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Cremmer–Gervais r-Matrices and the Cherednik Algebras of Type GL_2

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Abstract. We give an interpretation of the Cremmer–Gervais r-matrices for $\mathfrak{sl}_n$ in terms of actions of elements in the rational and trigonometric Cherednik algebras of type GL_2 on certain subspaces of their polynomial representations. This is used to compute the nilpotency index of the Jordanian r-matrices, thus answering a question of Gerstenhaber and Giaquinto. We also give an interpretation of the Cremmer–Gervais quantization in terms of the corresponding double affine Hecke algebra.

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1. Introduction

Let $\mathfrak{sl}_n$ denote the Lie algebra of traceless $n \times n$ matrices having entries in a field k. Let V denote the vector representation of $\mathfrak{sl}_n$ and let $r \in \mathfrak{sl}_n \wedge \mathfrak{sl}_n \subset End(V \otimes V)$ be a skew-symmetric linear operator. Define $r_{12} := r \otimes 1$, $r_{23} := 1 \otimes r$, and $r_{13} := P_{23} r_{12} P_{23}$ where P_{23} is the permutation operator on $V^{\otimes 3}$: $P_{23}(u \otimes v \otimes w) = u \otimes w \otimes v$. An important class of operators which arise in studying Lie bialgebras and Poisson–Lie groups are those satisfying the modified classical Yang–Baxter equation (MCYBE)

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] - \lambda(P_{123} - P_{213}) = 0$$

for some $\lambda \in k$ (see [3,8] for more details). The solutions to the MCYBE are called *classical r-matrices* and fall into two classes: those satisfying the MCYBE for λ nonzero (resp. zero) are called *quasitriangular* (resp. *triangular*).

In the early 1980s, Belavin and Drinfel'd successfully classified all quasitriangular r-matrices in the case when k is the field of complex numbers [1]. This classification gives us a solution space which we view as a disjoint union of quasi-projective subvarieties of $\mathbb{P}(\mathfrak{sl}_n \wedge \mathfrak{sl}_n)$. In contrast, the triangular r-matrices are more mysterious as there is not a constructive classification of them (except in the smaller cases of $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$, see [3, Chapter 3]), only a homological interpretation exists (due to the work of Stolin [15]: for details see [3, Section 3.1.D.] and [8, Section 3.5]), and there are currently few known examples.

In the paper [9], Gerstenhaber and Giaquinto investigate the behavior along the boundaries of the aforementioned quasi-projective varieties and show that the boundary points are all triangular r-matrices. In the same paper, they construct the most general class of known examples of triangular r-matrices, the so-called *generalized Jordanian r-matrices* $r_{J,n}$ (see [7,9]). They prove that the Jordanian r-matrices lie on the boundary of the component corresponding to the quasitriangular Cremmer–Gervais r-matrices (discussed in [6,7,9,11–13]) and conjecture that $r_{J,n}^3 = 0$.

In Theorem 4.3, we prove that the nilpotency index of $r_{J,n}$ is quite different than conjectured. We do this by first interpreting the quantum Cremmer–Gervais R-matrix R in terms of the double affine Hecke algebra (DAHA) $\mathcal{H}^{q,t}$ of type GL_2 (Theorem 2.2) and the classical Cremmer–Gervais r-matrix in terms of the degenerate DAHA $\mathcal{H}'_c$ (Theorem 3.4). Using Suzuki's embedding [16] of the rational Cherednik algebra $\mathcal{H}''_{-c}$ into $\mathcal{H}'_c$, we give a simple interpretation of both the Cremmer–Gervais and Jordanian r-matrices as operators on the polynomial representation of $\mathcal{H}''_{-c}$. Using the relations in $\mathcal{H}''_{-c}$, we find that the nilpotency index of $r_{J,n}$ is n when n is odd, and $2n-1$ when n is even. The conceptual difference between the two cases has a representation theoretic origin: the polynomial representation of the rational Cherednik algebra $\mathcal{H}''_c$ of type A_1 is reducible if and only if the deformation parameter c has the form $n/2$ for an odd positive integer n (a special case of Dunkl's theorem [2]).

