Su et al. *Fixed Point Theory and Applications* (2015) 2015:76
DOI 10.1186/s13663-015-0323-4

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Generalized contraction mapping principle and generalized best proximity point theorems in probabilistic metric spaces

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Abstract

The purpose of this paper is to introduce some basic definitions about fixed point and best proximity point in two classes of probabilistic metric spaces and to prove contraction mapping principle and relevant best proximity point theorems. The first class is the so-called *S*-probabilistic metric spaces. In *S*-probabilistic metric spaces, the generalized contraction mapping principle and generalized best proximity point theorems have been proved by authors. These results improve and extend the recent results of Su and Zhang (Fixed Point Theory Appl. 2014:170, 2014). The second class is the so-called Menger probabilistic metric spaces. In Menger probabilistic metric spaces, the contraction mapping principle and relevant best proximity point theorems have been proved by authors. These results also improve and extend the results of many authors. In order to get the results of this paper, some new methods have been used. Meanwhile some error estimate inequalities have been established.

Keywords: probabilistic metric spaces; contraction; fixed point; best proximity point; mathematical expectation; *b*-metric spaces

1 Introduction and preliminaries

Probabilistic metric spaces were introduced in 1942 by Menger [1]. In such spaces, the notion of distance between two points x and y is replaced by a distribution function $F_{x,y}(t)$. Thus one thinks of the distance between points as being probabilistic with $F_{x,y}(t)$ representing the probability that the distance between x and y is less than t. Sehgal, in his PhD thesis [2], extended the notion of a contraction mapping to the setting of Menger probabilistic metric spaces. For example, a mapping T is a probabilistic contraction if T is such that for some constant $0 < k < 1$, the probability that the distance between image points Tx and Ty is less than kt is at least as large as the probability that the distance between x and y is less than t.

In 1972, Sehgal and Bharucha-Reid proved the following result.

Theorem 1.1 (Sehgal and Bharucha-Reid [3]) *Let* $(E,F,\triangle)$ *be a complete Menger probabilistic metric space for which the triangular norm* $\triangle$ *is continuous and satisfies* $\triangle(a,b) = \min(a,b)$. *If* T *is a mapping of* E *into itself such that for some* $0 < k < 1$ *and all* $x,y \in E$,

$$F_{Tx,Ty}(t) \geq F_{x,y}\left(\frac{t}{k}\right), \quad \forall t > 0, \tag{1.1}$$

then T has a unique fixed point x^* in E, and for any given $x_0 \in X$, $T^n x_0$ converges to x^*.

The mapping T satisfying (1.1) is called a k-probabilistic contraction or a Sehgal contraction [3]. The fixed point theorem obtained by Sehgal and Bharucha-Reid is a generalization of the classical Banach contraction principle and is further investigated by many authors [2, 4–17]. Some results in this theory have found their applications to control theory, system theory and optimization problems.

Next we recall some well-known definitions and results in the theory of probabilistic metric spaces which are used later in this paper. For more details, we refer the reader to [8].

Definition 1.2 A triangular norm (shorter $\triangle$-norm) is a binary operation $\triangle$ on $[0,1]$ which satisfies the following conditions:

(a) $\triangle$ is associative and commutative;
(b) $\triangle$ is continuous;
(c) $\triangle(a,1) = a$ for all $a \in [0,1]$;
(d) $\triangle(a,b) \le \triangle(c,d)$ whenever $a \le c$ and $b \le d$ for each $a,b,c,d \in [0,1]$.

The following are the three basic $\triangle$-norms:
$\triangle_1(a,b) = \max(a+b-1,0)$;
$\triangle_2(a,b) = a \cdot b$;
$\triangle_3(a,b) = \min(a,b)$.
It is easy to check that the above three $\triangle$-norms have the following relations:

$$\triangle_1(a,b) \le \triangle_2(a,b) \le \triangle_3(a,b)$$

for any $a,b \in [0,1]$.

Definition 1.3 A function $F(t) : (-\infty,+\infty) \to [0,1]$ is called a distribution function if it is non-decreasing and left-continuous with $\lim_{t\to-\infty} F(t) = 0$. If in addition $F(0) = 0$, then F is called a distance distribution function.

Definition 1.4 A distance distribution function F satisfying $\lim_{t\to+\infty} F(t) = 1$ is called a Menger distance distribution function. The set of all Menger distance distribution functions is denoted by D^+. A special Menger distance distribution function is given by

$$H(t) = \begin{cases} 0, & t \le 0, \\ 1, & t > 0. \end{cases}$$

Definition 1.5 A probabilistic metric space is a pair (E,F), where E is a nonempty set, F is a mapping from $E \times E$ into D^+ such that, if $F_{x,y}$ denotes the value of F at the pair (x,y), the following conditions hold:

(PM-1) $F_{x,y}(t) = H(t)$ if and only if $x = y$;
(PM-2) $F_{x,y}(t) = F_{y,x}(t)$ for all $x,y \in E$ and $t \in (-\infty,+\infty)$;
(PM-3) $F_{x,z}(t) = 1$, $F_{z,y}(s) = 1$ implies $F_{x,y}(t+s) = 1$

for all $x,y,z \in E$ and $-\infty < t < +\infty$.

Definition 1.6 A Menger probabilistic metric space (abbreviated, Menger PM space) is a triple $(E, F, \triangle)$, where E is a nonempty set, $\triangle$ is a continuous t-norm and F is a mapping from $E \times E$ into D^+ such that, if $F_{x,y}$ denotes the value of F at the pair (x, y), the following conditions hold:

(MPM-1) $F_{x,y}(t) = H(t)$ if and only if $x = y$;

(MPM-2) $F_{x,y}(t) = F_{y,x}(t)$ for all $x, y \in E$ and $t \in (-\infty, +\infty)$;

(MPM-3) $F_{x,y}(t + s) \geq \triangle(F_{x,z}(t), F_{z,y}(s))$ for all $x, y, z \in E$ and $t > 0$, $s > 0$.

In 2014, authors gave a new definition of probabilistic metric space, the so-called S-probabilistic metric space. This definition reflects more probabilistic meaning and probabilistic background. In this definition, the triangle inequality changed to a new form.

