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Chern–Simons and Born–Infeld gravity theories and Maxwell algebras type

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Abstract Recently it was shown that standard odd- and even-dimensional general relativity can be obtained from a $(2n+1)$-dimensional Chern–Simons Lagrangian invariant under the B_{2n+1} algebra and from a $(2n)$-dimensional Born–Infeld Lagrangian invariant under a subalgebra $\mathcal{L}^{B_{2n+1}}$, respectively. Very recently, it was shown that the generalized Inönü–Wigner contraction of the generalized AdS–Maxwell algebras provides Maxwell algebras of types $\mathcal{M}_m$ which correspond to the so-called B_m Lie algebras. In this article we report on a simple model that suggests a mechanism by which standard odd-dimensional general relativity may emerge as the weak coupling constant limit of a $(2p+1)$-dimensional Chern–Simons Lagrangian invariant under the Maxwell algebra type $\mathcal{M}_{2m+1}$, if and only if $m \geq p$. Similarly, we show that standard even-dimensional general relativity emerges as the weak coupling constant limit of a $(2p)$-dimensional Born–Infeld type Lagrangian invariant under a subalgebra $\mathcal{L}^{\mathcal{M}_{\mathbf{2m}}}$ of the Maxwell algebra type, if and only if $m \geq p$. It is shown that when $m < p$ this is not possible for a $(2p+1)$-dimensional Chern–Simons Lagrangian invariant under the $\mathcal{M}_{2m+1}$ and for a $(2p)$-dimensional Born–Infeld type Lagrangian invariant under the $\mathcal{L}^{\mathcal{M}_{\mathbf{2m}}}$ algebra.

1 Introduction

The most general action for the metric satisfying the criteria of general covariance and second-order field equations for $d > 4$ is a polynomial of degree $[d/2]$ in the curvature known as the Lanczos–Lovelock gravity theory (LL) [1,2]. The LL Lagrangian in a d-dimensional Riemannian manifold can be defined as a linear combination of the dimensional continuation of all the Euler classes of dimension $2p < d$ [3,4]:

$$S = \int \sum_{p=0}^{[d/2]} \alpha_p L^{(p)} \tag{1}$$

where α_p are arbitrary constants and

$$L_p = \varepsilon_{a_1 a_2 \cdots a_d} R^{a_1 a_2} \cdots R^{a_{2p-1} a_{2p}} e^{a_{2p+1}} \cdots e^{a_d} \tag{2}$$

with $R^{ab} = d\omega^{ab} + \omega^a_c \omega^{cb}$. The expression (1) can be used both for even and for odd dimensions.

The large number of dimensionful constants in the LL theory α_p, $p = 0, 1, \ldots [d/2]$, which are not fixed from first principles, contrasts with the two constants of the Hilbert–Einstein action.

In Ref. [5] it was found that these parameters can be fixed in terms of the gravitational and the cosmological constants, and that the action in odd dimensions can be formulated as a Chern–Simons theory of the AdS group.

The closest one can get to a Chern–Simons theory in even dimensions is with the so-called Born–Infeld theories [5–8]. The Born–Infeld Lagrangian is obtained by a particular choice of the parameters in the Lovelock series, so that the Lagrangian is invariant only under local Lorentz rotations in the same way as the Hilbert–Einstein action.

If Chern–Simons theory is the appropriate odd-dimensional gauge theory and if Born–Infeld theory is the appropriate even-dimensional theories to provide a framework for the gravitational interaction, then these theories must satisfy the correspondence principle, namely they must be related to general relativity.

In Ref. [9] it was shown that the standard, odd-dimensional general relativity (without a cosmological constant) can be obtained from Chern–Simons gravity theory for a certain Lie algebra $\mathfrak{B}$ and recently it was found that standard, even-dimensional general relativity (without a cosmological constant) emerges as a limit of a Born–Infeld theory invariant under a certain subalgebra of the Lie algebra $\mathfrak{B}$ [10].

Very recently it was found that the so-called $\mathfrak{B}_m$ Lie algebra of Ref. [9] corresponds to Maxwell algebras type $\mathcal{M}_m$ [11]. In fact, it was shown that the generalized Inönü–Wigner contraction of the generalized AdS–Maxwell algebras provides Maxwell algebras types $M_{\mathbf{m}}$ which correspond to $B_{\mathbf{m}}$

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Lie algebra. These Maxwell algebras of type M_m can be obtained by an S-expansion resonant reduction of the AdS Lie algebra when we use $S_E^{(N)} = \{\lambda_\alpha\}_{\alpha=0}^{N+1}$ as a semigroup.

It is the purpose of this paper to show that standard odd general relativity emerges as the weak coupling constant limit of a $(\mathbf{2p+1})$-dimensional Chern–Simons Lagrangian invariant under the $M_{\mathbf{2m+1}}$ algebra, if and only if $m \geq p$. Similarly, we show that standard even general relativity emerges as the weak coupling constant limit of a $(\mathbf{2p})$-dimensional Born–Infeld type Lagrangian invariant under the $L^{\mathcal{M}_{\mathbf{2m}}}$ algebra, if and only if $m \geq p$. It is shown that when $m < p$ this is not possible for a $(2p+1)$-dimensional Chern–Simons Lagrangian invariant under the $M_{\mathbf{2m+1}}$ and for a $(\mathbf{2p})$-dimensional Born–Infeld type Lagrangian invariant under $L^{\mathcal{M}_{\mathbf{2m}}}$.

This paper is organized as follows: In Sect. 2 we briefly review some aspects of: (i) Lovelock gravity theory, (ii) the construction of the so-called $\mathcal{M}_{2n+1}$ algebra, and (iii) obtaining odd- and even-dimensional general relativity from Chern–Simons gravity theory and from Born–Infeld theory, respectively.

In Sect. 3 it is shown that the odd-dimensional Hilbert–Einstein Lagrangian can be obtained from a Chern–Simons Lagrangian in $(2p+1)$ dimensions invariant under the algebra $M_{\mathbf{2m+1}}$, if and only if $m \geq p$. However, this is not possible for Chern–Simons Lagrangian in $(\mathbf{2p+1})$ dimensions invariant under the $M_{\mathbf{2m+1}}$ algebra when $m < p$.

In Sect. 4 it is shown that the even-dimensional Hilbert–Einstein Lagrangian can be obtained from a Born–Infeld type Lagrangian in $(\mathbf{2p})$ dimensions invariant under the $L^{\mathcal{M}_{\mathbf{2m}}}$ subalgebra of the $M_{\mathbf{2m}}$ algebra, if and only if $m \geq p$. However, this is not possible for Born–Infeld type Lagrangians in $(\mathbf{2p})$ dimensions invariant under the $L^{\mathcal{M}_{\mathbf{2m}}}$ subalgebra when $m < p$.

Section 5 concludes the work with a comment about possible developments.

2 The Lovelock action, the $\mathcal{M}_{2n+1}$ algebra and general relativity

In this section we shall review some aspects of higher dimensional gravity, the construction of the so-called Maxwell algebra types, and obtaining odd- and even-dimensional general relativity from Chern–Simons gravity theory and from Born–Infeld theory, respectively. The main point of this section is to display the differences between the invariances of Lovelock action when odd and even dimensions are considered.

2.1 The Chern–Simons gravity

The Lovelock action is a polynomial of degree $[d/2]$ in curvature, which can be written in terms of the Riemann curvature and the vielbein e^a in the form of (1) and (2). In the first order formalism the Lovelock action is regarded as a functional of the vielbein and spin connection, and the corresponding field equations are obtained by varying with respect to e^a and ω^{ab} [5]:

$$\varepsilon_a = \sum_{p=0}^{[(d-1)/2]} \alpha_p (d-2p)\varepsilon_a^p = 0; \qquad \varepsilon_{ab} = \sum_{p=1}^{[(d-1)/2]} \alpha_p p(d-2p)\varepsilon_{ab}^p = 0 \tag{3}$$

where

$$\varepsilon_a^p := \varepsilon_{ab_1\cdots b_{d-1}} R^{b_1b_2}\cdots R^{b_{2p-1}b_{2p}} e^{b_{2p+1}}\cdots e^{b_{d-1}}, \tag{4}$$

$$\varepsilon_{ab}^p = \varepsilon_{aba_3\cdots a_d} R^{a_3a_4}\cdots R^{a_{2p-1}a_{2p}} T^{a_{2p+1}} e^{a_{2p+2}}\cdots e^{a_d}. \tag{5}$$

Here $T^a = de^a + \omega_b^a e^b$ is the torsion 2-form. Using the Bianchi identity one finds [5]

$$D\varepsilon_a = \sum_{p=1}^{[(d-1)/2]} \alpha_{p-1}(d-2p+2)(d-2p+1)e^b\varepsilon_{ba}^p. \tag{6}$$

Moreover

$$e^b\varepsilon_{ba} = \sum_{p=1}^{[(d-1)/2]} \alpha_p p(d-2p)e^b\varepsilon_{ba}^p. \tag{7}$$

From (6) and (7) one finds for $d = 2n-1$

$$\alpha_p = \alpha_0 \frac{(2n-1)(2\gamma)^p}{(2n-2p-1)} \binom{n-1}{p} \tag{8}$$

with $\alpha_0 = \frac{\kappa}{dl^{d-1}}$, $\gamma = -sign(\Lambda)\frac{l^2}{2}$, where for any number of dimensions l is a length parameter related to the cosmological constant by $\Lambda = \pm(d-1)(d-2)/2l^2$.

With these coefficients, the Lovelock action is a Chern–Simons $(2n-1)$-form invariant not only under standard local Lorentz rotations $\delta e^a = \kappa_b^a e^b$, $\delta\omega^{ab} = -D\kappa^{ab}$, but also under a local AdS boost [5].

2.2 Born–Infeld gravity

For $d = 2n$ it is necessary to write (6) in the form [5]

$$D\varepsilon_a = T^{\mathbf{b}} \sum_{p=1}^{[n-1]} 2\alpha_{p-1}(n-p+1)\mathcal{T}_{ab}^p - \sum_{p=1}^{[n-1]} 4\alpha_{p-1}(n-p+1)(n-p)e^b\varepsilon_{ba}^p \tag{9}$$

with

$$\mathcal{T}_{ab} = \frac{\delta L}{\delta R^{ab}} = \sum_{p=1}^{[(d-1)/2]} \alpha_p p \mathcal{T}_{ab}^p \tag{10}$$

where

$$\mathcal{T}_{ab}^p = \varepsilon_{aba_3 \cdots a_d} R^{a_3 a_4} \cdots R^{a_{2p-1} a_{2p}} T^{a_{2p+1}} e^{a_{2p+2}} \cdots e^{a_d}. \tag{11}$$

The comparison between (7) and (9) leads to [5]

$$\alpha_p = \alpha_0 (2\gamma)^p \binom{n}{p}. \tag{12}$$

With these coefficients the LL Lagrangian takes the form [5]

$$L = \frac{\kappa}{2n} \varepsilon_{a_1 a_2 \cdots a_d} \bar{R}^{a_1 a_2} \cdots \bar{R}^{a_{d-1} a_d}, \tag{13}$$

which is the Pfaffian of the 2-form $\bar{R}^{ab} = R^{ab} + \frac{1}{l^2} e^a e^b$ and can be formally written as the Born–Infeld like form [5,8]. The corresponding action, known as Born–Infeld action is invariant only under local Lorentz rotations.

The corresponding Born–Infeld action is given by [5,8]

$$S = \int \sum_{p=0}^{[d/2]} \frac{\kappa}{2n} \binom{n}{p} l^{2p-d+1} \varepsilon_{a_1 \cdots a_d} R^{a_1 a_2} \cdots R^{a_{2p-1} a_{2p}} e^{a_{2p+1}} \cdots e^{a_d} \tag{14}$$

where e^a corresponds to the 1-form *vielbein*, and $R^{ab} = \mathrm{d}\omega^{ab} + \omega_c^a \omega^{cb}$ to the Riemann curvature in the first order formalism.

The action (14) is off-shell invariant under the Lorentz–Lie algebra SO$(2n-1,1)$, whose generators $\tilde{\boldsymbol{J}}_{ab}$ of Lorentz transformations satisfy the commutation relationships

$$\left[\tilde{\boldsymbol{J}}_{ab}, \tilde{\boldsymbol{J}}_{cd}\right] = \eta_{cb} \tilde{\boldsymbol{J}}_{ad} - \eta_{ca} \tilde{\boldsymbol{J}}_{bd} + \eta_{db} \tilde{\boldsymbol{J}}_{ca} - \eta_{da} \tilde{\boldsymbol{J}}_{cb}.$$

The Levi-Civita symbol $\varepsilon_{a_1 \ldots a_{2n}}$ in (14) should be regarded as the only non-vanishing component of the symmetric, SO $(2n-1,1)$ invariant tensor of rank n, namely

$$\left\langle \tilde{\boldsymbol{J}}_{a_1 a_2} \cdots \tilde{\boldsymbol{J}}_{a_{2n-1} a_{2n}} \right\rangle = \frac{2^n}{n} \epsilon_{a_1 \cdots a_{2n}}. \tag{15}$$

In order to interpret the gauge field as the vielbein, one is *forced* to introduce a length scale l in the theory. To see why this happens, consider the following argument: Given that (i) the exterior derivative operator $\mathrm{d} = \mathrm{d}x^\mu \partial_\mu$ is dimensionless, and (ii) one always chooses Lie algebra generators T_A to be dimensionless as well, the 1-form connection fields $\boldsymbol{A} = A^A{}_\mu \boldsymbol{T}_A \mathrm{d}x^\mu$ must also be dimensionless. However, the vielbein $e^a = e^a_\mu \mathrm{d}x^\mu$ must have dimensions of length if it is to be related to the spacetime metric $g_{\mu\nu}$ through the usual equation $g_{\mu\nu} = e^a{}_\mu e^b{}_\nu \eta_{ab}$. This means that the 'true' gauge field must be of the form e^a/l, with l a length parameter.

Therefore, following Refs. [14,15], the 1-form gauge field $\boldsymbol{A}$ of the Chern–Simons theory is given in this case by

$$\boldsymbol{A} = \frac{1}{l} e^a \tilde{\boldsymbol{P}}_a + \frac{1}{2} \omega^{ab} \tilde{\boldsymbol{J}}_{ab}. \tag{16}$$

It is important to notice that once the length scale l is brought into the Born–Infeld theory, the Lagrangian splits into several sectors, each one of them proportional to a different power of l, as we can see directly in (14).

2.3 The Maxwell algebra type

2.3.1 *The S-expansion procedure*

In this subsection we shall review the main aspects of the S-expansion procedure and their properties introduced in Ref. [12].