2. The Yang–Baxter Equations and the Double Affine Hecke Algebra

Let k be a field of characteristic 0 and let $K = k(q, t^{1/2})$. We begin with a

DEFINITION 2.1. (see [4,5]) The double affine Hecke algebra $\mathcal{H}^{q,t}$ of type GL_2 is the associative K-algebra with generators $X_1^{\pm1}, X_2^{\pm1}, Y_1^{\pm1}, Y_2^{\pm1}, T$ and relations

$$\begin{aligned}
&X_j X_j^{-1} = X_j^{-1} X_j = Y_j Y_j^{-1} = Y_j^{-1} Y_j = 1,\\
&(T - t^{1/2})(T + t^{-1/2}) = 0, \quad T X_1 T = X_2, \quad T Y_2 T = Y_1,\\
&Y_2^{-1} X_1 Y_2 X_1^{-1} = T^2, \quad Y_1 Y_2 X_j = q X_j Y_1 Y_2,\\
&Y_j X_1 X_2 = q X_1 X_2 Y_j, \quad [Y_1, Y_2] = 0, \quad [X_1, X_2] = 0,
\end{aligned}$$

for $j = 1, 2$.

The K-vector space $K[X_1^{\pm1}, X_2^{\pm1}]$ can be made into a $\mathcal{H}^{q,t}$-module, called the *polynomial representation*, defined as follows. Let (12) act on $K[X_1^{\pm1}, X_2^{\pm1}]$ by swapping variables and let $S = \frac{1-(12)}{X_1 - X_2}$. For integers a, b define $\Gamma_{a,b}.f(X_1, X_2) :=$

$f(q^a X_1, q^b X_2)$. The double affine Hecke algebra $\mathcal{H}\mathcal{H}^{q,t}$ acts faithfully on $K[X_1^{\pm 1}, X_2^{\pm 1}]$ via

$$T \mapsto t^{1/2}(12) - (t^{1/2} - t^{-1/2})X_2 S,$$
$$Y_1 \mapsto T\Gamma_{0,1}(12), \quad Y_2 \mapsto \Gamma_{0,1}(12)T^{-1},$$

and the X's act via multiplication. For an operator $R \in \mathrm{End}_K(K[X_1^{\pm 1}, X_2^{\pm 1}])$, let R_{12}, R_{13}, R_{23} denote the corresponding operators on $K[X_1^{\pm 1}, X_2^{\pm 1}, X_3^{\pm 1}]$. In [10], Gerstenhaber and Giaquinto introduced the *modified quantum Yang–Baxter equation* (MQYBE). An operator is called a *modified quantum R-matrix* if it satisfies the MQYBE

$$R_{12}R_{13}R_{23} - R_{23}R_{13}R_{12} - \lambda\,(P_{123}R_{12} - P_{213}R_{23}) = 0$$

for some scalar λ. Here, P_{ijk} denotes the permutation on the variables $X_i \mapsto X_j \mapsto X_k \mapsto X_i$. Furthermore, R is called *unitary* if $R(12)R(12) = 1$.

The classical analogue of the MQYBE is called the *modified classical Yang–Baxter equation* (MCYBE). An operator r is called a *classical r-matrix* if it satisfies the MCYBE

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] - \lambda(P_{123} - P_{213}) = 0$$

for some scalar λ (for more details, see [3]). Furthermore, r is called *skew-symmetric* if $(12)r(12) = -r$. For shorthand, we will denote the left hand sides of the above equations by $\mathrm{MQYBE}_\lambda(R)$ and $\mathrm{MCYBE}_\lambda(r)$, respectively.

