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Strong convergence theorems of the Halpern-Mann's mixed iteration for a totally quasi- ϕ -asymptotically nonexpansive nonself multi-valued mapping in Banach spaces

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Abstract

In this paper, we introduce a class of totally quasi- ϕ -asymptotically nonexpansive nonself multi-valued mapping to modify the Halpern-Mann-type iteration algorithm for a totally quasi- ϕ -asymptotically nonexpansive nonself multi-valued mapping, which has the strong convergence under a limit condition only in the framework of Banach spaces. Our results are applied to study the approximation problem of solution to a system of equilibrium problems. Also, the results presented in the paper improve and extend the corresponding results of Chang *et al.* (Appl. Math. Comput. 218:7864-7870, 2012) and others.

Keywords: totally quasi- ϕ -asymptotically nonexpansive nonself multi-valued mapping; iterative sequence; Halpern and Mann-type iteration algorithm; nonexpansive retraction; generalized projection

1 Introduction and preliminaries

A Banach space X is said to be strictly convex if $\|\frac{x+y}{2}\| \leq 1$ for all $x, y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$. A Banach space is said to be uniformly convex if $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}, \{y_n\} \subset X$ with $\|x_n\| = \|y_n\| = 1$ and $\lim_{n \rightarrow \infty} \|\frac{x_n + y_n}{2}\| = 0$.

The norm of Banach space X is said to be Gâteaux differentiable, if, for each $x, y \in S(x)$, the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \quad (1.1)$$

exists, where $S(x) = \{x : \|x\| = 1, x \in X\}$. In this case, X is said to be smooth. The norm of Banach space X is said to be Fréchet differentiable, if, for each $x \in S(x)$, the limit (1.1) is attained uniformly for $y \in S(x)$ and the norm is uniformly Fréchet differentiable if the limit (1.1) is attained uniformly for $x, y \in S(x)$. In this case, X is said to be uniformly smooth.

Let D be a nonempty closed subset of a real Banach space X . A mapping $T : D \rightarrow D$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in D$. An element $p \in D$ is called a fixed point of a nonself multi-valued mapping $T : D \rightarrow X$ if $p \in Tp$. The set of fixed points of T is represented by $F(T)$.

A subset D of X is said to be retract of X , if there exists a continuous mapping $P : X \rightarrow D$ such that $Px = x$, for all $x \in X$. It is well known that every nonempty, closed, convex subset of a uniformly convex Banach space X is a retract of X . A mapping $P : X \rightarrow D$ is said to be a retraction, if $P^2 = P$. It follows that if a mapping P is a retraction, then $Py = y$ for all y in the range of P . A mapping $P : X \rightarrow D$ is said to be a nonexpansive retraction, if it is nonexpansive and it is a retraction from X to D .

Assume that X is a real Banach space with the dual X^* , D is a nonempty, closed, convex subset of X . We also denote by J the normalized duality mapping from X to 2^{X^*} which is defined by

$$J(x) = \{f^* \in X^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}, \quad x \in X,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

Next we assume that X is a smooth, strictly convex and reflexive Banach space and D is a nonempty, closed, convex subset of X . In the sequel, we always use $\phi : X \times X \rightarrow \mathbb{R}^+$ to denote the Lyapunov functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad x, y \in X. \quad (1.2)$$

It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \quad (1.3)$$

$$\phi(y, x) = \phi(y, z) + \phi(z, x) + 2\langle z - y, Jx - Jz \rangle, \quad x, y, z \in X, \quad (1.4)$$

and

$$\phi(x, J^{-1}(\lambda Jy + (1 - \lambda)Jz)) \leq \lambda\phi(x, y) + (1 - \lambda)\phi(x, z) \quad (1.5)$$

for all $\lambda \in [0, 1]$ and $x, y, z \in X$.

Following Alber [2], the generalized projection $\Pi_D : X \rightarrow D$ is defined by

$$\Pi_D(x) = \arg \inf_{y \in D} \phi(y, x), \quad \forall x \in X. \quad (1.6)$$

Lemma 1.1 (see [3]) *Let X be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of X such that $\{x_n\}$ and $\{y_n\}$ is bounded, if $\phi(x_n, y_n) \rightarrow 0$, then $\|x_n - y_n\| \rightarrow 0$.*

Many problems in nonlinear analysis can be reformulated as a problem of finding a fixed point of a nonexpansive mapping.