Definition 1.7 ([18]) An S-probabilistic metric space is a pair (E, F), where E is a nonempty set, F is a mapping from $E \times E$ into D^+ such that, if $F_{x,y}$ denotes the value of F at the pair (x, y), the following conditions hold:

(SPM-1) $F_{x,y}(t) = H(t)$ if and only if $x = y$;

(SPM-2) $F_{x,y}(t) = F_{y,x}(t)$ for all $x, y \in E$ and $t \in (-\infty, +\infty)$;

(SPM-3) $F_{x,y}(t) \geq F_{x,z}(t) * F_{z,y}(t)$, $\forall x, y, z \in E$, where $F_{x,z}(t) * F_{z,y}(t)$ is the convolution between $F_{x,z}(t)$ and $F_{z,y}(t)$ defined by

$$F_{x,z}(t) * F_{z,y}(t) = \int_0^{+\infty} F_{x,z}(t - u)\, dF_{z,y}(u).$$

Example ([18]) Let X be a nonempty set, S be a measurable space which consists of some metrics on the X, (Ω, P) be a complete probabilistic measure space and $f : \Omega \to S$ be a measurable mapping. It is easy to think that S is a random metric on the X, of course, (X, S) is a random metric space. The following expression of distribution functions $F_{x,y}(t)$, $F_{x,z}(t)$ and $F_{z,y}(t)$ is reasonable:

$$F_{x,y}(t) = P\bigl\{f^{-1}\bigl\{d \in S; d(x,y) < t\bigr\}\bigr\}$$

and

$$F_{x,z}(t) = P\bigl\{f^{-1}\bigl\{d \in S; d(x,z) < t\bigr\}\bigr\},$$

and

$$F_{z,y}(t) = P\bigl\{f^{-1}\bigl\{d \in S; d(z,y) < t\bigr\}\bigr\}$$

for all $x, y, z \in X$. Since

$$P\bigl\{f^{-1}\bigl\{d \in S; d(x,y) < t\bigr\}\bigr\} \geq P\bigl\{f^{-1}\bigl\{d \in S; d(x,z) + d(z,y) < t\bigr\}\bigr\}$$

and it follows from probabilistic theory that

$$P\bigl\{f^{-1}\bigl\{d \in S; d(x,z) + d(z,y) < t\bigr\}\bigr\} = F_{x,z}(t) * F_{z,y}(t).$$

Therefore

$$F_{x,y}(t) \geq F_{x,z}(t) * F_{z,y}(t), \quad \forall x, y, z \in X.$$

In addition, the conditions (SPM-1) and (SPM-2) are obvious.

In this paper, both the Menger probabilistic metric spaces and S-probabilistic metric spaces are included in the probabilistic metric spaces.

On the other hand, several problems can be changed as equations of the form $Tx = x$, where T is a given self-mapping defined on a subset of a metric space, a normed linear space, a topological vector space or some suitable space. However, if T is a non-self mapping from A to B, then the aforementioned equation does not necessarily admit a solution. In this case, it is contemplated to find an approximate solution x in A such that the error $d(x, Tx)$ is minimum, where d is the distance function. In view of the fact that $d(x, Tx)$ is at least $d(A, B)$, a best proximity point theorem guarantees the global minimization of $d(x, Tx)$ by the requirement that an approximate solution x satisfies the condition $d(x, Tx) = d(A, B)$. Such optimal approximate solutions are called best proximity points of the mapping T. Interestingly, best proximity point theorems also serve as a natural generalization of fixed point theorems, for a best proximity point becomes a fixed point if the mapping under consideration is a self-mapping. Research on best proximity point is an important topic in the nonlinear functional analysis and applications (see [19–31]).

Let A, B be two nonempty subsets of a complete metric space and consider a mapping $T : A \to B$. The best proximity point problem is whether we can find an element $x_0 \in A$ such that $d(x_0, Tx_0) = \min\{d(x, Tx) : x \in A\}$. Since $d(x, Tx) \geq d(A, B)$ for any $x \in A$, in fact, the optimal solution to this problem is the one for which the value $d(A, B)$ is attained.

Let A, B be two nonempty subsets of a metric space (X, d). We denote by A_0 and B_0 the following sets:

$$A_0 = \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\},$$
$$B_0 = \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\},$$

where $d(A, B) = \inf\{d(x, y) : x \in A \text{ and } y \in B\}$.

It is interesting to notice that A_0 and B_0 are contained in the boundaries of A and B, respectively, provided A and B are closed subsets of a normed linear space such that $d(A, B) > 0$ [19].

In order to study the best proximity point problems, we need the following notations.

Definition 1.8 ([30]) Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the *P-property* if and only if, for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

$$\begin{cases} d(x_1, y_1) = d(A, B), \\ d(x_2, y_2) = d(A, B) \end{cases} \quad \Rightarrow \quad d(x_1, x_2) = d(y_1, y_2).$$

In [31], the authors prove that any pair (A, B) of nonempty closed convex subsets of a real Hilbert space H satisfies the P-property.

In [25, 26], P-property was weakened to weak P-property. And an example that satisfies P-property but not weak P-property can be found there.

Definition 1.9 ([25, 26]) Let (A,B) be a pair of nonempty subsets of a metric space (X,d) with $A_0 \neq \emptyset$. Then the pair (A,B) is said to have the *weak P-property* if and only if, for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

$$\begin{cases} d(x_1,y_1) = d(A,B), \\ d(x_2,y_2) = d(A,B) \end{cases} \quad \Rightarrow \quad d(x_1,x_2) \leq d(y_1,y_2).$$

Recently, many best proximity point problems with applications have been discussed and some best proximity point theorems have been proved. For more details, we refer the reader to [27].

In 2014, authors established some definitions and basic concepts of best proximity point in the framework of probabilistic metric spaces.

Definition 1.10 ([18]) Let (E,F) be a probabilistic metric space, $A,B \subset E$ be two nonempty sets. Let

$$F_{A,B}(t) = \sup_{x\in A, y\in B} F_{x,y}(t), \quad \forall t \in (-\infty, +\infty),$$

which is said to be the probabilistic distance of A, B.

Example ([18]) Let X be a nonempty set and d_1, d_2 be two metrics defined on X with the probabilities $p_1 = 0.5$, $p_2 = 0.5$, respectively. Assume that

$$d_1(x,y) \leq d_2(x,y), \quad \forall x, y \in X.$$

For any $x, y \in X$, Table 1 is a discrete random variable with the distribution function

$$F_{x,y}(t) = \begin{cases} 0, & t \leq d_1(x,y), \\ 0.5, & d_1(x,y) < t \leq d_2(x,y), \\ 1, & d_2(x,y) < t. \end{cases}$$

Let A, B be two nonempty sets of X, Table 2 is also a discrete random variable with the distribution function

$$F_{A,B}(t) = \begin{cases} 0, & t \leq d_1(A,B), \\ 0.5, & d_1(A,B) < t \leq d_2(A,B), \\ 1, & d_2(A,B) < t, \end{cases}$$

Table 1 The random variable $d(x,y)$

$d_1(x,y)$	$d_2(x,y)$
0.5	0.5

Table 2 The random variable $d(A,B)$

$d_1(A,B)$	$d_2(A,B)$
0.5	0.5

where

$$d_i(A,B) = \inf_{x\in A,y\in B} d_i(x,y), \quad i = 1,2.$$

It is easy to see that

$$F_{A,B}(t) = \sup_{x\in A,y\in B} F_{x,y}(t), \quad \forall t \in (-\infty,+\infty).$$

Definition 1.11 ([18]) Let (E,F) be a probabilistic metric space, $A,B \subset E$ be two nonempty subsets and $T : A \to B$ be a mapping. We say that $x^* \in A$ is the best proximity point of the mapping T if the following equality holds:

$$F_{x^*,Tx^*}(t) = F_{A,B}(t), \quad \forall t \in (-\infty,+\infty).$$

Example ([18]) Let X be a nonempty set and d_1, d_2 be two metrics defined on X with the probabilities $p_1 = 0.5$, $p_2 = 0.5$, respectively. Let A, B be two nonempty sets of X and $T : A \to B$ be a mapping. Assume

$$d_1(x,y) \le d_2(x,y), \quad \forall x,y \in X.$$

If there exists a point $x^* \in A$ such that

$$d_1\bigl(x^*, Tx^*\bigr) = d_1(A,B),$$
$$d_2\bigl(x^*, Tx^*\bigr) = d_2(A,B),$$

then Table 3 is a discrete random variable with the distribution function

$$F_{x^*,Tx^*}(t) = \begin{cases} 0, & t \le d_1(x^*, Tx^*), \\ 0.5, & d_1(x^*, Tx^*) < t \le d_2(x^*, Tx^*), \\ 1, & d_2(x^*, Tx^*) < t. \end{cases}$$

It is obvious that $F_{x^*,Tx^*}(t) = F_{A,B}(t)$.

