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Characterization of Oblique Dual Frame Pairs

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Given a frame for a subspace $\mathcal{W}$ of a Hilbert space $\mathcal{H}$, we consider all possible families of oblique dual frame vectors on an appropriately chosen subspace $\mathcal{V}$. In place of the standard description, which involves computing the pseudoinverse of the frame operator, we develop an alternative characterization which in some cases can be computationally more efficient. We first treat the case of a general frame on an arbitrary Hilbert space, and then specialize the results to shift-invariant frames with multiple generators. In particular, we present explicit versions of our general conditions for the case of shift-invariant spaces with a single generator. The theory is also adapted to the standard frame setting in which the original and dual frames are defined on the same space.

1. INTRODUCTION

Frames are generalizations of bases which lead to redundant signal expansions [1–4]. A frame for a Hilbert space is a set of not necessarily linearly independent vectors that has the property that each vector in the space can be expanded in terms of these vectors. Frames were first introduced by Duffin and Schaeffer [1] in the context of nonharmonic Fourier series, and play an important role in the theory of nonuniform sampling [1, 2, 5, 6]. Recent interest in frames has been motivated in part by their utility in analyzing wavelet expansions [7, 8], and by their robustness properties [3, 8–13].

Frame-like expansions have been developed and used in a wide range of disciplines. Many connections between frame theory and various signal processing techniques have been recently discovered and developed. For example, the theory of frames has been used to study and design oversampled filter banks [14–17] and error correction codes [18]. Wavelet families have been used in quantum mechanics and many other areas of theoretical physics [8, 19].

One of the prime applications of frames is that they lead to expansions of vectors (or signals) in the underlying Hilbert space in terms of the frame elements. Specifically, if $\mathcal{H}$ is a separable Hilbert space and $\{f_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$, then any f in $\mathcal{H}$ can be expressed as

$$f = \sum_{k=1}^{\infty} \langle f, g_k \rangle f_k, \qquad (1)$$

for some dual frame $\{g_k\}_{k=1}^{\infty}$ for $\mathcal{H}$. In order to use this representation in practice, we need to be able to calculate the coefficients $\langle f, g_k \rangle$. A popular choice of $\{g_k\}_{k=1}^{\infty}$ is the minimal-norm dual frame, that is, the canonical dual frame. However, computing the minimal-norm dual is highly nontrivial in general. Another issue is that the frame $\{f_k\}_{k=1}^{\infty}$ might have a certain structure which is not shared by the minimal-norm dual. This complication appears, for example, if $\{f_k\}_{k=1}^{\infty}$ is a wavelet frame: there are cases where the canonical dual of a wavelet frame does not have the wavelet structure (cf. [8]). One way to circumvent these types of problems is to search for more general choices of duals. Usually, one requires additional constraints on the choice of $\{g_k\}_{k=1}^{\infty}$; for example, if $\{f_k\}_{k=1}^{\infty}$ has a shift-invariant structure, it is natural to require that $\{g_k\}_{k=1}^{\infty}$ also share this structure.

More recently, the traditional concept of frames has been broadened to include frames on subspaces. Oblique frame decompositions, which were suggested in [10, 20] and further developed in [21–23], allow for frame expansions in which (1) is restricted to signals f in a given closed subspace X of $\mathcal{H}$. The vectors $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are still required to be frames, but only for subspaces of $\mathcal{H}$; $\{f_k\}_{k=1}^{\infty}$ forms a frame for X and $\{g_k\}_{k=1}^{\infty}$ constitutes a frame for a possibly different subspace S such that $\mathcal{H} = X \oplus S^{\perp}$, where $S^{\perp}$ denotes the orthogonal complement of S in $\mathcal{H}$. By choosing $S = X = \mathcal{H}$, we recover the conventional dual frames; however, oblique dual frames allow for more freedom in the design since the analysis space S is not restricted to be equal to the synthesis space X as in traditional frame expansions. A further generalization of this concept leads to pseudoframes [24]. As in oblique dual frames, (1) is restricted to $f \in X$, but $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are no longer constrained to be frame sequences. Since, in this paper, we are interested in *frame decompositions*, we focus our attention on oblique dual frames which provide a general setting for frame analysis.

Given a frame $\{f_k\}_{k=1}^{\infty}$ for a subspace X, a complete characterization of all possible oblique dual frames on a subspace S has been obtained in [22, 24]. This characterization involves computing the pseudoinverse of the frame operator TT^*, where T is the preframe operator associated with the frame $\{f_k\}_{k=1}^{\infty}$. In many cases, computing this pseudoinverse is computationally demanding. An interesting question therefore is whether there is an alternative characterization for all oblique duals which does not necessarily involve the pseudoinverse of TT^*. Our main result, derived in Section 3, shows that the oblique dual frames can be characterized in an alternative way in which the pseudoinverse of TT^* is replaced by the pseudoinverse of HT^*, where H is an appropriately chosen operator. The advantage of this characterization is that there is freedom in choosing the operator H so that it can be tailored such that the pseudoinverse of HT^* is easier to compute than the pseudoinverse of TT^*. Concrete examples demonstrating this computational advantage have recently been explored in [25–27] in the context of Gabor expansions.

An important class of frames in signal processing applications are shift-invariant frames, which are generated by translates of a set of generators [6]. The advantage of these frames is that the corresponding frame expansion can be implemented using linear time-invariant (LTI) filters. In Section 4, we specialize our results to the case of shift-invariant frames. As we show, while the classical frame representation may involve ideal filters which cannot be implemented in practice, by using the proposed alternative representation, the ideal filters can often be replaced by non ideal realizable filters. Furthermore, our general conditions take a particular simple form in the case of a shift-invariant space generated by a single function.

Before proceeding to the detailed development, in the next section, we summarize the required mathematical preliminaries.

2. DEFINITIONS AND BASIC RESULTS

We now introduce some definitions and results that will be used throughout the paper.

Given a transformation T, we denote by $\mathcal{N}(T)$ and $\mathcal{R}(T)$ the null space and range space of T, respectively. The *Moore-Penrose pseudoinverse* of T is written as $T^\dagger$ and the adjoint is denoted by T^*. The inner product between vectors $x, y \in \mathcal{H}$ is denoted by $\langle x, y\rangle$, and is linear in the first argument. We use $\mathbb{R}$ and $\mathbb{Z}$ to denote the reals and integers, respectively. The complex conjugate of a complex function $f(x)$ is denoted by $\overline{f(x)}$. For a subspace $\mathcal{W}$ of a Hilbert space $\mathcal{H}$, $\mathcal{W}^\perp$ is the orthogonal complement of $\mathcal{W}$ in $\mathcal{H}$. Given a sequence of vectors $\{g_k\}_{k=1}^{\infty} \subset \mathcal{H}$, we let $\overline{\text{span}}\{g_k\}_{k=1}^{\infty}$ be the closure of the span of $\{g_k\}_{k=1}^{\infty}$, that is, the smallest closed subspace containing $\{g_k\}_{k=1}^{\infty}$ (the span of a set of vectors consists by definition of all finite linear combinations of the vectors with complex coefficients).

Projection operators play an important role in our development. Given closed subspaces $\mathcal{W}$ and $\mathcal{V}$ of a Hilbert space $\mathcal{H}$ such that $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp$ (a direct sum, not necessarily orthogonal), the oblique projection $E_{\mathcal{W}\mathcal{V}^\perp}$ onto $\mathcal{W}$ along $\mathcal{V}^\perp$ is defined as the unique operator satisfying

$$\begin{aligned} E_{\mathcal{W}\mathcal{V}^\perp} w &= w \quad \text{for any } w \in \mathcal{W}, \\ E_{\mathcal{W}\mathcal{V}^\perp} v &= 0 \quad \text{for any } v \in \mathcal{V}^\perp. \end{aligned} \tag{2}$$

Thus, $\mathcal{R}(E_{\mathcal{W}\mathcal{V}^\perp}) = \mathcal{W}$ and $\mathcal{N}(E_{\mathcal{W}\mathcal{V}^\perp}) = \mathcal{V}^\perp$. If $\mathcal{W} = \mathcal{V}$, then $E_{\mathcal{W}\mathcal{V}^\perp}$ is the orthogonal projection onto $\mathcal{W}$, which we denote by $P_{\mathcal{W}}$. On the other hand, *any* projection P (i.e., a bounded linear operator on $\mathcal{H}$ for which $P^2 = P$) leads to a decomposition of $\mathcal{H}$; in fact, as proved in, for example, [28, Proposition 38.4],

$$\mathcal{H} = \mathcal{R}(P) \oplus \mathcal{N}(P). \tag{3}$$

That is, there is a one-to-one correspondence between decompositions of $\mathcal{H}$ and projections on $\mathcal{H}$. Thus, our results in this paper obtained via the splitting assumption $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp$ could as well be formulated starting with a projection.