In the sequel, we denote the strong convergence and weak convergence of the sequence $\{x_n\}$ by $x_n \rightarrow x$ and $x_n \rightharpoonup x$, respectively.

Lemma 1.2 (see [2]) *Let X be a smooth, strictly convex, and reflexive Banach space and D be a nonempty, closed, convex subset of X . Then the following conclusions hold:*

- (a) $\phi(x, y) = 0$ if and only if $x = y$;
- (b) $\phi(x, \Pi_D y) + \phi(\Pi_D y, y) \leq \phi(x, y)$, $\forall x, y \in D$;
- (c) if $x \in X$ and $z \in D$, then $z = \Pi_D x$ if and only if $\langle z - y, Jx - Jz \rangle \geq 0$, $\forall y \in D$.

Remark 1.1 (see [4]) Let Π_D be the generalized projection from a smooth, reflexive and strictly convex Banach space X onto a nonempty, closed, convex subset D of X . Then Π_D is a closed and quasi- ϕ -nonexpansive from X onto D .

Remark 1.2 (see [4]) If H is a real Hilbert space, then $\phi(x, y) = \|x - y\|^2$, and Π_D is the metric projection of H onto D .

Definition 1.1 Let $P : X \rightarrow D$ be the nonexpansive retraction.

- (1) A nonself multi-valued mapping $T : D \rightarrow X$ is said to be quasi- ϕ -nonexpansive, if $F(T) \neq \Phi$, and

$$\phi(p, z_n) \leq \phi(p, x), \quad \forall x \in D, p \in F(T), z_n \in T(PT)^{n-1}x, \forall n \geq 1; \quad (1.7)$$

- (2) A nonself multi-valued mapping $T : D \rightarrow X$ is said to be quasi- ϕ -asymptotically nonexpansive, if $F(T) \neq \Phi$ and there exists a real sequence $k_n \subset [1, +\infty)$, $k_n \rightarrow 1$ (as $n \rightarrow \infty$) such that

$$\phi(p, z_n) \leq k_n \phi(p, x), \quad \forall x \in D, p \in F(T), z_n \in T(PT)^{n-1}x, \forall n \geq 1; \quad (1.8)$$

- (3) A nonself multi-valued mapping $T : D \rightarrow X$ is said to be totally quasi- ϕ -asymptotically nonexpansive, if $F(T) \neq \Phi$ and there exist nonnegative real sequences $\{v_n\}$, $\{\mu_n\}$, with $v_n, \mu_n \rightarrow 0$ (as $n \rightarrow \infty$) and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\zeta(0) = 0$ such that

$$\begin{aligned} \phi(p, z_n) &\leq \phi(p, x) + v_n \zeta[\phi(p, x)] + \mu_n, \\ \forall x \in D, p &\in F(T), z_n \in T(PT)^{n-1}x, \forall n \geq 1. \end{aligned} \quad (1.9)$$

Remark 1.3 From the definitions, it is obvious that a quasi- ϕ -nonexpansive nonself multi-valued mapping is a quasi- ϕ -asymptotically nonexpansive nonself multi-valued mapping, and a quasi- ϕ -asymptotically nonexpansive nonself multi-valued mapping is a totally quasi- ϕ -asymptotically nonexpansive nonself multi-valued mapping, but the converse is not true.

Now, we give an example of totally quasi- ϕ -asymptotically nonexpansive nonself multi-valued mapping.

Example 1.1 (see [4]) Let D be a unit ball in a real Hilbert space l^2 and let $T : D \rightarrow l^2$ be a nonself multi-valued mapping defined by

$$T : (x_1, x_2, \dots) \rightarrow (0, x_1^2, a_2 x_2, a_3 x_3, \dots) \in l^2, \quad \forall (x_1, x_2, \dots) \in D,$$

where $\{a_i\}$ is a sequence in $(0, 1)$ such that $\prod_{i=2}^{\infty} a_i = \frac{1}{2}$.

It is proved in [5] that

- (i) $\|Tx - Ty\| \leq 2\|x - y\|$, $\forall x, y \in D$;
(ii) $\|T^n x - T^n y\| \leq 2 \prod_{j=2}^n a_j$, $\forall x, y \in D$, $n \geq 2$.

