3 functions, presented in [4] as

$$\frac{1}{2} f_{A_3}\left(\frac{\langle 1245\rangle\langle 1567\rangle}{\langle 1257\rangle\langle 1456\rangle}, \frac{\langle 1235\rangle\langle 1456\rangle}{\langle 1256\rangle\langle 1345\rangle}, \frac{\langle 1234\rangle\langle 1257\rangle}{\langle 1237\rangle\langle 1245\rangle}\right) + \frac{1}{2} f_{A_3}\left(\frac{\langle 1345\rangle\langle 1567\rangle}{\langle 1357\rangle\langle 1456\rangle}, \frac{\langle 1235\rangle\langle 3456\rangle}{\langle 1356\rangle\langle 2345\rangle}, \frac{\langle 1234\rangle\langle 1357\rangle}{\langle 1237\rangle\langle 1345\rangle}\right)$$
$$+ \text{ dihedral} + \text{parity conjugate.} \quad (5.1)$$

[7] In case of any discrepancy between formulas in the text and the Mathematica file, the latter is authoritative.

As we emphasized in [4], the difference between $R_7^{(2)}$ and (5.1) is a weight-four polynomial in the functions $\mathrm{Li}_k(-x)$ for $k = 1, 2, 3$ (and π^2), with arguments x drawn from the 385 $\mathcal{X}$-coordinates of the Gr(4,7) cluster algebra.

The $B_3 \otimes \mathbb{C}^*$ component of the coproduct of $R_7^{(2)}$ was computed in [2]. We find that the Li_4 terms

$$
\begin{aligned}
&-\mathrm{Li}_4\left(-\tfrac{\langle 1234\rangle\langle 1256\rangle}{\langle 1236\rangle\langle 1245\rangle}\right)-\mathrm{Li}_4\left(-\tfrac{\langle 1234\rangle\langle 1257\rangle}{\langle 1237\rangle\langle 1245\rangle}\right)-\frac{1}{2}\mathrm{Li}_4\left(-\tfrac{\langle 1234\rangle\langle 1357\rangle}{\langle 1237\rangle\langle 1345\rangle}\right)-\frac{1}{2}\mathrm{Li}_4\left(-\tfrac{\langle 1234\rangle\langle 1456\rangle}{\langle 1246\rangle\langle 1345\rangle}\right)\\
&\qquad + \text{ dihedral} + \text{parity conjugate}, \qquad (5.2)
\end{aligned}
$$

must be added to eq. (5.1) in order to correctly reproduce the full coproduct of the amplitude.

At this stage we know that the difference between $R_7^{(2)}$ and eqs. (5.1) plus (5.2) is a product of Li_k functions of weight strictly less than four. Following the procedure outlined in [1] we find that the missing $\mathrm{Li}_2\,\mathrm{Li}_2$ terms (beyond the ones that we have already snuck in via eq. (3.2)) are

$$
\begin{aligned}
&\mathrm{Li}_2\left(\tfrac{\langle 3(17)(24)(56)\rangle}{\langle 1237\rangle\langle 3456\rangle}\right)\mathrm{Li}_2\left(\tfrac{\langle 1456\rangle\langle 3(17)(24)(56)\rangle}{\langle 1234\rangle\langle 1567\rangle\langle 3456\rangle}\right)+\mathrm{Li}_2\left(-\tfrac{\langle 1234\rangle\langle 1257\rangle}{\langle 1237\rangle\langle 1245\rangle}\right)\mathrm{Li}_2\left(-\tfrac{\langle 1234\rangle\langle 1457\rangle}{\langle 1247\rangle\langle 1345\rangle}\right)\\
&\quad-\frac{1}{2}\mathrm{Li}_2\left(-\tfrac{\langle 1234\rangle\langle 1257\rangle}{\langle 1237\rangle\langle 1245\rangle}\right)\mathrm{Li}_2\left(-\tfrac{\langle 1245\rangle\langle 1567\rangle}{\langle 1257\rangle\langle 1456\rangle}\right)-\mathrm{Li}_2\left(-\tfrac{\langle 1234\rangle\langle 1357\rangle}{\langle 1237\rangle\langle 1345\rangle}\right)\mathrm{Li}_2\left(-\tfrac{\langle 1345\rangle\langle 1567\rangle}{\langle 1357\rangle\langle 1456\rangle}\right)\\
&\quad-\mathrm{Li}_2\left(-\tfrac{\langle 1237\rangle\langle 1467\rangle}{\langle 1267\rangle\langle 1347\rangle}\right)\mathrm{Li}_2\left(-\tfrac{\langle 1236\rangle\langle 2567\rangle}{\langle 1267\rangle\langle 2356\rangle}\right)+\mathrm{Li}_2\left(-\tfrac{\langle 1236\rangle\langle 2567\rangle}{\langle 1267\rangle\langle 2356\rangle}\right)\mathrm{Li}_2\left(-\tfrac{\langle 2345\rangle\langle 3467\rangle}{\langle 2347\rangle\langle 3456\rangle}\right)\\
&\qquad + \text{ dihedral} + \text{parity conjugate}. \qquad (5.3)
\end{aligned}
$$