Let $S = \{\lambda_\alpha\}$ be an abelian semigroup with 2-selector $K_{\alpha\beta}^{\gamma}$ defined by

$$K_{\alpha\beta}^{\gamma} = \begin{cases} 1 & \text{when} \quad \lambda_\alpha \lambda_\beta = \lambda_\gamma \\ 0 & \text{otherwise,} \end{cases} \tag{17}$$

and $\mathfrak{g}$ a Lie (super)algebra with basis $\{\mathbf{T}_A\}$ and structure constant C_{AB}^{C},

$$[\mathbf{T}_A, \mathbf{T}_B] = C_{AB}^{C} \mathbf{T}_C. \tag{18}$$

Then it may be shown that the product $\mathfrak{G} = S \times \mathfrak{g}$ is also a Lie (super)algebra with structure constants $C_{(A,\alpha)(B,\beta)}^{(C,\gamma)} = K_{\alpha\beta}^{\gamma} C_{AB}^{C}$,

$$\left[\mathbf{T}_{(A,\alpha)}, \mathbf{T}_{(B,\beta)}\right] = C_{(A,\alpha)(B,\beta)}^{(C,\gamma)} \mathbf{T}_{(C,\gamma)}. \tag{19}$$

The proof is direct and may be found in Ref. [12].

Definition 1 Let S be an abelian semigroup and $\mathfrak{g}$ a Lie algebra. The Lie algebra $\mathfrak{G}$ defined by $\mathfrak{G} = S \times \mathfrak{g}$ is called the S-expanded algebra of $\mathfrak{g}$.

When the semigroup has a zero element $0_S \in S$, it plays a somewhat peculiar role in the S-expanded algebra. The above considerations motivate the following definition.

Definition 2 Let S be an abelian semigroup with a zero element $0_S \in S$, and let $\mathfrak{G} = S \times \mathfrak{g}$ be an S-expanded algebra. The algebra obtained by imposing the condition $0_S \mathbf{T}_A = 0$

on $\mathfrak{G}$ (or a subalgebra of it) is called a 0_S-reduced algebra of $\mathfrak{G}$ (or of the subalgebra).

An S-expanded algebra has a fairly simple structure. Interestingly, there are at least two ways of extracting smaller algebras from $S \times \mathfrak{g}$. The first one gives rise to a *resonant subalgebra*, while the second produces reduced algebras. In particular, a resonant subalgebra can be obtained as follows.

Let $g = \bigoplus_{p\in I} V_p$ be a decomposition of g in subspaces V_p, where I is a set of indices. For each $p, q \in I$ it is always possible to define $i_{(p,q)} \subset I$ such that

$$\left[V_p, V_q\right] \subset \bigoplus_{r\in i_{(p,q)}} V_r. \tag{20}$$

Now, let $S = \bigcup_{p\in I} S_p$ be a subset decomposition of the abelian semigroup S such that

$$S_p \cdot S_q \subset \bigcup_{r\in i_{(p,q)}} S_p. \tag{21}$$

When such a subset decomposition $S = \bigcup_{p\in I} S_p$ exists, then we say that this decomposition is in resonance with the subspace decomposition of g, $g = \bigoplus_{p\in I} V_p$.

The resonant subset decomposition is crucial in order to systematically extract subalgebras from the S-expanded algebra $G = S \times g$, as is proven in the following Theorem IV.2 of Ref. [12]: Let $g = \bigoplus_{p\in I} V_p$ be a subspace decomposition of g, with a structure described by (20), and let $S = \bigcup_{p\in I} S_p$ be a resonant subset decomposition of the abelian semigroup S, with the structure given in (21). Define the subspaces of $G = S \times g$,

$$W_p = S_p \times V_p, \quad p \in I. \tag{22}$$

Then

$$\mathfrak{G}_R = \bigoplus_{p\in I} W_p \tag{23}$$

is a subalgebra of $G = S \times g$.

Proof the proof may be found in Ref. [12].

Definition 3 The algebra $G_R = \bigoplus_{p\in I} W_p$ obtained is called a resonant subalgebra of the S-expanded algebra $G = S \times g$.

A useful property of the S-expansion procedure is that it provides us with an invariant tensor for the S-expanded algebra $\mathfrak{G} = S \times \mathfrak{g}$ in terms of an invariant tensor for $\mathfrak{g}$. As shown in Ref. [12] the theorem VII.2 provides us with a general expression for the invariant tensor for a 0_S-reduced algebra.

Theorem VII.2 of Ref. [12] Let S be an abelian semigroup with nonzero elements λ_i, $i = 0, \ldots, N$ and $\lambda_{N+1} = 0_S$. Let $\mathfrak{g}$ be a Lie (super)algebra of basis $\{\mathbf{T}_A\}$, and let $\langle \mathbf{T}_{A_n} \cdots \mathbf{T}_{A_n}\rangle$ be an invariant tensor for $\mathfrak{g}$. The expression

$$\langle \mathbf{T}_{(A_1,i_1)} \cdots \mathbf{T}_{(A_n,i_n)}\rangle = \alpha_j K^{j}_{i_a\cdots i_n} \langle \mathbf{T}_{A_1} \cdots \mathbf{T}_{A_n}\rangle, \tag{24}$$

where α_j are arbitrary constants, corresponds to an invariant tensor for the 0_S-reduced algebra obtained from $\mathfrak{G} = S \times \mathfrak{g}$.

Proof the proof may be found in Section 4.5 of Ref. [12].

2.3.2 S-expansion of $SO\,(2n, 2)$ algebra

Let us consider the S-expansion of the Lie algebra SO$(2n, 2)$ using the Abelian semigroup $S_{\mathrm{E}}^{(2n-1)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \ldots, \lambda_{2n}\}$ defined by the product

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{when } \alpha + \beta \leq 2n \\ \lambda_{2n}, & \text{when } \alpha + \beta > 2n \end{cases} \tag{25}$$

The λ_α elements are dimensionless, and they can be represented by the set of $2n \times 2n$ matrices $[\lambda_\alpha]^i_j = \delta^i_{j+\alpha}$, where $i, j = 1, \ldots, 2n-1$, $\alpha = 0, \ldots, 2n$, and δ stands for the Kronecker delta [9].

After extracting a resonant subalgebra and perfoming its $0_S(= \lambda_{2n})$-reduction, one finds a new Lie algebra, the so-called Maxwell algebra type $\mathcal{M}_{2n+1}$, which in Ref. [9] was called a $\mathfrak{B}_{2n+1}$ algebra, whose generators

$$J_{(ab,2k)} = \lambda_{2k} \otimes \tilde{J}_{ab}, \tag{26}$$

$$P_{(a,2k+1)} = \lambda_{2k+1} \otimes \tilde{P}_a, \tag{27}$$

with $k = 0, \ldots, n-1$, satisfy the commutation relationships [9]

$$[P_a, P_b] = Z^{(1)}_{ab}, \qquad [J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b \tag{28}$$

$$\left[J_{ab}, J_{cd}\right] = \eta_{cb} J_{ad} - \eta_{ca} J_{bd} + \eta_{db} J_{ca} - \eta_{da} J_{cb} \tag{29}$$

$$\left[J_{ab}, Z^{(i)}_c\right] = \eta_{bc} Z^{(i)}_a - \eta_{ac} Z^{(i)}_b, \tag{30}$$

$$\left[Z^{(i)}_{ab}, P_c\right] = \eta_{bc} Z^{(i)}_a - \eta_{ac} Z^{(i)}_b, \tag{31}$$

$$\left[Z^{(i)}_{ab}, Z^{(j)}_c\right] = \eta_{bc} Z^{(i+j)}_a - \eta_{ac} Z^{(i+j)}_b \tag{32}$$

$$\left[J_{ab}, Z^{(i)}_{cd}\right] = \eta_{cb} Z^{(i)}_{ad} - \eta_{ca} Z^{(i)}_{bd} + \eta_{db} Z^{(i)}_{ca} - \eta_{da} Z^{(i)}_{cb} \tag{33}$$

$$\left[Z^{(i)}_{ab,} Z^{(j)}_{cd}\right] = \eta_{cb} Z^{(i+j)}_{ad} - \eta_{ca} Z^{(i+j)}_{bd} + \eta_{db} Z^{(i+j)}_{ca} - \eta_{da} Z^{(i+j)}_{cb} \tag{34}$$

$$\left[P_a, Z^{(i)}_c\right] = Z^{(i+1)}_{ab}, \qquad \left[Z^{(i)}_a, Z^{(j)}_c\right] = Z^{(i+j+1)}_{ab}. \tag{35}$$

and where we have defined

$$J_{ab} = J_{(ab,0)} = \lambda_0 \otimes \tilde{J}_{ab}, \tag{36}$$

$$P_a = P_{(a,1)} = \lambda_1 \otimes \tilde{P}_a, \tag{37}$$

$$Z_{ab}^{(i)} = J_{(ab,2i)} = \lambda_{2i} \otimes \tilde{J}_{ab}, \tag{38}$$

$$Z_a^{(i)} = P_{(a,2i+1)} = \lambda_{2i+1} \otimes \tilde{P}_a, \tag{39}$$

with $i = 1, \ldots, n-1$.

We note that the commutation relations (29), (33), and (34) form a subalgebra of the $\mathcal{M}_{2n+1}$ algebra, which we will denote as $\mathfrak{L}^{\mathcal{M}_{2n+1}}$. This subalgebra can be obtained from an S-expansion of the Lorentz–Lie algebra using as a semigroup the sub-semigroup $S_0^{(2n-1)} = \{\lambda_0, \lambda_2, \lambda_4, \lambda_6, \ldots, \lambda_{2n}\}$ of semigroup $S_E^{(2n-1)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \ldots, \lambda_{2n}\}$. After extracting a resonant subalgebra and perfoming its $0_S(=\lambda_{2n})$-reduction, one finds the $\mathfrak{L}^{\mathcal{M}_{2n+1}}$ algebra, which is a subalgebra of the $\mathcal{M}_{2n+1}$ algebra, whose generators $J_{ab} = \lambda_0 \tilde{\boldsymbol{J}}_{ab}$, $Z_{ab}^{(1)} = \lambda_2 \tilde{\boldsymbol{J}}_{ab}$, $Z_{ab}^{(2)} = \lambda_4 \tilde{\boldsymbol{J}}_{ab}, \ldots, Z_{ab}^{(n)} = \lambda_{2n} \tilde{\boldsymbol{J}}_{ab}$ satisfy the commutation relationships (29), (33), and (34).

2.4 General relativity

2.4.1 Odd-dimensional general relativity

In Ref. [9], it was shown that the standard, odd-dimensional general relativity (without a cosmological constant) can be obtained from Chern–Simons gravity theory for the algebra $\mathcal{M}_{2n+1}$. The Chern–Simons Lagrangian is built from a $\mathcal{M}_{2n+1}$-valued, 1-form gauge connection A which depends on a scale parameter l, which can be interpreted as a coupling constant that characterizes different regimes within the theory. The field content induced by $\mathcal{M}_{2n+1}$ includes the vielbein e^a, the spin connection ω^{ab}, and extra bosonic fields $h^{a(i)}$ and $k^{ab(j)}$. The odd-dimensional Chern–Simons Lagrangian invariant under the $\mathcal{M}_{2n+1}$ algebra is given by [9]

$$L_{CS}^{(2n+1)} = \sum_{k=1}^{n} l^{2k-2} c_k \alpha_j \delta_{i_1+\cdots+i_{n+1}}^{j} \delta_{p_1+q_1}^{i_{k+1}} \cdots \delta_{p_{n-k}+q_{n-k}}^{i_n} \varepsilon_{a_1\cdots a_{2n+1}} \times R^{(a_1a_2,i_1)} \ldots R^{(a_{2k-1}a_{2k},i_k)} e^{(a_{2k+1},p_1)} e^{(a_{2k+2},q_1)} \ldots \ldots e^{(a_{2n-1},p_{n-k})} e^{(a_{2n},q_{n-k})} e^{(a_{2n+1},i_{n+1})}, \tag{40}$$

where

$$c_k = \frac{1}{2(n-k)+1} \binom{n}{k}$$

$$R^{(ab,2k)} = d\omega^{(ab,2k)} + \eta_{cd}\omega^{(ac,2i)}\omega^{(db,2j)}\delta_{i+j}^{k}.$$

In the $l \longrightarrow 0$ limit, the only nonzero term in (40) corresponds to the case $k=1$, whose only non-vanishing component occurs for $p = q_1 = \cdots = q_{2n-1} = 0$ and is proportional to the odd-dimensional Hilbert–Einstein Lagrangian [9]

$$L_{CS}^{(2n+1)}\Big|_{l=0} = \frac{n\alpha_{2n-1}}{2n-1} \varepsilon_{a_1\cdots a_{2n+1}} R^{a_1a_2} e^{a_3} \ldots e^{a_{2n+1}}. \tag{41}$$

2.4.2 Even-dimensional general relativity

In Ref. [10], it was recently shown that standard, even-dimensional general relativity (without a cosmological constant) emerges as a limit of a Born–Infeld theory invariant under the subalgebra $\mathfrak{L}^{\mathcal{M}_{2n+1}}$ of the Lie algebra $\mathcal{M}_{2n+1}$.