Let $\sqrt{k_1} = 2$, $\sqrt{k_n} = 2 \prod_{j=2}^n a_j$, $n \geq 2$. Then $\lim_{n \rightarrow \infty} k_n = 1$. Letting $v_n = k_n - 1$ ($n \geq 2$), $\zeta(t) = t$ ($t \geq 0$) and $\{\mu_n\}$ be a nonnegative real sequence with $\mu_n \rightarrow 0$, then from (i) and (ii) we have

$$\|T^n x - T^n y\|^2 \leq \|x - y\|^2 + v_n \zeta(\|x - y\|^2) + \mu_n, \quad \forall x, y \in D.$$

Since D is a unit ball in a real Hilbert space l^2 , it follows from Remark 1.2 that $\phi(x, y) = \|x - y\|^2$, $\forall x, y \in D$. The inequality above can be written as

$$\phi(T^n x, T^n y) \leq \phi(x, y) + v_n \zeta(\phi(x, y)) + \mu_n, \quad \forall x, y \in D.$$

Again since $0 \in D$ and $0 \in F(T)$, this implies that $F(T) \neq \emptyset$. From the inequality above, we get

$$\phi(p, z_n) \leq \phi(p, x) + v_n \zeta(\phi(p, x)) + \mu_n, \quad \forall p \in F(T), x \in D, z_n \in T(PT)^{n-1}x,$$

where P is the nonexpansive retraction. This shows that the mapping T defined above is a totally quasi- ϕ -asymptotically nonexpansive nonself multi-valued mapping.

Lemma 1.3 *Let X be a smooth, strictly convex and reflexive Banach space and D be a nonempty, closed, convex subset of X . Let $T : D \rightarrow X$ be a totally quasi- ϕ -asymptotically nonexpansive nonself multi-valued mapping with $\mu_1 = 0$. Then $F(T)$ is a closed and convex subset of D .*

Proof Let $\{x_n\}$ be a sequence in $F(T)$ such that $x_n \rightarrow p$. Since T is a totally quasi- ϕ -asymptotically nonexpansive nonself multi-valued mapping, we have

$$\phi(x_n, z) \leq \phi(x_n, p) + v_1 \zeta(\phi(x_n, p)), \quad z \in Tp, \forall n \in N.$$

Therefore,

$$\phi(p, z) = \lim_{n \rightarrow \infty} \phi(x_n, z) \leq \lim_{n \rightarrow \infty} \phi(x_n, p) + v_1 \zeta(\phi(x_n, p)) = \phi(p, p) = 0.$$

By Lemma 1.2, we obtain $p = z \in Tp$. So we have $p \in F(T)$. This implies that $F(T)$ is closed.

Let $p, q \in F(T)$ and $t \in (0, 1)$, and put $w = tp + (1 - t)q$. We prove that $w \in F(T)$. Indeed, in view of the definition of ϕ , let $\{u_n\}$ be a sequence generated by $u_1 \in Tw$, $u_2 \in T(PT)w$, $u_3 \in T(PT)^2w, \dots, u_n \in T(PT)^{n-1}w \subset TPu_{n-1}$, we have

$$\begin{aligned} \phi(w, u_n) &= \|w\|^2 - 2\langle w, Ju_n \rangle + \|u_n\|^2 \\ &= \|w\|^2 - 2\langle tp + (1 - t)q, Ju_n \rangle + \|u_n\|^2 \\ &= \|w\|^2 + t\phi(p, u_n) + (1 - t)\phi(q, u_n) - t\|p\|^2 - (1 - t)\|q\|^2. \end{aligned} \quad (1.10)$$

Since

$$\begin{aligned} &t\phi(p, u_n) + (1 - t)\phi(q, u_n) \\ &\leq t[\phi(p, w) + v_n \zeta[\phi(p, w)] + \mu_n] + (1 - t)[\phi(q, w) + v_n \zeta[\phi(q, w)] + \mu_n] \end{aligned}$$

$$\begin{aligned}
 &= t\{\|p\|^2 - 2\langle p, Jw \rangle + \|w\|^2 + v_n \zeta[\phi(p, w)] + \mu_n\} \\
 &\quad + (1-t)\{\|q\|^2 - 2\langle q, Jw \rangle + \|w\|^2 + v_n \zeta[\phi(q, w)] + \mu_n\} \\
 &= t\|p\|^2 + (1-t)\|q\|^2 - \|w\|^2 + tv_n \zeta[\phi(p, w)] + (1-t)v_n \zeta[\phi(q, w)] + \mu_n. \quad (1.11)
 \end{aligned}$$