It is clear that the notion of fixed point coincided with the notion of best proximity point when the underlying mapping is a self-mapping. Let (E,F) be a probabilistic metric space. Suppose that $A \subset E$ and $B \subset E$ are nonempty subsets. We define the following sets:

$$A_0 = \bigl\{x \in A : F_{x,y}(t) = F_{A,B}(t) \text{ for some } y \in B\bigr\},$$
$$B_0 = \bigl\{y \in A : F_{x,y}(t) = F_{A,B}(t) \text{ for some } x \in A\bigr\}.$$

Table 3 The random variable *d*(*x, *Tx**)**

$d_1(x^*, Tx^*)$	$d_2(x^*, Tx^*)$
0.5	0.5

Definition 1.12 ([18]) Let (A, B) be a pair of nonempty subsets of a probabilistic metric space (E, F) with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the P-property if and only if for any $x_1, x_2 \in A$ and $y_1, y_2 \in B$,

$$F_{x_1,y_1}(t) = F_{A,B}(t), \qquad F_{x_2,y_2}(t) = F_{A,B}(t) \quad \Rightarrow \quad F_{x_1,x_2}(t) = F_{y_1,y_2}(t).$$

Definition 1.13 ([18]) Let (A, B) be a pair of nonempty subsets of a probabilistic metric space (E, F) with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the weak P-property if and only if for any $x_1, x_2 \in A$ and $y_1, y_2 \in B$,

$$F_{x_1,y_1}(t) = F_{A,B}(t), \qquad F_{x_2,y_2}(t) = F_{A,B}(t) \quad \Rightarrow \quad F_{x_1,x_2}(t) \geq F_{y_1,y_2}(t).$$

Definition 1.14 ([3]) Let (E, F) be a probabilistic metric space.

(1) A sequence $\{x_n\}$ in E is said to converge to $x \in E$ if for any given $\varepsilon > 0$ and $\lambda > 0$, there must exist a positive integer $N = N(\varepsilon, \lambda)$ such that $F_{x_n,x}(\varepsilon) > 1 - \lambda$ whenever $n > N$.

(2) A sequence $\{x_n\}$ in E is called a Cauchy sequence if for any $\varepsilon > 0$ and $\lambda > 0$, there must exist a positive integer $N = N(\varepsilon, \lambda)$ such that $F_{x_n,x_m}(\varepsilon) > 1 - \lambda$, whenever $n, m > N$.

(3) $(E, F, \triangle)$ is said to be complete if each Cauchy sequence in E converges to some point in E.

We denote by $x_n \to x$ that $\{x_n\}$ converges to x. It is easy to see that $x_n \to x$ if and only if $F_{x_n,x}(t) \to H(t)$ for any given $t \in (-\infty, +\infty)$ as $n \to \infty$.

The purpose of this paper is to introduce some basic definitions about fixed point and best proximity point in two classes of probabilistic metric spaces and to prove contraction mapping principle and relevant best proximity point theorems. The first class is the so-called S-probabilistic metric spaces. In S-probabilistic metric spaces, the generalized contraction mapping principle and generalized best proximity point theorems have been proved by authors. These results improve and extend the recent results of Su and Zhang [18]. The second class is the so-called Menger probabilistic metric spaces. In Menger probabilistic metric spaces, the contraction mapping principle and relevant best proximity point theorems have been proved by authors. These results also improve and extend the results of many authors. In order to get the results of this paper, some new methods have been used. Meanwhile some error estimate inequalities have been established.

2 Contraction mapping principle in *S*-probabilistic metric spaces

Let (E, F) be an S-probabilistic metric space. For any $x, y \in E$, we define

$$d_F(x, y) = \int_0^{+\infty} t\, dF_{x,y}(t).$$

Since t is a continuous function and $F_{x,y}$ is a bounded variation function, so the above integral is well defined. In fact, the above integral is just the mathematical expectation of $F_{x,y}(t)$. Throughout this paper we assume that

$$d_F(x, y) = \int_0^{+\infty} t\, dF_{x,y}(t) < +\infty, \quad \forall x, y \in E$$

for all probabilistic metric spaces (E, F) presented in this paper.

Theorem 2.1 *Let (E,F) be an S-probabilistic metric space. For any $x,y \in E$, we define*

$$d_F(x,y) = \int_0^{+\infty} t\,dF_{x,y}(t).$$

Then $d_F(x,y)$ is a metric on E.

Proof Since $F_{x,y}(t) = H(t)$ ($\forall t \in R$) if and only if $x = y$. And

$$\int_0^{+\infty} t\,dH(t) = 0,$$

we know that the condition $d_F(x,y) = 0 \Leftrightarrow x = y$ holds. The condition $d_F(x,y) = d_F(y,x)$, for all $x,y \in E$, is obvious. Next we prove the triangle inequality. For any $x,y,z \in E$, from (SPM-3) we have

$$F_{x,y}(t) \geq \int_0^{+\infty} F_{x,z}(t-u)\,dF_{z,y}(u) = F_{x,z}(t) * F_{z,y}(t).$$

By using probabilistic theory we know that

$$\int_0^{+\infty} t\,dF_{x,y}(t) \leq \int_0^{+\infty} t\,dF_{x,z}(t) + \int_0^{+\infty} t\,dF_{z,y}(t),$$

which implies that

$$d_F(x,y) \leq d_F(x,z) + d_F(z,y).$$

This completes the proof. □

Now we prove the following generalized contraction mapping principle in the S-probabilistic metric spaces which is a generalized form of the result in [18].

Theorem 2.2 *Let (E,F) be a complete S-probabilistic metric space. Let $T : E \to E$ be a mapping satisfying the following condition*:

$$F_{Tx,Ty}\bigl(\psi(t)\bigr) \geq F_{x,y}\bigl(\phi(t)\bigr), \quad \forall x,y \in E, \forall t \in R = (-\infty,+\infty), \tag{2.1}$$

where $\psi(t)$, $\phi(t)$ are two functions which satisfy

(1) *$\psi(t)$, $\phi(t)$ are strictly monotone increasing and continuous*;
(2) *$\psi(t) < \phi(t)$ for all $t > 0$*;
(3) *$\psi(0) = \phi(0)$.*

Then T has a unique fixed point $x^* \in E$ and for any given $x_0 \in E$ the iterative sequence $x_{n+1} = Tx_n$ converges to x^*.