For $f \in L^1(\mathbb{R})$, we denote the Fourier transform by

$$\mathcal{F} f(\omega) = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x\omega} dx. \tag{4}$$

As usual, the Fourier transform is extended to a unitary operator on $L^2(\mathbb{R})$. For a sequence $c = \{c_k\} \in \ell^2$, we define the discrete-time Fourier transform as the 1-periodic function in $L^2(0, 1)$ given by

$$\mathcal{F} c(e^{2\pi i\omega}) = C(e^{2\pi i\omega}) = \sum_{k\in\mathbb{Z}} c_k e^{-2\pi i k\omega}. \tag{5}$$

The discrete-time convolution $a_k = c_k * d_k$ between two sequences $c, d \in \ell^2$ is defined by

$$a_k = \sum_{m\in\mathbb{Z}} c_m d_{k-m}. \tag{6}$$

The continuous-time convolution between two functions $\phi, \phi_1 \in L^2(\mathbb{R})$ is given by

$$\phi(x) * \phi_1(x) = \int_{-\infty}^{\infty} \phi(y)\phi_1(x-y) dy. \tag{7}$$

A set of vectors $\{f_k\}_{k=1}^{\infty}$ forms a *Bessel sequence* for a Hilbert space $\mathcal{H}$ if there exists a constant $B < \infty$ such that

$$\sum_{k=1}^{\infty} \left| \langle x, f_k\rangle \right|^2 \le B\|x\|^2, \tag{8}$$

for all $x \in \mathcal{H}$. The vectors $\{f_k\}_{k=1}^{\infty}$ form a *frame* for a Hilbert space $\mathcal{H}$ if there exist constants $A > 0$ and $B < \infty$ such that

$$A\|x\|^2 \leq \sum_{k=1}^{\infty} |\langle x, f_k\rangle|^2 \leq B\|x\|^2, \tag{9}$$

for all $x \in \mathcal{H}$ [3].

The preframe operator associated with a Bessel sequence $\{f_k\}_{k=1}^{\infty}$ is given by

$$T : \ell^2 \longrightarrow \mathcal{H}, \qquad T\{c_k\} = \sum_{k\in\mathbb{Z}} c_k f_k, \tag{10}$$

and its adjoint is given by

$$T^* : \mathcal{H} \longrightarrow \ell^2, \qquad T^* f = \{\langle f, f_k\rangle\}_{k=1}^{\infty}. \tag{11}$$

The assumption $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^{\perp}$ will play a crucial role throughout the paper. Lemma 1, proved by Tang (see [29, Theorem 2.3]), deals with this condition, and relies on the concept of the angle between two subspaces. The angle from $\mathcal{V}$ to $\mathcal{W}$ is defined as the unique number $\theta(\mathcal{V}, \mathcal{W}) \in [0, \pi/2]$ for which

$$\cos\theta(\mathcal{V}, \mathcal{W}) = \inf_{f\in\mathcal{V}, \|f\|=1} ||P_{\mathcal{W}} f||. \tag{12}$$

Lemma 1. *Given closed subspaces $\mathcal{V}, \mathcal{W}$ of a separable Hilbert space $\mathcal{H}$, the following are equivalent:*

(i) $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^{\perp}$;
(ii) $\mathcal{H} = \mathcal{V} \oplus \mathcal{W}^{\perp}$;
(iii) $\cos\theta(\mathcal{V}, \mathcal{W}) > 0$ *and* $\cos\theta(\mathcal{W}, \mathcal{V}) > 0$.

More information on the condition $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^{\perp}$ in general Hilbert spaces can be found in [22].

3. CHARACTERIZATION OF DUALS

3.1. Oblique dual frames

Let $\{f_k\}_{k=1}^{\infty}$ be a frame for a closed subspace $\mathcal{W} \subseteq \mathcal{H}$, and let $\{g_k\}_{k=1}^{\infty}$ be a frame for a closed subspace $\mathcal{V} \subseteq \mathcal{H}$ such that $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^{\perp}$. The vectors $\{g_k\}_{k=1}^{\infty}$ in $\mathcal{V}$ form an oblique dual frame of $\{f_k\}_{k=1}^{\infty}$ on $\mathcal{V}$ [10, 20–22] if

$$f = \sum_{k=1}^{\infty} \langle f, g_k\rangle f_k, \quad \forall f \in \mathcal{W}. \tag{13}$$

The terminology oblique dual frame originates from the relation of these frames with oblique projections, as incorporated in the following lemma [22].

Lemma 2. *Assume that $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are Bessel sequences in $\mathcal{H}$, let $\mathcal{W} = \overline{span}\{f_k\}_{k=1}^{\infty}$ and $\mathcal{V} = \overline{span}\{g_k\}_{k=1}^{\infty}$, with $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^{\perp}$. Then the following are equivalent.*

(i) $f = \sum_{k=1}^{\infty}\langle f, g_k\rangle f_k$, *for all* $f \in \mathcal{W}$.
(ii) $E_{\mathcal{W}\mathcal{V}^{\perp}} f = \sum_{k=1}^{\infty}\langle f, g_k\rangle f_k$, *for all* $f \in \mathcal{H}$.
(iii) $E_{\mathcal{V}\mathcal{W}^{\perp}} f = \sum_{k=1}^{\infty}\langle f, f_k\rangle g_k$, *for all* $f \in \mathcal{H}$.
(iv) $\langle E_{\mathcal{V}\mathcal{W}^{\perp}} f, g\rangle = \sum_{k=1}^{\infty}\langle f, f_k\rangle\langle g_k, g\rangle$, *for all* $f, g \in \mathcal{H}$.
(v) $\langle E_{\mathcal{W}\mathcal{V}^{\perp}} f, g\rangle = \sum_{k=1}^{\infty}\langle f, g_k\rangle\langle f_k, g\rangle$, *for all* $f, g \in \mathcal{H}$.

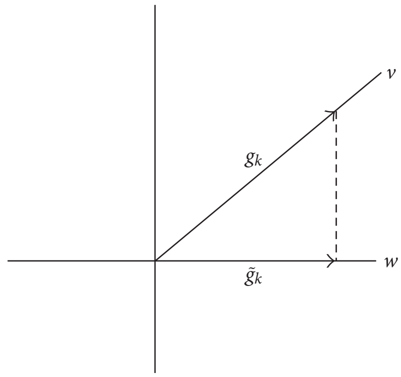

Figure 1: Geometrical interpretation of oblique dual frames. The vector $\tilde{g}_k$ is a dual vector in $\mathcal{W}$ and g_k is an oblique dual vector in $\mathcal{V}$.

In case the equivalent conditions are satisfied, $\{g_k\}_{k=1}^{\infty}$ is an oblique dual frame of $\{f_k\}_{k=1}^{\infty}$ on $\mathcal{V}$, and $\{f_k\}_{k=1}^{\infty}$ is an oblique dual frame of $\{g_k\}_{k=1}^{\infty}$ on $\mathcal{W}$. Furthermore, $\{f_k\}_{k=1}^{\infty}$ and $\{P_{\mathcal{W}} g_k\}_{k=1}^{\infty}$ are dual frames for $\mathcal{W}$ (in the sense of classical frame theory), and $\{g_k\}_{k=1}^{\infty}$ and $\{P_{\mathcal{V}} f_k\}_{k=1}^{\infty}$ are dual frames for $\mathcal{V}$.