The Born–Infeld Lagrangian is built from the curvature 2-form $S_0^{(2n-1)}$-expanded

$$F = \sum_{k=0}^{n-1} \frac{1}{2} F^{(ab,2k)} J_{(ab,2k)}, \tag{42}$$

where

$$F^{(ab,2k)} = d\omega^{(ab,2k)} + \eta_{cd}\omega^{(ac,2i)}\omega^{(db,2j)}\delta_{i+j}^{k} + \frac{1}{l^2} e^{(a,2i+1)} e^{(b,2j+1)} \delta_{i+j+1}^{k}, \tag{43}$$

which depends on a scale parameter l which can be interpreted as a coupling constant that characterizes different regimes within the theory. The field content induced by $L^{\mathcal{M}_{2n+1}}$ includes the vielbein e^a, the spin connection ω^{ab}, and extra bosonic fields $h^{a(i)} = e^{(a,2i+1)}$ and $k^{ab(i)} = \omega^{(ab,2i)}$, with $i = 1, \ldots, n-1$. The even-dimensional Born–Infeld gravity Lagrangian invariant under the $L^{\mathcal{M}_{2n+1}}$ algebra is given by [10]

$$L_{BI(2n)}^{\mathfrak{L}\mathcal{M}} = \sum_{k=1}^{n} l^{2k-2} \frac{1}{2n} \binom{n}{k} \alpha_j \delta_{i_1+\cdots+i_n}^{j} \delta_{p_1+q_1}^{i_{k+1}} \cdots \delta_{p_{n-k}+q_{n-k}}^{i_n} \varepsilon_{a_1\cdots a_{2n}} R^{(a_1a_2,i_1)} \ldots R^{(a_{2k-1}a_{2k},i_k)} e^{(a_{2k+1},p_1)} e^{(a_{2k+2},q_1)} \ldots e^{(a_{2n-1},p_{n-k})} e^{(a_{2n},q_{n-k})}. \tag{44}$$

where we can see that in the limit $l=0$ the only nonzero term corresponds to the case $k=1$, whose only nonzero component (corresponding to the case $p = q_1 = \cdots = q_{2n-2} = 0$) [10] is proportional to the even-dimensional Hilbert–Einstein Lagrangian

$$L_{BI(2n)}^{\mathfrak{L}\mathcal{M}}\Big|_{l=0} = \frac{1}{2}\alpha_{2n-2}\varepsilon_{a_1\cdots a_{2n}} R^{(a_1a_2,0)} e^{(a_3,1)} \cdots e^{(a_{2n},1)} = \frac{1}{2}\alpha_{2n-2}\varepsilon_{a_1\cdots a_{2n}} R^{a_1a_2} e^{a_3} \cdots e^{a_{2n}}. \tag{45}$$

3 Chern–Simons Lagrangians invariant under the Maxwell algebra type

In this section it is shown that the Hilbert–Einstein Lagrangian for an odd number of dimensions can be obtained from a

Chern–Simons Lagrangian in $(2p+1)$ dimensions invariant under the M_{2m+1} algebra, if and only if $m \geq p$. However, this is not possible when $m < p$ for Chern–Simons Lagrangians in $(2p+1)$ dimensions invariant under the M_{2m+1} algebra.

The 1-form gauge connection A is $\mathcal{M}_{2n+1}$-valued; it is given by

$$A = \sum_{k=0}^{n-1} \left[\frac{1}{2} \omega^{(ab,2k)} J_{(ab,2k)} + \frac{1}{l} e^{(a,2k+1)} P_{(a,2k+1)} \right], \quad (46)$$

and the 2-form curvature $F = dA + A^2$ is

$$F = \sum_{k=0}^{n-1} \left[\frac{1}{2} F^{(ab,2k)} J_{(ab,2k)} + \frac{1}{l} F^{(a,2k+1)} P_{(a,2k+1)} \right], \quad (47)$$

where

$$F^{(ab,2k)} = d\omega^{(ab,2k)} + \eta_{cd} \omega^{(ac,2i)} \omega^{(db,2j)} \delta^k_{i+j} + \frac{1}{l^2} e^{(a,2i+1)} e^{(b,2j+1)} \delta^k_{i+j+1}, \quad (48)$$

$$F^{(a,2k+1)} = de^{(a,2k+1)} + \eta_{bc} \omega^{(ab,2i)} e^{(c,2j)} \delta^k_{i+j}. \quad (49)$$

It is interesting to note that the Maxwell algebra type M_{2m+1} can be used to construct different odd-dimensional Chern–Simons Lagrangians. For example, if we consider a $S_E^{(3)}$-expansion of the AdS algebra SO (4, 2) and after extracting a resonant subalgebra and performing its 0_S-reduction, one finds the M_5 algebra in $D = 5$ dimensions. On the other hand, if we consider an $S_E^{(3)}$-expansion of the AdS algebra $SO(6,2)$ and after extracting a resonant subalgebra and performing its 0_S-reduction, one finds the M_5 algebra in $D = 7$ dimensions. In this way, the CS Lagrangians $L_{CS(5)}^{\mathcal{M}_5}$ and $L_{CS(7)}^{\mathcal{M}_5}$ are invariant under the same M_5 algebra, however, the indices of the generators T_a run over five and seven values, respectively.

These considerations allow the construction of gravitational theories in every odd number of dimensions. Nevertheless, as discussed below, only in some dimensions it is possible to obtain general relativity as the weak coupling constant limit of a Chern–Simons theory.

3.1 $(2+1)$-dimensional Chern–Simons Lagrangians invariant under $\mathcal{M}_7$-algebra

Before considering the Chern–Simons $(2n+1)$-dimensional Lagrangian, we study the case of the $\mathcal{M}_7$ algebra. The $\mathcal{M}_7$-algebra can be found by an S-expansion of the AdS algebra using as semigroup $S_E^{(5)}$. In fact, after extracting a resonant subalgebra and performing the 0_S reduction, one finds the $\mathcal{M}_7$-algebra whose generators satisfy the following commutation relations:

$$[P_a, P_b] = Z_{ab}^{(1)}, \qquad [J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \quad (50)$$

$$\left[J_{ab}, J_{cd}\right] = \eta_{cb} J_{ad} - \eta_{ca} J_{bd} + \eta_{db} J_{ca} - \eta_{da} J_{cb}, \quad (51)$$

$$\left[J_{ab}, Z_c^{(1)}\right] = \eta_{bc} Z_a^{(1)} - \eta_{ac} Z_b^{(1)}, \qquad \left[J_{ab}, Z_c^{(2)}\right] = \eta_{bc} Z_a^{(2)} - \eta_{ac} Z_b^{(2)}, \quad (52)$$

$$\left[Z_{ab}^{(1)}, P_c\right] = \eta_{bc} Z_a^{(1)} - \eta_{ac} Z_b^{(1)}, \qquad \left[Z_{ab}^{(2)}, P_c\right] = \eta_{bc} Z_a^{(2)} - \eta_{ac} Z_b^{(2)}, \quad (53)$$

$$\left[Z_{ab}^{(1)}, Z_c^{(1)}\right] = \eta_{bc} Z_a^{(2)} - \eta_{ac} Z_b^{(2)}, \qquad \left[P_a, Z_c^{(1)}\right] = Z_{ab}^{(2)}, \quad (54)$$

$$\left[J_{ab}, Z_{cd}^{(1)}\right] = \eta_{cb} Z_{ad}^{(1)} - \eta_{ca} Z_{bd}^{(1)} + \eta_{db} Z_{ca}^{(1)} - \eta_{da} Z_{cb}^{(1)}, \quad (55)$$

$$\left[J_{ab}, Z_{cd}^{(2)}\right] = \eta_{cb} Z_{ad}^{(2)} - \eta_{ca} Z_{bd}^{(2)} + \eta_{db} Z_{ca}^{(2)} - \eta_{da} Z_{cb}^{(2)}, \quad (56)$$

$$\left[Z_{ab}^{(1)}, Z_{cd}^{(1)}\right] = \eta_{cb} Z_{ad}^{(2)} - \eta_{ca} Z_{bd}^{(2)} + \eta_{db} Z_{ca}^{(2)} - \eta_{da} Z_{cb}^{(2)}, \quad (57)$$

$$\left[Z_{ab}^{(2)}, Z_c^{(1)}\right] = \left[Z_{ab}^{(2)}, Z_c^{(2)}\right] = \left[Z_{ab}^{(1)}, Z_c^{(2)}\right] = 0, \quad (58)$$

$$\left[Z_{ab}^{(2)}, Z_{cd}^{(2)}\right] = \left[Z_{ab}^{(1)}, Z_{cd}^{(2)}\right] = \left[P_a, Z_c^{(2)}\right] = 0, \quad (59)$$

$$\left[Z_a^{(1)}, Z_c^{(1)}\right] = \left[Z_a^{(1)}, Z_c^{(2)}\right] = \left[Z_a^{(2)}, Z_c^{(2)}\right] = 0. \quad (60)$$

Consider the construction of a 3-dimensional Chern–Simons Lagrangian invariant under $\mathcal{M}_7$. In fact, using Theorem VII.2 of Ref. [12], it is possible to show that the only non-vanishing components of an invariant tensor for the $\mathcal{M}_7$ algebra are given by

$$\langle J_{ab} J_{cd} \rangle_{\mathcal{M}_7} = \alpha_0 \left(\eta_{ad} \eta_{bc} - \eta_{ac} \eta_{bd}\right), \quad (61)$$

$$\left\langle J_{ab} Z_{cd}^{(1)} \right\rangle_{\mathcal{M}_7} = \alpha_2 \left(\eta_{ad} \eta_{bc} - \eta_{ac} \eta_{bd}\right), \quad (62)$$

$$\left\langle Z_{ab}^{(1)} Z_{cd}^{(1)} \right\rangle_{\mathcal{M}_7} = \left\langle J_{ab} Z_{cd}^{(2)} \right\rangle_{\mathcal{M}_7} = \alpha_4 \left(\eta_{ad} \eta_{bc} - \eta_{ac} \eta_{bd}\right), \quad (63)$$

$$\langle P_a P_c \rangle_{\mathcal{M}_7} = \alpha_2 \eta_{ac}, \quad (64)$$

$$\left\langle P_a Z_c^{(1)} \right\rangle_{\mathcal{M}_7} = \alpha_4 \eta_{ac}, \quad (65)$$

$$\langle J_{ab} P_c \rangle_{\mathcal{M}_7} = \alpha_1 \epsilon_{abc}, \quad (66)$$

$$\left\langle Z_{ab}^{(1)} P_c \right\rangle_{\mathcal{M}_7} = \left\langle J_{ab} Z_c^{(1)} \right\rangle_{\mathcal{M}_7} = \alpha_3 \epsilon_{abc}, \quad (67)$$

$$\left\langle Z_{ab}^{(1)} Z_c^{(1)} \right\rangle_{\mathcal{M}_7} = \left\langle J_{ab} Z_c^{(2)} \right\rangle_{\mathcal{M}_7} = \alpha_5 \epsilon_{abc}, \quad (68)$$

where α_0, α_1, α_2, α_3, α_5, and α_5 are arbitrary independent dimensionless constants. The 1-form gauge connection A is $\mathcal{M}_7$-valued; it is given by

$$A = \frac{1}{2} \omega^{ab} J_{ab} + \frac{1}{l} e^a P_a + \frac{1}{2} k^{(ab,1)} Z_{ab}^{(1)} + \frac{1}{l} h^{(a,1)} Z_a^{(1)} + \frac{1}{2} k^{(ab,2)} Z_{ab}^{(2)} + \frac{1}{l} h^{(a,2)} Z_a^{(2)}, \quad (69)$$

and the 2-form curvature is

$$
\begin{aligned}
F = {} & \frac{1}{2} R^{ab} J_{ab} + \frac{1}{l} T^a P_a + \frac{1}{2}\left(D_\omega k^{(ab,1)} + \frac{1}{l^2} e^a e^b\right) Z_{ab}^{(1)} \\
& + \frac{1}{l}\left(D_\omega h^{(a,1)} + k_b^{a(1)} e^b\right) Z_a^{(1)} \\
& + \frac{1}{2}\left(D_\omega k^{(ab,2)} + k_c^{a(1)} k^{cb(1)} + \frac{1}{l^2}\left[e^a h^{(b,1)} + h^{(a,1)} e^b\right]\right) Z_{ab}^{(2)} \\
& + \frac{1}{l}\left(D_\omega h^{(a,2)} + k_c^{a(2)} e^c + k_c^{a(1)} h^{(c,1)}\right) Z_a^{(2)}.
\end{aligned}
$$

Using the dual procedure of the S-expansion, we find that the 3-dimensional Chern–Simons Lagrangian invariant under the $\mathcal{M}_7$-algebra is given by

$$
\begin{aligned}
L_{CS(2+1)}^{\mathcal{M}_7} = {} & \frac{\boldsymbol{\alpha}_1}{l} \varepsilon_{abc}\left(R^{ab} e^c - d\left(\frac{1}{2}\omega^{ab} e^c\right)\right) \\
& + \frac{\boldsymbol{\alpha}_3}{l} \varepsilon_{abc}\left(R^{ab} h^{(c,1)} + \mathfrak{R}^{(ab,1)} e^c + \frac{1}{3l^2} e^a e^b e^c\right. \\
& \left. - \frac{d}{2}\left(\omega^{ab} h^{(c,1)} + k^{(ab,1)} e^c\right)\right) \\
& + \frac{\boldsymbol{\alpha}_5}{l} \varepsilon_{abc}\left(R^{ab} h^{(c,2)} + \mathfrak{R}^{(ab,1)} h^{(c,1)}\right. \\
& + \mathfrak{R}^{(ab,2)} e^c + \frac{1}{l^2} e^a e^b h^{(c,1)} \\
& \left. - \frac{d}{2}\left(\omega^{ab} h^{(c,2)} + k^{(ab,1)} h^{(c,1)} + k^{(ab,2)} e^c\right)\right) \\
& + \frac{\boldsymbol{\alpha}_0}{2}\left(\omega_b^a d\omega_a^b + \frac{2}{3}\omega_b^a \omega_c^b \omega_a^c\right) \\
& + \frac{\boldsymbol{\alpha}_2}{2}\left(\omega_b^a dk_a^{b(1)} + k_b^{a(1)} d\omega_a^b + 2\omega_b^a \omega_c^b k_a^{c(1)} + \frac{2}{l^2} e_a T^a\right) \\
& + \frac{\boldsymbol{\alpha}_4}{2}\left(\omega_b^a dk_a^{b(2)} + k_b^{a\ (2)} d\omega_a^b + 2\omega_b^a \omega_c^b k_a^{c(2)}\right. \\
& + k_b^{a(1)} dk_a^{b(1)} + 2\omega_b^a k_c^{b\ (1)} k_a^{c(1)} \\
& \left. + \frac{2}{l^2} e_a \mathfrak{T}^{(a,1)} + \frac{2}{l^2} h_a^{(1)} T^a\right)
\end{aligned} \tag{70}
$$

where

$$
\mathfrak{R}^{(ab,1)} = D_\omega k^{(ab,1)}, \tag{71}
$$

$$
\mathfrak{R}^{(ab,2)} = D_\omega k^{(ab,2)} + k_c^{a(1)} k^{cb(1)} \tag{72}
$$

$$
\mathfrak{T}^{(a,1)} = D_\omega h^{(a,1)} + k_c^{a(1)} e^c. \tag{73}
$$

The Lagrangian (70) is split into six independent pieces, each one proportional to α_1, α_3, α_5, α_0, α_2, and α_4. The term proportional to α_1 corresponds to the Chern–Simons Lagrangian for $ISO(2,1)$, which contains the Hilbert–Einstein term $\varepsilon_{abc} R^{ab} e^c$.