We also find the $\mathrm{Li}_3 \log$ terms

$$
\begin{aligned}
&\left(\frac{1}{2}\mathrm{Li}_3\left(-\tfrac{\langle 1267\rangle\langle 1456\rangle}{\langle 1246\rangle\langle 1567\rangle}\right)-\frac{1}{2}\mathrm{Li}_3\left(-\tfrac{\langle 1246\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1456\rangle}\right)\right)\log\left(\tfrac{\langle 1237\rangle\langle 1246\rangle}{\langle 1234\rangle\langle 1267\rangle}\right)\\
&+\left(-\frac{1}{2}\mathrm{Li}_3\left(-\tfrac{\langle 1237\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1357\rangle}\right)-\frac{1}{2}\mathrm{Li}_3\left(-\tfrac{\langle 1247\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1457\rangle}\right)+\mathrm{Li}_3\left(-\tfrac{\langle 1257\rangle\langle 1347\rangle}{\langle 1237\rangle\langle 1457\rangle}\right)+\mathrm{Li}_3\left(-\tfrac{\langle 1257\rangle\langle 1456\rangle}{\langle 1245\rangle\langle 1567\rangle}\right)\right.\\
&\left.+\frac{1}{2}\mathrm{Li}_3\left(-\tfrac{\langle 1267\rangle\langle 1456\rangle}{\langle 1246\rangle\langle 1567\rangle}\right)+\frac{1}{2}\mathrm{Li}_3\left(-\tfrac{\langle 1357\rangle\langle 1456\rangle}{\langle 1345\rangle\langle 1567\rangle}\right)-\mathrm{Li}_3\left(-\tfrac{\langle 1235\rangle\langle 1267\rangle\langle 1457\rangle}{\langle 1237\rangle\langle 1245\rangle\langle 1567\rangle}\right)\right)\log\left(\tfrac{\langle 1247\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1457\rangle}\right)\\
&+\left(-\mathrm{Li}_3\left(-\tfrac{\langle 1236\rangle\langle 1245\rangle}{\langle 1234\rangle\langle 1256\rangle}\right)-\frac{1}{2}\mathrm{Li}_3\left(-\tfrac{\langle 1237\rangle\langle 1245\rangle}{\langle 1234\rangle\langle 1257\rangle}\right)+\mathrm{Li}_3\left(-\tfrac{\langle 1247\rangle\langle 1256\rangle}{\langle 1245\rangle\langle 1267\rangle}\right)+\frac{1}{2}\mathrm{Li}_3\left(-\tfrac{\langle 1237\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1357\rangle}\right)\right.\\
&+\frac{1}{2}\mathrm{Li}_3\left(-\tfrac{\langle 1246\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1456\rangle}\right)+\mathrm{Li}_3\left(-\tfrac{\langle 1247\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1457\rangle}\right)-\frac{1}{2}\mathrm{Li}_3\left(-\tfrac{\langle 1257\rangle\langle 1456\rangle}{\langle 1245\rangle\langle 1567\rangle}\right)-\frac{1}{2}\mathrm{Li}_3\left(-\tfrac{\langle 1267\rangle\langle 1456\rangle}{\langle 1246\rangle\langle 1567\rangle}\right)\\
&\left.-\frac{1}{2}\mathrm{Li}_3\left(-\tfrac{\langle 1456\rangle\langle 2345\rangle}{\langle 1245\rangle\langle 3456\rangle}\right)-\mathrm{Li}_3\left(-\tfrac{\langle 1457\rangle\langle 2345\rangle}{\langle 1245\rangle\langle 3457\rangle}\right)+\mathrm{Li}_3\left(-\tfrac{\langle 1457\rangle\langle 2456\rangle}{\langle 1245\rangle\langle 4567\rangle}\right)\right)\log\left(\tfrac{\langle 1237\rangle\langle 1245\rangle}{\langle 1234\rangle\langle 1257\rangle}\right)\\
&\qquad + \text{ dihedral} + \text{parity conjugate}. \qquad (5.4)
\end{aligned}
$$

The remaining $\mathrm{Li}_2 \log\log$ and $\log\log\log\log$ terms which must be added to eqs. (5.1), (5.2), (5.3) and (5.4) in order to fully match the known symbol of $R_7^{(2)}$ are too numerous to display here and are recorded in the attached Mathematica file.