Lemma 2 leads to a simple geometric interpretation of the oblique dual frames. Given a classical dual $\{\tilde{g}_k\}_{k=1}^{\infty}$ of $\{f_k\}_{k=1}^{\infty}$, that is, a dual in $\mathcal{W}$, we can extend $\{\tilde{g}_k\}_{k=1}^{\infty}$ to an oblique dual on $\mathcal{V}$ by constructing the sequence $\{g_k\}_{k=1}^{\infty} \in \mathcal{V}$ whose orthogonal projection onto $\mathcal{W}$ is the sequence $\{\tilde{g}_k\}_{k=1}^{\infty}$. The corresponding vectors are $\{g_k\}_{k=1}^{\infty} = \{E_{\mathcal{V}\mathcal{W}^{\perp}}\tilde{g}_k\}_{k=1}^{\infty}$. This interpretation is illustrated in Figure 1.

Denoting by T the preframe operator of the frame $\{f_k\}_{k=1}^{\infty}$, it was shown in [22, 24] that the oblique dual frames of $\{f_k\}_{k=1}^{\infty}$ on $\mathcal{V}$ are the families

$$\{g_k\}_{k=1}^{\infty}=\left\{E_{\mathcal{V}\mathcal{W}^{\perp}}(TT^*)^{\dagger} f_k+h_k-\sum_{j=1}^{\infty}\left\langle (TT^*)^{\dagger} f_k, f_j\right\rangle h_j\right\}_{k=1}^{\infty}, \tag{14}$$

where $\{h_k\}_{k=1}^{\infty} \in \mathcal{V}$ is a Bessel sequence. The characterization (14) involves computing the pseudoinverse of TT^* which can be computationally demanding. An interesting question therefore is whether there is an alternative characterization for the duals which does not involve the pseudoinverse of TT^*. Our main result, Theorem 1, shows that the oblique dual frames can be characterized in an alternative way in which the pseudoinverse $(TT^*)^{\dagger}$ is replaced by $(HT^*)^{\dagger}$, where H is an appropriately chosen operator. The advantage of this characterization is that there is freedom in choosing the operator H so that it can be tailored such that $(HT^*)^{\dagger}$ is easier to compute than $(TT^*)^{\dagger}$. Furthermore, in this representation, the infinite sum is no longer required.

In Section 4, we specialize the results to the case of shift-invariant frames which are important in signal processing applications since frame expansions involving shift-invariant frames can be implemented using LTI filters.

3.2. Mathematical preliminaries

The proof of our main theorem is based on some general results from the theory of operators on Hilbert spaces. Therefore, before stating our result, we collect the needed facts in Lemma 4. We first present a well-known identity, which we will use in the sequel.

Lemma 3. *Let A and B be bounded operators with closed range. If $\mathcal{R}(B) = \mathcal{N}(A)^\perp$, $\mathcal{N}(AB) = \mathcal{N}(B)$, and $\mathcal{R}(AB) = \mathcal{R}(A)$, then*

$$(AB)^\dagger = B^\dagger A^\dagger. \tag{15}$$

Proof. The lemma is proven in a straightforward manner by showing that under the conditions of the lemma, $B^\dagger A^\dagger$ satisfies the Moore-Penrose conditions [30]. □

Lemma 4. *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be separable Hilbert spaces, and let $\mathcal{W}$, $\mathcal{V}$ be closed subspaces of $\mathcal{H}_2$ such that $\mathcal{H}_2 = \mathcal{W} \oplus \mathcal{V}^\perp$. Further, let $Y : \mathcal{H}_1 \to \mathcal{H}_2$ and $U : \mathcal{H}_1 \to \mathcal{H}_2$ be bounded operators with $\mathcal{R}(Y) = \mathcal{W}$, $\mathcal{R}(U) = \mathcal{V}$. Then the following hold.*

(i) *$\mathcal{R}(Y^*U) = \mathcal{R}(Y^*)$ and $(Y^*U)^\dagger$ is a bounded operator from $\mathcal{H}_1$ to $\mathcal{H}_1$.*

(ii) *$(Y^*U)^\dagger Y^*U$ is the orthogonal projection onto $\mathcal{N}(U)^\perp$.*

(iii) *The oblique projection onto $\mathcal{V}$ along $\mathcal{W}^\perp$ can be written as*

$$E_{\mathcal{V}\mathcal{W}^\perp} = U(Y^*U)^\dagger Y^*. \tag{16}$$

(iv) *The operator*

$$M = U(Y^*U)^\dagger \tag{17}$$

is independent of the choice of the bounded operator $U : \mathcal{H}_1 \to \mathcal{H}_2$, as long as $\mathcal{R}(U) = \mathcal{V}$.

(v) *The bounded operators $U : \mathcal{H}_1 \to \mathcal{V}$ for which $UY^* = E_{\mathcal{V}\mathcal{W}^\perp}$ are the operators having the form $E_{\mathcal{V}\mathcal{W}^\perp}(HY^*)^\dagger H$, where $H : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded operator with closed range, satisfying that $\mathcal{H}_1 = \mathcal{N}(H) \oplus \mathcal{R}(Y^*)$.*

For the proof, see the appendix.

We note that Lemma 4(iii) provides an explicit method for computing the oblique projection $E_{\mathcal{V}\mathcal{W}^\perp}$; it is especially convenient if we choose $\mathcal{H}_1 = \ell^2$, in which case Y^*U becomes an operator on ℓ^2.

3.3. Oblique dual families

We now present our main result, which provides an alternative characterization of all oblique duals.

Theorem 1. *Let $\{f_k\}_{k=1}^\infty$ be a frame for a subspace $\mathcal{W} \subseteq \mathcal{H}$ with preframe operator T, and let $\mathcal{V}$ be a closed subspace such that $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp$. Then the oblique dual frames of $\{f_k\}_{k=1}^\infty$ on $\mathcal{V}$ are precisely the families*

$$\{g_k\}_{k=1}^\infty = \left\{E_{\mathcal{V}\mathcal{W}^\perp}(HT^*)^\dagger h_k\right\}_{k=1}^\infty, \tag{18}$$

where $\{h_k\}_{k=1}^\infty$ is a frame sequence with preframe operator H, satisfying that $\mathcal{N}(H) \oplus \mathcal{R}(T^*) = \ell^2$. Alternatively,

$$\{g_k\}_{k=1}^\infty = \left\{B(T^*B)^\dagger E_{\mathcal{R}(T^*)\mathcal{S}}\delta_k\right\}_{k=1}^\infty, \tag{19}$$

where $B : \ell^2 \to \mathcal{H}$ is any bounded operator with $\mathcal{R}(B) = \mathcal{V}$, $\mathcal{S}$ is a closed subspace of ℓ^2 such that $\ell^2 = \mathcal{R}(T^*) \oplus \mathcal{S}$, and $\{\delta_k\}_{k=1}^\infty$ is the canonical orthonormal basis for ℓ^2.

Note that from Lemma 4(iv), it follows that the families defined by (19) differ only in the choice of $\mathcal{S}$.

Proof. The proof of the theorem relies on the following lemma.