Proof By using probabilistic theory we know that for all $x,y \in E$,

$$\psi^{-1}\biggl(\int_0^{+\infty} t\,dF_{Tx,Ty}(t)\biggr) = \int_0^{+\infty} \psi^{-1}(t)\,dF_{Tx,Ty}(t) = \int_0^{+\infty} t\,dF_{Tx,Ty}\bigl(\psi(t)\bigr),$$
$$\phi^{-1}\biggl(\int_0^{+\infty} t\,dF_{x,y}(t)\biggr) = \int_0^{+\infty} \phi^{-1}(t)\,dF_{x,y}(t) = \int_0^{+\infty} t\,dF_{x,y}\bigl(\phi(t)\bigr),$$

which together with (2.1) implies that

$$\psi^{-1}\big(d_F(Tx, Ty)\big) \le \phi^{-1}\big(d_F(x,y)\big) \tag{2.2}$$

for all $x, y \in E$.

For any given $x_0 \in X$, define an iterative sequence as follows:

$$x_1 = Tx_0, \qquad x_2 = Tx_1, \qquad \ldots, \qquad x_{n+1} = Tx_n, \qquad \ldots.$$

Then, for each integer $n \ge 1$, from (2.2) we get

$$\psi^{-1}\big(d_F(x_{n+1}, x_n)\big) = \psi^{-1}\big(d_F(Tx_n, Tx_{n-1})\big) \le \phi^{-1}\big(d_F(x_n, x_{n-1})\big). \tag{2.3}$$

Using condition (2) we have

$$d_F(x_{n+1}, x_n) \le d_F(x_n, x_{n-1})$$

for all $n \ge 1$. Hence the sequence $d(x_{n+1}, x_n)$ is decreasing, and consequently there exists $r \ge 0$ such that

$$d_F(x_{n+1}, x_n) \to r$$

as $n \to \infty$. By using conditions (2) and (3) we know $r = 0$.

In what follows, we show that $\{x_n\}$ is a Cauchy sequence in the metric space (E, d_F). Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{n_k}\}$, $\{x_{m_k}\}$ with $n_k > m_k > k$ such that

$$d_F(x_{n_k}, x_{m_k}) \ge \varepsilon \tag{2.4}$$

for all $k \ge 1$. Further, corresponding to m_k we can choose n_k in such a way that it is the smallest integer with $n_k > m_k$ satisfying (2.4). Then

$$d_F(x_{n_k-1}, x_{m_k}) < \varepsilon. \tag{2.5}$$

From (2.4) and (2.5), we have

$$\varepsilon \le d_F(x_{n_k}, x_{m_k}) \le d_F(x_{n_k}, x_{n_k-1}) + d_F(x_{n_k-1}, x_{m_k}) < d_F(x_{n_k}, x_{n_k-1}) + \varepsilon.$$

Letting $k \to \infty$, we get

$$\lim_{k\to\infty} d_F(x_{n_k}, x_{m_k}) = \varepsilon. \tag{2.6}$$

By using the triangular inequality we have

$$d_F(x_{n_k}, x_{m_k}) \le d_F(x_{n_k}, x_{n_k-1}) + d_F(x_{n_k-1}, x_{m_k-1}) + d_F(x_{m_k-1}, x_{m_k}),$$

$$d_F(x_{n_k-1}, x_{m_k-1}) \le d_F(x_{n_k-1}, x_{n_k}) + d_F(x_{n_k}, x_{m_k}) + d_F(x_{m_k}, x_{m_k-1}).$$

Letting $k \to \infty$ in the above two inequalities and applying (2.6), we have

$$\lim_{k\to\infty} d_F(x_{n_k-1}, x_{m_k-1}) = \varepsilon.$$

Since

$$\psi\bigl(d_F(x_{n_k}, x_{m_k})\bigr) \le \phi\bigl(d_F(x_{n_k-1}, x_{m_k-1})\bigr),$$

by using condition (2) we know $\varepsilon = 0$, this is a contradiction. This shows that $\{x_n\}$ is a Cauchy sequence in the metric space (E, d_F).

We prove that the sequence $\{x_n\}$ is also a Cauchy sequence in an S-probabilistic space (E, F), that is, we need to prove

$$\lim_{n\to\infty} F_{x_n, x_{n+m}}(t) = H(t). \tag{2.7}$$

If not, there must exist the numbers $t_0 > 0$, $0 < \lambda_0 < 1$ and subsequences $\{n_k\}$, $\{m_k\}$ of $\{n\}$ such that $F_{x_{n_k}, x_{n_k+m_k}}(t_0) \le \lambda_0$ for all $k \ge 1$. In this case, we have

$$\begin{aligned} d_F(x_{n_k}, x_{n_k+m_k}) &= \int_0^{+\infty} t\, dF_{x_{n_k}, x_{n_k+m_k}}(t) \\ &= \int_0^{t_0} t\, dF_{x_{n_k}, x_{n_k+m_k}}(t) \\ &\quad + \int_{t_0}^{+\infty} t\, dF_{x_{n_k}, x_{n_k+m_k}}(t) \\ &\ge \int_{t_0}^{+\infty} t\, dF_{x_{n_k}, x_{n_k+m_k}}(t) \\ &\ge t_0\bigl(1 - F_{x_{n_k}, x_{n_k+m_k}}(t_0)\bigr) \\ &\ge t_0(1 - \lambda_0) > 0. \end{aligned}$$

This is a contradiction.

From (2.7) we know that $\{x_n\}$ is a Cauchy sequence in a complete S-probabilistic metric space (E, F). Hence there exists a point $x^* \in E$ such that $\{x_n\}$ converges to x^* in the meaning of

$$\lim_{n\to\infty} F_{x_n, x^*}(t) = H(t), \quad \forall t \ge 0.$$

Therefore

$$\lim_{n\to\infty} F_{x_n, Tx^*}\bigl(\psi(t)\bigr) \ge \lim_{n\to\infty} F_{x_{n-1}, x^*}\bigl(\phi(t)\bigr) = H(t), \quad \forall t \ge 0,$$

which implies that

$$\lim_{n\to\infty} F_{x_n, Tx^*}(t) = H(t), \quad \forall t \ge 0.$$

We claim that x^* is a fixed point of T. In fact, for any $t > 0$, it follows from condition (SPM-3) that

$$\begin{aligned} F_{x^*,Tx^*}(t) &\geq \int_0^{+\infty} F_{x^*,x_n}(t-u)\,dF_{x_n,Tx^*}(u) \\ &\geq \int_0^{\frac{t}{2}} F_{x^*,x_n}(t-u)\,dF_{x_n,Tx^*}(u) \\ &= F_{x^*,x_n}\left(\frac{t}{2}\right)\left(F_{x_n,Tx^*}\left(\frac{t}{2}\right) - 0\right) \to 1 \end{aligned}$$

as $n \to \infty$, which implies $F_{x^*,Tx^*}(t) = H(t)$ and hence $x^* = Tx^*$. The x^* is a fixed point of T. If there exists another fixed point x^{**} of T, we observe

$$F_{x^*,x^{**}}(t) = F_{Tx^*,Tx^{**}}(t) \geq F_{x^*,x^{**}}\left(\frac{t}{h}\right),$$

which implies $F_{x^*,x^{**}}(t) = H(t)$, $\forall t \in R$, and hence $x^* = x^{**}$. Then the fixed point of T is unique. Meanwhile, for any given x_0, the iterative sequence $x_n = T^n x_0$ converges to x^*. This completes the proof. □

