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Completely positive compact operators on non-commutative symmetric spaces

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Abstract Under natural conditions, it is shown that a completely positive operator between two non-commutative symmetric spaces of τ-measurable operators which is dominated in the sense of complete positivity by a completely positive compact operator is itself compact.

Keywords Completely positive operators · Compact operators · Non-commutative symmetric spaces

Mathematics Subject Classification (2000) Primary 46L52; Secondary 46E30 · 47B60

1 Introduction

It was shown in [9] that each positive operator from the Banach lattice E to the Banach lattice F, which is dominated by a positive compact operator, is itself compact, provided the norms on F and the Banach dual E^* are order continuous. Special cases of particular interest occur when E is an abstract M-space and F is an abstract L-space or the cases that $E = L^p$, $1 < p \leq \infty$ and $F = L^q$, $1 \leq q < \infty$. Full details of this theorem, and its subsequent development may be found in the monographs [1,13,21].

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In the setting of non-commutative L^p-spaces, a non-commutative version of this (so-called) compact majorisation theorem was given by Neuhardt [15,16]. In this setting, key technical difficulties arise from the fact that there are no non-commutative counterparts to the arguments of [9] based on properties of order-bounded operators between Banach lattices and technical lattice arguments characterising approximately order-bounded sets in Banach lattices. The key new idea introduced by Neuhardt was to consider domination in the sense of complete positivity, a notion which goes back to Stinespring [18], and to replace arguments in the Banach lattice case based on formulae for the infimum of positive linear operators between Banach lattices by representation theorems for C^*-algebras and linear functionals, using the assumption of complete positivity.

The purpose of the present paper is to place Neuhardt's theorem within the more general framework of symmetric spaces of τ-measurable operators affiliated with a semi-finite von Neumann algebra. The principal result of the paper (Theorem 5.5) is that if E, F are strongly symmetric spaces on the positive half-line, if $0 \leq S, T : E(\tau) \to F(\sigma)$ are completely positive operators with S dominated by T in the sense of complete positivity, then S is compact provided T is compact and order continuous and the norms on F and the Köthe dual $E^{\times}$ are order continuous. This result uses Neuhardt's theorem in the special case that $E = L^{\infty}$ and $F = L^1$ together with a characterisation of compact subsets of non-commutative spaces with order continuous norm (Proposition 4.6) in terms of sets of uniformly absolutely continuous norm.

2 Notation and preliminaries

Throughout this paper $\mathcal{M}$ will denote a von Neumann algebra on some Hilbert space $\mathcal{H}$. Unless otherwise stated, it will be assumed throughout that $\mathcal{M}$ is equipped with a fixed semifinite faithful normal trace τ. For standard facts concerning von Neumann algebras, we refer to [19]. The identity in $\mathcal{M}$ is denoted by $\mathbf{1}$ and we denote by $\mathcal{P}(\mathcal{M})$ the complete lattice of all (self-adjoint) projections in $\mathcal{M}$. A linear operator $x : \mathcal{D}(x) \to \mathcal{H}$, with domain $\mathcal{D}(x) \subseteq \mathcal{H}$, is said to be *affiliated with* $\mathcal{M}$ if $ux = xu$ for all unitary u in the commutant $\mathcal{M}'$ of $\mathcal{M}$. For any self-adjoint operator x on $\mathcal{H}$, its spectral measure is denoted by e^x. A self-adjoint operator x is affiliated with $\mathcal{M}$ if and only if $e^x(B) \in \mathcal{P}(\mathcal{M})$ for any Borel set $B \subseteq \mathbb{R}$. The closed and densely defined operator x, affiliated with $\mathcal{M}$, is called τ*-measurable* if and only if there exists a number $s \geq 0$ such that $\tau\left(e^{|x|}(s, \infty)\right) < \infty$. The collection of all τ-measurable operators is denoted by $S(\tau)$. With the sum and product defined as the respective closures of the algebraic sum and product, it is well known that $S(\tau)$ is a *-algebra. For $\epsilon, \delta > 0$, we denote by $V(\epsilon, \delta)$ the set of all $x \in S(\tau)$ for which there exists an orthogonal projection $p \in P(\mathcal{M})$ such that $p(\mathcal{H}) \subseteq \mathcal{D}(x)$, $\|xp\|_{\mathcal{B}(\mathcal{H})} \leq \epsilon$ and $\tau(\mathbf{1} - p) \leq \delta$. The sets $\{V(\epsilon, \delta) : \epsilon, \delta > 0\}$ form a base at 0 for a metrizable Hausdorff topology on $S(\tau)$, which is called the *measure topology*. Equipped with this topology, $S(\tau)$ is a complete topological $*$-algebra. These facts and their proofs can be found in the papers [14,20].

For $x \in S(\tau)$, the *singular value function* $\mu(\cdot; x) = \mu(\cdot; |x|)$ is defined by

$$\mu(t; x) = \inf\left\{s \geq 0 : \tau\left(e^{|x|}(s, \infty)\right) \leq t\right\}, \quad t \geq 0.$$

It follows directly that the singular value function $\mu(x)$ is a decreasing, right-continuous function on the positive half-line $[0, \infty)$. Moreover, $\mu(uxv) \leq \|u\|\|v\|\mu(x)$ for all $u, v \in \mathcal{M}$ and $x \in S(\tau)$ and and $\mu(f(x)) = f(\mu(x))$ whenever $0 \leq x \in S(\tau)$ and f is an increasing continuous function on $[0, \infty)$ which satisfies $f(0) = 0$.

It should be observed that a sequence $\{x_n\}_{n=1}^{\infty}$ in $S(\tau)$ converges to zero for the measure topology if and only if $\mu(t; x_n) \to 0$ as $n \to \infty$ for all $t > 0$.

If m denotes Lebesgue measure on the semiaxis $[0, \infty)$, and if we consider $L^{\infty}(m)$ as an Abelian von Neumann algebra acting via multiplication on the Hilbert space $\mathcal{H} = L^2(m)$, with the trace given by integration with respect to m, then $S(m)$ consists of all measurable functions on $[0, \infty)$ which are bounded except on a set of finite measure, and for $f \in S(m)$, the generalized singular value function $\mu(f)$ is precisely the classical decreasing rearrangement of the function $|f|$, which is usually denoted by f^*. In this setting, convergence for the measure topology coincides with the usual notion of convergence in measure. If $\mathcal{M} = \mathcal{L}(\mathcal{H})$ and τ is the standard trace, then $S(\tau) = \mathcal{M}$, the measure topology coincides with the operator norm topology.