Lemma 5 (see [22]). *Let $\{f_k\}_{k=1}^\infty$ be a frame for $\mathcal{W}$, and let $\mathcal{V}$ be a closed subspace such that $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp$. Let $\{\delta_k\}_{k=1}^\infty$ be the canonical orthonormal basis for ℓ^2. The oblique dual frames for $\{f_k\}_{k=1}^\infty$ on $\mathcal{V}$ are the families $\{g_k\}_{k=1}^\infty = \{V\delta_k\}_{k=1}^\infty$, where $V : \ell^2 \to \mathcal{V}$ is a bounded operator for which $VT^* = E_{\mathcal{V}\mathcal{W}^\perp}$.*

By Lemmas 4 and 5, we can characterize the oblique dual frames on $\mathcal{V}$ along $\mathcal{W}^\perp$ as all families of the form

$$\{g_k\}_{k=1}^\infty = \left\{E_{\mathcal{V}\mathcal{W}^\perp}(HT^*)^\dagger H\delta_k\right\}_{k=1}^\infty, \tag{20}$$

where $H : \ell^2 \to \mathcal{H}$ is a bounded operator with closed range, satisfying that $\ell^2 = \mathcal{N}(H) \oplus \mathcal{R}(T^*)$. Such an operator has the form $H\{c_j\}_{j=1}^\infty = \sum_{j=1}^\infty c_j h_j$ with $\{h_k\}_{k=1}^\infty \in \mathcal{H}$ a frame sequence. By inserting this expression for H in (20), we get

$$\{g_k\}_{k=1}^\infty = \left\{E_{\mathcal{V}\mathcal{W}^\perp}(HT^*)^\dagger h_k\right\}_{k=1}^\infty. \tag{21}$$

From Lemma 4(iii), we can write $E_{\mathcal{V}\mathcal{W}^\perp}$ as

$$E_{\mathcal{V}\mathcal{W}^\perp} = MT^*, \tag{22}$$

where $M = B(T^*B)^\dagger$. Substituting (22) into (18), we have that

$$g_k = MT^*(HT^*)^\dagger H\delta_k = ME_{\mathcal{R}(T^*)\mathcal{S}}\delta_k, \tag{23}$$

with $\mathcal{S} = \mathcal{N}(H)$, thus completing the proof. □

In the special case in which $\mathcal{W} = \mathcal{H}$, Theorem 1 implies that the classical dual frames of $\{f_k\}_{k=1}^\infty$ are the families

$$\{g_k\}_{k=1}^\infty = \left\{(HT^*)^\dagger h_k\right\}_{k=1}^\infty, \tag{24}$$

where $\{h_k\}_{k=1}^\infty$ is a frame sequence, satisfying that $\mathcal{N}(H) \oplus \mathcal{R}(T^*) = \ell^2$. This should be compared with the known characterization [31]

$$\{g_k\}_{k=1}^\infty = \left\{(TT^*)^\dagger f_k + h_k - \sum_{j=1}^\infty \left\langle (TT^*)^{-1} f_k, f_j\right\rangle h_j\right\}_{k=1}^\infty, \tag{25}$$

where $\{h_k\}_{k=1}^\infty \in \mathcal{H}$ is a Bessel sequence.

Note that if $\{f_k\}_{k=1}^\infty$ is a Riesz basis, then $\mathcal{R}(T^*) = \ell^2$, that is, the condition $\mathcal{N}(H) \oplus \mathcal{R}(T^*) = \ell^2$ is satisfied if and only if H is injective. However, if $\{f_k\}_{k=1}^{\infty}$ is overcomplete, then $\mathcal{R}(T^*)$ is a subspace of ℓ^2; the more redundant the frame is, the "smaller" $\mathcal{R}(T^*)$ is, that is, the larger the kernel of H is forced to be.

In [25–27], it is shown that using the characterization (24) in a finite-dimensional setting can lead to Gabor expansions that are computationally much more efficient than conventional Gabor expansions. Furthermore, by proper choice of H, one can improve the condition number of HT^*. Specifically, consider the case in which we are given the Gabor expansion of a finite-length signal, and the goal is to reconstruct the signal from these samples. Instead of using the minimal-norm dual for reconstruction, corresponding to $(TT^*)^{\dagger}T$, it is suggested to use a nonminimal norm dual of the form $(HT^*)^{\dagger}H$, where H is chosen such that HT^* is efficient to compute. For example, if T is a frame operator corresponding to a Gabor frame with a Gaussian window $\phi[k] = e^{-k^2/\sigma_1^2}$ for some $\sigma_1^2 > 0$, then we can choose H as a frame operator corresponding to a Gabor frame with a Gaussian window $h[k] = e^{-k^2/\sigma_2^2}$, where σ_2 is chosen such that the effective spread of $h[k]$ is equal to a. If L/b is divisible by a, where L is the length of the signal and a and b are the shifts along the time and frequency axes, respectively, then the matrix HT^* is invertible for any choice of σ_2. Because of the limited spread of $h[k]$, the matrix HT^* can be computed very efficiently, resulting in an efficient method for reconstructing the signal from its Gabor coefficients.

One more advantage of the approach is that for large values of σ_1, the matrix TT^* can be poorly conditioned. By appropriately selecting the spread σ_2 of $h[k]$, it is possible to improve the condition number of HT^*, leading to a more stable reconstruction algorithm.

3.4. Minimal-norm duals

We now use the representation of Theorem 1 to develop alternative forms of the minimal-norm oblique duals.

Given a bounded operator B with $\mathcal{R}(B) = \mathcal{V}$, the minimal-norm oblique dual vectors of $\{f_k\}_{k=1}^{\infty}$ on $\mathcal{V}$ along $\mathcal{W}$, that is, the oblique dual vectors leading to coefficients with minimal ℓ^2 norm, can be written as [10, 20]

$$g_k = B(T^*B)^{\dagger}\delta_k. \tag{26}$$

The representation (26) follows from Theorem 1 if we choose $\mathcal{S} = \mathcal{N}(T)$. Indeed, in this case, $E_{\mathcal{R}(T^*)\mathcal{S}} = P_{\mathcal{R}(T^*)}$. Since $\mathcal{N}((T^*B)^{\dagger}) = \mathcal{R}(T^*B)^{\perp} = \mathcal{R}(T^*)^{\perp}$, (19) reduces to (26). Alternatively, it was shown in [22] that the minimal-norm oblique duals can be expressed as

$$g_k = E_{\mathcal{V}\mathcal{W}^{\perp}}(TT^*)^{\dagger}f_k. \tag{27}$$

This characterization also follows from Theorem 1, with $H = T$. More generally, we can obtain this characterization by choosing H as an arbitrary operator with $\mathcal{N}(H) = \mathcal{N}(T)$, as incorporated in the following theorem.

Theorem 2. *Let $\{f_k\}_{k=1}^{\infty}$ be a frame for a subspace $\mathcal{W} \subseteq \mathcal{H}$ with preframe operator T, and let $\mathcal{V}$ be a closed subspace such that $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^{\perp}$. Then the minimal-norm oblique dual frames of $\{f_k\}_{k=1}^{\infty}$ on $\mathcal{V}$ can be expressed as*

$$\{g_k\}_{k=1}^{\infty} = \left\{E_{\mathcal{V}\mathcal{W}^{\perp}}(HT^*)^{\dagger}h_k\right\}_{k=1}^{\infty}, \tag{28}$$

where $\{h_k\}_{k=1}^{\infty}$ is a frame sequence with preframe operator H, satisfying that $\mathcal{N}(H) = \mathcal{N}(T)$. Alternatively,

$$\{g_k\}_{k=1}^{\infty} = \left\{B(T^*B)^{\dagger}\delta_k\right\}_{k=1}^{\infty}, \tag{29}$$

where B is a bounded operator with $\mathcal{R}(B) = \mathcal{V}$ and $\{\delta_k\}_{k=1}^{\infty}$ is the canonical orthonormal basis for ℓ^2.

Proof. The proof of the theorem follows from the fact that if $T : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded operator with closed range, then the operator

$$M = (UT^*)^{\dagger}U \tag{30}$$

is independent of the choice of the bounded operator $U : \mathcal{H}_1 \to \mathcal{H}_2$, as long as $\mathcal{N}(U) = \mathcal{N}(T)$ and the range of U is closed. Indeed, since $\mathcal{R}(U^*) = \mathcal{N}(U)^{\perp} = \mathcal{N}(T)^{\perp} = \mathcal{R}(T^*)$, we have that $\mathcal{H}_1 = \mathcal{R}(T^*) \oplus \mathcal{R}(U^*)^{\perp}$. From Lemma 4, it then follows that the pseudoinverse $(UT^*)^{\dagger}$ is a well-defined bounded operator. Because U is bounded with $\mathcal{N}(U) = \mathcal{N}(T)$, it can be expressed as $U = XT$ for a bounded operator $X : \mathcal{H}_2 \to \mathcal{H}_2$ with $\mathcal{N}(X) = \mathcal{R}(T)^{\perp}$. In particular, we can choose

$$X = UT^{\dagger}. \tag{31}$$

From Lemma 3, it then follows that

$$(UT^*)^{\dagger} = (XTT^*)^{\dagger} = (TT^*)^{\dagger}X^{\dagger}. \tag{32}$$

Therefore,

$$(UT^*)^{\dagger}U = (TT^*)^{\dagger}X^{\dagger}XT = (TT^*)^{\dagger}P_{\mathcal{N}(X)^{\perp}}T = (TT^*)^{\dagger}T, \tag{33}$$

thus completing the proof. □

If $\mathcal{V} = \mathcal{W}$, then the vectors g_k defined by Theorem 2 are the conventional minimal-norm dual frame vectors. Thus, Theorem 2 provides an alternative method for computing the conventional dual frame vectors, which are typically given by

$$g_k = (TT^*)^{\dagger}f_k = T(T^*T)^{\dagger}\delta_k. \tag{34}$$

By using Theorem 2, we may choose B so that $(T^*B)^{\dagger}$ is easier to compute than $(T^*T)^{\dagger}$; alternatively, we may choose H such that $(HT^*)^{\dagger}$ can be evaluated more efficiently than $(TT^*)^{\dagger}$.

