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Superfield theories on S^3 and their localization

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Abstract: We consider the superfield formulation of supersymmetric gauge and matter field theories on a three-dimensional sphere with rigid $\mathcal{N}=2$ supersymmetry, as well as with $\mathcal{N}>2$. The construction is based on a supercoset SU(2|1)/U(1) containing S^3 as the bosonic subspace. We derive an explicit form of SU(2|1)/U(1) supervielbein and covariant derivatives, and use them to construct classical superfield actions for gauge and matter supermultiplets in this superbackground. We then apply superfield methods for computing one-loop partition functions of these theories and demonstrate how the localization technique works directly in the superspace.

Keywords: Supersymmetric gauge theory, Extended Supersymmetry, Superspaces, Chern-Simons Theories

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Contents

1 Introduction

Supersymmetric field theories on curved backgrounds with rigid supersymmetries are in an intermediate position between locally supersymmetric field theories coupled to supergravity and those in flat space. Although these theories describe field dynamics in curved space-time, they share many properties of corresponding field theories in flat space, in particular, when the theory is (super)conformal and the background is conformally flat. In such cases results of quantum computations performed in curved (compact) backgrounds can be extrapolated to the flat-space field theory.

For field theories on curved backgrounds with rigid supersymmetry there is a special tool which allows one to compute quantum objects, such as the partition function, correlators or Wilson loops *exactly*, beyond the perturbation theory. This is the so-called localization method (see e.g. [1] for a review and references) whose efficiency was exploited by Pestun [2] for studying non-perturbative aspects of four-dimensional superconformal field theories on S^4. Subsequently, this technique was extended to field theories in diverse dimensions and to other interesting curved supersymmetric backgrounds. It has proved to be one of the most powerful approaches to study quantum dynamics of supersymmetric field theories non-perturbatively.

These developments brought into the foreground the problem of a systematic construction of classical actions for field models on curved backgrounds with rigid supersymmetry, which until recently was mainly of an academic interest. Within the component field formulation, the systematic approach for solving this problem was developed in [3–6]. The prescription is to couple a supersymmetric field model to off-shell supergravity (which requires the presence of auxiliary fields) and then to 'freeze' a supergravity background such that it preserves some number of supersymmetries. In the limit of large Plank mass the gravity fluctuations decouple and one is left with the field theory model on the fixed curved background which, by construction, respects the supersymmetries of the background.

Within the superfield formulation of supergravity and supersymmetric field theories (see, e.g., [7, 8]) the prescription of [3] is carried out straightforwardly, since the superfield formulations include all the necessary auxiliary fields which automatically receive correct values when one fixes the superfield background. So, in superspace one can, in principle, construct any field theory on curved background with rigid supersymmetries when the corresponding superfield actions in flat superspace are available and a curved superbackground possessing superisometries is chosen.

The problem is to solve superfield supergravity constraints for a given superbackground and to find an explicit form of the superfield objects, such as supervielbeins and superconnections, which encode its geometry. This problem is drastically simplified when the background superspace has the structure of a supercoset manifold G/H (as e.g. a supersphere, or an AdS superspace) with G being the isometry supergroup and H being its stability subgroup. In these cases the superbackground geometry is described by Cartan superforms on G/H, which satisfy corresponding Maurer-Cartan equations. The derivation of an explicit form of the G/H Cartan superforms as series expansions in powers of Grassmann-odd coordinates is carried out by conventional group-theoretical methods.

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Once this is done, it is straightforward to consider field models in such a curved superspace. We will follow exactly this strategy and develop basic methods for studying some classical and quantum aspects of such theories.

We will mainly consider three-dimensional gauge and matter field theories with $\mathcal{N}=2$ supersymmetry, i.e. with four supercharges, on the round S^3 sphere, but will also discuss $\mathcal{N}=2$ superfield formulations of $\mathcal{N}=4,6$ and 8 supersymmetric theories. The appropriate superspace with four Grassmann-odd directions, whose bosonic subspace is S^3, is the supercoset SU(2|1)/U(1). For this supercoset we construct explicitly all the basic geometric objects such as supervielbeins, superconnection, supertorsion and supersymmetric covariant derivatives. We consider superfield actions on SU(2|1)/U(1) which are, in fact, Euclidean counterparts of superfield models in an AdS_3 superspace considered in [9–12]. Next, we develop methods of quantum one-loop computations for such superfield theories and show how to apply the localization technique to the Chern-Simons theory in $\mathcal{N}=2$ superspace which was considered originally in [13, 14] employing conventional component fields.