The real vector space $S_h(\tau) = \{x \in S(\tau) : x = x^*\}$ is a partially ordered vector space with the ordering defined by setting $x \geq 0$ if and only if $\langle x\xi, \xi\rangle \geq 0$ for all $\xi \in \mathcal{D}(x)$. The positive cone in $S_h(\tau)$ will be denoted by $S(\tau)_+$. If $0 \leq x_\alpha \uparrow_\alpha \leq x$ holds in $S(\tau)_+$, then $\sup_\alpha x_\alpha$ exists in $S(\tau)_+$. The trace τ extends to $S(\tau)_+$ as a non-negative extended real-valued functional which is positively homogeneous, additive, unitarily invariant and normal. This extension is given by $\tau(x) = \int_0^{\infty} \mu(t; x)\, dt$, $x \in S(\tau)_+$, and satisfies $\tau(x^*x) = \tau(xx^*)$ for all $x \in S(\tau)$. It should be observed that if f is an increasing continuous function on $[0, \infty)$ satisfying $f(0) = 0$, then

$$\tau(f(|x|)) = \int_0^{\infty} \mu(t; f(|x|))\, dt = \int_0^{\infty} f(\mu(t; x))\, dt \tag{2.1}$$

for all $x \in S(\tau)$.

If $1 \leq p < \infty$, we set $L^p(\tau) = \left\{x \in S(\tau) : \tau\left(|x|^p\right) < \infty\right\}$. Note that it follows from (2.1) that $L^p(\tau)$ is also given by $L^p(\tau) = \{x \in S(\tau) : \mu(x) \in L^p(m)\}$, where m denotes Lebesgue measure on $[0, \infty)$. The space $L^p(\tau)$ is a linear subspace of $S(\tau)$ and the functional $x \longmapsto \|x\|_{L^p(\tau)} = \tau\left(|x|^p\right)^{1/p}$, $x \in L^p(\tau)$, is a norm. It should be observed that $\|x\|_{L^p(\tau)} = \|\mu(x)\|_{L^p(m)}$ for all $x \in L^p(\tau)$. Equipped with this norm, $L^p(\tau)$ is a Banach space. In this setting, we also have that $L^{\infty}(\tau) = \mathcal{M}$.

In the commutative setting, the spaces $L^p(\tau)$ are the familiar Lebesgue spaces. In the special case that $\mathcal{M}$ is $\mathcal{B}(\mathcal{H})$ equipped with standard trace, the corresponding L^p-spaces are the Schatten classes $\mathfrak{S}_p$. As is well known, the space $L^1(\tau)$ may be identified with the von Neumann algebra predual of $\mathcal{M}$ with respect to trace duality. If $x \in S(\tau)$, then the projection onto the closure of the range of $|x|$ is called the *support* of x and is denoted by $s(x)$. We set $\mathcal{F}(\tau) = \{x \in \mathcal{M} : \tau(s(x)) < \infty\}$.

If $(\mathcal{N}, \sigma)$ is a semifinite von Neumann algebra, possibly on some different Hilbert space, if $x \in S(\tau)$ and $y \in S(\sigma)$, then x is said to be *submajorised* by y (in the sense of Hardy, Littlewood and Polya) if and only if $\int_0^t \mu(s; x)ds \leq \int_0^t \mu(s; y)ds$ for all $t \geq 0$. We write $x \prec\prec y$, or equivalently, $\mu(x) \prec\prec \mu(y)$ (with respect to Lebesgue measure on $(0, \infty)$).

We shall need, in particular, the submajorization inequality $\mu(xy) \prec\prec \mu(x)\mu(y)$ whenever $x, y \in S(\tau)$. A proof of this inequality may be found in [3] in the case that $\mathcal{M}$ is non-atomic, and this latter assumption may be removed by a standard argument (see [10]). If this inequality is combined with an inequality of Hardy (see [12, Chapter II.2.18]), then it is easily seen that if $x, y, z \in S(\tau)$, then $\mu(xyz) \prec\prec \mu(x)\mu(y)\mu(z)$. For further details and proofs, we refer the reader to [4,6,10].

A linear subspace $E \subseteq S(\tau)$ is called an $\mathcal{M}$-bimodule if $uxv \in E$ whenever $x \in E$ and $u, v \in \mathcal{M}$. If the $\mathcal{M}$-bimodule E is equipped with a norm $\|\cdot\|_E$ which satisfies $\|uxv\|_E \leq \|u\|_{\mathcal{B}(\mathcal{H})}\|v\|_{\mathcal{B}(\mathcal{H})}\|x\|_E, \quad x \in E, u, v \in \mathcal{M}$, then E is called a normed $\mathcal{M}$-bimodule (of τ-measurable operators). If $E \subseteq S(\tau)$ is an $\mathcal{M}$-bimodule, and if $x \in S(\tau)$, then $x \in E \iff |x| \in E \iff x^* \in E$; and if $y \in E$ is such that $|x| \leq |y|$, then $x \in E$. Further, if E is a normed $\mathcal{M}$-bimodule, then $\||x|\|_E = \|x\|_E$ and $\|x^*\|_E = \|x\|_E$ for all $x \in E$ and $\|x\|_E \leq \|y\|_E$ whenever $x, y \in E$ satisfy $|x| \leq |y|$. A normed $\mathcal{M}$-bimodule which is a Banach space is called a Banach $\mathcal{M}$-bimodule.It is easily seen that $\mathcal{F}(\tau)$ is an $\mathcal{M}$-bimodule and that each of the Banach spaces $\mathcal{M}$, $L^1(\tau)$, $L^1(\tau)\cap\mathcal{M}$, $L^1(\tau)+\mathcal{M}$ are Banach $\mathcal{M}$-bimodules. If $E \subseteq S(\tau)$ is a normed $\mathcal{M}$-bimodule, then E will be called *symmetrically normed* if $x \in E$, $y \in S(\tau)$ and $\mu(y) \leq \mu(x)$ imply that $y \in E$ and $\|y\|_E \leq \|x\|_E$; *strongly symmetrically normed* if E is symmetrically normed and its norm has the additional property that $\|y\|_E \leq \|x\|_E$ whenever $x, y \in E$ satisfy $y \prec\prec x$; *fully symmetric* if E is symmetrically normed and if, whenever $x \in E$ and $y \in S(\tau)$ satisfy $y \prec\prec x$ then $y \in E$ and $\|y\|_E \leq \|x\|_E$.

If a strongly symmetrically normed space is Banach, then it will be simply called a *strongly symmetric space*. It may be shown that any strongly symmetrically normed space, with the property that $\bigvee_{x\in E} s(x) = \mathbf{1}$ (which will always be assumed) satisfies $\mathcal{F}(\tau) \subseteq E \subseteq L^1(\tau)) + \mathcal{M}$, with continuous inclusions (where $\mathcal{F}(\tau)$ is equipped with the $L^1 \cap L^\infty$-norm). If, in addition, E is a Banach space, then $L^1(\tau) \cap \mathcal{M} \subseteq E$, with continuous embedding. If $E \subseteq S(\tau)$ is a strongly symmetrically normed $\mathcal{M}$-bimodule, then the embedding of E into $S(\tau)$ is continuous from the norm topology of E to the measure topology on $S(\tau)$. A wide class of strongly symmetrically normed $\mathcal{M}$-bimodules may be constructed as follows. If $E \subseteq S(m)$ is a strongly symmetrically normed space, set $E(\tau) = \{x \in S(\tau) : \mu(x) \in E\}, \quad \|x\|_{E(\tau)} := \|\mu(x)\|_E$. It may be shown as in [4] (see [5]) that $(E(\tau), \|\cdot\|_{E(\tau)})$ is a strongly symmetrically normed $\mathcal{M}$-bimodule and is a Banach $\mathcal{M}$-bimodule if E is a Banach space.