Theorem 2.3 *Let $(E, F, \triangle)$ be a complete Menger probabilistic metric space. Assume*

$$\triangle\left(F_{x,z}\left(\frac{t}{2}\right), F_{z,y}\left(\frac{t}{2}\right)\right) \geq \int_0^{+\infty} F_{x,z}(t-u)\,dF_{z,y}(u) \tag{2.8}$$

for all $x, y, z \in E$, $t > 0$. Let $T : E \to E$ be a mapping satisfying the following conditions:

$$F_{Tx,Ty}\big(\psi(t)\big) \geq F_{x,y}\big(\phi(t)\big), \quad \forall x, y \in E, \forall t \in R = (-\infty, +\infty), \tag{2.9}$$

where $\psi(t)$, $\phi(t)$ are two functions which satisfy

(1) *$\psi(t)$, $\phi(t)$ are strictly monotone increasing and continuous*;

(2) *$\psi(t) < \phi(t)$ for all $t > 0$*;

(3) *$\psi(0) = \phi(0)$.*

Then T has a unique fixed point $x^* \in E$ and for any given $x_0 \in E$ the iterative sequence $x_{n+1} = Tx_n$ converges to x^*.

Proof From (2.8) we know that $(E, F, \triangle)$ is an S-probabilistic metric space. This together with (2.9), by using Theorem 2.2, proves the conclusion. □

3 Best proximity point theorems in *S*-probabilistic spaces

We first define the notion of P-operator $P : B_0 \to A_0$, which is very useful for the proof of the theorem. From the definitions of A_0 and B_0, we know that for any given $y \in B_0$, there exists an element $x \in A_0$ such that $F_{x,y}(t) = F_{A,B}(t)$. Because (A, B) has the weak P-property, so such x is unique. We denote by $x = Py$ the P-operator from B_0 into A_0.

Theorem 3.1 *Let (E, F) be a complete S-probabilistic metric space. Let (A, B) be a pair of nonempty subsets in E and A_0 be a nonempty closed subset. Suppose that (A, B) satisfies the* *weak P-property. Let $T : A \to B$ be a mapping satisfying the following condition*:

$$F_{Tx,Ty}\big(\psi(t)\big) \geq F_{x,y}\big(\phi(t)\big), \quad \forall x, y \in E, \forall t \in R = (-\infty, +\infty),$$

where $\psi(t)$, $\phi(t)$ are two functions which satisfy

(1) *$\psi(t)$, $\phi(t)$ are strictly monotone increasing and continuous*;

(2) *$\psi(t) < \phi(t)$ for all $t > 0$*;

(3) *$\psi(0) = \phi(0)$*.

Assume that $T(A_0) \subset B_0$. Then T has a unique best proximity point $x^* \in A$ and for any given $x_0 \in E$ the iterative sequence $x_{n+1} = PTx_n$ converges to x^*.

Proof Since the pair (A, B) has the weak P-property, so we have

$$F_{PTx_1,PTx_2}\big(\psi(t)\big) \geq F_{Tx_1,Tx_2}\big(\psi(t)\big) \geq F_{x_1,x_2}\big(\phi(t)\big), \quad \forall t > 0$$

for any $x_1, x_2 \in A_0$. This shows that $PT : A_0 \to A_0$ is a contraction from a complete S-probabilistic metric subspace A_0 into itself. Using Theorem 2.2, we know that PT has a unique fixed point x^* and for any given $x_0 \in E$ the iterative sequence $x_{n+1} = PTx_n$ converges to x^*. Since $PTx^* = x^*$ if and only if $F_{x^*,Tx^*}(t) = F_{A,B}(t)$, so the point x^* is the unique best proximity point of $T : A \to B$. This completes the proof. □

Theorem 3.2 *Let $(E, F, \triangle)$ be a complete Menger probabilistic metric space. Assume that*

$$\triangle\left(F_{x,z}\left(\frac{t}{2}\right), F_{z,y}\left(\frac{t}{2}\right)\right) \geq \int_0^{+\infty} F_{x,z}(t-u)\, dF_{z,y}(u) \tag{3.1}$$

for all $x, y, z \in E$, $t > 0$. Let (A, B) be a pair of nonempty subsets in E and A_0 be a nonempty closed subset. Suppose that (A, B) satisfies the weak P-property. Let $T : A \to B$ be a mapping satisfying the following condition:

$$F_{Tx,Ty}\big(\psi(t)\big) \geq F_{x,y}\big(\phi(t)\big), \quad \forall x, y \in E, \forall t \in R = (-\infty, +\infty),$$

where $\psi(t)$, $\phi(t)$ are two functions which satisfy

(1) *$\psi(t)$, $\phi(t)$ are strictly monotone increasing and continuous*;

(2) *$\psi(t) < \phi(t)$ for all $t > 0$*;

(3) *$\psi(0) = \phi(0)$*.

Assume that $T(A_0) \subset B_0$. Then T has a unique best proximity point $x^* \in A$ and for any given $x_0 \in E$ the iterative sequence $x_{n+1} = PTx_n$ converges to x^*.

Proof From (3.1) we know that $(E, F, \triangle)$ is an S-probabilistic metric space. By using Theorem 3.1, the conclusion is proved. □

4 Contraction mapping principle in Menger probabilistic metric spaces

Let $(E, F, \triangle)$ be a Menger probabilistic metric space. For any $x, y \in E$, we define

$$d_F(x, y) = \int_0^{+\infty} t\, dF_{x,y}(t).$$

Since t is a continuous function and $F_{x,y}$ is a bounded variation functions, so the above integral is well defined. In fact, the above integer is just the mathematical expectation of $F_{x,y}(t)$. Throughout this paper we assume that

$$d_F(x,y) = \int_0^{+\infty} t\,dF_{x,y}(t) < +\infty, \quad \forall x,y \in E$$

for all Menger probabilistic metric spaces $(E,F,\triangle)$.

In 1973, Czerwik [32] presented a notable generalization of the classical Banach fixed point theorem in the so-called b-metric spaces.

Definition 4.1 Let E be a nonempty set and $s > 1$ be a given real number. A function $d : E \times E \to R^+$ is called a b-metric provided that, for all $x,y,z \in E$,

(BM-1) $d(x,y) = 0$ if and only if $x = y$;

(BM-2) $d(x,y) = d(y,x)$;

(BM-3) $d(x;y) \leq s(d(x,z) + d(z,y))$.

(E,d) is called a b-metric space with coefficient s.

The notions of topology including the convergence, completeness and Cauchy sequence are similar to those of metric spaces. Now, we are in a position to present the interesting result of our paper as follows.