4. FRAME SEQUENCES IN SHIFT-INVARIANT SPACES

We now consider frames of translates in shift-invariant spaces. The importance of this class of frames stems from the fact that the corresponding frame expansions can be implemented using LTI filters.

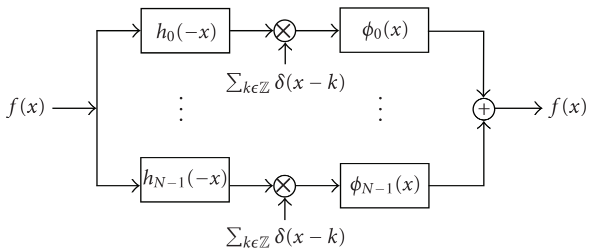

Figure 2: Filter bank representation of a shift-invariant frame expansion.

4.1. Shift-invariant frames

A *shift-invariant* frame with multiple generators is a frame $\{f_{kj}\}_{k\in\mathbb{Z},j\in J}$ of the form

$$\{f_{kj}\}_{k\in\mathbb{Z},j\in J} = \{\phi_j(x-k)\}_{k\in\mathbb{Z},j\in J} \triangleq \{T_k\phi_j\}_{k\in\mathbb{Z},j\in J}, \quad (35)$$

where J is an index set, $\phi_j \in L^2(\mathbb{R})$ and we define the translation operator acting on functions in $L^2(\mathbb{R})$ by $T_k f(x) = f(x-k)$, $x\in\mathbb{R}$, $k\in\mathbb{Z}$. The corresponding space

$$\mathcal{W} := \overline{\operatorname{span}}\{T_k\phi_j\}_{k\in\mathbb{Z},j\in J} = \left\{\sum_{k\in\mathbb{Z},j\in J} c_{kj}T_k\phi_j : \{c_{kj}\}\in\ell^2\right\} \quad (36)$$

is said to be *shift-invariant*.

A shift-invariant frame expansion of the form $f = \sum_{j=0}^{N-1}\sum_{k\in\mathbb{Z}}\langle f, h_{kj}\rangle\phi_{kj}$, where $h_{kj} = T_k h_j$ and $\phi_{kj} = T_k\phi_j$, can be implemented using a bank of LTI filters, as depicted in Figure 2. To see this, we first note that for fixed j, the coefficients

$$c_{kj} = \langle f, h_{kj}\rangle = \int_{-\infty}^{\infty} f(x)h_j(x-k)dx, \quad k\in\mathbb{Z}, \quad (37)$$

can be expressed as samples at $x = k$ of a convolution integral

$$c_{kj} = \int_{-\infty}^{\infty} f(x)h_j(k-x)dx = f(x) * g(x)|_{x=k}, \quad k\in\mathbb{Z}, \quad (38)$$

where $g(x) = h_j(-x)$. Thus, the sequence c_{kj} can be viewed as samples at $x = k$ of the output of an LTI filter with impulse response $h_j(-x)$, with $f(x)$ as its input. Next, we note that the sum $\sum_{k\in\mathbb{Z}} c_{kj}\phi_j(x-k)$ can be expressed as a convolution

$$\sum_{k\in\mathbb{Z}} c_{kj}\phi_j(x-k) = p(x) * \phi_j(x), \quad (39)$$

where $p(x)$ is the modulated impulse train

$$p(x) = \sum_{k\in\mathbb{Z}} c_{kj}\delta(x-k). \quad (40)$$

4.2. Shift-invariant duals

Having defined shift-invariant frames, our goal now is to obtain shift-invariant oblique dual frames via Theorem 1.

For $\phi_j, h_j \in L^2(\mathbb{R})$, $j\in J$, we let

$$\mathcal{W} = \overline{\operatorname{span}}\{T_k\phi_j\}_{k\in\mathbb{Z},j\in J}, \qquad \mathcal{V} = \overline{\operatorname{span}}\{T_k h_j\}_{k\in\mathbb{Z},j\in J}. \quad (41)$$

We further denote by T and H the preframe operators of the sequences $\{T_k\phi_j\}_{k\in\mathbb{Z},j\in J}$ and $\{T_k h_j\}_{k\in\mathbb{Z},j\in J}$, respectively. Throughout the section, we make the following assumptions:

(i) $L^2(\mathbb{R}) = \mathcal{W}\oplus\mathcal{V}^\perp$;
(ii) $\ell^2 = \mathcal{R}(T^*)\oplus\mathcal{N}(H)$.

Note that if $\{T_k\phi_j\}_{k\in\mathbb{Z},j\in J}$ is a frame sequence, then these conditions can be formulated entirely in terms of the operators T and H via

$$L^2(\mathbb{R}) = \mathcal{R}(T)\oplus\mathcal{R}(H)^\perp, \qquad \ell^2 = \mathcal{N}(T)\oplus\mathcal{N}(H)^\perp. \quad (42)$$

This formulation shows that in general, the two conditions are unrelated. In fact, if $\{T_k\phi_j\}_{k\in\mathbb{Z},j\in J}$ and $\{T_k h_j\}_{k\in\mathbb{Z},j\in J}$ are frames for $L^2(\mathbb{R})$, then the first condition holds; but if, for example, $\{T_k\phi_j\}_{k\in\mathbb{Z},j\in J}$ is a Riesz basis and $\{T_k h_j\}_{k\in\mathbb{Z},j\in J}$ is overcomplete, then the second condition does not hold. On the other hand, if $\{T_k\phi_j\}_{k\in\mathbb{Z},j\in J}$ and $\{T_k h_j\}_{k\in\mathbb{Z},j\in J}$ are Riesz sequences, then the second condition holds; but in case one of these sequences spans $L^2(\mathbb{R})$ and the other does not, then the first condition is not satisfied.

Theorem 3. *Let $\phi_j, h_j \in L^2(\mathbb{R})$, $j\in J$, and assume that $\{T_k\phi_j\}_{k\in\mathbb{Z},j\in J}$ and $\{T_k h_j\}_{k\in\mathbb{Z},j\in J}$ are frame sequences. Then, under assumptions* (i) *and* (ii)*, the sequence*

$$\{g_{kj}\}_{k\in\mathbb{Z},j\in J} = \left\{E_{\mathcal{V}\mathcal{W}^\perp}(HT^*)^\dagger T_k h_j\right\}_{k\in\mathbb{Z},j\in J} = \{T_k g_j\}_{k\in\mathbb{Z},j\in J} \quad (43)$$

is a shift-invariant oblique dual frame of $\{T_k\phi_j\}_{k\in\mathbb{Z},j\in J}$ on $\mathcal{V}$, with $\{g_j\}_{j\in J} = \{E_{\mathcal{V}\mathcal{W}^\perp}(HT^*)^\dagger h_j\}_{j\in J}$.