Next we turn to the problem of fixing "beyond-the-symbol" terms, given by numerical constants (in this application, rational numbers times π^k) times functions of weight $4-k$. The terms proportional to π^2 may be deduced by computing the full differential of all of the terms we have accumulated so far, and subtracting the result from the known analytic formula for $dR_7^{(2)}$ [3, 14]. The result is a linear combination (with rational coefficients) of terms like $\pi^2\log(a_1)d\log a_2$ for various $\mathcal{A}$-coordinates a_1, a_2. This can be integrated analytically to a linear combination of terms like $\pi^2\,\mathrm{Li}_2(-x_i)$ and $\pi^2\log(x_j)\log(x_k)$ with all arguments being $\mathcal{X}$-coordinates. In this manner we find that the $\pi^2\,\mathrm{Li}_2$ terms in our representation of $R_7^{(2)}$ are given by

$$\frac{7\pi^2}{48}\,\mathrm{Li}_2\left(-\frac{\langle 1247\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1457\rangle}\right)-\frac{\pi^2}{8}\,\mathrm{Li}_2\left(-\frac{\langle 1(23)(45)(67)\rangle}{\langle 1234\rangle\langle 1567\rangle}\right)+\text{dihedral}+\text{parity conjugate},\tag{5.5}$$

while the $\pi^2\log\log$ terms are again somewhat too numerous to efficiently display here.

At this point we have constructed a function which agrees with $R_7^{(2)}$ up to a single overall additive constant.[8] This constant, expected to be a rational number times π^4, can be determined by the requirement that $R_7^{(2)}\to R_6^{(2)}$ smoothly in the collinear limit. We choose to parameterize the $6\parallel 7$ collinear limit following [14] by replacing

$$Z_7\to Z_7(t)=Z_6-t(\alpha Z_1+\beta Z_5)+t^2Z_2,\tag{5.6}$$

with α and β being arbitrary parameters, and then taking the limit $t\to 0$. As long as the starting point $(Z_1,\ldots,Z_7)$ is inside the positive domain and α and β are chosen to be positive, then there exists a finite $t_0>0$ such that $(Z_1,\ldots,Z_6,Z_7(t))$ lies in the positive domain for all $0<t<t_0$. Then the collinear limit[9]

$$\lim_{t\to 0^+}R_7^{(2)}(Z_1,\ldots,Z_6,Z_7(t))=R_6^{(2)}(Z_1,\ldots,Z_6),\tag{5.7}$$

together with the known formula [1] for $R_6^{(2)}$, determines the overall additive constant in $R_7^{(2)}$.

Each cross-ratio appearing our formula for $R_7^{(2)}$ approaches either 0, ∞, or a finite value in the limit $t\to 0^+$, so it is a simple matter to compute the limit of the formula using the asymptotic behavior of the polylogarithm functions

$$\mathrm{Li}_2(-1/t)\sim-\frac{1}{2}\log^2 t-\frac{\pi^2}{2},\tag{5.8}$$

$$\mathrm{Li}_3(-1/t)\sim+\frac{1}{6}\log^3 t+\frac{\pi^2}{6}\log t,\tag{5.9}$$

$$\mathrm{Li}_4(-1/t)\sim-\frac{1}{24}\log^4 t-\frac{\pi^2}{12}\log^2 t-\frac{7\pi^4}{360}\,,\tag{5.10}$$

together with the asymptotic expansions (when x, t and a are positive)

$$L_{2,2}(x,t)\sim 0,\tag{5.11}$$

[8] There are no $\zeta(3)\log$ terms since $dR_n^{(2)}$ is known [14] to not contain any terms proportional to $\zeta(3)$.