If $E \subseteq S(\tau)$ is a strongly symmetrically normed $\mathcal{M}$-bimodule, set

$$E^\times = \{y \in S(\tau) : \sup\{\tau(|xy|) : x \in E, \|x\|_E \leq 1\} < \infty\}$$

and

$$\|y\|_{E^\times} = \sup\{\tau(|xy|) : x \in E, \|x\|_E \leq 1\}, \quad y \in E^\times.$$

If $y \in S(\tau)$, then

$$y \in E^{\times} \Longleftrightarrow \sup \left\{ \int\limits_{[0,\infty)} \mu(x)\mu(y)dm : x \in E, \|x\|_E \leq 1 \right\} < \infty,$$

in which case, the latter quantity is equal to $\|y\|_{E^{\times}}$. The space $(E^{\times}, \|\cdot\|_{E^{\times}})$ is a normed Banach $\mathcal{M}$-bimodule. If $y \in E^{\times}$, define $\phi_y : E \to \mathbb{C}$ by $\phi_y(x) = \tau(xy), x \in E$. The Banach $\mathcal{M}$-bimodule $E^{\times}$ has the following properties(see [6]): (i) $\phi_y \in E^*$ and the map $y \to \phi_y \in E^*, y \in E^{\times}$ is an isometry; (ii) $E^{\times}$ has the Fatou property, that is, $0 \leq y_\alpha \uparrow_\alpha \subseteq E^{\times}, \sup_\alpha \|y_\alpha\|_{E^{\times}} < \infty \Longrightarrow y = \sup_\alpha y_\alpha$ exists in $E^{\times}$ and $\|y\|_{E^{\times}} = \sup_\alpha \|y_\alpha\|_{E^{\times}}$; (iii) $E^{\times}$ is fully symmetric; (iv) If $E \subseteq S(m)$ is a strongly symmetrically normed space, then $E^{\times}(\tau) = E(\tau)^{\times}$.

3 Completely positive mappings

Let A be a C^*-algebra of operators acting in some Hilbert space $\mathcal{H}$. It is assumed that the identity operator $\mathbf{1}$ is an element of A. Denote by $M_n(A)$ the set of all $n \times n$-matrices $a = [a_{ij}]_{i,j=1}^n$ with entries from A. With the obvious definitions of addition, scalar multiplication and matrix multiplication, together with the $*$-operation defined by setting $(a^*)_{ij} = a_{ji}^*$, the set $M_n(A)$ is an involutive algebra. If $a = [a_{ij}]_{i,j=1}^n \in M_n(A)$, then a induces, in the obvious manner, a bounded linear operator on the n-fold direct product Hilbert space $\mathcal{H}_n = \sum_{i=1}^n \oplus \mathcal{H}$. Equipped with the corresponding operator norm, $M_n(A)$ is then a C^*-algebra with identity. The element $a \in M_n(A)$ is positive if and only if there exists $b \in M_n(A)$ such that $a = b^*b$. We denote by $e_{i,j}$, $1 \leq i, j \leq n$, the usual matrix basis for $M_n(\mathbb{C})$. Note that if $a = [a_{ij}]_{i,j=1}^n \in M_n(A)$, then $a = \sum_{i,j=1}^n a_{ij} \otimes e_{ij}$. It is shown in [19] that an element $a = [a_{ij}]_{i,j=1}^n$ of $M_n(A)$ is positive if and only if it is a sum of matrices of the form $[a_i^* a_j]_{i,j=1}^n$ with $a_1, \ldots a_n \in A$.

We suppose that $(\mathcal{M}, \tau)$ is a semifinite von Neumann algebra acting in some Hilbert space $\mathcal{H}$. For each $n \in \mathbb{N}$, we let $\mathcal{M}_n = \mathcal{M} \overline{\otimes} M_n(\mathbb{C})$ be the von Neumann algebra tensor product acting in the tensor product Hilbert space $\mathcal{H} \otimes \mathbb{C}^n$ and with the tensor product trace $\tau_n := \tau \otimes \mathrm{tr}_n$. Here tr_n denotes the standard matrix trace. As is well known (see, for example [19, Chapter IV, Proposition 1.6]), the von Neumann algebra tensor product $\mathcal{M}_n$ coincides with the algebraic tensor product $\mathcal{M} \otimes M_n(\mathbb{C})$ and may be identified with the space $M_n(\mathcal{M})$ of all $n \times n$-matrices $[x_{ij}]_{i,j=1}^n$ with values in $\mathcal{M}$.

Given $x \in S(\tau)$ and $y \in M_n(\mathbb{C})$, the linear operator $x \odot y : \mathcal{D}(x) \otimes \mathbb{C}^n \to \mathcal{H} \otimes \mathbb{C}^n$ is defined by setting $(x \odot y)(\xi \otimes \eta) = x\xi \otimes y\eta$ for all $\xi \in \mathcal{D}(x)$ and $\eta \in \mathbb{C}^n$. The operator $x \odot y$ is pre-closed, affiliated with $\mathcal{M} \otimes M_n(\mathbb{C})$ and the domain $\mathcal{D}(x) \otimes \mathbb{C}^n$ is τ_n-dense. Consequently, its closure, denoted by $x \otimes y$, is τ_n-measurable. Observing that $(x_1 + x_2) \otimes y_1 = x_1 \otimes y_1 + x_2 \otimes y_1$, $x_1 \otimes (y_1 + y_2) = x_1 \otimes y_1 + x_1 \otimes y_2$ and $(x_1 \otimes x_2)(y_1 \otimes y_2) = x_1 x_2 \otimes y_1 y_2$ for all $x_1, x_2 \in S(\tau)$ and $y_1, y_2 \in M_n(\mathbb{C}^n)$, it is easily verified that the linear subspace of $S(\tau_n)$ generated by the operators $x \otimes y, x \in S(\tau)$ and $y \in M_n(\mathbb{C}^n)$, is a subalgebra of $S(\tau_n)$ which may be identified with the tensor product $S(\tau) \otimes M_n(\mathbb{C}^n) \cong M_n(S(\tau))$. We claim that $S(\tau_n) = S(\tau) \otimes M_n(\mathbb{C}^n)$. Indeed, defining $p_i = \mathbf{1} \otimes e_{ii}$, $1 \le i \le n$, it is clear that the map $x \longmapsto x \otimes e_{11}$, $x \in \mathcal{M}$, is a trace preserving $*$-isomorphism from $\mathcal{M}$ onto the reduced von Neumann algebra $p_1 \mathcal{M}_n p_1$, which has a unique extension to a trace preserving $*$-isomorphism from $S(\tau)$ onto $S(p_1 \mathcal{M}_n p_1, \tau_n)$, given by $x \longmapsto x \otimes e_{11}$, $x \in S(\tau)$. Furthermore, it should be observed that $S(p_1 \mathcal{M}_n p_1, \tau_n) = p_1 S(\tau_n) p_1$. Given $x \in S(\tau_n)$ and $1 \le i, j \le n$, we have $(\mathbf{1} \otimes e_{1i}) x (\mathbf{1} \otimes e_{j1}) \in p_1 S(\tau_n) p_1$ and so, there exists $x_{ij} \in S(\tau)$ such that $(\mathbf{1} \otimes e_{1i}) x (\mathbf{1} \otimes e_{j1}) = x_{ij} \otimes p_1$. Consequently,

$$p_i x p_j = (\mathbf{1} \otimes e_{i1}) \left(x_{ij} \otimes p_1\right) \left(\mathbf{1} \otimes e_{1j}\right) = x_{ij} \otimes e_{ij}$$

and hence,

$$x = \sum_{i,j=1}^{n} x_{ij} \otimes e_{ij} \in S(\tau) \otimes M_n\left(\mathbb{C}^n\right). \tag{3.1}$$