Substituting (1.11) into (1.10) and simplifying it, we have

$$\phi(w, u_n) \leq tv_n \zeta[\phi(p, w)] + (1-t)v_n \zeta[\phi(q, w)] + \mu_n \rightarrow 0 \quad (\text{as } n \rightarrow \infty).$$

Hence, $u_n \rightarrow w$ holds, which yields $u_{n+1} \rightarrow w$. Since TP is closed and $u_{n+1} \in T(PT)^n w \subset TPu_n$, we have $w \in TPw$. It follows from $w \in D$ that $w \in Tw$, i.e., $w \in F(T)$. This implies that $F(T)$ is convex. This completes the proof of Lemma 1.3. \square

Definition 1.2 (see [1]) A nonself mapping $T : D \rightarrow X$ is said to be uniformly L -Lipschitz continuous, if there exists a constant $L > 0$, such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L\|x - y\|, \quad \forall x, y \in D, \forall n \geq 1. \quad (1.12)$$

Definition 1.3 A nonself multi-valued mapping $T : D \rightarrow X$ is said to be uniformly L -Lipschitz continuous, if there exists a constant $L > 0$, such that

$$d(T(PT)^{n-1}x, T(PT)^{n-1}y) \leq L\|x - y\|, \quad \forall x, y \in D, \forall n \geq 1, \quad (1.13)$$

where $d(\cdot, \cdot)$ is Hausdorff metric.

Strong and weak convergence of asymptotically nonexpansive self or nonself mappings, relatively nonexpansive, quasi- ϕ -nonexpansive and quasi- ϕ -asymptotically nonexpansive self or nonself mappings have been considered extensively by several authors in the setting of Hilbert or Banach spaces (see [1–4, 6–24]). In recent years, by hybrid projection methods, strong and weak convergence problems for totally quasi- ϕ quasi- ϕ -asymptotically nonexpansive nonself and multi-valued mapping, respectively, was also studied by Kim *et al.* (see [6, 7]), Li *et al.* (see [8]), Chang *et al.* (see [9]) and Yang *et al.* (see [10]).

Inspired by specialists above, the purpose of this paper is to modify the Halpern-Mann's mixed type iteration algorithm for a totally quasi- ϕ -asymptotically nonexpansive nonself multi-valued mapping, which has the strong convergence under a limit condition only in the framework of Banach spaces. As an application, we utilize our results to study the approximation problem of solution to a system of equilibrium problems. The results presented in the paper improve and extend the corresponding results of Chang *et al.* [1, 11–13], Hao *et al.* [14], Guo *et al.* [15], Yildirim *et al.* [16], Thianwan [17], Nilsrakoo *et al.* [18], Pathak *et al.* [19], Qin *et al.* [20], Su *et al.* [21], Wang [22, 23], Yang *et al.* [24] and others.

2 Main results

Theorem 2.1 Let X be a real uniformly smooth and uniformly convex Banach space, D be a nonempty, closed, convex subset of X . Let $P : X \rightarrow D$ be the nonexpansive retraction. Let

$T : D \rightarrow X$ be a totally quasi- ϕ -asymptotically nonexpansive nonself multi-valued mapping with sequence $\{\nu_n\}, \{\mu_n\}$ ($\mu_1 = 0$), with $\nu_n, \mu_n \rightarrow 0$ (as $n \rightarrow \infty$) and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\zeta(0) = 0$ such that T is uniformly L -Lipschitz continuous. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ and $\{\beta_n\}$ be a sequence in $(0, 1)$ satisfying the following conditions:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$;
- (ii) $0 < \lim_{n \rightarrow \infty} \inf \beta_n \leq \lim_{n \rightarrow \infty} \sup \beta_n < 1$.

Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_1 \in X \text{ is arbitrary; } & D_1 = D, \\ y_n = J^{-1}[\alpha_n Jx_1 + (1 - \alpha_n)(\beta_n Jx_n + (1 - \beta_n)Jz_n)], & z_n \in T(PT)^{n-1}x_n, \\ D_{n+1} = \{z \in D_n : \phi(z, y_n) \leq \alpha_n \phi(z, x_1) + (1 - \alpha_n)\phi(z, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{D_{n+1}}x_1 \quad (n = 1, 2, \dots), \end{cases} \quad (2.1)$$

where $\xi_n = \nu_n \sup_{p \in F(T)} \zeta(\phi(p, x_n)) + \mu_n$, $\Pi_{D_{n+1}}$ is the generalized projection of X onto D_{n+1} . If $F(T) \neq \emptyset$, then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_1$.