THEOREM 2.2. *The operator $R = (12)Y_2(12)Y_2^{-1}$ is unitary and satisfies the modified quantum Yang–Baxter equation*

$$R_{12}R_{13}R_{23} - R_{23}R_{13}R_{12} = \left(1 - t^{-1}\right)^2 (P_{123}R_{12} - P_{213}R_{23}).$$

Proof. It is obvious that R is unitary. To show R satisfies the MQYBE, we first set $v = (X_1 + X_2)S$. This operator satisfies $\mathrm{MQYBE}_1(v) = 0$, $\mathrm{MCYBE}_1(v) = 0$, and $v^2 = 0$. Therefore, it follows that $\mathrm{MQYBE}_{\lambda^2}(\exp(\lambda v)) = 0$ for all scalars λ. Setting $F = \Gamma_{0,-1}$ and $\hat{R} = \exp((1 - t^{-1})v)$, we have $R = F_{21}^{-1}\hat{R}F_{12}$. Furthermore

(i) $F_{12}F_{13}F_{23} = F_{23}F_{13}F_{12}$,
(ii) $\hat{R}_{12}F_{23}F_{13} = F_{13}F_{23}\hat{R}_{12}$,
(iii) $\hat{R}_{23}F_{12}F_{13} = F_{13}F_{12}\hat{R}_{23}$.

So R is obtained by twisting $\hat{R}$ by F and hence satisfies the MQYBE (cf. [14, Thm. 1]). □

3. Semiclassical Limit

Setting $q=e^h$ and $t=e^{ch}$, we view K as a subfield of $k[[c,h]]$ and the above formulae for the Y's in the polynomial representation become

$$Y_1 \mapsto 1+h\left(X_1\frac{\partial}{\partial X_1}+cX_2S+\frac{c}{2}\right)+O(h^2)$$
$$Y_2 \mapsto 1+h\left(X_2\frac{\partial}{\partial X_2}-cX_2S-\frac{c}{2}\right)+O(h^2)$$

Define y_i as the coefficient of h in Y_i in the above expressions. The operators y_1, y_2 obey commutation relations which motivate the following

DEFINITION 3.1. (see Cherednik [5]) The degenerate (or trigonometric) double affine Hecke algebra $\mathcal{H}'_c$ of type GL_2 is the $k(c)$-algebra having generators y_1, y_2, $X_1^{\pm 1}$, $X_2^{\pm 1}$, and (12) and relations

$$(12)^2=1, \quad [X_1,X_2]=0, \quad (12)X_1(12)=X_2,$$
$$(12)y_1-y_2(12)=c, \quad [y_1,y_2]=0,$$
$$[y_i,X_j]=\begin{cases} X_i+c(12)X_1 & \text{if } i=j \\ -c(12)X_1 & \text{if } i\neq j\end{cases}$$

Remark 3.2. If $\mathrm{MQYBE}_\lambda(1+hr+O(h^2))=0$, then λ is of the form $\epsilon h^2+O(h^3)$ for some scalar ϵ and $\mathrm{MCYBE}_\epsilon(r)=0$. Furthermore, if $1+hr+O(h^2)$ is unitary, then r is skew-symmetric.

One can readily obtain the following

LEMMA 3.3. $R=1+h\,(y_1-y_2-c(12))+O(h^2)$

Using Remark 3.2 together with Lemma 3.3, we have

COROLLARY 3.4. *R, y_1, y_2 as above*

(i) $r:=y_1-y_2-c(12)$ *is skew-symmetric*
(ii) $MCYBE_{c^2}(r)=0.$

In this situation, r is called the *semiclassical limit* of R. Since r is homogeneous, it follows that for any natural number n, we can restrict the action of r to the subspace $k(c)^{n,n}$ of $k(c)[X_1^{\pm},X_2^{\pm}]$ spanned by the monomials $X_1^aX_2^b$ with $0\leq a,b\leq n-1$. Doing this yields

$$r_n=2\sum_{1\leq k<l\leq n}(k-l-c)e_{kk}\wedge e_{ll}+2c\sum_{1\leq k<l\leq n}e_{kl}\wedge e_{lk}+4c\sum_{1\leq i<k<j\leq n}e_{i+j-k,j}\wedge e_{ki}.$$