This proves the claim. Identifying $S(\tau) \otimes M_n(\mathbb{C}^n)$ with $M_n(S(\tau))$, the elements x_{ij} in (3.1) are the matrix elements of x; we also write $x = \left[x_{ij}\right]_{i,j=1}^{n}$. The proposition which now follows may be established using standard arguments, and the details of proof will be omitted.

Proposition 3.1 *Suppose that* $x = [x_{ij}]_{i,j=1}^{n} \in M_n(S(\tau))$.

(i) $\mu(x_{ij}) = \mu(x_{ij} \otimes e_{ij}) = \mu(p_i x p_j) \le \mu(x)$ *for all* $1 \le i, j \le n$.
(ii) $x \in L^1(\tau_n)$ *if and only if* $x_{ij} \in L^1(\tau)$ *for all* $1 \le i, j \le 1$, *in which case* $\tau_n(x) = \sum_{i=1}^{n} \tau(x_{ii})$.

If $E \subseteq S(m)$ is any strongly symmetric space on $[0, \infty)$, and if $x = \sum_{i,j=1}^{n} x_{ij} \otimes e_{ij} \in S(\tau_n)$, then $x \in E(\tau_n)$ if and only if $x_{ij} \in E(\tau)$ for all $1 \le i, j \le n$. We may therefore identify $E(\tau_n)$ with the space of all $n \times n$ matrices with entries in $E(\tau)$ and will write $E(\tau_n) = M_n(E(\tau))$. With this identification, observe that if $x = \sum_{i,j=1}^{n} x_{ij} \otimes e_{ij} \in E(\tau_n)$ and if $y = \sum_{i,j=1}^{n} y_{ij} \otimes e_{ij} \in S(\tau_n)$ then $y \in E^{\times}(\tau_n) = E(\tau_n)^{\times}$ if and only if $y_{ij} \in E(\tau)^{\times}$, $1 \le i, j \le n$, in which case

$$xy = \sum_{i,j=1}^{n} \left(\sum_{k=1}^{n} x_{ik} y_{kj} \right) \otimes e_{ij} \in L^1(\tau_n) \text{ and } \tau_n(xy) = \sum_{i=1}^{n} \sum_{k=1}^{n} \tau(x_{ik} y_{ki}).$$

Observe that if $x \in S(\tau)$, then $x \ge 0$ if and only if there exists $z \in S(\tau)$ such that $x = z^* z$. We shall need the following simple result, which is proved exactly as in [19, Lemma IV.3.1].

Lemma 3.2 *If* $x \in S(\tau_n)$, *then* $x \ge 0$ *if and only if* x *is a sum of elements of the form* $\sum_{i,j=1}^{n} x_i^* x_j \otimes e_{ij}$ *with* $x_i \in S(\tau)$, $1 \le i \le n$.

The preceding lemma now yields the following criterion for positivity in spaces $E(\tau_n)$ in terms of the trace τ.

Proposition 3.3 *Suppose that $E \subseteq S(m)$ is a strongly symmetric space on $[0, \infty)$ and suppose that $x \in E(\tau_n)$. If $x = \sum_{i,j=1}^{n} x_{ij} \otimes e_{ij} \in M_n(E(\tau))$, then $x \geq 0$ if and only if $\sum_{i,j=1}^{n} \tau(y_i x_{ij} y_j^*) \geq 0$ for all choices $y_i, y_2, \ldots, y_n \in S(\tau)$ such that $y_i^* y_j \in E^\times(\tau)$, $1 \leq i, j \leq n$.*

Proof It will suffice to show that $\tau_n(xy) \geq 0$ for all $0 \leq y \in E^\times(\tau)$. If $0 \leq y \in E^\times(\tau_n)$, then it may be assumed by Lemma 3.2 that there exist $y_1, y_2, \ldots, y_n \in S(\tau)$ such that $y = \sum_{i,j=1}^{n} y_i^* y_j \otimes e_{ij}$. It follows, in particular, that $\mu(y_i^* y_i) = \mu(y_i y_i^*) \in E^\times$, $1 \leq i \leq n$, and so also, for all $1 \leq i, j \leq n$,

$$\mu(y_i)\mu(y_j^*) \leq \frac{1}{2}\left(\mu(y_i)^2 + \mu(y_j^*)^2\right) = \frac{1}{2}\left(\mu(y_i^* y_i) + \mu(y_j y_j^*)\right) \in E^\times.$$

From the submajorisation

$$\mu(y_i x_{ij} y_j^*) \prec\prec \mu(x_{ij})\mu(y_i)\mu(y_j), \quad 1 \leq i, j \leq n,$$

and the fact that $\mu(x_{ij}) \in E$, it follows that $\mu(y_i x_{ij} y_j^*) \in L^1(m)$ so that $y_i x_{ij} y_j^* \in L^1(\tau)$, $1 \leq i, j \leq n$. Since it is clear that $x_{ij} y_j^* y_i \in L^1(\tau)$, it follows from the first assertion of [6] Proposition 3.4 that $\tau(y_i x_{ij} y_j^*) = \tau(x_{ij} y_j^* y_i)$ for all $1 \leq i, j \leq n$. By the remarks preceding Lemma 3.2, it now follows that

$$\tau_n(xy) = \sum_{i=1}^{n} \sum_{j=1}^{n} \tau(x_{ij} y_j^* y_i) = \sum_{i,j=1}^{n} \tau(y_i x_{ij} y_j^*)$$

and this sufffices to complete the proof. □

Suppose now that $(\mathcal{N}, \sigma)$ is a semifinite von Neumann algebra, that $E \subseteq S(\tau)$, $F \subseteq S(\sigma)$ are strongly symmetric spaces and that $T : E \to F$ is a linear mapping. For each $n \in \mathbb{N}$, we let $T_n : M_n(E) \to M_n(F)$ be defined by setting