Proof (I) First, we prove that D_n are closed and convex subsets in D .

In fact, by Lemma 1.3, $F(T)$ is closed and convex in D . By the assumption, $D_1 = D$ is closed and convex. Suppose that D_n is closed and convex for some $n \geq 1$. In view of the definition of ϕ , we have

$$\begin{aligned} D_{n+1} &= \{z \in D_n : \phi(z, y_n) \leq \alpha_n \phi(z, x_1) + (1 - \alpha_n)\phi(z, x_n) + \xi_n\} \\ &= \{z \in D : \phi(z, y_n) \leq \alpha_n \phi(z, x_1) + (1 - \alpha_n)\phi(z, x_n) + \xi_n\} \cap D_n \\ &= \{z \in D : 2\alpha_n \langle z, Jx_1 \rangle + 2(1 - \alpha_n)\langle z, Jx_n \rangle - 2\langle z, Jy_n \rangle \\ &\quad \leq \alpha_n \|x_1\|^2 + (1 - \alpha_n)\|x_n\|^2 - \|y_n\|^2\} \cap D_n. \end{aligned}$$

This shows that D_{n+1} is closed and convex. The conclusions are proved.

(II) Next, we prove that $F(T) \subset D_n$ for all $n \geq 1$.

It is obvious that $F(T) \subset D_1$. Suppose that $F(T) \subset D_n$, $w_n = J^{-1}(\beta_n Jx_n + (1 - \beta_n)Jz_n)$ and $z_n \in T(PT)^{n-1}x_n$. Hence, for any $u \in F(T) \subset D_n$, by (1.5), we have

$$\begin{aligned} \phi(u, y_n) &= \phi(u, J^{-1}(\alpha_n Jx_1 + (1 - \alpha_n)Jw_n)) \\ &\leq \alpha_n \phi(u, x_1) + (1 - \alpha_n)\phi(u, w_n), \end{aligned} \quad (2.2)$$

and

$$\begin{aligned} \phi(u, w_n) &= \phi(u, J^{-1}(\beta_n Jx_n + (1 - \beta_n)Jz_n)) \\ &\leq \beta_n \phi(u, x_n) + (1 - \beta_n)\phi(u, z_n) \\ &\leq \beta_n \phi(u, x_n) + (1 - \beta_n)\{\phi(u, x_n) + \nu_n \zeta[\phi(u, x_n)] + \mu_n\} \\ &= \phi(u, x_n) + (1 - \beta_n)\nu_n \zeta[\phi(u, x_n)] + (1 - \beta_n)\mu_n. \end{aligned} \quad (2.3)$$

Therefore, we have

$$\begin{aligned}\phi(u, y_n) &\leq \alpha_n \phi(u, x_1) + (1 - \alpha_n) [\phi(u, x_n) + (1 - \beta_n) v_n \zeta [\phi(u, x_n)] + (1 - \beta_n) \mu_n] \\ &\leq \alpha_n \phi(u, x_1) + (1 - \alpha_n) \phi(u, x_n) + v_n \sup_{p \in F(T)} \zeta [\phi(p, x_n)] \\ &= \alpha_n \phi(u, x_1) + (1 - \alpha_n) \phi(u, x_n) + \xi_n,\end{aligned}\quad (2.4)$$

where $\xi_n = v_n \sup_{p \in F(T)} \zeta (\phi(p, x_n)) + \mu_n$. This shows that $u \in D_{n+1}$ and so $F(T) \subset D_n$. The conclusion is proved.

(III) Now we prove that $\{x_n\}$ converges strongly to some point p^* .

Since $x_n = \Pi_{D_n} x_1$, from Lemma 1.2(c), we have

$$\langle x_n - y, Jx_1 - Jx_n \rangle \geq 0, \quad \forall y \in D_n.$$

Again since $F(T) \subset D_n$, we have

$$\langle x_n - u, Jx_1 - Jx_n \rangle \geq 0, \quad \forall u \in F(T).$$

It follows from Lemma 1.2(b) that, for each $u \in F(T)$ and for each $n \geq 1$,

$$\phi(x_n, x_1) = \phi(\Pi_{D_n} x_1, x_1) \leq \phi(u, x_1) - \phi(u, x_n) \leq \phi(u, x_1). \quad (2.5)$$

Therefore, $\{\phi(x_n, x_1)\}$ is bounded, and so is $\{x_n\}$. Since $x_n = \Pi_{D_n} x_1$ and $x_{n+1} = \Pi_{D_{n+1}} x_1 \in D_{n+1} \subset D_n$, we have $\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1)$. This implies that $\{\phi(x_n, x_1)\}$ is nondecreasing. Hence $\lim_{n \rightarrow \infty} \phi(x_n, x_1)$ exists.