Varying the Lagrangian (70) we have

$$
\begin{aligned}
\delta L_{CS(2+1)}^{\mathcal{M}_7} = {} & \frac{1}{l}\varepsilon_{abc}\left(\alpha_1 R^{ab} + \frac{\alpha_3}{l^2} e^a e^b + \alpha_3 \mathfrak{R}^{(ab,1)} + \mathfrak{R}^{(ab,2)}\right)\delta e^c \\
& + \frac{1}{l}\varepsilon_{abc}\left(\alpha_3 R^{ab} + \alpha_5 \mathfrak{R}^{(ab,1)} + \frac{\alpha_5}{l^2} e^a e^b\right)\delta h^{(c,1)} \\
& + \frac{1}{l}\varepsilon_{abc}\left(\alpha_5 R^{ab}\right)\delta h^{(c,2)} + \frac{1}{l}\varepsilon_{abc}\delta\omega^{ab} \\
& \times\left(\alpha_1 T^c + \alpha_3 D_\omega h^{(c,1)} + \alpha_5 D_\omega h^{(c,2)}\right) \\
& + \frac{1}{l}\varepsilon_{acd}\delta\omega^{ab}\left(\alpha_3 e_b k^{(cd,1)} + \alpha_5 h_b^{;(1)} k^{(cd,1)} + \alpha_5 e_b k^{(cd,2)}\right) \\
& + \frac{1}{l}\varepsilon_{abc}\delta k^{(ab,1)}\left(\alpha_3 T^c + \alpha_5 D_\omega h^{(c,1)}\right) \\
& + \frac{1}{l}\varepsilon_{acd}\delta k^{(ab,1)}\left(2\alpha_5 k_b^{c,(1)} e^d\right) \\
& + \frac{1}{l}\varepsilon_{abc}\delta k^{(ab,2)}\left(\alpha_5 T^c\right) \\
& + \frac{\alpha_0}{2}\left(\delta L_3^{\text{Lorentz}}\right) + \frac{\alpha_2}{2}\left(\delta L_3^{\text{Lorentz}}\left(k^{(1)}\right)\right) \\
& + \frac{\alpha_4}{2}\left(\delta L_3^{\text{Lorentz}}\left(k^{(2)}\right)\right) + \frac{\alpha_4}{2}\left(\delta L_3^{\text{Lorentz}}\left(k^{(1)} k^{(1)}\right)\right) \\
& + \delta e_a\left(\frac{\alpha_4}{l^2}\mathfrak{T}^{(a,1)} + \frac{2\alpha_2}{l^2} T^a\right) + \delta\omega^{ab}\left(\frac{\alpha_2}{l^2} e_a e_b + \frac{\alpha_4}{l^2} e_b h_a^{(1)}\right) \\
& + \delta h_a^{;(1)}\left(\frac{2\alpha_4}{l^2} T^a\right) + \delta k^{(ab,1)}\left(\frac{\alpha_4}{l^2} e_b e_a\right),
\end{aligned}
$$

where $L_3^{\text{Lorentz}} = \omega d\omega + \frac{2}{3}\omega^3$.

If we consider the case where $k^{(ab,1)} = k^{(ab,2)} = 0$, $h^{(a,1)} = 0$ and $h^{(a,2)} = 0$) with the condition $\alpha_1 = \alpha_3 = \alpha_5 = 0$, we have

$$
\begin{aligned}
\delta L_{CS\,(2+1)}^{\mathcal{M}_7} & = \frac{\alpha_0}{2}\left(\delta L_3^{\text{Lorentz}}\right) + \frac{\alpha_2}{2l^2}\delta\omega^{ab}\left(e_a e_b\right) + \frac{\alpha_2}{2l^2}\delta e^a\left(T_a\right) \\
& = \boldsymbol{\alpha}_0 \delta\omega^{ab}\left(R_{ab}\right) + \frac{\alpha_2}{2l^2}\delta\omega^{ab}\left(e_a e_b\right) + \frac{\alpha_2}{2l^2}\delta e^a\left(T_a\right).
\end{aligned}
$$

Choosing $\alpha_0 = \alpha_2$ we find that $\delta L_{CS(2+1)}^{\mathcal{M}_7} = 0$ leads to the following equations of motion:

$$
R^{ab} + \frac{1}{l^2} e^a e^b = 0, \tag{74}
$$

$$
T_a = 0. \tag{75}
$$

which correspond to the equations of general relativity with a cosmological constant in $(2+1)$ dimensions.

3.2 (4 + 1)-dimensional Chern–Simons Lagrangian invariant under $\mathcal{M}_7$-algebra

The only non-vanishing components of an invariant tensor for the $\mathcal{M}_7$ algebra are given by

In $D = 5$, the only non-vanishing components of an invariant tensor for the $\mathcal{M}_7$ algebra are given by

$$\left\langle J_{ab} J_{cd} P_f \right\rangle_{\mathcal{M}_7} = \frac{4}{3} l^3 \alpha_1 \epsilon_{abcdf}. \tag{76}$$

$$\left\langle J_{ab} J_{cd} Z_f^{(1)} \right\rangle_{\mathcal{M}_7} = \frac{4}{3} l^3 \alpha_3 \epsilon_{abcdf}. \tag{77}$$

$$\left\langle J_{ab} Z_{cd}^{(1)} P_f \right\rangle_{\mathcal{M}_7} = \frac{4}{3} l^3 \alpha_3 \epsilon_{abcdf}. \tag{78}$$

$$\left\langle J_{ab} J_{cd} Z_f^{(2)} \right\rangle_{\mathcal{M}_7} = \frac{4}{3} l^3 \alpha_5 \epsilon_{abcdf}. \tag{79}$$

$$\left\langle J_{ab} Z_{cd}^{(1)} Z_f^{(1)} \right\rangle_{\mathcal{M}_7} = \frac{4}{3} l^3 \alpha_5 \epsilon_{abcdf}, \tag{80}$$

where α_1, α_3 and α_5 are arbitrary independent constant of dimensions $[length]^{-3}$. Using the dual procedure of S-expansion, we find that the 5-dimensional Chern–Simons Lagrangian invariant under the $\mathcal{M}_7$-algebra is given by

$$\begin{aligned}
L_{(4+1)}^{\mathcal{M}_7} &= \alpha_1 \varepsilon_{abcdf} \left(l^2 R^{ab} R^{cd} e^f \right) \\
&\alpha_3 \varepsilon_{abcdf} \left(l^2 R^{ab} R^{cd} h^{(f,1)} + 2 l^2 R^{ab} \mathfrak{R}^{(cd,1)} e^f + \frac{2}{3} R^{ab} e^c e^d e^f \right) \\
&\alpha_5 \varepsilon_{abcdf} \left(l^2 R^{ab} R^{cd} h^{(f,2)} + 2 l^2 R^{ab} \mathfrak{R}^{(cd,1)} h^{(f,1)} \right. \\
&+ 2 l^2 R^{ab} \mathfrak{R}^{(cd,2)} e^f \\
&+ l^2 \mathfrak{R}^{(ab,1)} \mathfrak{R}^{(cd,1)} e^f + 2 R^{ab} e^c e^d h^{(f,1)} \\
&\left. + \frac{2}{3} \mathfrak{R}^{(ab,1)} e^c e^d e^f + \frac{1}{5 l^2} e^a e^b e^c e^d e^f \right).
\end{aligned} \tag{81}$$

Varying the Lagrangian (81) we have

$$\begin{aligned}
\delta L_{(4+1)}^{\mathcal{M}_7} =& \varepsilon_{abcdf} \left(\boldsymbol{\alpha}_1 l^2 R^{ab} R^{cd} + 2 \boldsymbol{\alpha}_3 l^2 R^{ab} \mathfrak{R}^{(cd,1)} \right. \\
&+ 2 \boldsymbol{\alpha}_3 R^{ab} e^c e^d + 2 \boldsymbol{\alpha}_5 l^2 R^{ab} \mathfrak{R}^{(cd,2)} \\
&+ \boldsymbol{\alpha}_5 l^2 \mathfrak{R}^{(ab,1)} \mathfrak{R}^{(cd,1)} + 4 \boldsymbol{\alpha}_5 R^{ab} e^c h^{(d,1)} \\
&\left. + 2 \boldsymbol{\alpha}_5 \mathfrak{R}^{(ab,1)} e^c e^d + \frac{1}{l^2} \boldsymbol{\alpha}_5 e^a e^b e^c e^d \right) \delta e^f \\
&+ \varepsilon_{abcdf} \left(\boldsymbol{\alpha}_3 l^2 R^{ab} R^{cd} + 2 \boldsymbol{\alpha}_5 l^2 R^{ab} \mathfrak{R}^{(cd,1)} \right. \\
&\left. + 2 \boldsymbol{\alpha}_5 R^{ab} e^c e^d \right) \delta h^{(f,1)} \\
&+ \varepsilon_{abcdf} \boldsymbol{\alpha}_5 l^2 R^{ab} R^{cd} \delta h^{(f,2)} + 2 \varepsilon_{abcdf} \boldsymbol{\alpha}_5 l^2 \delta k^{(ab,2)} R^{cd} T^f \\
&+ \varepsilon_{abcdf} \delta k^{(ab,1)} \left(2 \boldsymbol{\alpha}_3 l^2 R^{cd} T^f \right. \\
&+ 2 \boldsymbol{\alpha}_5 l^2 R^{cd} D_\omega h^{(f,1)} + 2 \boldsymbol{\alpha}_5 e^c e^d T^f \\
&\left. + 2 \boldsymbol{\alpha}_5 l^2 D_\omega k^{(cd,1)} T^f \right) + \varepsilon_{acdfg} \delta k^{(ab,1)} \\
&\times \left(4 \alpha_5 l^2 k_b^{c,(1)} R^{df} e^g + 2 \alpha_5 l^2 R_b^c k^{(df,1)} T^g \right) \\
&+ \varepsilon_{abcdf} \delta \omega^{ab} \left[2 \boldsymbol{\alpha}_1 l^2 R^{cd} T^f + 2 \boldsymbol{\alpha}_3 l^2 R^{cd} D_\omega h^{(f,1)} \right. \\
&+ 2 \boldsymbol{\alpha}_3 l^2 \mathfrak{R}^{(cd,1)} T^f \\
&- 2 \alpha_3 l^2 k^{(cd,1)} R^{fg} e_g + 2 \boldsymbol{\alpha}_3 e^c e^d T^f + 2 \boldsymbol{\alpha}_5 l^2 R^{cd} D_\omega h^{(f,2)} \\
&+ 2 \boldsymbol{\alpha}_5 l^2 \mathfrak{R}^{(cd,1)} D_\omega h^{(f,1)} \\
&- 2 \alpha_5 l^2 k^{(cd,1)} R^{fg} h_g^{;(1)} + 2 \boldsymbol{\alpha}_5 l^2 D_\omega k^{(cd,2)} T^f \\
&- 2 \alpha_5 l^2 k^{(cd,2)} R^{fg} e_g \\
&+ 4 \alpha_5 l^2 \mathfrak{R}_g^{c,(1)} k^{(gd,1)} e^f + 2 \alpha_5 l^2 k_g^{c,(1)} k^{(gd,1)} T^f \\
&\left. + 4 \boldsymbol{\alpha}_5 e^c T^d h^{(f,1)} + 2 \boldsymbol{\alpha}_5 e^c e^d D_\omega h^{(f,1)} \right] \\
&+ \varepsilon_{acdfg} \delta \omega^{ab} \left(2 \alpha_3 l^2 e_b R^{cd} k^{(fg,1)} + 2 \alpha_5 l^2 h_b^{;(1)} R^{cd} k^{(fg,1)} \right. \\
&+ 2 \alpha_5 l^2 e_b R^{cd} k^{(fg,2)} \\
&- 2 \alpha_5 l^2 \mathfrak{R}_b^{c,(1)} k^{(df,1)} e^g + 2 \alpha_5 k_b^{c,(1)} \mathfrak{R}^{(df,1)} e^g \\
&\left. + e_b k^{(cd,1)} \mathfrak{R}^{(fg,1)} + 2 \alpha_5 e_b k^{(cd,1)} e^f e^g \right).
\end{aligned}$$

When a solution without matter $\left(k^{(ab,1)} = 0, k^{(ab,2)} = 0, h^{(a,1)} = 0, h^{(a,2)} = 0 \right)$ is singled out, we are left with

$$\begin{aligned}
\delta L_{(4+1)}^{\mathcal{M}_7} = \varepsilon_{abcdf} \Bigg[& \left(\boldsymbol{\alpha}_1 l^2 R^{ab} R^{cd} + 2 \boldsymbol{\alpha}_3 R^{ab} e^c e^d \right. \\
& \left. + \frac{1}{l^2} \boldsymbol{\alpha}_5 e^a e^b e^c e^d \right) \delta e^f \\
& + \left(\boldsymbol{\alpha}_3 l^2 R^{ab} R^{cd} + 2 \boldsymbol{\alpha}_5 R^{ab} e^c e^d \right) \delta h^{(f,1)} \\
& + 2 \boldsymbol{\alpha}_5 l^2 \delta k^{(ab,2)} R^{cd} T^f + \boldsymbol{\alpha}_5 l^2 R^{ab} R^{cd} \delta h^{(f,2)} \\
& + \delta k^{(ab,1)} \left(2 \boldsymbol{\alpha}_3 l^2 R^{cd} T^f + 2 \boldsymbol{\alpha}_5 e^c e^d T^f \right) \\
& + \delta \omega^{ab} \left(2 \boldsymbol{\alpha}_1 l^2 R^{cd} T^f + 2 \boldsymbol{\alpha}_3 e^c e^d T^f \right) \Bigg].
\end{aligned}$$

So when α_1 and α_5 vanish we finally get

$$\begin{aligned}
\delta L_{(4+1)}^{\mathcal{M}_7} =& \varepsilon_{abcdf} \left(2 \boldsymbol{\alpha}_3 R^{ab} e^c e^d \right) \delta e^f \\
&+ \varepsilon_{abcdf} \left(\boldsymbol{\alpha}_3 l^2 R^{ab} R^{cd} \right) \delta h^{(f,1)} \\
&+ \varepsilon_{abcdf} \delta k^{(ab,1)} \left(2 \boldsymbol{\alpha}_3 l^2 R^{cd} T^f \right) + \delta \omega^{ab} \left(2 \boldsymbol{\alpha}_3 e^c e^d T^f \right).
\end{aligned} \tag{82}$$

Therefore, if we impose the torsionlessness condition, we see that the Chern–Simons Lagrangian in $D = 5$ invariant under M_7 leads to the same equations of motion as the Chern–Simons Lagrangian in $D = 5$ invariant under M_5 [9]. From (82), like in Ref. [9], we see that in the limit where $l = 0$ the extra constraints just vanish, and $\delta L_{CS} = 0$ leads to the Hilbert–Einstein dynamics in vacuum,

$$\begin{aligned}
\delta L_{CS\,(4+1)}^{\mathcal{M}_7} =& \varepsilon_{abcdf} \left(2 \boldsymbol{\alpha}_3 R^{ab} e^c e^d \right) \delta e^f \\
&+ \varepsilon_{abcdf} \delta \omega^{ab} \left(2 \boldsymbol{\alpha}_3 e^c e^d T^f \right).
\end{aligned} \tag{83}$$