$$T_n\left([x_{ij}]_{i,j=1}^{n}\right) = \left[T(x_{ij})\right]_{i,j=1}^{n}.$$

for all $[x_{ij}]_{i,j=1}^{n} \in M_n(E)$. The mapping T is said to be *c*ompletely positive if and only if $T_n \geq 0$ for every $n \in \mathbb{N}$, that is, T_n maps $M_n(E) \cap S(\tau_n)_+$ into $M_n(F) \cap S(\sigma_n)_+$, for each $n \in \mathbb{N}$. Denote by $CP(E, F)$ the collection of all completely positive maps $T : E \to F$. If $T \in CP(E, F)$, then we will write $0 \leq_{\text{cp}} T : E \to F$. If $S, T \in CP(E, F)$, then we write $0 \leq_{\text{cp}} S \leq_{\text{cp}} \leq T$ if and only if $0 \leq_{\text{cp}} T - S$. Note that, if $E, F \subseteq S(m)$ are strongly symmetric spaces on $[0, \infty)$, then via the identifications $E(\tau_n) = M_n(E(\tau))$, $F(\sigma_n) = M_n(F(\sigma))$, each of the mappings T_n defined above induces a linear mapping from $E(\tau_n)$ to $F(\sigma_n)$. Without risk of confusion, we continue to denote these mappings by T_n. In this setting, the linear mapping $T : E(\tau) \to F(\sigma)$ is completely positive if and only if each of the mappings $T_n : E(\tau_n) \to F(\sigma_n)$ are positive in the usual sense, that is, $0 \leq T_n(x) \in F(\sigma_n)$ whenever $0 \leq x \in E(\tau_n)$.

The key result on which the main results of this paper are based is the following majorisation theorem, due to Neuhardt [15,16].

Theorem 3.4 (Neuhardt) *If $S, T : \mathcal{M} \to L^1(\sigma)$ are linear maps, if $0 \le_{cp} S \le_{cp} T$ and if T is compact, then S is compact.*

It was shown by Stinespring [18, Theorem 3], that a positive linear mapping from a *commutative* C^*-algebra $\mathcal{A}$ to another C^*-algebra $\mathcal{B}$ is necessarily completely positive. As noted in [18], the converse is false, even in the case that $\mathcal{A} = \mathcal{B} = M_2(\mathbb{C})$. See also [19] Proposition 3.9 of Chapter IV. Using Lemma 3.2 and Proposition 3.3, a modification of the proof given in [19] Proposition IV.3.9 yields the following variant. The details are omitted.

Proposition 3.5 *Let $(\mathcal{M}, \tau)$ be a commutative von Neumann algebra and let $(\mathcal{N}, \sigma)$ be any semi-finite von Neumann algebra. Suppose that $E, F \subseteq S(m)$ are strongly symmetric spaces. If $0 \le T : E(\tau) \to F(\sigma)$, then T is completely positive.*

It is worth remarking that the preceding proposition shows that the commutative specialisation of Neuhardt's theorem actually coincides with an important special case of one of the main results of [9].

It should be observed that, if $E \subseteq S(\tau)$ is a strongly symmetric space, and if $\varphi \in E^*$, then $\varphi \in E^\times$ if and only if, whenever $x_\alpha \downarrow_\alpha 0$ in E, it follows that $\varphi(x_\alpha) \to_\alpha 0$. See [6, Theorem 5.11]. It will now be convenient to make the following definition. Suppose that $E \subseteq S(\tau)$, $F \subseteq S(\sigma)$ are strongly symmetric spaces. The continuous linear mapping $T : E \to F$ will be called *order continuous* if $\sigma(T(x_\alpha)z) \to_\alpha 0$ whenever $x_\alpha \downarrow_\alpha 0$ in E and $0 \le z \in F^\times$. Using the above remark, it follows that if $T : E \to F$ is order continuous, and if $T^* : E^* \to F^*$ denotes the Banach adjoint mapping, then $T^*z \in E^\times$ whenever $z \in F^\times$. In this case, the restriction of the adjoint T^* to the Köthe dual $F^\times$ will be denoted by $T^\times$ so that $T^\times : F^\times \to E^\times$. It is clear that, if $T \ge 0$, then $T^\times \ge 0$. If $0 \le T : E \to F$, then T is order continuous if and only if $x_\alpha \downarrow_\alpha 0$ in E implies $Tx_\alpha \downarrow_\alpha 0$ in F. It follows, in particular, that if $0 \le S, T : E \to F$ are positive linear mappings with $0 \le S \le T$, then S is order continuous if T is order continuous.

Lemma 3.6 *If $E, F \subseteq S(m)$ are strongly symmetric spaces and if $0 \le T : E(\tau) \to F(\sigma)$ is order continuous, then $\sigma_n(zT_nx) = \tau_n(x\left(T^\times\right)_n z)$ for all $n \in \mathbb{N}$, $x \in E(\tau_n)$ and $z \in F^\times(\sigma_n) = F(\sigma_n)^\times$,*

Proof If $x = [x_{ij}]_{i,j=1}^n \in E(\tau_n) = M_n(E(\tau))$ and if $z \in F(\sigma_n)^\times = F^\times(\sigma_n) = M_n(F(\sigma)^\times)$, then, using the remarks concerning the trace preceding Lemmas 3.2, it follows that

$$\sigma_n(zT_nx) = \sum_{i,j=1}^n \sigma(z_{ij}Tx_{ji}) = \sum_{i,j=1}^n \tau(x_{ij}T^\times z_{ji}) = \tau_n(x\left(T^\times\right)_n z).$$

□

Proposition 3.7 *Suppose that $E, F \subseteq S(m)$ are strongly symmetric spaces. If $T : E(\tau) \to F(\sigma)$ is order continuous, and if $n \in \mathbb{N}$, then*

(i) $T_n : E(\tau_n) \to F(\sigma_n)$ *is order continuous;*
(ii) $(T_n)^\times = (T^\times)_n$*;*
(iii) T *is completely positive if and only if* $T^\times$ *is completely positive.*

Proof (i) Suppose that $x_\alpha \downarrow_\alpha 0$ in $E(\sigma_n)$ and that $z \in F(\sigma_n)^\times = F^\times(\sigma_n)$. If $z = [z_{ij}]_{i,j=1}^n \in F(\sigma_n)^\times = F^\times(\sigma_n) = M_n(F^\times)$, then, using the fact that T is order continuous so that $T^\times : F(\sigma)^\times \to E(\sigma)^\times$, it follows that $(T^\times)_n z = [T^\times z_{ij}]_{i,j=1}^n \in M_n(E(\sigma)^\times) = M_n(E^\times(\sigma)) = E^\times(\sigma_n) = E(\sigma_n)^\times$. Consequently, $\tau_n(x_\alpha (T^\times)_n z) \to_\alpha 0$. From Lemma 3.6, it now follows that $\sigma_n(zT_n x_\alpha) \to_\alpha 0$, and from this it follows that T_n is order continuous.

(ii) This is now a straightforward reformulation of Lemma 3.6.

(iii) Observe that $T \geq_{\text{cp}} 0 \Longleftrightarrow T_n \geq 0 \Longleftrightarrow (T_n)^\times \geq 0$. Since $(T_n)^\times = (T^\times)_n$, as follows from (ii), the assertion of (iii) now follows readily.

□

4 Characterisations of compactness

Throughout this section, $(\mathcal{M}, \tau)$ will denote a semifinite von Neumann algebra.