Theorem 4.2 *Let $(E,F,\triangle_1)$ be a Menger probabilistic metric space, where $\triangle_1(a,b) = \max\{a+b-1,0\}$. For any $x,y \in E$, define*

$$d_F(x,y) = \int_0^{+\infty} t\,dF_{x,y}(t).$$

Then $d_F(x,y)$ is a b-metric with $s = 2$ on E.

Proof Since $F_{x,y}(t) = H(t)$ $(\forall t \in R)$ if and only if $x = y$, and

$$\int_0^{+\infty} t\,dH(t) = 0.$$

We know that condition (BM-1) holds and condition (BM-2) is obvious. Next we prove condition (BM-3). For any $x,y,z \in E$, from (PM-3)

$$F_{x,y}(t) \geq \triangle_1\left(F_{x,z}\left(\frac{t}{2}\right), F_{z,y}\left(\frac{t}{2}\right)\right), \quad \forall t \in R = (-\infty,+\infty),$$

by using the property of Lebesgue-Stieltjes integral we have

$$\begin{aligned} d_F(x,y) &= \int_0^{+\infty} t\,dF_{x,y}(t) \leq \int_0^{+\infty} t\,d\triangle_1\left(F_{x,z}\left(\frac{t}{2}\right), F_{z,y}\left(\frac{t}{2}\right)\right) \\ &= \int_0^{+\infty} t\,d\max\left(F_{x,z}\left(\frac{t}{2}\right) + F_{z,y}\left(\frac{t}{2}\right) - 1, 0\right) \\ &= \int_0^{+\infty} t\,d\left(F_{x,z}\left(\frac{t}{2}\right) + F_{z,y}\left(\frac{t}{2}\right)\right) \end{aligned}$$

$$
\begin{aligned}
&= \int_0^{+\infty} t\,dF_{x,z}\left(\frac{t}{2}\right) + \int_0^{+\infty} t\,dF_{z,y}\left(\frac{t}{2}\right)\\
&= 2\int_0^{+\infty} \frac{t}{2}\,dF_{x,z}\left(\frac{t}{2}\right) + 2\int_0^{+\infty} \frac{t}{2}\,dF_{z,y}\left(\frac{t}{2}\right)\\
&= 2\int_0^{+\infty} u\,dF_{x,z}(u) + 2\int_0^{+\infty} u\,dF_{z,y}(u)\\
&= 2d_F(x,z) + 2d_F(z,y).
\end{aligned}
$$

This completes the proof. □

Theorem 4.3 *Let $(E,F,\triangle_1)$ be a complete Menger probabilistic metric space, where $\triangle_1(a,b) = \max\{a+b-1,0\}$. Let $T : E \to E$ be a mapping satisfying the following condition:*

$$
F_{Tx,Ty}(t) \geq F_{x,y}\left(\frac{t}{h}\right), \quad \forall x,y \in E, \forall t \in R = (-\infty,+\infty), \tag{4.1}
$$

where $0 < h < 1$ is a constant. Then T has a unique fixed point $x^* \in E$ and for any given $x_0 \in E$ the iterative sequence $x_{n+1} = Tx_n$ converges to x^*. Further, the error estimate inequality

$$
\int_0^{+\infty} t\,dF_{T^n x_0,x^*}(t) \leq \frac{2h^{L[\frac{n}{L}]}}{1-2h^L} \max_{0\leq i<L} \int_0^{+\infty} t\,dF_{T^i x_0,T^{i+1}x_0}(t)
$$

holds for some positive integer L provided $h^L < \frac{1}{2}$.

Proof For any $x,y \in E$, from (4.1), by using the property of Lebesgue-Stieltjes integral we have

$$
\begin{aligned}
d_F(Tx,Ty) &= \int_0^{+\infty} t\,dF_{Tx,Ty}(t)\\
&\leq \int_0^{+\infty} t\,dF_{x,y}\left(\frac{t}{h}\right) = h\int_0^{+\infty} \frac{t}{h}\,dF_{x,y}\left(\frac{t}{h}\right)\\
&= h\int_0^{+\infty} u\,dF_{x,y}(u) = hd_F(x,y).
\end{aligned}
$$

Further, for any positive integer l, we have

$$
d_F\left(T^l x, T^l y\right) \leq hd_F\left(T^{l-1}x, T^{l-1}y\right) \leq hd_F\left(T^{l-2}x, T^{l-2}y\right) \leq \cdots \leq h^l d_F(x,y).
$$

Choose a sufficiently large integer L such that $2h^L < \frac{1}{2}$, then

$$
d_F\left(T^L x, T^L y\right) \leq h^L d_F(x,y) = gd_F(x,y), \quad \forall x,y \in E,
$$

where $g = h^L$ and $0 < g < \frac{1}{2}$. For any given $x_0 \in E$, define $x_{n+1} = T^L x_n$ for all $n = 0,1,2,\ldots$. Observe that

$$
\begin{aligned}
d_F(x_n,x_{n+m}) &\leq 2d_F(x_n,x_{n+1}) + 2d_F(x_{n+1},x_{n+m})\\
&\leq 2d_F(x_n,x_{n+1}) + 4d_F(x_{n+1},x_{n+2})
\end{aligned}
$$

$$
\begin{aligned}
&+ 4d_F(x_{n+2}, x_{n+m}) \\
&\leq \left(2g^n + 2^2 g^{n+1} + 2^3 g^{n+2} + \cdots + 2^m g^{n+m-1}\right) d_F(x_0, x_1).
\end{aligned}
\tag{4.2}
$$

Since $0 < g < \frac{1}{2}$, we have

$$
\left(2g^n + 2^2 g^{n+1} + 2^3 g^{n+2} + \cdots + 2^m g^{n+m-1}\right) d_F(x_0, x_1) \to 0
$$

as $n \to \infty$. Hence

$$
\int_0^{+\infty} t\, dF_{x_n, x_{n+m}}(t) = d_F(x_n, x_{n+m}) \to 0
$$

as $n \to \infty$. We claim that

$$
\lim_{n\to\infty} F_{x_n, x_{n+m}} = H(t). \tag{4.3}
$$

If not, there exist numbers $t_0 > 0$, $0 < \lambda_0 < 1$ and two subsequences $\{n_k\}$, $\{m_k\}$ of $\{n\}$ such that $F_{x_{n_k}, x_{n_k+m_k}}(t_0) \leq \lambda_0$ for all $k \geq 1$. In this case, we have

$$
\begin{aligned}
d_F(x_{n_k}, x_{n_k+m_k}) &= \int_0^{+\infty} t\, dF_{x_{n_k}, x_{n_k+m_k}}(t) \\
&= \int_0^{t_0} t\, dF_{x_{n_k}, x_{n_k+m_k}}(t) + \int_{t_0}^{+\infty} t\, dF_{x_{n_k}, x_{n_k+m_k}}(t) \\
&\geq \int_{t_0}^{+\infty} t\, dF_{x_{n_k}, x_{n_k+m_k}}(t) \geq t_0\left(1 - F_{x_{n_k}, x_{n_k+m_k}}(t_0)\right) \\
&\geq t_0(1 - \lambda_0) > 0.
\end{aligned}
$$