By the construction of $\{D_n\}$, for any $m \geq n$, we have $D_m \subset D_n$ and $x_m = \Pi_{D_m} x_1 \in D_n$. This shows that

$$\phi(x_m, x_n) = \phi(x_m, \Pi_{D_n} x_1) \leq \phi(x_m, x_1) - \phi(x_n, x_1) \rightarrow 0 \quad (\text{as } n \rightarrow \infty).$$

It follows from Lemma 1.1 that $\lim_{n \rightarrow \infty} \|x_m - x_n\| = 0$. Hence $\{x_n\}$ is a Cauchy sequence in D . Since D is complete, without loss of generality, we can assume that $\lim_{n \rightarrow \infty} x_n = p^*$ (some point in D).

By the assumption, it is easy to see that

$$\lim_{n \rightarrow \infty} \xi_n = \lim_{n \rightarrow \infty} \left[v_n \sup_{p \in F(T)} \zeta (\phi(p, x_n)) + \mu_n \right] = 0. \quad (2.6)$$

(IV) Now we prove that $p^* \in F(T)$.

Since $x_{n+1} \in D_{n+1}$, from (2.1) and (2.6), we have

$$\phi(x_{n+1}, y_n) \leq \alpha_n \phi(x_{n+1}, x_1) + (1 - \alpha_n) \phi(x_{n+1}, x_n) + \xi_n \rightarrow 0. \quad (2.7)$$

Since $x_n \rightarrow p^*$, it follows from (2.7) and Lemma 1.1 that

$$y_n \rightarrow p^*. \quad (2.8)$$

Since $\{x_n\}$ is bounded and T is a totally quasi- ϕ -asymptotically nonexpansive nonself multi-valued mapping, we have

$$\phi(p, z_n) \leq \phi(p, x_n) + \nu_n \zeta[\phi(p, x_n)] + \mu_n, \quad \forall x \in D, \forall n, i \geq 1, p \in F(T).$$

This implies that $\{z_n\}$ is also bounded.

By condition (ii), we have

$$\begin{aligned} \|w_n\| &= \|J^{-1}(\beta_n Jx_n + (1 - \beta_n)Jz_n)\| \\ &\leq \beta_n \|x_n\| + (1 - \beta_n) \|z_n\| \\ &\leq \|x_n\| + \|z_n\|, \end{aligned}$$

this implies that $\{w_n\}$ is also bounded.

In view of $\alpha_n \rightarrow 0$, from (2.1), we have

$$\lim_{n \rightarrow \infty} \|Jy_n - Jw_n\| = \lim_{n \rightarrow \infty} \alpha_n \|Jx_1 - Jw_n\| = 0. \quad (2.9)$$

Since J^{-1} is uniformly continuous on each bounded subset of X^* , it follows from (2.8) and (2.9) that

$$w_n \rightarrow p^*. \quad (2.10)$$

Since J is uniformly continuous on each bounded subset of X , we have

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \|Jw_n - Jp^*\| \\ &= \lim_{n \rightarrow \infty} \|(\beta_n Jx_n + (1 - \beta_n)Jz_n) - Jp^*\| \\ &= \lim_{n \rightarrow \infty} \|\beta_n (Jx_n - Jp^*) + (1 - \beta_n)(Jz_n - Jp^*)\| \\ &= \lim_{n \rightarrow \infty} (1 - \beta_n) \|Jz_n - Jp^*\|. \end{aligned} \quad (2.11)$$