Lemma 4.1 *Suppose that* $E \subseteq S(\tau)$ *is a strongly symmetric space. If* $x \in E$ *and if* $0 \leq h \in E^\times$*, then* $h^{1/2}xh^{1/2} \in L^1(\tau)$ *and* $\|h^{1/2}xh^{1/2}\|_1 \leq 4\|x\|_E\|h\|_{E^\times}$.

Proof That $h^{1/2}xh^{1/2} \in L^1(\tau)$ follows by adapting the argument used to establish the second assertion of [6, Proposition 3.4]. To establish the norm estimate, suppose first that $0 \leq x \in E$. Observe that, if $0 \leq z \in \mathcal{M}$ satisfies $0 \leq z \leq \mathbf{1}$, then, using the first assertion of [6, Proposition 3.4],

$$\tau(h^{1/2}xh^{1/2}z) = \tau(xh^{1/2}zh^{1/2}) \leq \|x\|_E\|h^{1/2}zh^{1/2}\|_{E^\times} \leq \|x\|_E\|h\|_{E^\times}$$

since $h^{1/2}zh^{1/2} \leq h$. This suffices to establish the estimate in the case that $x \geq 0$. The general case that $x \in E$ follows by writing $x = Re(x)^+ - Re(x)^- + i(Im(x)^+ - Im(x)^-)$ and noting that $Re(x)^\pm \leq |Re(x)|$, $Im(x)^\pm \leq |Im(x)|$. The estimate then follows readily from the inequalities $\|Re(x)\|_E, \|Im(x)\|_E \leq \|x\|_E$. □

Let $E \subseteq S(\tau)$ be a strongly symmetric space. A bounded subset $\mathcal{K} \subseteq E$ is said to have *uniformly absolutely continuous norm* if and only if $\sup\{\|e_n x e_n\|_{E(\tau)} : x \in \mathcal{K}\} \to 0$ as $n \to \infty$ for all sequences $\{e_n\}_{n=1}^\infty \subseteq P(\mathcal{M})$ for which $e_n \downarrow_n 0$. See, for example, [8,17]. If $\mathcal{K} \subseteq E$ is of uniformly absolutely continuous norm, then $\mathcal{K}$ is contained in the set E^{oc} of elements of order continuous norm. See [8]. Here

$$E^{oc} = \{x \in E : |x| \geq x_\alpha \downarrow_\alpha 0 \Longrightarrow \|x_\alpha\|_E \downarrow_\alpha 0\}.$$

We shall need the following convergence criterion.

Proposition 4.2 *If* $\{x_n\}_{n=1}^\infty$ *is a sequence in* E^{oc}*, then the following statements are equivalent.*

(i) $\|x_n\|_E \to_n 0$ *as* $n \to \infty$*;*
(ii) $x_n \to_n 0$ *with respect to the measure topology and the set* $\{x_n : n \in \mathbb{N}\}$ *is of uniformly absolutely continuous norm.*

The preceding proposition is proved in [8, Theorem 6.11]. In the case that the von Neumann algebra $\mathcal{M}$ does not contain any minimal projections and $E = E(\tau)$, with $E \subset S(m)$, has order continuous norm, this proposition may also be found in [3]. See also [17].

Before proceeding, some additional preparation is needed. Let $E \subseteq S(\tau)$ be a symmetrically normed $\mathcal{M}$-bimodule. It follows from the theory of ordered Banach spaces that E^* is a (complex) ordered Banach space with a generating cone, and that there exists a constant K_E such that, whenever $x \in E$,

$$\|x\|_E \leq K_E \sup\{\psi(|x|) : 0 \leq \psi \in E^*, \|\psi\| \leq 1\}$$

See, for example, Ando [2]. We remark that, in the present setting, the constant K_E may be taken to be 4 (see [8]), and we will use this remark in what follows.

Lemma 4.3 *Suppose that* $E \subseteq S(\tau)$ *is a normed* $\mathcal{M}$*-bimodule. If* $0 \leq \psi \in E^*$*, if* $0 \leq x \in E$ *and* $0 \leq y \in \mathcal{M}$ *and if* $xy = u|xy|$ *is the polar decomposition, then*

$$\psi(|xy|) \leq \psi(u^*xu)^{1/2}\psi(yxy)^{1/2}.$$

Proof Using the fact that $\psi \geq 0$, a standard Cauchy-Schwartz type argument readily implies that

$$\psi(|xy|) = \psi(u^*x^{1/2}x^{1/2}y) \leq \psi(u^*xu)^{1/2}\psi(yxy)^{1/2}.$$

□

Lemma 4.4 *Suppose that* $E \subseteq S(\tau)$ *is a symmetrically normed* $\mathcal{M}$*-bimodule. If* $0 \leq x \in E$ *and* $0 \leq y \in \mathcal{M}$*, then*

$$\|xy\|_E \leq 4\|x\|_E^{1/2}\|yxy\|_E^{1/2}$$

and

$$\|xy\|_E \leq 4\|x\|_E^{1/2}\|x^{1/2}y^2x^{1/2}\|_E^{1/2}.$$

Proof Let $xy = u^*|xy|$ be the polar decomposition. Applying Lemma 4.3, it follows that

$$\begin{aligned}\|xy\|_E &\leq 4\sup\{\psi(|xy|) : 0 \leq \psi \in E^*, \|\psi\| \leq 1\}\\ &\leq 4\sup\{\psi(u^*xu)^{1/2}\psi(yxy)^{1/2} : 0 \leq \psi \in E^*, \|\psi\| \leq 1\}\\ &\leq 4\|x\|_E^{1/2}\sup\{\psi(yxy)^{1/2} : 0 \leq \psi \in E^*, \|\psi\| \leq 1\}\\ &\leq 4\|x\|_E^{1/2}\|yxy\|_E^{1/2}\end{aligned}$$

The second assertion follows directly from the first by observing that

$$\mu(yxy) = \mu((x^{1/2}y)^* x^{1/2}y) = \mu(x^{1/2}y(x^{1/2}y)^*) = \mu(x^{1/2}y^2x^{1/2}).$$

Since E is symmetrically normed, it follows that $\|yxy\|_E = \|x^{1/2}y^2x^{1/2}\|_E$ and the assertion follows. □

We note that a related inequality is given in [3] in the case that $\mathcal{M}$ is non-atomic and the norm on E is order continuous. We may now prove the following characterisation of sets of absolutely continuous norm contained in the positive cone of E which will be needed in what follows.