This is a contradiction. From (4.3) we know that $\{x_n\}$ is a Cauchy sequence in a complete Menger probabilistic metric space $(E, F, \triangle_1)$. Hence there exists a point $x^* \in E$ such that $\{x_n\}$ converges to x^* in the meaning of

$$
\lim_{n\to\infty} F_{x_n, x^*} = H(t).
$$

We claim that x^* is a fixed point of T^L. In fact, for any $t \in R$, it follows from condition (PM-3) and the property of $\triangle$-norm that

$$
\begin{aligned}
F_{x^*, T^L x^*}(t) &\geq \triangle_1\left(F_{x^*, x_n}\left(\frac{t}{2}\right), F_{x_n, T^L x^*}\left(\frac{t}{2}\right)\right) \\
&= \triangle_1\left(F_{x^*, x_n}\left(\frac{t}{2}\right), F_{T^L x_{n-1}, T^L x^*}\left(\frac{t}{2}\right)\right) \\
&\geq \triangle_1\left(F_{x^*, x_n}\left(\frac{t}{2}\right), F_{x_{n-1}, x^*}\left(\frac{t}{2g}\right)\right) \to H(t)
\end{aligned}
$$

as $n \to \infty$, which implies $F_{x^*, T^L x^*}(t) = H(t)$ and hence $x^* = T^L x^*$. The x^* is a fixed point of T^L. If there exists another fixed point x^{**} of T^L, we observe that

$$
F_{x^*, x^{**}}(t) = F_{T^L x^*, T^L x^{**}}(t) \geq F_{x^*, x^{**}}\left(\frac{t}{g}\right),
$$

which implies $F_{x^*,x^{**}}(t) = H(t)$, $\forall t \in R$, *i.e.*, $x^* = x^{**}$. Hence the fixed point of T^L is unique. On the other hand, it follows from $x^* = T^L x^*$ that $Tx^* = T^L(Tx^*)$, by the uniqueness we know $x^* = Tx^*$. Then x^* is the unique fixed point of T. Now we prove that, for any given x_0, the iterative sequence $x_n = T^n x_0$ converges to x^*. Observe that any positive integer n can be expressed as $n = mL + i$, where m, i are some positive integers and $0 \leq i < L$. In this case, $T^n x_0 = T^{mL} T^i x_0 \to x^*$. Finally, we prove the error estimate formula. Let $m \to \infty$ in inequality (4.2), we get

$$d_F(x_n, x^*) \leq \frac{2g^n}{1-2g} d_F(x_0, x_1). \tag{4.4}$$

Since $x_n = T^{nL} x_0$ for all $n \geq 0$, the above inequality (4.4) can be rewritten as follows:

$$d_F(T^{nL} x_0, x^*) \leq \frac{2g^n}{1-2g} d_F(x_0, x_1).$$

Because any positive integer n can be expressed as $n = mL + i$, where m, i are some positive integers and $0 \leq i < L$, we can get the following inequality:

$$d_F(T^n x_0, x^*) = d_F(T^{mL} T^i x_0, x^*) \leq \frac{2g^m}{1-2g} d_F(T^i x_0, T^{i+1} x_0),$$

where $m = [\frac{n}{L}]$ and $i = 1, 2, 3, \ldots, L-1$. Finally we can get the error estimate formula

$$\begin{aligned} d_F(T^n x_0, x^*) &\leq \frac{2g^{[\frac{n}{L}]}}{1-2g} \max_{0 \leq i < L} \bigl(d_F(T^i x_0, T^{i+1} x_0)\bigr) \\ &= \frac{2h^{L[\frac{n}{L}]}}{1-2h^L} \max_{0 \leq i < L} \bigl(d_F(T^i x_0, T^{i+1} x_0)\bigr) \\ &= \frac{2h^{L[\frac{n}{L}]}}{1-2h^L} \max_{0 \leq i < L} \int_0^{+\infty} t \, dF_{T^i x_0, T^{i+1} x_0}(t). \end{aligned}$$

That is,

$$\int_0^{+\infty} t \, dF_{T^n x_0, x^*}(t) \leq \frac{2h^{L[\frac{n}{L}]}}{1-2h^L} \max_{0 \leq i < L} \int_0^{+\infty} t \, dF_{T^i x_0, T^{i+1} x_0}(t).$$

This completes the proof. □

Theorem 4.4 *Let* $(E, F, \triangle)$ *be a complete Menger probabilistic metric space. Assume* $\triangle(a,b) \geq \triangle_1(a,b) = \max\{a+b-1, 0\}$. *Let* $T : E \to E$ *be a mapping satisfying the following condition*:

$$F_{Tx,Ty}(t) \geq F_{x,y}\left(\frac{t}{h}\right), \quad \forall x, y \in E, \forall t \in R = (-\infty, +\infty),$$

where $0 < h < 1$ *is a constant. Then* T *has a unique fixed point* $x^* \in E$ *and for any given* $x_0 \in E$ *the iterative sequence* $x_{n+1} = Tx_n$ *converges to* x^*. *Further, the error estimate inequality*

$$\int_0^{+\infty} t \, dF_{T^n x_0, x^*}(t) \leq \frac{2h^{L[\frac{n}{L}]}}{1-2h^L} \max_{0 \leq i < L} \int_0^{+\infty} t \, dF_{T^i x_0, T^{i+1} x_0}(t)$$

holds for some positive integer L *provided* $h^L < \frac{1}{2}$.

Proof Since $\triangle(a,b) \geq \triangle_1(a,b) = \max\{a+b-1, 0\}$, if $(E, F, \triangle)$ is a complete Menger probabilistic metric space, so is $(E, F, \triangle_1)$. By using Theorem 2.3, we get the conclusion of Theorem 4.4. This completes the proof. □

Corollary 4.5 (Sehgal and Bharucha-Reid [3], 1972) *Let* $(E, F, \min)$ *be a complete Menger probabilistic metric space. Let* $T : E \to E$ *be a mapping satisfying the following condition*:

$$F_{Tx,Ty}(t) \geq F_{x,y}\left(\frac{t}{h}\right), \quad \forall x, y \in E, \forall t \in R = (-\infty, +\infty),$$

where $0 < h < 1$ *is a constant. Then* T *has a unique fixed point* $x^* \in E$ *and for any given* $x_0 \in E$ *the iterative sequence* $x_{n+1} = Tx_n$ *converges to* x^*. *Further, the error estimate inequality*

$$\int_0^{+\infty} t\, dF_{T^n x_0, x^*}(t) \leq \frac{2h^{L[\frac{n}{L}]}}{1-2h^L} \max_{0 \leq i < L} \int_0^{+\infty} t\, dF_{T^i x_0, T^{i+1} x_0}(t)$$

holds for some positive integer L *provided* $h^L < \frac{1}{2}$.

5 Best proximity point theorems in Menger probabilistic metric spaces

We first define the notion of P-operator $P : B_0 \to A_0$, which is useful for our best proximity point theorem. From the definitions of A_0 and B_0, we know that for any given $y \in B_0$, there exists an element $x \in A_0$ such that $F_{x,y}(t) = F_{A,B}(t)$. Because (A, B) has the weak P-property, so such x is unique. We denote by $x = Py$ the P-operator from B_0 into A_0.