Here, $e_{ij}\wedge e_{kl}$ is the operator

$$X_1^aX_2^b\mapsto\frac{1}{2}\left(\delta_{j,a+1}\delta_{l,b+1}X_1^{i-1}X_2^{k-1}-\delta_{l,a+1}\delta_{j,b+1}X_1^{k-1}X_2^{i-1}\right).$$

Remark 3.5. The formula for r_n above suggests that we can view it as being in $\mathfrak{gl}_n \wedge \mathfrak{gl}_n$. Thus, we have a one-parameter family of solutions to the MCYBE over $\mathfrak{gl}_n$. Setting $c = -\frac{n}{2}$ is the only instance that r_n will be in $\mathfrak{sl}_n \wedge \mathfrak{sl}_n$.

As mentioned in the introduction, the skew-symmetric solutions to the MCYBE

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] - \lambda(P_{123} - P_{213}) = 0$$

fall into two classes: those solutions satisfying the MCYBE with λ nonzero (resp. zero) are called *quasitriangular* (resp. *triangular*) *r-matrices*. The quasitriangular r-matrices (over $\mathbb{C}$) were classified in the early 1980's by Belavin and Drinfel'd using combinatorial objects on the Dynkin graph, called BD-triples [1]. The classification tells us that the solution space of quasitriangular r-matrices may be viewed as a disjoint union of quasi-projective subvarieties of $\mathbb{P}(\mathfrak{sl}_n \wedge \mathfrak{sl}_n)$: the aforementioned subvarieties are indexed by the BD-triples. In this case r_n corresponds to the maximal BD-triple obtained by deleting an extremal node. It is the so-called Cremmer–Gervais r-matrix (discussed in [6,7,9,11–13]).

THEOREM 3.6. *When $c = -\frac{n}{2}$, r_n is the Cremmer–Gervais r-matrix.*

Proof. Apply the Lie algebra automorphism $e_{ij} \mapsto -e_{n+1-j,n+1-i}$ of $\mathfrak{gl}_n$ to the formula for r_n above, then multiply the result by $-\frac{1}{n}$, and finally set $c = -\frac{n}{2}$; we obtain the same formula for the Cremmer—Gervais r-matrix r_{CG} as it appears in [9]. □

4. Connections with the Rational Cherednik Algebra

We begin this section by recalling a

DEFINITION 4.1. (see Cherednik [5]) The rational Cherednik algebra (over $k(c)$) $\mathcal{H}''_c$ of type GL_2 has generators (12), x_1, x_2, u_1, u_2 and relations

$$(12)^2 = 1, \quad (12)x_1(12) = x_2, \quad (12)u_1(12) = u_2,$$
$$[x_1, x_2] = 0, \quad [u_1, u_2] = 0,$$
$$[u_i, x_j] = \begin{cases} 1 - c(12) & \text{if } i = j \\ c(12) & \text{if } i \neq j \end{cases}$$

The polynomial representation $k(c)[x_1, x_2]$ of $\mathcal{H}''_c$ is defined where (12) permutes the variables, x_1 and x_2 act via multiplication, and the u_i act by the Dunkl operators

$$u_i \mapsto \frac{\partial}{\partial x_i} + c(-1)^i S.$$

In [16], Suzuki shows that there is an algebra embedding $\psi:\mathcal{H}''_{-c}\to\mathcal{H}'_c$ defined on generators by

$$\begin{aligned}(12)&\mapsto(12)\\ x_1&\mapsto X_1\\ x_2&\mapsto X_2\\ u_1&\mapsto X_1^{-1}\left(y_1+\frac{c}{2}-c(12)\right)\\ u_2&\mapsto X_2^{-1}\left(y_2+\frac{c}{2}\right)\end{aligned}$$

Using this algebra embedding, we see that the Cremmer–Gervais r-matrix has an interpretation in the rational Cherednik algebra. Here, r_n corresponds to $x_1u_1-x_2u_2\in\mathcal{H}''_{-c}$.