Proposition 4.5 *Suppose that $E \subseteq S(\tau)$ is a strongly symmetric space. If E has order continuous norm and if $\mathcal{K} \subseteq E_+$ is bounded, then the following statements are equivalent.*

(i) *$\mathcal{K}$ is of uniformly absolutely continuous norm.*
(ii) *For all $e_n \downarrow_n 0 \subseteq P(\mathcal{M})$, $\sup\{\|xe_n\|_E : x \in \mathcal{K}\} \to_n 0$.*
(iii) *For all $e_n \downarrow_n 0 \subseteq P(\mathcal{M})$, $\sup\{\|e_n x\|_E : x \in \mathcal{K}\} \to_n 0$.*

Proof Since $\|e_n x\|_E = \|(e_n x)^*\|_E = \|xe_n\|_E$ for all $x \in \mathcal{K} \subseteq E_+$ and all n, the equivalence (ii)⟺(iii) is clear. Further, since $\|e_n x e_n\|_E \leq \|e_n x\|_E$ for all $x \in E$ and all n, the implication (i)⟹(ii) is also clear. The implication (i)⟹(ii) now follows by observing that

$$\|xe_n\|_E \leq 4\left(\sup_{x\in\mathcal{K}} \|x\|_E^{1/2}\right) \|e_n x e_n\|_E^{1/2}$$

for all $x \in \mathcal{K}$ and for all n, as follows from Lemma 4.4. □

We note that the above proposition fails if the assumption that $\mathcal{K}$ lies in the positive cone of E is omitted. The proposition which follows characterises norm compactness in spaces of order continuous norm in terms of sets of uniformly absolutely continuous norm

Proposition 4.6 *Suppose that $E \subseteq S(\tau)$ is strongly symmetric. Suppose that $\mathcal{K} \subseteq E$ is bounded, that the norm on E is order continuous and consider the following statements.*

(i) *$\mathcal{K}$ is relatively compact.*
(ii) *$\mathcal{K}$ has uniformly absolutely continuous norm and $\mathcal{K}$ is relatively compact for the measure topology.*
(iii) *$\mathcal{K}$ has uniformly absolutely continuous norm and $h^{1/2}\mathcal{K}h^{1/2}$ is relatively compact in $L^1(\tau)$ for all $0 \leq h \in E^\times$.*

The implications (i)⟺(ii) ⟹(iii) are always valid. If, in addition $\mathcal{K} \subseteq E_+$, then all three statements are equivalent.

Proof The equivalence (i)$\Longleftrightarrow$(ii) follows from Proposition 4.2.

(i)$\Longrightarrow$(iii). Let $0 \leq h \in E^{\times}$. It follows from Lemma 4.1 that the map

$$x \to h^{1/2} x h^{1/2} : E \to L^1(\tau)$$

is continuous. Consequently, the set $h^{1/2}\mathcal{K}h^{1/2} \subset L^1(\tau)$ is relatively compact in $L^1(\tau)$ whenever $\mathcal{K}$ is relatively compact in E.

We now assume, in addition, that $\mathcal{K} \subseteq E_+$. It will suffice to show the implication (iii)$\Longrightarrow$(ii). Assume then that $\mathcal{K}$ is of uniformly absolutely continuous norm and that $h^{1/2}\mathcal{K}h^{1/2}$ is relatively compact in $L^1(\tau)$ for all $0 \leq h \in E^{\times}$. Since the measure topology is metrizable, it will suffice to show that any sequence $\{x_n\}_{n=1}^{\infty} \subset \mathcal{K}$ has a convergent subsequence. To this end, we set $e := \sup_n s(x_n)$. Using the order continuity of the norm on E, it follows from [8], Lemma 6.9 and Lemma 6.10, that e is a σ-finite projection. We let $\{f_n\}_{n=1}^{\infty} \subseteq P(\mathcal{M})$ be any sequence of mutually disjoint projections such that $\tau(f_n) < \infty$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} f_n = e$. By considering the von Neumann algebra $e\mathcal{M}e$, we may assume that $e = \mathbf{1}$.We now set $h := \sum_{n=1}^{\infty} 2^{-n} \|f_n\|_{E^{\times}}^{-1} f_n \in E^{\times}$ and note that $s(h) = \mathbf{1} = s(h^{1/2})$. We may suppose that the sequence $\{h^{1/2}x_n h^{1/2}\}_{n=1}^{\infty}$ is Cauchy in measure. For each $k \in \mathbb{N}$, we set $e_k := e^{h^{1/2}}(1/k, \infty)$ and note that $\mathbf{1} - e_k \downarrow_k 0$. We observe that, for all $k, m, n \in \mathbb{N}$,

$$x_n - x_m = (\mathbf{1} - e_k)(x_n - x_m) + e_k(x_n - x_m)(\mathbf{1} - e_k) + e_k(x_n - x_m)e_k$$

and let $V(\epsilon, \delta)$ be any neighbourhood of 0 for the measure topology. Using the fact that the sequence $\{x_n\}_{n=1}^{\infty} \subseteq E_+$ is of uniformly absolutely absolutely continuous norm, it follows from Proposition 4.5 together with the continuity of the embedding of E into $S(\tau)$ that there exists $k_0 \in \mathbb{N}$ such that

$$(\mathbf{1} - e_k)(x_n - x_m) \in V(\epsilon/3, \delta/3), \quad (x_n - x_m)(\mathbf{1} - e_k) \in V(\epsilon/3, \delta/3)$$

for all $n, m \in \mathbb{N}$ and all $k \geq k_0$. It now suffices to show, for each fixed $k \in \mathbb{N}$, that $e_k(x_n - x_m)e_k \in V((\epsilon/3, \delta/3)$ for all sufficiently large $m, n \in \mathbb{N}$. Since $e_k h^{1/2} = h^{1/2} e_k \geq k^{-1} e_k$, there exists $z_k \in e_k \mathcal{M} e_k$ such that $z_k h^{1/2} e_k = e_k h^{1/2} z_k = e_k$. It follows that

$$e_k(x_n - x_m)e_k = z_k h^{1/2} e_k (x_n - x_m) e_k h^{1/2} z_k = z_k e_k h^{1/2}(x_n - x_m) h^{1/2} e_k z_k.$$

Since the sequence $\{h^{1/2}x_n h^{1/2}\}_{n=1}^{\infty}$ is Cauchy for the measure topology, it follows from the fact that multiplication is continuous for the measure topology that $e_k(x_n - x_m)e_k \in V(\epsilon/3, \delta/3)$ for all sufficiently large n, m and this suffices to conclude the proof of the proposition. □

It should be noted that several of the characterisations of compactness in this section go back to [11] in the case of (commutative) L^p-spaces and may be found in [16] in the case of the Haagerup L^p-spaces.

5 Domination by completely positive compact operators

We begin with several immediate consequences of the characterisations given in the preceding section. Throughout this section, $(\mathcal{M}, \tau)$ and $(\mathcal{N}, \sigma)$ will denote semifinite von Neumann algebras, acting in (possibly different) Hilbert spaces. Suppose first that $E \subseteq S(\tau)$, $F \subseteq S(\sigma)$ are strongly symmetric spaces. If $T : E \to F$ is a linear mapping, then, for all $0 \leq h \in F^{\times}$, the mapping $M(h)T$ is defined by setting

$$M(h)T(x) := h^{1/2}(Tx)h^{1/2}, \quad x \in E$$

The following result now follows directly from Proposition 4.6.