Theorem 5.1 *Let* $(E, F, \triangle_1)$ *be a complete Menger probabilistic metric space, where* $\triangle_1(a,b) = \max\{a+b-1, 0\}$. *Let* (A, B) *be a pair of nonempty subsets in* E *and* A_0 *be a nonempty closed subset. Suppose that* (A, B) *satisfies the weak* P*-property. Let* $T : A \to B$ *be a mapping satisfying the following condition*:

$$F_{Tx,Ty}(t) \geq F_{x,y}\left(\frac{t}{h}\right), \quad \forall x, y \in A, \forall t \in R = (-\infty, +\infty),$$

where $0 < h < 1$ *is a constant. Assume* $T(A_0) \subset B_0$. *Then* T *has a unique best proximity point* $x^* \in A$ *and for any given* $x_0 \in E$ *the iterative sequence* $x_{n+1} = PTx_n$ *converges to* x^*. *Further, the error estimate inequality*

$$\int_0^{+\infty} t\, dF_{(PT)^n x_0, x^*}(t) \leq \frac{2h^{L[\frac{n}{L}]}}{1-2h^L} \max_{0 \leq i < L} \int_0^{+\infty} t\, dF_{(PT)^i x_0, (PT)^{i+1} x_0}(t)$$

holds for some positive integer L *provided* $h^L < \frac{1}{2}$.

Proof Since the pair (A, B) has the weak P-property, we have

$$F_{PTx_1, PTx_2}(t) \geq F_{Tx_1, Tx_2}(t) \geq F_{x_1, x_2}\left(\frac{t}{h}\right), \quad \forall t \in R = (-\infty, +\infty)$$

for any $x_1, x_2 \in A_0$. This shows that $PT : A_0 \to A_0$ is a contraction from a complete Menger probabilistic metric subspace A_0 into itself. Using Theorem 4.3, we know that PT has a unique fixed point x^* and for any given $x_0 \in E$ the iterative sequence $x_{n+1} = PTx_n$ converges to x^*. Further, the error estimate inequality

$$\int_0^{+\infty} t\,dF_{(PT)^n x_0, x^*}(t) \le \frac{2h^{L[\frac{n}{L}]}}{1-2h^L} \max_{0\le i<L} \int_0^{+\infty} t\,dF_{(PT)^i x_0,(PT)^{i+1}x_0}(t)$$

holds for some positive integer L provided $h^L < \frac{1}{2}$. Since $PTx^* = x^*$ if and only if $F_{x^*,Tx^*}(t) = F_{A,B}(t)$, so the point x^* is the unique best proximity point of $T : A \to B$. This completes the proof. □

Theorem 5.2 *Let* $(E, F, \triangle)$ *be a complete Menger probabilistic metric space. Assume* $\triangle(a,b) \ge \triangle_1(a,b) = \max\{a+b-1, 0\}$. *Let* (A,B) *be a pair of nonempty subsets in E and* A_0 *be a nonempty closed subset. Suppose that* (A,B) *satisfies the weak P-property. Let* $T : A \to B$ *be a mapping satisfying the following condition*:

$$F_{Tx,Ty}(t) \ge F_{x,y}\left(\frac{t}{h}\right), \quad \forall x,y \in A, \forall t \in R = (-\infty, +\infty),$$

where $0 < h < 1$ *is a constant. Assume* $T(A_0) \subset B_0$. *Then* T *has a unique best proximity point* $x^* \in A$ *and for any given* $x_0 \in E$ *the iterative sequence* $x_{n+1} = PTx_n$ *converges to* x^*. *Further, the error estimate inequality*

$$\int_0^{+\infty} t\,dF_{(PT)^n x_0, x^*}(t) \le \frac{2h^{L[\frac{n}{L}]}}{1-2h^L} \max_{0\le i<L} \int_0^{+\infty} t\,dF_{(PT)^i x_0,(PT)^{i+1}x_0}(t)$$

holds for some positive integer L provided $h^L < \frac{1}{2}$.

Proof Because $\triangle(a,b) \ge \triangle_1(a,b) = \max\{a+b-1, 0\}$, by using Theorem 5.1, we can get the conclusion of Theorem 3.2. This completes the proof. □

Corollary 5.3 *Let* $(E, F, \min)$ *be a complete Menger probabilistic metric space. Let* (A,B) *be a pair of nonempty subsets in E and* A_0 *be a nonempty closed subset. Suppose that* (A,B) *satisfies the weak P-property. Let* $T : A \to B$ *be a mapping satisfying the following condition*:

$$F_{Tx,Ty}(t) \ge F_{x,y}\left(\frac{t}{h}\right), \quad \forall x,y \in A, \forall t \in R = (-\infty, +\infty),$$

where $0 < h < 1$ *is a constant. Assume that* $T(A_0) \subset B_0$. *Then* T *has a unique best proximity point* $x^* \in A$ *and for any given* $x_0 \in E$ *the iterative sequence* $x_{n+1} = PTx_n$ *converges to* x^*. *Further, the error estimate inequality*

$$\int_0^{+\infty} t\,dF_{(PT)^n x_0, x^*}(t) \le \frac{2h^{L[\frac{n}{L}]}}{1-2h^L} \max_{0\le i<L} \int_0^{+\infty} t\,dF_{(PT)^i x_0,(PT)^{i+1}x_0}(t)$$

holds for some positive integer L provided $h^L < \frac{1}{2}$.

Corollary 5.4 *Let* $(E, F, \triangle_2)$ *be a complete Menger probabilistic metric space, where* $\triangle_2(a,b) = a \cdot b$. *Let* (A,B) *be a pair of nonempty subsets in E and* A_0 *be a nonempty closed* *subset. Suppose that* (A, B) *satisfies the weak P-property. Let* $T : A \to B$ *be a mapping satisfying the following condition*:

$$F_{Tx,Ty}(t) \geq F_{x,y}\left(\frac{t}{h}\right), \quad \forall x, y \in A, \forall t \in R = (-\infty, +\infty),$$

where $0 < h < 1$ *is a constant. Assume that* $T(A_0) \subset B_0$. *Then* T *has a unique best proximity point* $x^* \in A$ *and for any given* $x_0 \in E$ *the iterative sequence* $x_{n+1} = PTx_n$ *converges to* x^*. *Further, the error estimate inequality*

$$\int_0^{+\infty} t\, dF_{(PT)^n x_0, x^*}(t) \leq \frac{2h^{L[\frac{n}{L}]}}{1 - 2h^L} \max_{0 \leq i < L} \int_0^{+\infty} t\, dF_{(PT)^i x_0, (PT)^{i+1} x_0}(t)$$

holds for some positive integer L provided $h^L < \frac{1}{2}$.

Remark The research for probabilistic metric spaces (probabilistic normed spaces) and relevant fixed point theory is an important topic. Many relevant results have been given by some authors. However, the profound relationship with the probabilistic theory has not been studied closely. The S-probabilistic metric spaces and relevant probabilistic methods will play an important role in the theory and applications.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Acknowledgements
This project is supported by the National Natural Science Foundation of China under Grant (11071279).

Received: 3 March 2015 Accepted: 8 May 2015 Published online: 30 May 2015

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