Similarly, when the cosmological constant is not considered and a solution without matter is singled out, the strict limit where the coupling constant l equals zero yields just the

Hilbert–Einstein term in the Lagrangian,

$$L^{\mathcal{M}_7}_{CS\,(4+1)} = \frac{2}{3}\alpha_3 \varepsilon_{abcdf} R^{ab} e^c e^d e^f. \tag{84}$$

3.3 (6 + 1)-dimensional Chern–Simons Lagrangian invariant under $\mathcal{M}_5$-algebra

Now, consider a Chern–Simons action (6 + 1)-dimensional invariant under the $\mathcal{M}_5$-algebra. The 1-form gauge connection A is $\mathcal{M}_5$-valued; it is given by

$$A = \frac{1}{2}\omega^{ab} J_{ab} + \frac{1}{l} e^a P_a + \frac{1}{2} k^{ab} Z_{ab} + \frac{1}{l} h^a Z_a, \tag{85}$$

and the 2-form of the curvature is given by

$$F = \frac{1}{2} R^{ab} J_{ab} + \frac{1}{l} T^a P_a + \frac{1}{2}\left(D_\omega k^{ab} + \frac{1}{l^2} e^a e^b\right) \times Z_{ab} + \frac{1}{l}\left(D_\omega h^a + k^a_b e^b\right) Z_a. \tag{86}$$

Using the dual procedure of S-expansion, we find that the 7-dimensional Chern–Simons Lagrangian invariant under the $\mathcal{M}_5$-algebra is given by

$$L^{\mathcal{M}_5}_{(6+1)} = \frac{\alpha_1}{l}\varepsilon_{abcdefg}\left(R^{ab}R^{cd}R^{ef}e^g\right) + \frac{\alpha_3}{l}\varepsilon_{abcdefg}\left(R^{ab}R^{cd}R^{ef}h^g + 3R^{ab}R^{cd}D_\omega k^{ef}e^g + \frac{1}{l^2}R^{ab}R^{cd}e^e e^f e^g\right), \tag{87}$$

where $\alpha_1 = \lambda_1\kappa$, $\alpha_3 = \lambda_3\kappa$. From here we see that the Hilbert–Einstein term is not present in the Lagrangian. This result holds for all $D = p$-dimensional Chern–Simons Lagrangians invariant under an algebra $\mathcal{M}_m$ if $p > m$.

Varying the Lagrangian we have

$$\begin{aligned}\delta L^{\mathcal{M}_5}_{(6+1)} &= \frac{\mathbf{1}}{l}\varepsilon_{abcdefg}\left(\alpha_1 R^{ab}R^{cd}R^{ef} + 3\alpha_3 R^{ab}R^{cd}D_\omega k^{ef} + \frac{3}{l^2}\alpha_3 R^{ab}R^{cd}e^e e^f\right)\delta e^g \\ &+ \frac{\mathbf{1}}{l}\varepsilon_{abcdefg}\left(\alpha_3 R^{ab}R^{cd}R^{ef}\right)\delta h^g \\ &+ \frac{\mathbf{1}}{l}\varepsilon_{abcdefg}\delta\omega^{ab}\left(3\alpha_1 R^{cd}R^{ef}T^g + 3\alpha_3 R^{cd}R^{ef}D_\omega h^g + 6\alpha_3 R^{cd}D_\omega k^{ef}T^g + \frac{6}{l^2}\alpha_3 R^{cd}e^e e^f T^g\right) + \frac{\mathbf{1}}{l}\varepsilon_{acdefgh}\delta\omega^{ab}\left(3e_b R^{cd}R^{ef}k^{gh}\right) \\ &+ \frac{\mathbf{1}}{l}\varepsilon_{abcdefg}\delta k^{ab}\left(3\alpha_3 R^{cd}R^{ef}T^g\right),\end{aligned} \tag{88}$$

from which we can see that it is not possible to obtain the Hilbert–Einstein dynamics.

In fact, imposing the torsionlessness condition and if we consider the case where $k^{ab} = 0$, $h^a = 0$ with $\alpha_1 = 0$ we find

$$\delta L^{\mathfrak{B}_5}_{(6+1)} = \frac{\alpha_3}{l^2}\varepsilon_{abcdefg}R^{ab}R^{cd}e^e e^f \delta e^g + \frac{\alpha_3}{l}\varepsilon_{abcdefg}R^{ab}R^{cd}R^{ef}\delta h^g, \tag{89}$$

which obviously does not correspond to the dynamics of general relativity.

3.4 (6 + 1)-dimensional Chern–Simons Lagrangian invariant under $\mathcal{M}_7$-algebra

Consider the 7-dimensional Chern–Simons Lagrangian invariant under the AdS algebra

$$\begin{aligned}L^{AdS}_{CS\,(6+1)} &= \kappa\left[\varepsilon_{abcdefg}\left(\frac{1}{l}R^{ab}R^{cd}R^{ef}e^g + \frac{1}{l^3}R^{ab}R^{cd}e^e e^f e^g + \frac{3}{5l^5}R^{ab}e^c e^d e^e e^f e^g + \frac{1}{7l^7}e^a e^b e^c e^d e^e e^f e^g\right)\right] \\ &+ \beta_{2,2}\left[R^a_b R^b_a + \frac{2}{l^2}\left(T^a T_a - R^{ab}e_a e_b\right)\right] \times \left(\omega^c_d d\omega^d_c + \frac{2}{3}\omega^c_f\,\omega^f_g\,\omega^g_c + \frac{2}{l^2}e_c T^c\right) \\ &+ \beta_4\left[\left(\omega^a_b d\omega^b_c d\omega^c_d d\omega^d_a + \frac{8}{5}\omega^a_b\omega^b_c\omega^c_d d\omega^d_e d\omega^e_a + \frac{4}{5}\omega^a_b d\omega^b_c\omega^c_d\omega^d_e d\omega^e_a + 2\omega^a_b\omega^b_c\omega^c_d\omega^d_e\omega^e_f d\omega^f_a + \frac{4}{7}\omega^a_b\omega^b_c\omega^c_d\omega^d_e\omega^e_f\omega^f_g\,\omega^g_a\right)\right. \\ &\left. + \frac{1}{l^2}4T_a R^a_b R^b_c e^c + \frac{1}{l^4}\left[2\left(R^{ab}e_a e_b + T^a T_a\right)T^c e_c\right]\right].\end{aligned}$$

Using the dual procedure of the S-expansion, we find that the 7-dimensional Chern–Simons Lagrangian invariant under the $\mathcal{M}_7$-algebra is given by

$$\begin{aligned}L^{\mathcal{M}_7}_{CS\,(6+1)} &= \alpha_1 l^4 \varepsilon_{abcdefg}R^{ab}R^{cd}R^{ef}e^g + \alpha_3\varepsilon_{abcdefg} \times \left(l^4 R^{ab}R^{cd}R^{ef}h^{(g,1)} + 3l^4 R^{ab}R^{cd}\mathfrak{R}^{(ef,1)}e^g + l^2 R^{ab}R^{cd}e^e e^f e^g\right) \\ &+ \alpha_5\varepsilon_{abcdefg}\left(l^4 R^{ab}R^{cd}R^{ef}h^{(g,2)} + 3l^4 R^{ab}\mathfrak{R}^{(cd,1)}\mathfrak{R}^{(ef,1)}e^g + 3l^4 R^{ab}R^{cd}\mathfrak{R}^{(ef,2)}e^g \right. \\ &+ 3l^4 R^{ab}R^{cd}\mathfrak{R}^{(ef,1)}h^{(g,1)} + 2l^2 R^{ab}\mathfrak{R}^{(cd,1)}e^e e^f e^g \\ &\left. + 3l^2 R^{ab}R^{cd}e^e e^f h^{(g,1)} + \frac{3}{5}R^{ab}e^c e^d e^e e^f e^g\right) \\ &+ \alpha_{0\{2,2\}}l^5\left[\left(R^a_{\ b}R^b_{\ a}\right)L^{\text{Lorentz}}_3\right]\end{aligned}$$

$$
\begin{aligned}
&+\alpha_{2\{2,2\}}l^5\Big[\left(R^a_{\ b}R^b_{\ a}\right)\left(L_3^{\text{Lorentz}}\left(k^{(1)}\right)+\frac{2}{l^2}e_cT^c\right)\\
&+2\left(R^a_{\ b}\mathfrak{R}^{b(1)}_{\ a}\right)L_3^{\text{Lorentz}}\\
&+\frac{2}{l^2}\left(T^aT_a-R^{ab}e_ae_b\right)L_3^{\text{Lorentz}}\Big]\\
&+\alpha_{4\{2,2\}}l^5\Big[\left(R^a_{\ b}R^b_{\ a}\right)\Big(L_3^{\text{Lorentz}}\left(k^{(2)}\right)\\
&+L_3^{\text{Lorentz}}\left(k^{(1)}k^{(1)}\right)+\frac{2}{l^2}e_c\mathfrak{T}^{c(1)}+\frac{2}{l^2}h_c^{(1)}T^c\Big)\\
&+2\left(R^a_{\ b}\mathfrak{R}^{b(1)}_{\ a}\right)\left(L_3^{\text{Lorentz}}\left(k^{(1)}\right)+\frac{2}{l^2}e_cT^c\right)\\
&+\left(\mathfrak{R}^{a(1)}_{\ b}\mathfrak{R}^{b(1)}_{\ a}\right)L_3^{\text{Lorentz}}\\
&+2\left(R^a_{\ b}\mathfrak{R}^{b(2)}_{\ a}\right)L_3^{\text{Lorentz}}+\frac{2}{l^2}\left(T^aT_a-R^{ab}e_ae_b\right)\\
&\times\left(L_3^{\text{Lorentz}}\left(k^{(1)}\right)+\frac{2}{l^2}e_cT^c\right)\\
&+\frac{2}{l^2}\left(2T^a\mathfrak{T}^{(1)}_a-2R^{ab}e_ah_b^{(1)}-\mathfrak{R}^{(ab,1)}e_ae_b\right)L_3^{\text{Lorentz}}\Big]\\
&+\alpha_{0\{4\}}l^5\left[L_7^{\text{Lorentz}}\right]+\alpha_{2\{4\}}l^5\Big[L_7^{\text{Lorentz}}\left(k^{(1)}\right)\\
&+\frac{1}{l^2}4T_aR^a_{\ b}R^b_{\ c}e^c\Big]\\
&+\alpha_{4\{4\}}l^5\Big[L_7^{\text{Lorentz}}\left(k^{(2)}\right)+L_7^{\text{Lorentz}}\left(k^{(1)}k^{(1)}\right)\\
&+\frac{4}{l^2}\Big(T_aR^a_{\ b}R^b_{\ c}h^{(c,1)}+\mathfrak{T}^{(1)}_aR^a_{\ b}R^b_{\ c}e^c\\
&+T_aR^a_{\ b}\mathfrak{R}^{b(1)}_{\ c}e^c+T_a\mathfrak{R}^{a(1)}_{\ b}R^b_{\ c}e^c\Big)\\
&+\frac{1}{l^4}\left[2\left(R^{ab}e_ae_b+T^aT_a\right)T^ce_c\right]\Big].
\end{aligned}
\tag{90}
$$

The Lagrangian (90) is split into nine independent pieces, each one proportional to α_1, α_3, α_5, $\alpha_{0\{2,2\}}$, $\alpha_{2\{2,2\}}$, $\alpha_{4\{2,2\}}$, $\alpha_{0\{4\}}$, $\alpha_{2\{4\}}$, and $\alpha_{4\{4\}}$. The term proportional to α_1 corresponds to the Chern–Simons Lagrangian for the ISO(6, 1) group. The Hilbert–Einstein term $\varepsilon_{abcdefg}R^{ab}e^ce^de^ee^fe^g$ appears in the term proportional to α_5.

Varying the Lagrangian (90) for the case $\alpha_{0\{2,2\}}=\alpha_{2\{2,2\}}=\alpha_{4\{2,2\}}=\alpha_{0\{4\}}=\alpha_{2\{4\}}=\alpha_{4\{4\}}=0$, we have

$$
\begin{aligned}
\delta L_{CS\,(6+1)}^{\mathcal{M}_7}&=\varepsilon_{abcdefg}\Big(\boldsymbol{\alpha}_1l^4R^{ab}R^{cd}R^{ef}+3\boldsymbol{\alpha}_3l^4R^{ab}R^{cd}\mathfrak{R}^{(ef,1)}\\
&+3\boldsymbol{\alpha}_3l^2R^{ab}R^{cd}e^ee^f\\
&+3\boldsymbol{\alpha}_5l^4R^{ab}\mathfrak{R}^{(cd,1)}\mathfrak{R}^{(ef,1)}+3\boldsymbol{\alpha}_5l^4R^{ab}R^{cd}\mathfrak{R}^{(ef,2)}\\
&+6\boldsymbol{\alpha}_5l^2R^{ab}\mathfrak{R}^{(cd,1)}e^ee^f\\
&+6\boldsymbol{\alpha}_5l^2R^{ab}R^{cd}e^eh^{(f,1)}+3\boldsymbol{\alpha}_5R^{ab}e^ce^de^ee^f\Big)\delta e^g\\
&+\varepsilon_{abcdefg}\Big(\boldsymbol{\alpha}_3l^4R^{ab}R^{cd}R^{ef}+3\boldsymbol{\alpha}_5l^4R^{ab}R^{cd}\mathfrak{R}^{(ef,1)}\\
&+3\boldsymbol{\alpha}_5l^2R^{ab}R^{cd}e^ee^f\Big)\delta h^{(g,1)}\\
&+\varepsilon_{abcdefg}\left(\boldsymbol{\alpha}_5l^4R^{ab}R^{cd}R^{ef}\right)\delta h^{(g,2)}\\
&+\varepsilon_{abcdefg}\delta\omega^{ab}\Big(3\boldsymbol{\alpha}_1l^4R^{cd}R^{ef}T^g\\
&+3\boldsymbol{\alpha}_3l^4R^{cd}R^{ef}\mathfrak{T}^{(g,1)}+6\boldsymbol{\alpha}_3l^4R^{cd}\mathfrak{R}^{(ef,1)}T^g\\
&+6\boldsymbol{\alpha}_3l^2R^{cd}e^ee^fT^g+3\boldsymbol{\alpha}_5l^4R^{cd}R^{ef}\mathfrak{T}^{(g,2)}\\
&+3\boldsymbol{\alpha}_5l^4\mathfrak{R}^{(cd,1)}\mathfrak{R}^{(ef,1)}T^g\\
&+2\boldsymbol{\alpha}_5l^4k^{(cd,1)}k^{(ef,1)}T^g+6\boldsymbol{\alpha}_5l^4R^{cd}\mathfrak{R}^{(ef,2)}T^g\\
&+6\boldsymbol{\alpha}_5l^4R^{cd}\mathfrak{R}^{(ef,1)}\left(\mathfrak{T}^{(g,1)}+k_h^{g(1)}e^h\right)\\
&+6\boldsymbol{\alpha}_5l^2\mathfrak{R}^{(cd,1)}e^ee^fT^g+3\boldsymbol{\alpha}_5l^2R^{cd}e^ee^f\mathfrak{T}^{(g,1)}\\
&+3\boldsymbol{\alpha}_5e^ce^de^ee^fT^g\Big)\\
&+\varepsilon_{abcdefg}\delta k^{(ab,1)}\Big(3\boldsymbol{\alpha}_3l^4R^{cd}R^{ef}T^g+6\boldsymbol{\alpha}_5l^4R^{cd}\mathfrak{R}^{(ef,1)}T^g\\
&+3\boldsymbol{\alpha}_5l^4R^{cd}R^{ef}\left(\mathfrak{T}^{(g,1)}+k_h^{g(1)}e^h\right)+2\boldsymbol{\alpha}_5l^2R^{cd}e^ee^fT^g\Big)\\
&+\varepsilon_{abcdefg}\delta k^{(ab,2)}\left(3\boldsymbol{\alpha}_5l^4R^{cd}R^{ef}T^g\right).
\end{aligned}
$$