Proof. We first show that

$$T_k HT^* = HT^* T_k. \quad (44)$$

Indeed, for any $f\in\mathcal{H}$,

$$\begin{aligned} HT^*T_k f &= \sum_{mj}\langle T_k f, T_m\phi_j\rangle T_m h_j = \sum_{mj}\langle f, T_{m-k}\phi_j\rangle T_m h_j \\ &= \sum_{mj}\langle f, T_m\phi_j\rangle T_{m+k}h_j = T_k HT^* f. \end{aligned} \quad (45)$$

Now, $h_j = Ha_j$ for some a_j. From assumption (ii), we can express a_j as $a_j = a_{Hj} + a_{Tj}$, where $a_{Hj} \in \mathcal{N}(H)$ and $a_{Tj} \in \mathcal{R}(T^*)$. Therefore, $h_j = Ha_j = Ha_{Tj}$. But since $a_{Tj} \in \mathcal{R}(T^*)$, we have that $a_{Tj} = T^*b_j$ for some $b_j \in \mathcal{N}(T^*)^\perp = \mathcal{R}(T) = \mathcal{W}$. We conclude that $h_j = HT^*b_j$ for some $b_j \in \mathcal{W}$, and

$$g_{kj} = E_{\mathcal{V}\mathcal{W}^\perp}(HT^*)^\dagger T_k HT^* b_j. \tag{46}$$

Substituting (44) into (46), we have that

$$g_{kj} = E_{\mathcal{V}\mathcal{W}^\perp}(HT^*)^\dagger HT^* T_k b_j = E_{\mathcal{V}\mathcal{W}^\perp} P T_k b_j, \tag{47}$$

where P is an orthogonal projection onto $\mathcal{N}(HT^*)^\perp$. But, by assumption (ii), $\mathcal{N}(HT^*) = \mathcal{N}(T^*) = \mathcal{R}(T)^\perp = \mathcal{W}^\perp$, so that $P = P_{\mathcal{W}}$. Since $E_{\mathcal{V}\mathcal{W}^\perp}P_{\mathcal{W}} = E_{\mathcal{V}\mathcal{W}^\perp}$, (47) reduces to

$$g_{kj} = E_{\mathcal{V}\mathcal{W}^\perp} T_k b_j. \tag{48}$$

Now, it was shown in [22, Corollary 4.2] that if $\mathcal{W}$ and $\mathcal{V}$ are shift-invariant, then $E_{\mathcal{V}\mathcal{W}^\perp}T_k = T_k E_{\mathcal{V}\mathcal{W}^\perp}$, which from (47) implies that

$$g_{kj} = T_k E_{\mathcal{V}\mathcal{W}^\perp} b_j = T_k g_j, \tag{49}$$

where $g_j = E_{\mathcal{V}\mathcal{W}^\perp}(HT^*)^\dagger h_j$. □

4.3. Single generator

An important special case of a shift-invariant frame is a frame of the form $\{T_k\phi\}_{k\in\mathbb{Z}}$, with a single generator ϕ. These frames are especially easy to analyze. In particular, as the following proposition shows, one can immediately characterize the generators that create a frame for their closed linear span ($\{T_k\phi\}_{k\in\mathbb{Z}}$ cannot be a frame for all of $L^2(\mathbb{R})$, cf. [32]).

Proposition 1 (see [4, 33]). *Let $\phi \in L^2(\mathbb{R})$,*

$$\Phi(e^{2\pi i\omega}) = \sum_{k\in\mathbb{Z}} |\hat{\phi}(\omega+k)|^2, \qquad \mathcal{N}(\Phi) = \{\omega : \Phi(e^{2\pi i\omega}) = 0\}. \tag{50}$$

Then $\{T_k\phi\}_{k\in\mathbb{Z}}$ is a frame sequence with bounds A, B if and only if

$$A \le \Phi(e^{2\pi i\omega}) \le B, \quad \text{a.e. on } \{\omega : \Phi(\omega) \ne 0\}. \tag{51}$$

It turns out that for single-generated systems, the conditions $L^2(\mathbb{R}) = \mathcal{W} \oplus \mathcal{V}^\perp$ and $\ell^2 = \mathcal{R}(T^*) \oplus \mathcal{N}(H)$ of the previous section are also easy to verify. Suppose that $\{T_k\phi\}_{k\in\mathbb{Z}}$ and $\{T_kh\}_{k\in\mathbb{Z}}$ are frame sequences, and let

$$\mathcal{W} := \overline{\operatorname{span}}\{T_k\phi\}_{k\in\mathbb{Z}}, \qquad \mathcal{V} := \overline{\operatorname{span}}\{T_kh\}_{k\in\mathbb{Z}}. \tag{52}$$

The following proposition, proved in [22], provides an easily verifiable condition on the generators ϕ and h such that $L^2(\mathbb{R}) = \mathcal{W} \oplus \mathcal{V}^\perp$.

Proposition 2. *Let $\phi, h \in L^2(\mathbb{R})$, and assume that $\{T_k\phi\}_{k\in\mathbb{Z}}$ and $\{T_kh\}_{k\in\mathbb{Z}}$ are frame sequences. Define Φ and $\mathcal{N}(\Phi)$ as in (50), and introduce $\Psi, \mathcal{N}(\Psi)$similarly for the function h. Then the following are equivalent:*

(i) $L^2(\mathbb{R}) = \mathcal{W} \oplus \mathcal{V}^\perp$;
(ii) $\mathcal{N}(\Phi) = \mathcal{N}(\Psi)$ and there exists a constant $A > 0$ such that

$$A \le \left| \sum_{k\in\mathbb{Z}} \hat{\phi}(\omega+k)\overline{\hat{h}(\omega+k)} \right| \quad \text{on } \{\omega : \Phi(e^{2\pi i\omega}) \ne 0\}. \tag{53}$$

We now show that the second condition $\ell^2 = \mathcal{R}(T^*) \oplus \mathcal{N}(H)$ is actually contained in the first condition $L^2(\mathbb{R}) = \mathcal{W} \oplus \mathcal{V}^\perp$. Thus, only the first condition needs to be verified, which can be done in a straightforward way by using Proposition 2.

Proposition 3. *Assume that T and H are preframe operators of shift-invariant frames $\{T_k\phi\}$ and $\{T_kh\}$, respectively. Define Φ and $\mathcal{N}(\Phi)$ as in (50), and introduce $\Psi, \mathcal{N}(\Psi)$ similarly for the function h. Then, $\mathcal{R}(T^*) \oplus \mathcal{N}(H) = \ell^2$ if and only if $\mathcal{N}(\Phi) = \mathcal{N}(\Psi)$.*

Proof. It was shown in [22, Lemma 4.7] that the range of the adjoint of the preframe operator associated to any single-generated shift-invariant frame is

$$\mathcal{R}(T^*) = \{c \in \ell^2 : C(e^{2\pi i\omega}) = 0 \text{ on } \mathcal{N}(\Phi)\}. \tag{54}$$

Applying this result to the preframe operator H, it follows that

$$\begin{aligned}\mathcal{N}(H) = \mathcal{R}(H^*)^\perp &= \{c \in \ell^2 : C(e^{2\pi i\omega}) = 0 \text{ on } \mathcal{N}(\Psi)\}^\perp \\ &= \{c \in \ell^2 : C(e^{2\pi i\omega}) = 0 \text{ on } \mathcal{N}(\Psi)^c\}.\end{aligned} \tag{55}$$

Thus, if $\mathcal{N}(\Psi) = \mathcal{N}(\Phi)$, then $\mathcal{N}(H) = \mathcal{R}(T^*)^\perp$ and $\ell^2 = \mathcal{N}(H) \oplus \mathcal{R}(T^*)$.

Conversely, suppose that $\mathcal{R}(T^*) \oplus \mathcal{N}(H) = \ell^2$. We now show that if we identify $\mathcal{N}(\Phi), \mathcal{N}(\Psi)$ with subsets of $[0,1]$, then $\mathcal{N}(\Phi) \cup \mathcal{N}(\Psi)^c = [0,1]$ and $\mathcal{N}(\Phi) \cap \mathcal{N}(\Psi)^c = \emptyset$; this implies that $\mathcal{N}(\Phi) = \mathcal{N}(\Psi)$.