By condition (ii), we have

$$\lim_{n \rightarrow \infty} \|Jz_n - Jp^*\| = 0.$$

Since J is uniformly continuous, this shows that

$$\lim_{n \rightarrow \infty} z_n = p^*. \quad (2.12)$$

Again by the assumptions that $T : D \rightarrow X$ be uniformly L -Lipschitz continuous, thus we have

$$\begin{aligned} &d(T(PT)^n x_n, T(PT)^{n-1} x_n) \\ &\leq d(T(PT)^n x_n, T(PT)^n x_{n+1}) + d(T(PT)^n x_{n+1}, x_{n+1}) \\ &\quad + \|x_{n+1} - x_n\| + d(x_n, T(PT)^{n-1} x_n) \\ &\leq (L + 1) \|x_{n+1} - x_n\| + d(T(PT)^n x_{n+1}, x_{n+1}) + d(x_n, T(PT)^{n-1} x_n). \end{aligned} \quad (2.13)$$

We get $\lim_{n \rightarrow \infty} d(T(PT)^n x_n, T(PT)^{n-1} x_n) = 0$, since $\lim_{n \rightarrow \infty} z_n = p^*$ and $\lim_{n \rightarrow \infty} x_n = p^*$.

In view of the continuity of TP , it yields $p^* \in TPp^*$. We have $p^* \in C$, which implies that $p^* \in Tp^*$. We have $p^* \in F(T)$.

(V) Finally, we prove that $p^* = \Pi_{F(T)} x_1$ and so $x_n \rightarrow \Pi_{F(T)} x_1 = p^*$.

Let $w = \Pi_{F(T)} x_1$. Since $w \in F(T) \subset D_n$ and $x_n = \Pi_{D_n} x_1$, we have $\phi(x_n, x_1) \leq \phi(w, x_1)$. This implies that

$$\phi(p^*, x_1) = \lim_{n \rightarrow \infty} \phi(x_n, x_1) \leq \phi(w, x_1), \quad (2.14)$$

which yields $p^* = w = \Pi_{F(T)} x_1$. Therefore, $x_n \rightarrow \Pi_{F(T)} x_1$. The proof of Theorem 3.1 is completed. \square

By Remark 1.3, the following corollary is obtained.

Corollary 2.1 Let $X, D, \{\alpha_n\}, \{\beta_n\}$ be the same as in Theorem 2.1. Let $T : D \rightarrow X$ be a quasi- ϕ -asymptotically nonexpansive nonself multi-valued mapping with sequence $k_n \subset [1, +\infty)$, $k_n \rightarrow 1$, $T : D \rightarrow X$ be uniformly L -Lipschitz continuous.

Suppose $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_1 \in X \text{ is arbitrary; } & D_1 = D, \\ y_n = J^{-1}[\alpha_n Jx_1 + (1 - \alpha_n)(\beta_n Jx_n + (1 - \beta_n)Jz_n)] & (i \geq 1), z_n \in T(PT)^{n-1} x_n, \\ D_{n+1} = \{z \in D_n : \phi(z, y_n) \leq \alpha_n \phi(z, x_1) + (1 - \alpha_n)\phi(z, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{D_{n+1}} x_1 & (n = 1, 2, \dots), \end{cases} \quad (2.15)$$

where $\xi_n = (k_n - 1) \sup_{p \in F(T)} \phi(p, x_n)$, $\Pi_{D_{n+1}}$ is the generalized projection of X onto D_{n+1} . If $F(T) \neq \emptyset$, then $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_1$.

Corollary 2.2 Let $X, D, \{\alpha_n\}, \{\beta_n\}$ be the same as in Theorem 2.1. Let $T : D \rightarrow X$ be a quasi- ϕ -nonexpansive nonself multi-valued mapping, $T : D \rightarrow X$ be uniformly L -Lipschitz continuous.

Suppose $\{x_n\}$ is a sequence generated by

$$\begin{cases} x_1 \in X \text{ is arbitrary; } & D_1 = D, \\ y_n = J^{-1}[\alpha_n Jx_1 + (1 - \alpha_n)(\beta_n Jx_n + (1 - \beta_n)Jz_n)] & (i \geq 1), z_n \in T(PT)^{n-1} x_n, \\ D_{n+1} = \{z \in D_n : \phi(z, y_n) \leq \alpha_n \phi(z, x_1) + (1 - \alpha_n)\phi(z, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{D_{n+1}} x_1 & (n = 1, 2, \dots), \end{cases} \quad (2.16)$$

where $\xi_n = (k_n - 1) \sup_{p \in F(T)} \phi(p, x_n)$, $\Pi_{D_{n+1}}$ is the generalized projection of X onto D_{n+1} . If $F(T) \neq \emptyset$, then $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_1$.

3 Application

First, we present an example of a quasi- ϕ -nonexpansive nonself multi-valued mapping.