[9] We caution the reader that our normalization convention for $R_7^{(2)}$ agrees with that of [14], which differs by a factor of four from that of [1], so the $R_6^{(2)}$ appearing on the right-hand side of eq. (5.7) should be four times the function $R_6^{(2)}$ given in the latter reference.

$$L_{2,2}(x,1/t) \sim \frac{1}{4}\,\mathrm{Li}_2(-x)\log^2 t + \mathrm{Li}_3(-x)\log t + \mathrm{Li}_4(-x) + \frac{\pi^2}{12}\,\mathrm{Li}_2(-x), \tag{5.12}$$

$$L_{2,2}(1/t, a/t^2) \sim -\frac{5}{24}\log^4 t + \frac{1}{3}\log a \log^3 t - \frac{1}{8}\log^2 a \log^2 t + \frac{\pi^4}{24}\log^2 t - \frac{\pi^2}{24}\log^2 a - \frac{\pi^4}{30}, \tag{5.13}$$

where $\sim$ signifies the omission of terms which vanish as powers of t (or powers of t times powers of $\log t$). We have taken the limit of $R_7^{(2)}$ by choosing various random initial kinematic points in the positive domain with all momentum twistors having integer entries. Then, after taking the limit $t \to 0^+$, the two sides of eq. (5.7) can be evaluated numerically with arbitrary precision. In this manner we find that that we have to add $-\frac{13}{36}\pi^4$ to our formula for $R_7^{(2)}$ in order for eq. (5.7) to be satisfied.

6 The function $R_7^{(2)}$

Several very different ingredients have gone into the construction of our formula for $R_7^{(2)}$, from Caron-Huot's calculation of symbols via an extension of superspace to the mathematical structure of cluster algebras. As an independent test that all of these ingredients have been put together correctly it is reassuring to compare our result to numerical values for $R_7^{(2)}$ obtained in [17] via the Wilson loop approach to scattering amplitudes in SYM theory.

To get an intuition for a function it is often useful to see a plot of it, such as figure 11 of [17] which shows $R_7^{(2)}$ evaluated on the "symmetric line", the locus where

$$(u_{14}, u_{25}, u_{36}, u_{47}, u_{15}, u_{26}, u_{37}) = (u, u, u, u, u, u, u) \tag{6.1}$$

in terms of

$$u_{ij} = \frac{\langle i\, i{+}1\, j{+}1\, j{+}2\rangle\langle i{+}1\, i{+}2\, j\, j{+}1\rangle}{\langle i\, i{+}1\, j\, j{+}1\rangle\langle i{+}1\, i{+}2\, j{+}1\, j{+}2\rangle}. \tag{6.2}$$

When the seven momentum vectors of the scattering particles are required to lie in four spacetime dimensions, the u_{ij} are not free (indeed they cannot be, since the dimension of $\mathrm{Conf}_7(\mathbb{P}^3)$ is only six) but are constrained to satisfy a particular seventh-order polynomial equation called the Gram determinant constraint. The symmetric line intersects the Gram locus only at isolated points (specifically, at the roots of $(1+u)(1-4u+3u^2+u^3)^2$). The authors of [17] evaded this constraint by allowing the momenta to lie in arbitrary dimension. By making use of momentum twistor machinery our result for $R_7^{(2)}$ is solidly tied to four-dimensional kinematics, although we anticipate that it should not be very difficult to relax this constraint.

Until that is done we are therefore unable to provide a plot of our $R_7^{(2)}$ formula along the symmetric line. Instead we display in figure 1 a plot of this function along the line segment

$$(u_{14}, u_{25}, u_{36}, u_{47}, u_{15}, u_{26}, u_{37}) = \left(u, u, u, u, u, u, \frac{(1-u-u^2)^2}{1-2u^2}\right) \tag{6.3}$$

which satisfies the Gram constraint for all u and which lies in the positive domain for $0 < u < u_0 = 0.35689586789\ldots$, this number being the smallest positive root of $1 - 4u + 3u^2 + u^3 = 0$.

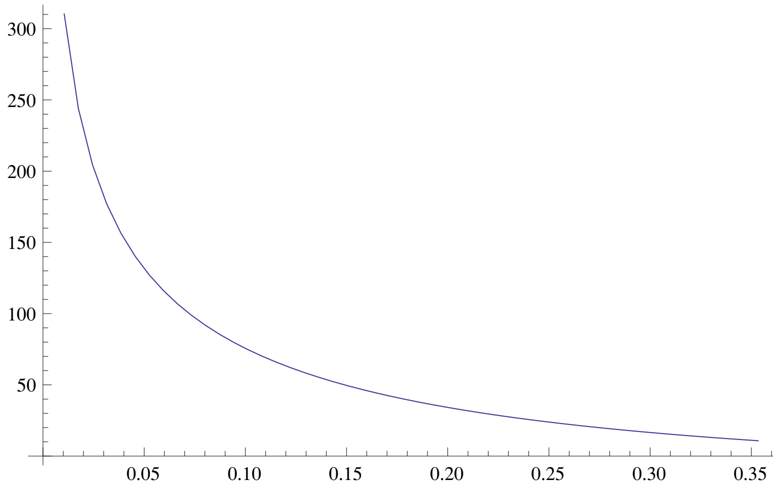

Figure 1. The two-loop seven-point MHV remainder function $R_7^{(2)}$ evaluated along the line segment parameterized by eq. (6.3) between $u = 0$ and the boundary of the positive domain at $u \approx 0.3569$.