Proposition 5.1 *Suppose that the norm on F is order continuous. If $0 \leq T : E \to F$ is a positive linear mapping, then the following statements are equivalent.*

- (i) *T is compact.*
- (ii) *T maps the unit ball of E into a set of uniformly absolutely continuous norm in F and the mapping $M(h)T : E \to L^1(\sigma)$ is compact, for all $0 \leq h \in F^{\times}$.*

Corollary 5.2 *Suppose that the norm on F is order continuous. If $0 \leq T : \mathcal{M} \to F$ is a positive linear mapping, then the following statements are equivalent.*

- (i) *T is compact.*
- (ii) *The mapping $M(h)T : \mathcal{M} \to L^1(\sigma)$ is compact for all $0 \leq h \in F^{\times}$.*

Proof It needs only be observed that the image under T of the positive part of the unit ball of $\mathcal{M}$ is contained in the order interval

$$[0, T(\mathbf{1})] = \{z \in F(\sigma) : 0 \leq z \leq T(\mathbf{1})\}.$$

Since the norm on F is order continuous, this clearly implies that the order interval $[0, T(\mathbf{1})]$ is of uniformly absolutely continuous norm. The assertion of the corollary now follows from Proposition 5.1. □

Lemma 5.3 *If $0 \leq T : E \to F$ is completely positive, and if $0 \leq h \in F^{\times}$, then $0 \leq M(h)T : E \to L^1(\sigma)$ is completely positive.*

Proof It needs only be observed that, if $n \in \mathbb{N}$ and if $0 \leq [a_{ij}]_{i,j=1}^{n} \in M_n(S(\sigma)) = S(\sigma_n)$, then $0 \leq [h^{1/2}a_{ij}h^{1/2}]_{i,j=1}^{n} \in M_n(S(\sigma)) = S(\sigma_n)$. If $a_{ij} = a_i^* a_j$ with $a_1, a_2, \ldots, a_n \in S(\sigma)$, then $h^{1/2}a_{ij}h^{1/2} = (a_i h^{1/2})^* a_j h^{1/2}$, and the assertion now follows from Lemma 3.2. □

Proposition 5.4 *Suppose that the norm on F is order continuous, and that $S, T : \mathcal{M} \to F$ are linear mappings. If $0 \leq_{cp} S \leq_{cp} T$ and if T is compact then S is compact.*

Proof By Corollary 5.2, it suffices to show that the map $M(h)S : \mathcal{M} \to L^1(\sigma)$ is compact for all $0 \leq h \in F^{\times}$. Suppose then that $0 \leq h \in F^{\times}$. As follows from Lemma 5.3, $0 \leq_{cp} M(h)S \leq_{cp} M(h)T : \mathcal{M} \to L^1(\sigma)$. That $M(h)S$ is compact now follows from Theorem 3.4. □

For the remainder of this section, it will be assumed that $E, F \subseteq S(m)$ are strongly symmetric spaces. Recall (see [6]) that if $E \subseteq S(m)$ has order continuous norm then the norm on $E(\tau)$ is order continuous and the Banach dual $E(\tau)^*$ coincides with the Köthe dual $E(\tau)^\times = E^\times(\tau)$. Note that, if F has order continuous norm and if $T : E(\tau) \to F(\sigma)$ is an order continuous linear mapping, then the Banach adjoint T^* coincides with the mapping $T^\times$. We may now state the principal result of this paper.

Theorem 5.5 *Let $0 \leq S, T : E(\tau) \to F(\sigma)$ be linear mappings and suppose that $0 \leq_{\text{cp}} S \leq_{\text{cp}} T$. If T is order continuous, if the norms on $E^\times$ and F are order continuous, and if T is compact, then S is compact.*

Proof Since T is compact, it follows from Corollary 5.1 that $T(B(E_+) \subseteq F(\sigma)$ is of uniformly absolutely continuous norm. Since $0 \leq S \leq T$ it follows also that $S(B(E_+))$, and hence also $S(B(E))$, is of uniformly absolutely continuous norm. To show that S is compact, it follows again from Corollary 5.1 that it suffices to show that $M(h)S : E \to L^1(\sigma)$ is compact for all $0 \leq h \in F(\sigma)^\times$. Now observe that the map $M(h)T : E \to L^1(\sigma)$ is order continuous. Indeed, if $0 \leq h \in F^\times(\sigma)$, and if $x_\alpha \downarrow_\alpha 0 \subseteq E(\tau)$ then $Tx_\alpha \downarrow_\alpha 0 \subseteq F(\sigma)$ and this implies that $h^{1/2}(Tx_\alpha)h^{1/2} \downarrow_\alpha 0$ holds in $L^1(\sigma)$. Since $0 \leq S \leq T$, it follows that $0 \leq M(h)S \leq M(h)T$ and so also $M(h)S : E \to L^1(\sigma)$ is order continuous. Consequently

$$0 \leq (M(h)S)^*, (M(h)T)^* : \mathcal{M} \to E^\times$$

and so $(M(h)S)^* = (M(h)S)^\times$ and $(M(h)T)^* = (M(h)T)^\times$. It follows from Proposition 3.7 (iii) that

$$0 \leq_{\text{cp}} (M(h)S)^* = (M(h)S)^\times \leq_{\text{cp}} (M(h)T)^\times = (M(h)T)^*.$$

Further, by Schauder's theorem, $(M(h)T)^* : \mathcal{M} \to E^\times$ is compact. Since the norm on $E^\times$ is order continuous, it follows from Proposition 5.4 that $(M(h)S)^* : \mathcal{M} \to E^\times$ is compact. Again using Schauder's theorem, it follows that $M(h)S : E \to L^1(\sigma)$ is compact, and this completes the proof of the Theorem. □

Corollary 5.6 *Suppose that $0 \leq S, T : E(\tau) \to E(\tau)$ are linear mappings which satisfy $0 \leq_{cp} S \leq_{cp} T$. If the norms on $E, E^\times$ are order continuous and if T is compact, then S is compact.*

Proof To apply the preceding Theorem, we need only note that, since the norm on $E(\tau)$ is order-continuous, then each positive linear map on $E(\tau)$ is necessarily order-continuous. □

Finally, suppose that $\tau(\mathbf{1}) < \infty$, and that $0 \leq S, T : E(\tau) \to E(\tau)$ are linear mappings with T compact which satisfy $0 \leq_{cp} S \leq_{cp} T$. It can be shown that if the norm on E is order continuous or if E has the Fatou property and the norm on $E^\times$ is order continuous, then S^2 is compact. This is a non-commutative counterpart to a well known theorem of Aliprantis and Burkinshaw [1], Meyer-Nieberg [13], and Zaanen [21]. Further, it may be shown that a completely positive operator on the predual of a finite von Neumann algebra which is dominated in the sense of complete positivity by a Dunford–Pettis operator, is itself a Dunford–Pettis operator.

In the case of abstract L-spaces this was first proved in [9]. In fact, this result continues to hold for the non-commutative counterparts of separable Lorentz spaces and certain Orlicz spaces. In addition, a completely positive mapping from a semifinite von Neumann algebra $\mathcal{M}$ to any non-commutative space $F(\sigma)$ with order continuous norm can be expressed uniquely as the sum of a completely positive compact operator and a completely positive operator which dominates no non-zero compact mapping, in the sense of complete positivity. This is shown in [16] in the case that F is an L^p-space, $1 \leq p < \infty$ and in the Banach lattice setting again goes back to [9]. A similar decomposition holds for Dunford–Pettis operators in the case of finite von Neumann algebras. The details will appear elsewhere.

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