In [9], Gerstenhaber and Giaquinto provide the largest known class of examples of triangular r-matrices, the so-called *generalized Jordanian r-matrices* (also discussed in [7]). They demonstrate that the Jordanian r-matrices lie on the boundary of the orbit $SL_n.r_n$, where SL_n acts via the adjoint action. One can translate this into the setting of the rational Cherednik algebra. Here,

$$e^{\tau\cdot ad(u_1+u_2)}.(x_1u_1-x_2u_2)=x_1u_1-x_2u_2+\tau(u_1-u_2).$$

Therefore, we have the following

COROLLARY 4.2. *$u_1-u_2\in\mathcal{H}''_{-c}$ is a boundary solution to the MCYBE (in particular, $MCYBE_0(u_1-u_2)=0$). Restricting its action to the linear subspace $k(c)^{n,n}$ and setting $c=-n/2$ corresponds to the Jordanian r-matrix $r_{J,n}$.*

As an operator on the polynomial representation of $\mathcal{H}''_c$,

$$u_1-u_2=\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2}-2cS.$$

This gives us a one-parameter family of triangular r-matrices.

In [9], Gerstenhaber and Giaquinto conjecture that $r_{J,n}^3=0$. We use the above interpretation to show that this is not true and to compute the nilpotency index.

THEOREM 4.3. (*Gerstenhaber–Giaquinto Conjecture*) *The nilpotency index of $r_{J,n}\in\mathcal{H}''_{n/2}$ is n when n is odd and $2n-1$ when n is even.*

Proof. Set $x=\frac{1}{2}(x_1-x_2)$, $x'=\frac{1}{2}(x_1+x_2)$. We have $[u_1-u_2,x']=0$ and $(u_1-u_2)x^m=(m-n[[m]])x^{m-1}$. Here $[[m]]=1$ if m is odd and $[[m]]=0$ otherwise. Observe that

$$x_1^ix_2^j=(-1)^jx^{i+j}+(-1)^j(i-j)x^{i+j-1}x'+\cdots.$$

One computes that in the case when n is odd, we have

$$(u_1-u_2)^{n-1}x_2^{n-1}=(n-1)(-2)(n-3)(-4)\cdots(3-n)(2)(1-n)\neq0$$

and for all $0 \le i, j \le n-1$, $(u_1 - u_2)^n (x_1^i x_2^j) = 0$. In the case when n is even, one computes

$$(u_1 - u_2)^{2(n-1)} (x_1^{n-1} x_2^{n-1}) = (2n-2)(n-3)(2n-4)(n-5) \cdots (3-n)(2)(1-n) \neq 0$$

and for all $1 \le i, j \le n-1$, $(u_1 - u_2)^{2n-1} (x_1^i x_2^j) = 0$. $\square$

Remarks. For $n \geqslant 2$, the Cremmer–Gervais r-matrix r_n is not nilpotent except when $c = -1/2$ and $n = 2$. One can see this by viewing r_n as the operator

$$X_1 \frac{\partial}{\partial X_1} - X_2 \frac{\partial}{\partial X_2} + c\,(X_1 + X_2)\,S$$

and verifying $r_n^2(X_1 - X_2) = (1+2c)(X_1 - X_2)$ and $r_n^2(X_1^2 - X_2^2) = 4(1+c)(X_1^2 - X_2^2)$. So in this particular case, nilpotency is only a boundary condition.

The conceptual difference between the even and odd cases in Theorem 4.3 has a representation theoretic origin: the polynomial representation of the rational Cherednik algebra $\mathcal{H}_c''$ of type A_1 is reducible if and only if the deformation parameter c has the form $n/2$ for an odd positive integer n (a special case of Dunkl's theorem [2]).

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