The endpoint of this line segment at $u = u_0$ is rather special:[10] it touches the symmetric line at the tip of the positive domain. At this special point we find

$$R_7^{(2)}(u_0, u_0, u_0, u_0, u_0, u_0, u_0) = 10.4368451968\ldots . \tag{6.4}$$

At a conveniently chosen non-symmetric point point satisfying the Gram constraint we find for example

$$R_7^{(2)}\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{121}{224}\right) = 23.8717248322\ldots . \tag{6.5}$$

Both of these values are consistent with numerical results obtained using the Wilson loop computation of [17].[11] At all points in the positive domain where we have evaluated $R_7^{(2)}$, we have always found it take positive values, supporting the conjecture of [47].

7 Conclusion

We have described an algorithm for bootstrapping explicit analytic formulas for the two-loop n-point MHV remainder functions $R_n^{(2)}$ in SYM theory from known results in the literature for the symbol [14] and the differential [3] of $R_n^{(2)}$. The algorithm expresses these amplitudes as linear combination of A_3 cluster polylogarithm functions [4] and (products of) classical polylogarithm functions $\mathrm{Li}_k(-x)$ with arguments x drawn from the set of $\mathcal{X}$-coordinates [6] for the $\mathrm{Gr}(4, n)$ cluster algebra. Each building block utilized in the construction is manifestly real-valued and singularity-free inside the positive domain, ensuring that the generated formula for $R_n^{(2)}$ manifests this property as well.

[10] This point is a close analog to the special point $(u_{14}, u_{25}, u_{36}) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ in six-particle kinematics.

[11] We thank A. Brandhuber, P. Heslop and G. Travaglini for correspondence and for providing us with their results at these kinematic points, which match eqs. (6.4) and (6.5) to 0.003%, within the estimated margin of error of their numerical calculation.

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As a sample application of this algorithm we have constructed an explicit analytic representation for $R_7^{(2)}$. We would like to emphasize that we have put almost no effort into optimizing our result, instead opting to see what we get by treating this as nothing more than a "shift-enter" computation. Although we were somewhat surprised that the answer came out as compact as it did, we anticipate that our result for $R_7^{(2)}$ will not be the "best" representation possible but hope that it may provide a useful starting point for further exploration of the structure of this amplitude. In that sense we suspect our representation for $R_7^{(2)}$ may be more similar to the DDS formula [48, 49] than to the GSVV formula [1] for $R_6^{(2)}$.

Let us end by speculating about some possible ways in which our representation for $R_7^{(2)}$ (and $R_n^{(2)}$ more generally) ought to be improved. As a general statement, it is our hope that amplitudes should admit natural functional representations which are as canonical as possible[12] and that any unexplained ambiguity in how to write an amplitude should be a cause for disappointment. This is because our ultimate dream is that it should be possible to formulate a list of physical and mathematical constraints which determine SYM theory amplitudes uniquely, and any free parameter appearing in the representation of some amplitude represents a lost opportunity to manifest some otherwise hidden property it satisfies.

For example, we find it suboptimal that (as mentioned in [4]) the non-classical part of $R_7^{(2)}$ may be expressed in many different ways in terms of A_3 functions. It would be fantastic if one could identify some particular A_3 subalgebras inside the $\mathrm{Gr}(4,7)$ cluster algebra (or $\mathrm{Gr}(4,n)$ more generally) which are for some reason preferred for expressing two-loop MHV amplitudes. Moreover it would be nice if all of the classical terms tabulated in section 4 could be absorbed into an appropriate redefinition of the A_3 function given in eq. (3.2) so that the complete formula for $R_7^{(2)}$, or even all $R_n^{(2)}$, could be written as a simple linear combination of suitably defined A_3 functions and nothing else. If this magic A_3 function were positive-valued inside the positive domain, it would furthermore manifest the conjectured [47] positivity of $R_n^{(2)}$ itself. It would be ideal if this could be done for all n in a way which manifests collinear limits, with the various A_3 functions appearing in $R_n^{(2)}$ tending either to zero or to $n{-}1$-point A_3 functions in the collinear limit. Finally, perhaps it is not the A_3 function but something else which is the right building block for realizing all of these dreams.

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[12] Given the various known, but classifiable, polylogarithm identities — the 5-term A_2 identity for Li_2, the 40-term D_4 identity for Li_3, and other possibly existent identities not yet discovered.

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