We first show that $\mathcal{R}(T^*) \cap \mathcal{N}(H) = \{0\}$ implies that $\mathcal{N}(\Phi) \cup \mathcal{N}(\Psi)^c = [0,1]$. To see this, we note that if $c \in \mathcal{R}(T^*) \cap \mathcal{N}(H)$, then from (55), we have that $C(e^{2\pi i\omega}) = 0$ on $\mathcal{N}(\Phi) \cup \mathcal{N}(\Psi)^c$. Now, suppose that $\mathcal{N}(\Phi) \cup \mathcal{N}(\Psi)^c$ was just a subset of $[0,1]$; then we could construct a function $C(e^{2\pi i\omega}) = \sum_k c_k e^{-2\pi ik\omega}$ which is zero on the subset, but nonzero on the rest of $[0,1]$. Since $C(e^{2\pi i\omega}) = 0$ on $\mathcal{N}(\Phi) \cup \mathcal{N}(\Psi)^c$, we have that $c \in \mathcal{R}(T^*) \cap \mathcal{N}(H) = \{0\}$, and therefore $C(e^{2\pi i\omega})$ is forced to be zero on $[0,1]$. This contradiction shows that indeed $\mathcal{N}(\Phi) \cup \mathcal{N}(\Psi)^c = [0,1]$.

Next, we show that $\mathcal{R}(T^*) + \mathcal{N}(H) = \ell^2$ implies that $\mathcal{N}(\Phi) \cap \mathcal{N}(\Psi)^c = \emptyset$. If $\mathcal{R}(T^*) + \mathcal{N}(H) = \ell^2$, then any $c \in \ell^2$ can be written as $c = c_1 + c_2$, where $c_1 \in \mathcal{R}(T^*)$ and $c_2 \in \mathcal{N}(H)$. This in turn implies that

$$\begin{aligned} C(e^{2\pi i\omega}) &= C_1(e^{2\pi i\omega}) + C_2(e^{2\pi i\omega}), \\ C_1(e^{2\pi i\omega}) &= 0 \quad \text{on } \mathcal{N}(\Phi), \\ C_2(e^{2\pi i\omega}) &= 0 \quad \text{on } \mathcal{N}(\Psi)^c. \end{aligned} \tag{56}$$

From (56), we conclude that $C(e^{2\pi i\omega}) = 0$ on $\mathcal{N}(\Phi)\cap\mathcal{N}(\Psi)^c$. Thus, if $\mathcal{R}(T^*) + \mathcal{N}(H) = \ell^2$, then (56) implies that for any $c \in \ell^2$, its discrete-time Fourier transform satisfies $C(e^{2\pi i\omega}) = 0$ on $\mathcal{N}(\Phi) \cap \mathcal{N}(\Psi)^c$, from which we conclude that $\mathcal{N}(\Phi) \cap \mathcal{N}(\Psi)^c = \emptyset$. □

Combining our results leads to the following characterization of all oblique duals in the single-generated shift-invariant case.

Theorem 4. *Let $\phi, h \in L^2(\mathbb{R})$, let*

$$\Phi(e^{2\pi i\omega}) = \sum_{k\in\mathbb{Z}} \left|\hat{\phi}(\omega+k)\right|^2, \qquad \Psi(e^{2\pi i\omega}) = \sum_{k\in\mathbb{Z}} \left|\hat{h}(\omega+k)\right|^2, \tag{57}$$

and let

$$\mathcal{N}(\Phi) = \{\omega : \Phi(e^{2\pi i\omega}) = 0\}, \qquad \mathcal{N}(\Psi) = \{\omega : \Psi(e^{2\pi i\omega}) = 0\}. \tag{58}$$

Suppose that $\{T_k\phi\}_{k\in\mathbb{Z}}$ is a frame sequence so that

$$A \le \Phi(e^{2\pi i\omega}) \le B, \quad \text{a.e. on } \{\omega : \Phi(\omega) \ne 0\} \tag{59}$$

for some $A > 0$. Then, the sequence

$$\{g_k\}_{k\in\mathbb{Z}} = \left\{E_{\mathcal{V}\mathcal{W}^\perp}(HT^*)^\dagger T_k h\right\}_{k\in\mathbb{Z}} = \{T_k g\}_{k\in\mathbb{Z}} \tag{60}$$

is a shift-invariant oblique dual frame of $\{T_k\phi\}_{k\in\mathbb{Z}}$ on $\mathcal{V}$, with $g = E_{\mathcal{V}\mathcal{W}^\perp}(HT^*)^\dagger h$, if and only if

$$\alpha \le \Psi(e^{2\pi i\omega}) \le \beta, \quad \text{a.e. on } \{\omega : \Psi(\omega) \ne 0\} \tag{61}$$

for some $\alpha > 0$, $\mathcal{N}(\Phi) = \mathcal{N}(\Psi)$, and there exists a constant $C > 0$ such that

$$C \le \left|\sum_{k\in\mathbb{Z}} \hat{\phi}(\omega+k)\overline{\hat{h}(\omega+k)}\right| \quad \text{on } \{\omega : \Phi(e^{2\pi i\omega}) \ne 0\}. \tag{62}$$

4.3.1. LTI representation of minimal-norm duals

We now develop an LTI representation of the minimal-norm duals of a single-generated shift-invariant frame.

We have seen in Theorem 2 that the minimal-norm oblique duals can be characterized as $g_k = B(T^*B)^\dagger\delta_k$, where $B : \ell^2 \to \mathcal{H}$ is a bounded operator with range $\mathcal{V}$ such that $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp$. Suppose now that we let T be the preframe operator of a shift-invariant frame $\{T_k\phi\}_{k\in\mathbb{Z}}$ for $\mathcal{W}$ and choose B as the preframe operator of a shift-invariant frame $\{T_kb\}_{k\in\mathbb{Z}}$. Proposition 2 provides necessary and sufficient conditions on $\hat{b}(\omega)$ such that $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp$. Given a generator $b(x)$ satisfying these conditions, we now show how to implement the operator $B(T^*B)^\dagger$ using LTI filters.

Lemma 6. *Let $\phi, b \in L^2(\mathbb{R})$, and assume that $\{T_k\phi\}_{k\in\mathbb{Z}}$ and $\{T_kb\}_{k\in\mathbb{Z}}$ are frame sequences with preframe operators T and B, respectively. Then, the operator $B(T^*B)^\dagger : \ell^2 \to \mathcal{H}$ can be implemented using the block diagram of Figure 3, where*

$$A(e^{j2\pi\omega}) = \begin{cases} \dfrac{1}{\sum_{k\in\mathbb{Z}} \hat{\phi}(\omega+k)\hat{b}(\omega+k)}, & \Phi(e^{2\pi i\omega}) \ne 0, \\ 0, & \Phi(e^{2\pi i\omega}) = 0. \end{cases} \tag{63}$$

Proof. We first show that if $c = (T^*B)^\dagger d$, then the sequence c_k can be obtained by filtering the sequence d_k with the filter $A(e^{j2\pi\omega})$. To this end, we note that if $d = T^*Bg$, then d can be obtained by filtering the sequence g_k with a filter

$$H(e^{j2\pi\omega}) = \sum_{k\in\mathbb{Z}} \hat{\phi}(\omega+k)\overline{\hat{b}(\omega+k)}. \tag{64}$$

Indeed,

$$\begin{aligned} d_k &= \sum_{m\in\mathbb{Z}} \int \phi(x-k) g_m b(x-m)dx \\ &= \sum_{m\in\mathbb{Z}} g_m \int \phi(x)b(x+k-m)dx = g_k * h_k, \end{aligned} \tag{65}$$

where $h_k = \int \phi(x)b(x+k)dx$. Now, we can express h_k as $h_k = f(k)$, where

$$f(x) = \int \phi(y)b(y+x)dy = \phi(x) * b(-x). \tag{66}$$

It then follows that h_k are the samples at the points $x = k$ of the function $f(x)$ whose Fourier transform is given by $\hat{f}(\omega) = \hat{\phi}(\omega)\overline{\hat{b}(\omega)}$. Therefore,

$$H(e^{j2\pi\omega}) = \sum_{k\in\mathbb{Z}} \hat{f}(\omega+k) = \sum_{k\in\mathbb{Z}} \hat{\phi}(\omega+k)\overline{\hat{b}(\omega+k)}. \tag{67}$$

Thus, $(T^*B)^\dagger$ is equivalent to filtering the input sequence with the filter $A(e^{j2\pi\omega})$. To conclude the proof, we note that if $f = Bg$, then $f(x) = \sum_{k\in\mathbb{Z}} g_k b(x-k)$, which is equivalent to modulating the sequence g_k by an impulse train, and filtering the modulated sequence with a filter with impulse response $b(x)$. □

Lemma 6 can be used to develop an efficient method for reconstructing a signal $g(x)$ in $\mathcal{W}$ from coefficients $c = T^*g$. Specifically, the reconstruction is obtained as $g = B(T^*B)^\dagger c$ which is the output of the block diagram in Figure 3 with the sequence c as its input. Now, the kth coefficient c_k can be written as

$$c_k = \langle f_k, g\rangle = \int f(t-x)g(x) = g(x) * f(-x)|_{x=k}, \tag{68}$$

and thus can be obtained by filtering the input signal $g(x)$ with a filter with impulse response $f(-x)$ and frequency response $\hat{f}(\omega)$, and then sampling the output at $x = k$.