In the event that (i) α_1 and α_3 are zero, (ii) the torsionlessness condition is imposed, and (iii) $k^{(ab,1)}=0$, $k^{(ab,2)}=0$, $h^{(a,1)}=0$, $h^{(a,2)}=0$, it is found that

$$
\begin{aligned}
\delta L_{CS\,(6+1)}^{\mathcal{M}_7}&=\varepsilon_{abcdefg}\left(3\boldsymbol{\alpha}_5R^{ab}e^ce^de^ee^f\right)\delta e^g\\
&+\varepsilon_{abcdefg}\left(3\boldsymbol{\alpha}_5l^2R^{ab}R^{cd}e^ee^f\right)\delta h^{(g,1)}\\
&+\varepsilon_{abcdefg}\left(\boldsymbol{\alpha}_5l^4R^{ab}R^{cd}R^{ef}\right)\delta h^{(g,2)}.
\end{aligned}
\tag{91}
$$

Here we see that in the limit $l\to 0$ the Lagrangian leads to the Hilbert–Einstein term

$$
L_{CS\,(6+1)}^{\mathcal{M}_7}=\frac{3}{5}\boldsymbol{\alpha}_5\varepsilon_{abcdefg}R^{ab}e^ce^de^ee^fe^g,
\tag{92}
$$

and the condition $\delta L_{CS\,(6+1)}^{\mathcal{M}_7}=0$ leads to the Einstein equations,

$$
\begin{aligned}
\delta L_{CS\,(6+1)}^{\mathcal{M}_7}&=\varepsilon_{abcdefg}\left(3\boldsymbol{\alpha}_5R^{ab}e^ce^de^ee^f\right)\delta e^g\\
&+\varepsilon_{abcdefg}\delta\omega^{ab}\left(3\boldsymbol{\alpha}_5e^ce^de^ee^fT^g\right).
\end{aligned}
\tag{93}
$$

The results show that the $(2p+1)$-dimensional Chern–Simons actions invariant under the algebra $\mathcal{M}_{2m+1}$ does not always lead to the action of general relativity. Indeed, for certain values of m it is impossible to obtain the Hilbert–Einstein term in the $(2p+1)$-dimensional Chern–Simons Lagrangian invariant under $\mathcal{M}_{2m+1}$. This is because to obtain the Hilbert–Einstein term, the presence is necessary of the $\langle \boldsymbol{J}_{a_1a_2}\boldsymbol{Z}_{a_3a_4}\cdots Z_{a_{2p-1}a_{2p}}\boldsymbol{P}_{2p+1}\rangle$ component of the invariant tensor, which is given by

$$
\begin{aligned}
&\langle \boldsymbol{J}_{a_1a_2}\boldsymbol{Z}_{a_3a_4}\cdots\boldsymbol{Z}_{a_{2p-1}a_{2p}}\boldsymbol{P}_{a_{2p+1}}\rangle_{\mathcal{M}_{2m+1}}\\
&=\begin{cases} l^{2p-1}\alpha_{2p-1}\langle J_{a_1a_2}\cdots J_{a_{2p-1}a_{2p}}P_{a_{2p+1}}\rangle_{AdS}, & \text{if } m\geq p\\ 0, & \text{if } m<p.\end{cases}
\end{aligned}
\tag{94}
$$

This observation leads us to state the following theorem.

Theorem 4 *Let $\mathcal{M}_{2m+1}$ be the Maxwell type algebra, which is obtained from the AdS algebra by a resonant reduced $S_E^{(2m-1)}$-expansion. If $L_{CS(2p+1)}^{\mathcal{M}_{2m+1}}$ is a Chern–Simons Lagrangian $(2p+1)$-dimensional invariant under the $\mathcal{M}_{2m+1}$-algebra, then the $(2p+1)$-dimensional Chern–Simons Lagrangian leads to the Hilbert–Einstein Lagrangian in a certain limit of the coupling constant l, if and only if $m \geq p$.*

The following table shows a set of Chern–Simons Lagrangians $L_{CS(2p+1)}^{\mathcal{M}_{2m+1}}$, invariant under the Lie algebra $\mathcal{M}_{2m+1}$, that flow into the general relativity Lagrangian in a certain limit:

$\mathcal{M}_3$	$L_{CS\,(3)}^{\mathcal{M}_3}$						
$\mathcal{M}_5$	$L_{CS\,(3)}^{\mathcal{M}_5}$	$L_{CS\,(5)}^{\mathcal{M}_5}$					
$\mathcal{M}_7$	$L_{CS\,(3)}^{\mathcal{M}_7}$	$L_{CS\,(5)}^{\mathcal{M}_7}$	$L_{CS\,(7)}^{\mathcal{M}_7}$				
$\vdots$		$\vdots$					
$\vdots$		$\vdots$					
$\mathcal{M}_{2n-1}$	$L_{CS\,(3)}^{\mathcal{M}_{2n-1}}$	$L_{CS\,(5)}^{\mathcal{M}_{2n-1}}$	$L_{CS\,(7)}^{\mathcal{M}_{2n-1}}$	$\cdots$	$\cdots$	$L_{CS\,(2n-1)}^{\mathcal{M}_{2n-1}}$	
$\mathcal{M}_{2n+1}$	$L_{CS\,(3)}^{\mathcal{M}_{2n+1}}$	$L_{CS\,(5)}^{\mathcal{M}_{2n+1}}$	$L_{CS\,(7)}^{\mathcal{M}_{2n+1}}$	$\cdots$	$\cdots$	$L_{CS\,(2n-1)}^{\mathcal{M}_{2n+1}}$	$L_{CS\,(2n+1)}^{\mathcal{M}_{2n+1}}$

(95)

It is interesting to note that for each dimension D of spacetime, we have the Lagrangian $L_{CS(D)}$ invariant under the algebra $\mathcal{M}_{2n+1}$ that contains all other D-dimensional Lagrangian with values in an algebra $\mathcal{M}_{2m+1}$ with $m < n$. So it is always possible to obtain an action of a lower algebra from the appropriate fields.

4 Born–Infeld Lagrangians invariant under the subalgebra $\mathfrak{L}^{\mathcal{M}}$

In this section is shown that the even-dimensional Hilbert–Einstein Lagrangian can be obtained from a Born–Infeld Lagrangian in $(2p)$ dimensions invariant under the subalgebra $L^{\mathcal{M}_{2m}}$ of the algebra M_{2m}, if and only if $m \geq p$. However, this is not possible when $m < p$ for a Born–Infeld Lagrangian in $(2p)$ dimensions invariant under the subalgebra $L^{\mathcal{M}_{2m}}$.

4.1 Born–Infeld Lagrangian in $D = 4$ invariant under $\mathfrak{L}^{\mathcal{M}_5}$

Following the definitions of Ref. [12], let us consider the S-expansion of the Lie algebra $SO\,(3,1)$ using as a semigroup the sub-semigroup $S_0^{(3)} = \{\lambda_0, \lambda_2, \lambda_4\}$ of the semigroup $S_{\mathrm{E}}^{(3)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4\}$. After performing its $0_S(=\lambda_4)$-reduction, one finds a new Lie algebra, call it $L^{\mathcal{M}_5}$, which is a subalgebra of the so-called M_5 algebra, whose generators $J_{ab} = \lambda_0 \tilde{J}_{ab}$ and $Z_{ab} = \lambda_2 \tilde{J}_{ab}$ satisfy the commutation relationships

$$\begin{aligned} \left[J_{ab}, J_{cd}\right] &= \eta_{cb} J_{ad} - \eta_{ca} J_{bd} + \eta_{db} J_{ca} - \eta_{da} J_{cb}, \\ \left[J_{ab}, Z_{cd}\right] &= \eta_{cb} Z_{ad} - \eta_{ca} Z_{bd} + \eta_{db} Z_{ca} - \eta_{da} Z_{cb}, \\ \left[Z_{ab}, Z_{cd}\right] &= 0. \end{aligned} \tag{96}$$

In order to write down a Born–Infeld, we start from the 2-form of the $L^{\mathcal{M}_5}$ curvature F

$$F = \frac{1}{2} R^{ab} J_{ab} + \frac{1}{2}\left(D_\omega k^{ab} + \frac{1}{l^2} e^a e^b\right) Z_{ab}. \tag{97}$$

Using Theorem VII.2 of Ref. [12], it is possible to show that the only non-vanishing components of an invariant tensor for the $\mathfrak{L}^{\mathcal{M}_5}$ algebra are given by

$$\langle J_{ab} J_{cd} \rangle_{\mathfrak{L}^{\mathcal{M}_5}} = \alpha_0 l^2 \varepsilon_{abcd}, \tag{98}$$

$$\langle J_{ab} Z_{cd} \rangle_{\mathfrak{L}^{\mathcal{M}_5}} = \alpha_2 l^2 \varepsilon_{abcd} \tag{99}$$

where α_0 and α_2 are arbitrary independent constants of dimensions $[\text{length}]^{-2}$.

Using the dual procedure of S-expansion in terms of the Maurer–Cartan forms [13], we find that the 4-dimensional Born–Infeld Lagrangian invariant under the $\mathfrak{L}^{\mathcal{M}_5}$ algebra is given by [10]

$$\begin{aligned} L_{BI(4)}^{\mathfrak{L}^{\mathcal{M}_5}} &= \frac{\alpha_0}{4} \epsilon_{abcd} l^2 R^{ab} R^{cd} \\ &+ \frac{\alpha_2}{2} \epsilon_{abcd} \left(R^{ab} e^c e^d + l^2 D_\omega k^{ab} R^{cd}\right). \end{aligned} \tag{100}$$

Here we can see that the Lagrangian (100) is split into two independent pieces, one proportional to α_0 and the other to α_2. The term proportional to α_0 corresponds to the Euler invariant. The piece proportional to α_2 contains the Hilbert–Einstein term $\varepsilon_{abcd} R^{ab} e^c e^d$ plus a boundary term which contains, besides the usual curvature R^{ab}, a bosonic matter field k^{ab}.

Unlike the Born–Infeld Lagrangian the coupling constant l^2 does not appear explicitly in the Hilbert–Einstein term but accompanies the remaining elements of the Lagrangian. This allows one to recover the 4-dimensional Hilbert–Einstein Lagrangian in the limit where l equals zero.

The variation of the Lagrangian, modulo boundary terms, is given by

$$\delta L_{BI(4)}^{\mathfrak{L}^{\mathcal{M}_5}} = \varepsilon_{abcd}\left(\alpha_2 R^{ab}e^c\right)\delta e^d + \varepsilon_{abcd}\delta\omega^{ab}\left(\alpha_2 T^c e^d + \alpha_2 k_e^c R^{ed}\right), \tag{101}$$

from which we see that to recover the field equations of general relativity it is not necessary to impose the limit $l = 0$. $\delta L_{BI(4)}^{\mathfrak{L}^{\mathcal{M}}} = 0$ leads to the dynamics of relativity when considering the case of a solution without matter ($k^{ab} = 0$). This is possible only in 4 dimensions. However, to recover the field equations of general relativity in dimensions greater than 4, it is necessary to take the limit of the coupling constant l.

4.2 Born–Infeld Lagrangian in $D = 4$ invariant under $\mathfrak{L}^{\mathcal{M}_7}$ algebra

Now, we consider the Born–Infeld Lagrangian in $D = 4$ invariant under $\mathfrak{L}^{\mathcal{M}_7}$ algebra whose generators satisfy the following commutation relations:

$$\left[J_{ab}, J_{cd}\right] = \eta_{cb}J_{ad} - \eta_{ca}J_{bd} + \eta_{db}J_{ca} - \eta_{da}J_{cb}, \tag{102}$$

$$\left[J_{ab}, Z_{cd}^{(1)}\right] = \eta_{cb}Z_{ad}^{(1)} - \eta_{ca}Z_{bd}^{(1)} + \eta_{db}Z_{ca}^{(1)} - \eta_{da}Z_{cb}^{(1)}, \tag{103}$$

$$\left[J_{ab}, Z_{cd}^{(2)}\right] = \eta_{cb}Z_{ad}^{(2)} - \eta_{ca}Z_{bd}^{(2)} + \eta_{db}Z_{ca}^{(2)} - \eta_{da}Z_{cb}^{(2)}, \tag{104}$$

$$\left[Z_{ab,}^{(1)} Z_{cd}^{(1)}\right] = \eta_{cb}Z_{ad}^{(2)} - \eta_{ca}Z_{bd}^{(2)} + \eta_{db}Z_{ca}^{(2)} - \eta_{da}Z_{cb}^{(2)}, \tag{105}$$

$$\left[Z_{ab,}^{(1)} Z_{cd}^{(2)}\right] = 0 = \left[Z_{ab,}^{(2)} Z_{cd}^{(2)}\right]. \tag{106}$$

The curvature 2-form $S_0^{(3)}$-expanded and reduced is

$$F = \frac{1}{2}R^{ab}J_{ab} + \frac{1}{l}T^a P_a + \frac{1}{2}\left(D_\omega k^{(ab,1)} + \frac{1}{l^2}e^a e^b\right)Z_{ab}^{(1)} + \frac{1}{2}\left(D_\omega k^{(ab,2)} + k_c^{a(1)}k^{cb(1)} + \frac{1}{l^2}\left[e^a h^{(b,1)} + h^{(a,1)}e^b\right]\right)Z_{ab}^{(2)}. \tag{107}$$

Using theorem VII.2 of Ref. [12], it is possible to show that the only non-vanishing components of an invariant tensor for the $\mathfrak{L}^{\mathcal{M}_7}$ algebra are given by

$$\langle J_{ab}J_{cd}\rangle_{\mathfrak{L}^{\mathcal{M}_7}} = \alpha_0\varepsilon_{abcd}, \tag{108}$$

$$\left\langle J_{ab}Z_{cd}^{(1)}\right\rangle_{\mathfrak{L}^{\mathcal{M}_7}} = \alpha_2\varepsilon_{abcd}, \tag{109}$$

$$\left\langle J_{ab}Z_{cd}^{(2)}\right\rangle_{\mathfrak{L}^{\mathcal{M}_7}} = \left\langle Z_{ab}^{(1)}Z_{cd}^{(1)}\right\rangle_{\mathfrak{L}^{\mathcal{M}_7}} = \alpha_4\varepsilon_{abcd}, \tag{110}$$

where α_0, α_2 and α_4 are arbitrary dimensionless independent constants.