Example 3.1 (see [4]) Let H be a real Hilbert space, D be a nonempty closed and convex subset of H and $f : D \times D \rightarrow R$ be a bifunction satisfying the conditions: (A1) $f(x, x) = 0$,

$\forall x \in D$; (A2) $f(x, y) + f(y, x) \leq 0, \forall x, y \in D$; (A3) for each $x, y, z \in D, \lim_{t \rightarrow 0} f(tz + (1-t)x, y) \leq f(x, y)$; (A4) for each given $x \in D$, the function $y \mapsto f(x, y)$ is convex and lower semicontinuous. The 'so-called' equilibrium problem for f is to find a $x^* \in D$ such that $f(x^*, y) \geq 0, \forall y \in D$. The set of its solutions is denoted by $EP(f)$.

Let $r > 0, x \in H$ and define a mapping $T_r : D \rightarrow D \subset H$ as follows:

$$T_r(x) = \left\{ z \in D, f(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in D \right\}, \quad \forall x \in D \subset H. \quad (3.1)$$

Then (1) T_r is single-valued, so $z = T_r(x)$; (2) T_r is a relatively nonexpansive nonself mapping, therefore it is a closed quasi- ϕ -nonexpansive nonself mapping; (3) $F(T_r) = EP(f)$ and $F(T_r)$ is a nonempty and closed convex subset of D ; (4) $T_r : D \rightarrow D$ is a nonexpansive. Since $F(T_r)$ nonempty, it is a quasi- ϕ -nonexpansive nonself mapping from D to H , where $\phi(x, y) = \|x - y\|^2, x, y \in H$.

In this section we utilize Corollary 2.1 to study a modified Halpern iterative algorithm for a system of equilibrium problems. We have the following result.

Theorem 3.1 *Let H be a real Hilbert space, D be a nonempty closed and convex subset of H , $\{\alpha_n\}, \{\beta_n\}$ be the same as in Theorem 2.1. Let $f : D \times D \rightarrow R$ be a bifunction satisfying conditions (A1)-(A4) as given in Example 3.1. Let $T_r : D \rightarrow D \subset H$ be mapping defined by (3.1), i.e.,*

$$T_r(x) = \left\{ z \in D, f(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in D \right\}, \quad \forall x \in D \subset H.$$

Let $\{x_n\}$ be the sequence generated by

$$\begin{cases} x_1 \in D \text{ is arbitrary; } & D_1 = D, \\ f(u_n, y) + \frac{1}{r} \langle y - u_n, u_n - x_n \rangle \geq 0, & \forall y \in D, r > 0, \\ y_n = \alpha_n x_1 + (1 - \alpha_n)[\beta_n x_n + (1 - \beta_n)u_n], \\ D_{n+1} = \{z \in D_n : \|z - y_{n,i}\|^2 \leq \alpha_n \|z - x_1\|^2 + (1 - \alpha_n)\|z - x_n\|^2\}, \\ x_{n+1} = \Pi_{D_{n+1}} x_1 \quad (n = 1, 2, \dots). \end{cases} \quad (3.2)$$

If $F(T_r) \neq \emptyset$, then $\{x_n\}$ converges strongly to $\Pi_{F(T_r)} x_1$, which is a common solution of the system of equilibrium problems for f .

Proof In Example 3.1, we have pointed out that $u_n = T_r(x_n), F(T_r) = EP(f)$ is nonempty and convex, T_r is a quasi- ϕ -nonexpansive nonself mapping. Since $F(T_r)$ is nonempty, and so T_r is a quasi- ϕ -nonexpansive mapping and T_r is uniformly 1-Lipschitzian mapping. Hence (3.1) can be rewritten as follows:

$$\begin{cases} x_1 \in H \text{ is arbitrary; } & D_1 = D, \\ y_n = \alpha_n x_1 + (1 - \alpha_n)[\beta_n x_n + (1 - \beta_n)z_n], & z_n \in T_r x_n, \\ D_{n+1} = \{z \in D_n : \|z - y_n\|^2 \leq \alpha_n \|z - x_1\|^2 + (1 - \alpha_n)\|z - x_n\|^2\}, \\ x_{n+1} = \Pi_{D_{n+1}} x_1 \quad (n = 1, 2, \dots). \end{cases} \quad (3.3)$$

Therefore, the conclusion of Theorem 3.1 can be obtained from Corollary 2.1. \square

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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