The advantage of this reconstruction is that given the samples c, we have freedom in choosing the filter $\hat{b}(\omega)$ so that it can be tailored such that the filters $\hat{b}(\omega)$ and $A(e^{2\pi i\omega})$ are easy to implement.

Note that if the signal $g(x)$ does not lie in the space $\mathcal{W}$ spanned by the signals $\{f(x-k)\}$, then the output

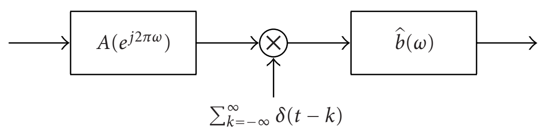

Figure 3: Filter-based implementation of the oblique dual frame vectors.

of the block diagram of Figure 3 will be equal to $P_{\mathcal{W}}g(x)$. This follows immediately from the fact that $B(T^*B)^\dagger T^* = T(T^*T)^\dagger T^* = P_{\mathcal{R}(T)} = P_{\mathcal{W}}$.

A similar idea was first introduced in [34] in the context of consistent sampling. In that setting, it was suggested to choose a filter $\hat{b}(\omega)$ that spans a space $\mathcal{V}$, different from the sampling space $\mathcal{W}$, that is easy to implement, and then use a discrete-time correction filter in order to compensate for the mismatch between the sampling filter and the reconstruction filter. Here we use a similar idea where the essential difference is that in the scheme of Figure 3, the overall reconstruction is equivalent to an orthogonal projection onto the reconstruction space, while the scheme of [34] is equivalent to an oblique projection.

5. CONCLUSION

We have obtained a complete characterization of the oblique dual frames associated with a frame for a subspace of a Hilbert space. Compared to the use of the classical dual frame, this leads to considerable freedom in the design. In [25, 26], we demonstrated that these results can lead to much more efficient representations in the case of finite-dimensional spaces; we believe that the results presented here will lead to similar gains in the general case. As an important special case, we considered frame expansions in shift-invariant spaces. For the case of a single generator, our general conditions take a particular simple form.

APPENDIX

PROOF OF LEMMA 4

We prove each part of the lemma separately.

(i) By Lemma 1, $\mathcal{H}_2 = \mathcal{V} \oplus \mathcal{W}^\perp$; since $\mathcal{W}^\perp = \mathcal{R}(Y)^\perp = \mathcal{N}(Y^*)$, this implies that

$$\mathcal{R}(Y^*U) = Y^*\mathcal{V} = \mathcal{R}(Y^*), \tag{A.1}$$

where we use the notation $Y^*\mathcal{V}$ to denote the image of the space $\mathcal{V}$ under the operator Y^*. By assumption $\mathcal{R}(Y) = \mathcal{W}$, which is closed, this implies that $\mathcal{R}(Y^*)$ is closed, from which we conclude using (A.1) that $\mathcal{R}(Y^*U)$ is closed. The fact that $\mathcal{R}(Y^*U)$ is closed and Y^*U is bounded implies that $(Y^*U)^\dagger$ is a bounded operator from $\mathcal{H}_1$ into $\mathcal{H}_1$.

(ii) It is well known that $(Y^*U)^\dagger Y^*U$ is the orthogonal projection onto $\mathcal{N}(Y^*U)^\perp$. Now,

$$\mathcal{N}(Y^*U)^\perp = \mathcal{R}(U^*Y) = \mathcal{R}(U^*) = \mathcal{N}(U)^\perp, \tag{A.2}$$

where we used the fact that from (i), $\mathcal{R}(U^*Y) = \mathcal{R}(U^*)$, which is closed.

(iii) Suppose that $x \in \mathcal{V}$. Then $x = Uy$ for some $y \in \mathcal{N}(U)^\perp$ so that

$$U(Y^*U)^\dagger Y^*x = U(Y^*U)^\dagger Y^*Uy = Uy = x. \tag{A.3}$$

On the other hand, if $x \in \mathcal{W}^\perp = \mathcal{R}(Y)^\perp = \mathcal{N}(Y^*)$, then $U(Y^*U)^\dagger Y^*x = 0$. These calculations show that $U(Y^*U)^\dagger Y^*$ has the properties characterizing $E_{\mathcal{V}\mathcal{W}^\perp}$.

(iv) Suppose that $U, Z : \mathcal{H}_1 \to \mathcal{H}_2$ are bounded operators with $\mathcal{R}(U) = \mathcal{R}(Z) = \mathcal{V}$. Then, $Z = UX$ for some bounded operator $X : \mathcal{H}_1 \to \mathcal{H}_1$ with $\mathcal{R}(X) = \mathcal{N}(U)^\perp$ (in particular, we can choose $X = U^\dagger Z$. Indeed, since U is a bounded operator with closed range, $U^\dagger$ is bounded. Furthermore, using the fact that $\mathcal{R}(Z) = \mathcal{R}(U) = \mathcal{N}(U^\dagger)^\perp$, we have $\mathcal{R}(X) = \mathcal{R}(U^\dagger) = \mathcal{N}(U)^\perp$).

With $Z = UX$, we have that $(Y^*Z)^\dagger = (Y^*UX)^\dagger$. To simplify $(Y^*UX)^\dagger$, we use Lemma 3, from which it follows that

$$(Y^*UX)^\dagger = X^\dagger(Y^*U)^\dagger. \tag{A.4}$$

Therefore,

$$\begin{aligned} Z(Y^*Z)^\dagger &= UXX^\dagger(Y^*U)^\dagger \\ &= UP_{\mathcal{R}(X)}(Y^*U)^\dagger = U(Y^*U)^\dagger. \end{aligned} \tag{A.5}$$

(v) If $\mathcal{H}_1 = \mathcal{N}(H) \oplus \mathcal{R}(Y^*)$, then

$$\mathcal{H}_1 = \mathcal{R}(H^*)^\perp \oplus \mathcal{R}(Y^*) = \mathcal{R}(H^*) \oplus \mathcal{R}(Y^*)^\perp. \tag{A.6}$$

Applying (ii) with Y replaced by H^* and U replaced by Y^* shows that $(HY^*)^\dagger HY^* = P_{\mathcal{W}}$. Since $E_{\mathcal{V}\mathcal{W}^\perp}P_{\mathcal{W}} = E_{\mathcal{V}\mathcal{W}^\perp}$, we have that $E_{\mathcal{V}\mathcal{W}^\perp}(HY^*)^\dagger HY^* = E_{\mathcal{V}\mathcal{W}^\perp}$.

On the other hand, if $U : \mathcal{H}_1 \to \mathcal{V}$ satisfies that $UY^* = E_{\mathcal{V}\mathcal{W}^\perp}$, then it follows from [21, Proposition 3.4] that $\mathcal{N}(U) \oplus \mathcal{R}(Y^*) = \mathcal{H}_1$. By taking $H = U$,

$$E_{\mathcal{V}\mathcal{W}^\perp}(HY^*)^\dagger H = E_{\mathcal{V}\mathcal{W}^\perp}(E_{\mathcal{V}\mathcal{W}^\perp})^\dagger U = P_{\mathcal{V}}U = U. \tag{A.7}$$

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