Using the dual procedure of S-expansion in terms of the Maurer–Cartan forms [13], we find that the 4-dimensional Born–Infeld Lagrangian invariant under the $\mathfrak{L}^{\mathcal{M}_7}$ algebra is given by

$$L_{BI(4)}^{\mathfrak{L}^{\mathcal{M}_7}} = \frac{\alpha_0}{4}\epsilon_{abcd}R^{ab}R^{cd} + \frac{\alpha_2}{2}\epsilon_{abcd}\left(\mathfrak{R}^{(ab,1)}R^{cd} + \frac{1}{l^2}R^{ab}e^c e^d\right) + \frac{\alpha_4}{4}\epsilon_{abcd}\left(\mathfrak{R}^{(ab,1)}\mathfrak{R}^{(cd,1)} + \mathfrak{R}^{(ab,2)}R^{cd} + \frac{2}{l^2}\mathfrak{R}^{(ab,1)}e^c e^d + \frac{4}{l^2}R^{ab}h^{(c,1)}e^d + \frac{1}{l^4}e^a e^b e^c e^d\right), \tag{111}$$

where

$$\mathfrak{R}^{(ab,1)} = D_\omega k^{(ab,1)}, \tag{112}$$

$$\mathfrak{R}^{(ab,2)} = D_\omega k^{(ab,2)} + k_c^{a(1)}k^{cb(1)}. \tag{113}$$

The Lagrangian (111) is split into three independent pieces, each one proportional to α_0, α_2, and α_4 respectively. The term proportional to α_0 corresponds to the Euler invariant. The piece proportional to α_2 contains the Hilbert–Einstein term $\varepsilon_{abcd}R^{ab}e^c e^d$ plus a boundary term which contains, besides the usual curvature R^{ab}, a bosonic matter field $k^{(ab,1)}$.

The variation of the Lagrangian, modulo boundary terms, is given by

$$\begin{aligned}\delta L_{BI(4)}^{\mathfrak{L}^{\mathcal{M}_7}} &= \varepsilon_{abcd}\left(\frac{\alpha_2}{l^2}R^{ab}e^c + \frac{\alpha_4}{l^2}\mathfrak{R}^{(ab,1)}e^c + \frac{\alpha_4}{l^2}R^{ab}h^{(c,1)} + \frac{\alpha_4}{l^4}e^a e^b e^c\right)\delta e^d \\ &+ \varepsilon_{abcd}\left(\frac{\alpha_4}{l^2}R^{ab}e^c\right)\delta h^{(d,1)} + \varepsilon_{abcd}\delta\omega^{ab}\Big(\alpha_2 k_e^{c,(1)}R^{de} + \frac{\alpha_2}{l^2}T^c e^d + \frac{\alpha_4}{2}k_e^{c,(2)}R^{de} \\ &+ \frac{\alpha_4}{l^2}\left(D_\omega h^{(c,1)}e^d - h^{(c,1)}T^d\right)\Big) \\ &+ \varepsilon_{acde}\delta\omega^{ab}\left(\alpha_2 k_b^{c,(1)}R^{de} + \alpha_4 k_b^{c,(1)}\mathfrak{R}^{(de,1)} + \frac{\alpha_4}{2}k_b^{c,(2)}R^{de} + \frac{\alpha_4}{l^2}k_b^{c,(1)}e^d e^e\right) \\ &+ \varepsilon_{abcd}\delta k^{(ab,1)}\left(\frac{\alpha_4}{l^2}T^c e^d\right) \\ &+ \varepsilon_{acde}\delta k^{(ab,1)}\left(\alpha_4\omega_b^c\mathfrak{R}^{(de,1)} + \frac{\alpha_4}{2}k_b^{c,(1)}R^{de}\right)\end{aligned} \tag{114}$$

where $\mathfrak{T}^{(a,1)} = D_\omega h^{(a,1)} + k_c^{a(1)}e^c$. If we consider the case where $k^{(ab,1)} = h^{(a,1)} = 0$, we have

$$\delta L_{BI(4)}^{\mathfrak{L}^{\mathcal{M}_7}} = \varepsilon_{abcd}\left(\frac{\alpha_2}{l^2}R^{ab}e^c + \frac{\alpha_4}{l^4}e^a e^b e^c\right)\delta e^d + \varepsilon_{abcd}\left(\frac{\alpha_4}{l^2}R^{ab}e^c\right)\delta h^{(d,1)} + \varepsilon_{abcd}\delta\omega^{ab}\left(\frac{\alpha_2}{l^2}T^c e^d\right) + \varepsilon_{abcd}\delta k^{(ab,1)}\left(\frac{\alpha_4}{l^2}T^c e^d\right), \tag{115}$$

from which

$$\varepsilon_{abcd}R^{ab}e^c = 0, \tag{116}$$

$$\varepsilon_{abcd}T^c e^d = 0. \tag{117}$$

That is, we have obtained the Hilbert–Einstein dynamics in a vacuum without any restriction on the coupling constant l.

4.3 Born–Infeld Lagrangian in $D = 4$ invariant under $\mathfrak{L}^{\mathcal{M}_{2n+1}}$

The generators of the $\mathcal{M}_{2n+1}$ algebra satisfy the commutation relation (46–35). The corresponding 1-form gauge connection A and the curvature 2-form, the $\mathcal{M}_{2n+1}$-valued $F = dA + A^2$ are given in (46) and (47). The generators of the $\mathfrak{L}^{\mathcal{M}_{2n+1}}$ algebra satisfy the following commutation relation:

$$\begin{aligned}\left[J_{ab}, J_{cd}\right] &= \eta_{cb}J_{ad} - \eta_{ca}J_{bd} + \eta_{db}J_{ca} - \eta_{da}J_{cb}\\ \left[J_{ab}, Z_{cd}^{(i)}\right] &= \eta_{cb}Z_{ad}^{(i)} - \eta_{ca}Z_{bd}^{(i)} + \eta_{db}Z_{ca}^{(i)} - \eta_{da}Z_{cb}^{(i)},\\ \left[Z_{ab}^{(i)}, Z_{cd}^{(j)}\right] &= \eta_{cb}Z_{ad}^{(i+j)} - \eta_{ca}Z_{bd}^{(i+j)} + \eta_{db}Z_{ca}^{(i+j)} - \eta_{da}Z_{cb}^{(i+j)}.\end{aligned} \tag{118}$$

We bear in mind that the nonzero components of the invariant tensor are given by

$$\left\langle J_{(ab,2i)}J_{(cd,2j)}\right\rangle = \alpha_{2i+2j}\varepsilon_{abcd}, \tag{119}$$

where α_{2i+2j} are arbitrary independent dimensionless constants and where we have defined

$$J_{ab} = \lambda_0\tilde{J}_{ab} = J_{(ab,0)}$$
$$Z_{ab}^{(i)} = \lambda_{2i}\tilde{J}_{ab} = J_{(ab,2i)}$$

with $i = 1, \ldots, n-1$.

Using the same procedure as used in the previous cases, we found that the Lagrangian $L_{BI(4)}^{\mathfrak{L}^{\mathcal{M}_{2n+1}}}$ is given by

$$L_{BI(4)}^{\mathfrak{L}^{\mathcal{M}_{2n+1}}} = \frac{\alpha_{2i+2j}}{4}\varepsilon_{abcd}F^{(ab,2i)}F^{(cd,2j)}. \tag{120}$$

Varying the Lagrangian and considering the case without matter, $(k^{(ab,i)} = h^{(a,j)} = 0)$, we have

$$\delta L_{BI(4)}^{\mathfrak{L}^{\mathcal{M}_{2n+1}}} = \varepsilon_{abcd}\left(\frac{\alpha_2}{l^2}R^{ab}e^c + \frac{\alpha_4}{l^4}e^a e^b e^c\right)\delta e^d + \varepsilon_{abcd}\left(\frac{\alpha_{i+1}}{l^2}R^{ab}e^c\right)\delta h^{(d,i)} \tag{121}$$

$$+ \varepsilon_{abcd}\delta\omega^{ab}\left(\frac{\alpha_2}{l^2}T^c e^d\right) + \varepsilon_{abcd}\delta k^{(ab,i)}\left(\frac{\alpha_{i+1}}{l^2}T^c e^d\right) \tag{122}$$

and the equations leading to the field equations of general relativity

$$\varepsilon_{abcd}R^{ab}e^c = 0, \tag{123}$$

$$\varepsilon_{abcd}T^c e^d = 0. \tag{124}$$

4.4 Born–Infeld Lagrangian in $D = 6$ invariant under $\mathfrak{L}^{\mathcal{M}_{2n}}$ algebra

It should be noted that the $L^{\mathcal{M}_{2n+1}}$ algebra has the property of being identical to the $L^{\mathcal{M}_{2n}}$ algebra. However, they have different origins: The $L^{\mathcal{M}_{2n+1}}$ algebra corresponds to a reduced $S_0^{(2n-1)}$-expansion of the Lorentz algebra, as we have seen previously, and the $L^{\mathcal{M}_{2n}}$ algebra corresponds to a reduced $S_0^{(2n-2)}$-expansion of the Lorentz algebra, where the semigroup $S_0^{(2n-2)}$ is a sub-semigroup of the semigroup $S_E^{(2n-2)} = \{\lambda_i\}_{i=0}^{2n-1}$.

It is also interesting to note that the $L^{\mathcal{M}_{2n}}$ algebra can be used to construct different even-dimensional Born–Infeld type Lagrangians. For example, if we consider a reduced $S_0^{(4)}$-expansion of the Lorentz algebra $SO(3,1)$, the $L^{\mathcal{M}_6}$ algebra in $D = 4$ dimensions is obtained, and if we consider a reduced $S_0^{(4)}$-expansion of the Lorentz algebra $SO(5,1)$ then we get the $L^{\mathcal{M}_6}$ algebra in $D = 6$ dimensions. In this way, the Lagrangians $L_{BI(4)}^{\mathfrak{L}^{\mathcal{M}_6}}$ and $L_{BI(6)}^{\mathfrak{L}^{\mathcal{M}_6}}$, are invariant under the same algebra $L^{\mathcal{M}_6}$, but the indices in the generators J_{ab} run over four and six values, respectively.

These considerations allow the construction of gravitational theories in every even number of dimensions. However, as discussed below, only in some dimensions it is possible to obtain general relativity as a weak coupling constant limit of Born–Infeld theory.

4.4.1 Born–Infeld Lagrangian in $D = 6$ invariant under $\mathfrak{L}^{\mathcal{M}_4}$

The Born–Infeld Lagrangian invariant under the Lorentz algebra is given by

$$L_{BI}^{(6)} = \frac{\kappa}{6}\epsilon_{abcdef}\left(R^{ab}R^{cd}R^{ef} + \frac{3}{l^2}R^{ab}R^{cd}e^e e^f + \frac{3}{l^4}R^{ab}e^c e^d e^e e^f + \frac{1}{l^6}e^a e^b e^c e^d e^e e^f\right). \tag{125}$$

Following the definitions of Ref. [12], let us consider the S-expansion of the Lie algebra $SO(5,1)$ using $S_0^{(2)} =$

$\{\lambda_0, \lambda_2, \lambda_3\}$ as a sub-semigroup of $S_E^{(2)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$. After performing its 0_S-reduction, one finds the $L^{\mathcal{M}_4}$ algebra which corresponds to a subalgebra of the M_4 algebra. The new algebra is generated by $\{J_{ab}, Z_{ab}\}$, where these new generators can be written as

$$\lambda_0 \otimes \tilde{J}_{ab} = J_{ab}, \tag{126}$$
$$\lambda_2 \otimes \tilde{J}_{ab} = Z_{ab}. \tag{127}$$

In this case, $\tilde{J}_{ab}$ corresponds to the original generator of $SO(5,1)$ and the λ_α belong to a finite abelian semigroup $S_0^{(2)}$. Using the invariant tensors

$$\langle J_{ab} J_{cd} J_{ef} \rangle_{\mathfrak{L}^{\mathcal{M}_4}} = \frac{4}{3}\alpha_0 \varepsilon_{abcdef}, \tag{128}$$
$$\langle J_{ab} J_{cd} Z_{ef} \rangle_{\mathfrak{L}^{\mathcal{M}_4}} = \frac{4}{3}\alpha_2 \varepsilon_{abcdef}, \tag{129}$$

we find that the 6-dimensional Born–Infeld Lagrangian invariant under the $\mathfrak{L}^{\mathcal{M}_4}$ algebra is given by

$$L_{BI-(6)}^{\mathfrak{L}^{\mathcal{M}_4}} = \frac{\alpha_0}{6}\epsilon_{abcdef} R^{ab} R^{cd} R^{ef} + \frac{\alpha_2}{2}\epsilon_{abcdef} \times \left(\mathfrak{R}^{ab} R^{cd} R^{ef} + \frac{1}{l^2} R^{ab} R^{cd} e^e e^f\right) \tag{130}$$

where $\mathfrak{R}^{ab} = D_\omega k^{ab}$.

Note that in this case the S-expansion procedure caused the Hilbert–Einstein term to disappear. This means that the case of a 6-dimensional Born–Infeld Lagrangian invariant under $\mathfrak{L}^{\mathcal{M}_4}$ does not lead to general relativity in any limit.

4.4.2 *Born–Infeld Lagrangian in $D = 6$ invariant under $\mathfrak{L}^{\mathcal{M}_6}$ algebra*

In this case the curvature 2-form is given by [10]

$$F = \frac{1}{2} R^{ab} J_{ab} + \frac{1}{2}\left(D_\omega k^{(ab,1)} + \frac{1}{l^2} e^a e^b\right) Z_{ab}^{(1)} + \frac{1}{2}\left(D_\omega k^{(ab,2)} + k_c^{a(1)} k^{cb(1)} + \frac{1}{l^2}\left[e^a h^{(b,1)} + h^{(a,1)} e^b\right]\right) Z_{ab}^{(2)}. \tag{131}$$

Using the invariant tensors

$$\langle J_{ab} J_{cd} J_{ef} \rangle_{\mathfrak{L}^{\mathcal{M}_6}} = \frac{4}{3} l^4 \alpha_0 \varepsilon_{abcdef}, \tag{132}$$
$$\langle J_{ab} J_{cd} Z_{ef} \rangle_{\mathfrak{L}^{\mathcal{M}_6}} = \frac{4}{3} l^4 \alpha_2 \varepsilon_{abcdef}, \tag{133}$$
$$\left\langle J_{ab} J_{cd} Z_{ef}^{(2)} \right\rangle_{\mathfrak{L}^{\mathcal{M}_6}} = \left\langle J_{ab} Z_{cd}^{(1)} Z_{cd}^{(1)} \right\rangle_{\mathfrak{L}^{\mathcal{M}_6}} = \frac{4}{3} l^4 \alpha_4 \varepsilon_{abcdef},$$

we find that the 6-dimensional Born–Infeld Lagrangian invariant under $\mathfrak{L}^{\mathcal{M}_6}$ algebra is given by

$$L_{BI-(6)}^{\mathfrak{L}^{\mathcal{M}_6}} = \frac{\alpha_0}{6}\epsilon_{abcdef} l^4 R^{ab} R^{cd} R^{ef} + \frac{\alpha_2}{2}\epsilon_{abcdef} \times \left(l^4 \mathfrak{R}^{(ab,1)} R^{cd} R^{ef} + l^2 R^{ab} R^{cd} e^e e^f\right) + \frac{\alpha_4}{2}\epsilon_{abcdef}\left(l^4 \mathfrak{R}^{(ab,1)} \mathfrak{R}^{(cd,1)} R^{ef} + l^4 \mathfrak{R}^{(ab,2)} R^{cd} R^{ef} + 2R l^{4ab} R^{cd} h^{(e,1)} e^f + l^2 \mathfrak{R}^{(ab,1)} R^{cd} e^e e^f + R^{ab} e^c e^d e^e e^f\right).$$

Varying the Lagrangian and considering the case without matter, $k^{(ab,1)} = h^{(a,1)} = 0$, we have

$$\varepsilon_{abcdef} R^{ab} e^c e^d e^e = 0, \tag{134}$$
$$\varepsilon_{abcdef} T^c e^d e^e = 0. \tag{135}$$

which are the Einstein equations in vacuum. Note that if in the Lagrangian $L_{BI-(6)}^{\mathfrak{L}^{\mathcal{M}_6}}$ we take the limit $l = 0$, we obtain the Hilbert–Einstein term.

$$L_{BI-(6)}^{\mathfrak{L}^{\mathcal{M}_6}} = \frac{\alpha_4}{2}\epsilon_{abcdef} R^{ab} e^c e^d e^e e^f.$$

4.5 Born–Infeld Lagrangian in $D = 2n$ invariant under $\mathfrak{L}^{\mathcal{M}_{2n}}$

The generators of the algebra $\mathfrak{L}^{\mathcal{M}_{2n}}$ satisfy the following commutation relations:

$$\begin{aligned}
\left[J_{ab}, J_{cd}\right] &= \eta_{cb} J_{ad} - \eta_{ca} J_{bd} + \eta_{db} J_{ca} - \eta_{da} J_{cb} \\
\left[J_{ab}, Z_{cd}^{(i)}\right] &= \eta_{cb} Z_{ad}^{(i)} - \eta_{ca} Z_{bd}^{(i)} + \eta_{db} Z_{ca}^{(i)} - \eta_{da} Z_{cb}^{(i)} \\
\left[Z_{ab}^{(i)}, Z_{cd}^{(j)}\right] &= \eta_{cb} Z_{ad}^{(i+j)} - \eta_{ca} Z_{bd}^{(i+j)} + \eta_{db} Z_{ca}^{(i+j)} - \eta_{da} Z_{cb}^{(i+j)},
\end{aligned} \tag{136}$$

Theorem VII.2 of Ref. [9] allows us to see that the only nonzero components of the tensor invariant are given by

$$\langle J_{(a_1 a_2, i_1)} \cdots J_{(a_{2n-1} a_{2n}, i_n)} \rangle = \frac{2^{n-1} l^{2n-2}}{n} \alpha_j \delta_{i_1 + \cdots + i_n}^{j} \varepsilon_{a_1 \cdots a_{2n}}, \tag{137}$$

where $j = 0, \ldots, 2n-2$ and α_j are arbitrary independent constants of dimensions $[\text{length}]^{2-2n}$.

In this case, the curvature 2-form is given by

$$F = \sum_{k=0}^{n-1} \frac{1}{2} F^{(ab,2k)} J_{(ab,2k)} \tag{138}$$

where

$$F^{(ab,2k)} = d\omega^{(ab,2k)} + \eta_{cd}\omega^{(ac,2i)}\omega^{(db,2j)}\delta^k_{i+j} + \frac{1}{l^2}e^{(a,2i+1)}e^{(b,2j+1)}\delta^k_{i+j+1}. \quad (139)$$

Using the dual procedure of the S-expansion in terms of the Maurer–Cartan forms [13], we find that the $2n$-dimensional Born–Infeld Lagrangian invariant under the $\mathfrak{L}^{\mathcal{M}_{2n}}$ algebra is given by

$$L_{BI(2n)}^{\mathfrak{L}^{\mathcal{M}}_{2n}} = \sum_{k=1}^{n} l^{2k-2}\frac{1}{2n}\binom{n}{k}\alpha_j\delta^j_{i_1+\cdots+i_n}\delta^{i_{k+1}}_{p_1+q_1}\cdots\delta^{i_n}_{p_{n-k}+q_{n-k}} \varepsilon_{a_1\cdots a_{2n}}R^{(a_1a_2,i_1)}\cdots R^{(a_{2k-1}a_{2k},i_k)}e^{(a_{2k+1},p_1)} e^{(a_{2k+2},q_1)}\cdots e^{(a_{2n-1},p_{n-k})}e^{(a_{2n},q_{n-k})}. \quad (140)$$

In the $l\to 0$ limit, the only surviving term in (140) is given by $k=1$:

$$\begin{aligned} L_{BI(2n)}^{\mathfrak{L}^{\mathcal{M}}_{2n}}\Big|_{l=0} &= \frac{1}{2}\alpha_j\delta^j_{i+k_1+\cdots+k_{2n-2}}\varepsilon_{a_1\cdots a_{2n}}R^{(a_1a_2,i)}e^{(a_3,k_1)}\cdots e^{(a_{2n},k_{2n-2})}\\ &= \frac{1}{2}\alpha_j\delta^j_{2p+2q_1+1+\cdots+2_{q_{2n-2}}+1}\varepsilon_{a_1\cdots a_{2n}}R^{(a_1a_2,2p)} \times e^{(a_3,2q_1+1)}\cdots e^{\left(a_{2n},2_{q_{2n-2}}+1\right)}\\ &= \frac{1}{2}\alpha_j\delta^j_{2(p+q_1+\cdots+q_{2n-2})+2n-2}\varepsilon_{a_1\cdots a_{2n}}R^{(a_1a_2,2p)} \times e^{(a_3,2q_1+1)}\cdots e^{\left(a_{2n},2_{q_{2n-2}}+1\right)}. \end{aligned} \quad (141)$$

The only non-vanishing component of this expression occurs for $p=q_1=\cdots=q_{2n-2}=0$, namely

$$\begin{aligned} L_{BI(2n)}^{\mathfrak{L}^{\mathcal{M}}_{2n}}\Big|_{l=0} &= \frac{1}{2}\alpha_{2n-2}\varepsilon_{a_1\cdots a_{2n}}R^{(a_1a_2,0)}e^{(a_3,1)}\cdots e^{(a_{2n},1)}\\ &= \frac{1}{2}\alpha_{2n-2}\varepsilon_{a_1\cdots a_{2n}}R^{a_1a_2}e^{a_3}\cdots e^{a_{2n}}, \end{aligned} \quad (142)$$

which is proportional to the Hilbert–Einstein Lagrangian.

The results show that the 2p-dimensional Born–Infeld action invariant under the algebra $\mathfrak{L}^{\mathcal{M}_{2m}}$ does not always lead to the action of general relativity. Indeed, for certain values of m it is impossible to obtain the Hilbert–Einstein term in the 2p-dimensional Born–Infeld **type** Lagrangian invariant under $\mathfrak{L}^{\mathcal{M}_{2m}}$. This is because to obtain the Hilbert–Einstein term, the presence is necessary of the $\langle J_{a_1a_2}Z_{a_3a_4}\cdots Z_{a_{2p-1}a_{2p}}\rangle$ component of the invariant tensor, which is given by

$$\langle J_{a_1a_2}Z_{a_3a_4}\cdots Z_{a_{2p-1}a_{2p}}\rangle_{\mathfrak{L}^{\mathcal{M}_{2m}}} = \begin{cases} l^{2p-2}\alpha_{2p-2}\langle J_{a_1a_2}\cdots J_{a_{2p-1}a_{2p}}\rangle_{\mathfrak{L}}, & \text{if } m\geq p\\ 0, & \text{if } m<p. \end{cases} \quad (143)$$

This observation leads us to state the following theorem.

Theorem 5 *Let $\mathfrak{L}^{\mathcal{M}_{2m}}$ be the algebra obtained from the Lorentz algebra by a reduced $S_0^{(2m-2)}$-expansion, which corresponds to a subalgebra of the $\mathcal{M}_{2m}$ algebra. If $L_{BI-2p}^{\mathfrak{L}^{\mathcal{M}_{2m}}}$ is a Born–Infeld type $(2p)$-dimensional Lagrangian built from the curvature 2-form of $\mathfrak{L}^{\mathcal{M}_{2m}}$ F, then the $2p$-dimensional Lagrangian of Born–Infeld type leads to the Lagrangian of general relativity, in a certain limit of the coupling constant l, if and only if $m\geq p$.*

The following table shows a set of Born–Infeld type Lagrangians $L_{BI-2p}^{\mathfrak{L}^{\mathcal{M}_{2n}}}$, invariant under the Lie algebra $\mathfrak{L}^{\mathcal{M}_{2n}}$, that flow into the Lagrangian of general relativity in a certain limit:

$\mathfrak{L}^{\mathcal{M}_4}$	$L_{BI\ (4)}^{\mathfrak{L}^{\mathcal{M}_4}}$					
$\mathfrak{L}^{\mathcal{M}_6}$	$L_{BI\ (4)}^{\mathfrak{L}^{\mathcal{M}_6}}$	$L_{BI\ (6)}^{\mathfrak{L}^{\mathcal{M}_6}}$				
$\mathfrak{L}^{\mathcal{M}_8}$	$L_{BI\ (4)}^{\mathfrak{L}^{\mathcal{M}_8}}$	$L_{BI\ (6)}^{\mathfrak{L}^{\mathcal{M}_8}}$	$L_{BI\ (8)}^{\mathfrak{L}^{\mathcal{M}_8}}$			
$\vdots$	$\vdots$					
$\vdots$	$\vdots$					
$\mathfrak{L}^{\mathcal{M}_{2n-2}}$	$L_{BI\ (4)}^{\mathfrak{L}^{\mathcal{M}_{2n-2}}}$	$L_{BI\ (6)}^{\mathfrak{L}^{\mathcal{M}_{2n-2}}}$	$L_{BI\ (8)}^{\mathfrak{L}^{\mathcal{M}_{2n-2}}}$	$\cdots\ \cdots$	$L_{BI\ (2n-2)}^{\mathfrak{L}^{\mathcal{M}_{2n-2}}}$	
$\mathfrak{L}^{\mathcal{M}_{2n}}$	$L_{BI\ (4)}^{\mathfrak{L}^{\mathcal{M}_{2n}}}$	$L_{BI\ (6)}^{\mathfrak{L}^{\mathcal{M}_{2n}}}$	$L_{BI\ (8)}^{\mathfrak{L}^{\mathcal{M}_{2n}}}$	$\cdots\ \cdots$	$L_{BI\ (2n-2)}^{\mathfrak{L}^{\mathcal{M}_{2n}}}$	$L_{BI\ (2n)}^{\mathfrak{L}^{\mathcal{M}_{2n}}}$

(144)

It is interesting to note that for each number of dimensions D of spacetime, we see that the Lagrangian $L_{BI(D)}$ invariant under the $\mathfrak{L}^{\mathcal{M}_{2n}}$ algebra contains all other D-dimensional Lagrangian evaluated in an $\mathfrak{L}^{\mathcal{M}_{2m}}$ algebra with $m<n$. So it is always possible to obtain an action of a lower algebra from the appropriate fields.

It is also of interest to note that it was found that, analogously to what happens in the case of 3-dimensional Chern–Simons gravity, in 4 dimensions it is not necessary to take the limit $l=0$ to result in general relativity.

5 Comments and possible developments

In the present work we have shown the following.

(i) Standard odd-dimensional general relativity (without a cosmological constant) emerges as the weak coupling constant limit of a $(2p+1)$-dimensional Chern–Simons Lagrangian invariant under the M_{2m+1} algebra, if and only if $m \geq p$.
(ii) When $m < p$, it is impossible to obtain odd-dimensional general relativity from a $(2p+1)$-dimensional Chern–Simons Lagrangian invariant under the M_{2m+1} algebra.
(iii) Standard even-dimensional general relativity (without a cosmological constant) emerges as the weak coupling constant limit of a $(2p)$-dimensional Born–Infeld type Lagrangian invariant under the $\mathcal{L}^{\mathcal{M}_{2m}}$ algebra, if and only if $m \geq p$.
(iv) When $m < p$, it is impossible to obtain even-dimensional general relativity from a $(2p)$-dimensional Born–Infeld type Lagrangian invariant under the $\mathcal{L}^{\mathcal{M}_{2m}}$ algebra.

The toy model and procedure considered here could play an important role in the context of supergravity in higher dimensions. In fact, it seems likely that it is possible to recover standard odd- and even-dimensional supergravity from Chern–Simons and Born–Infeld gravity theories, in a way very similar to the one shown here. In this way, the procedure sketched here could provide us with valuable information of what the underlying geometric structure of supergravity could be. This work is in progress.

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