The superspace and superfield techniques allow us to make several simple observations about field theories on S^3 with rigid supersymmetries. For instance, we find that the supervolume of SU(2|1)/U(1) vanishes,

$$\int d^3x d^2\theta d^2\bar{\theta}\, E = 0\,, \tag{1.1}$$

where $E = \mathrm{Ber} E_M{}^A$ is the Berezinian of the SU(2|1)/U(1) supervielbein. In particular, for the $\mathcal{N}=2$ super-Yang-Mills theory this fact trivializes the problem of finding critical points, i.e. the values of (super)fields for which the SYM action vanishes,

$$0 = S_{\rm SYM} \propto \mathrm{tr}\int d^3x d^2\theta d^2\bar{\theta}\, E\, G^2 \quad \Rightarrow \quad G = const. \tag{1.2}$$

Here G is the superfield strength of the $\mathcal{N}=2$ gauge superfield $V(x,\theta,\bar{\theta})$. In components this superfield starts with the scalar $\sigma(x)$ which is part of the $\mathcal{N}=2$, $d=3$ gauge supermultiplet. As we will show, in a certain supersymmetric gauge the vanishing of the SU(2|1)/U(1) supervolume also trivializes the contribution of the gauge supermultiplet into the SYM partition function which acquires non-trivial structure due to Faddeev-Popov and Nielsen-Kallosh ghosts.

We will also show that the geometry of the supercoset SU(2|1)/U(1) is superconformally flat. This property is useful for extending quantum superfield methods from flat superspace to SU(2|1)/U(1).

When constructing an $\mathcal{N}=4$ supersymmetric extension of the $\mathcal{N}=2$ SYM theory on S^3 by adding to the latter a chiral matter superfield, we come across the fact that when the chiral superfield carries a non-zero $\mathcal{N}=2$ R-charge, the invariance of the SYM action under $\mathcal{N}=4$ supersymmetry, in general, requires the presence of a Chern-Simons term (see section 4.1 for details). In the component formulation this fact was first noticed in [15] using $\mathrm{SU}(2)_{\rm R}$ symmetry arguments. In this paper we will present an explicit form of the $\mathcal{N}=4$ supersymmetry transformations on S^3, which to the best of our knowledge have not been given in the literature before.

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Finally, we point out that the superfield approach is quite useful at the quantum level. The localization method effectively reduces functional integrals to the problem of computing one-loop determinants of operators of quadratic fluctuations of bosonic and fermionic fields around critical points (see, e.g., [16] for a review). As a rule, these one-loop determinants are given by simple elementary functions since many bosonic and fermionic modes cancel against each other due to supersymmetry. As we will show, in the superfield gauge theories on S^3 the one-loop determinants correspond to supersymmetric operators acting on superfields propagating on the coset SU(2|1)/U(1). For such operators the pairing of bosonic and fermionic modes is automatic, since the gauge fixing is supersymmetric. This is a useful feature of the superspace approach.

The main part of this paper is organized as follow. In section 2 we consider the geometry of the supercoset SU(2|1)/U(1). In particular, we construct in a suitable chiral basis the supervielbeins, supercurvature, supercovariant derivatives and the Killing supervector. The geometry of SU(2|1)/U(1) is shown to be superconformally flat. In section 3 we introduce classical $\mathcal{N} = 2$ superfield actions for gauge and matter fields on SU(2|1)/U(1). Section 4 is devoted to constructing $\mathcal{N} = 2$ superfield actions for models with extended supersymmetry, such as $\mathcal{N} = 4$ SYM and Gaiotto-Witten theories, $\mathcal{N} = 8$ SYM and $\mathcal{N} = 6$ ABJM theory. In section 5 we develop superfield methods of one-loop quantum computations in $\mathcal{N} = 2$ SYM and chiral matter models on SU(2|1)/U(1) and use them, in particular, for computing one-loop partition functions. In section 6 we consider how the localization techniques works for the $\mathcal{N} = 2$ Chern-Simons theory in the superfield form. Section 7 is devoted to discussions of the results and perspectives. In appendices we collect details of direct computations of determinants of supersymmetric operators and revisit component field calculations of the SYM partition function.

2 SU(2|1)/U(1) supergeometry

2.1 $su(2|1)$ superalgebra

We are interested in field theories on S^3 which are invariant under the SU(2|1) supergroup. We would like to describe these theories in a superspace whose isometries include SU(2|1).[1] So, we need a superspace with three bosonic variables x^m, $m = 1, 2, 3$ and four Grassmann-odd variables θ^μ, $\bar\theta^\mu$, $\mu = 1, 2$ such that its bosonic body is the sphere S^3. The SU(2)×SU(2) isometry of S^3 naturally embeds into the supergroup SU(2|1) × SU(2), so one can realize the superspace in question as the supercoset

$$\frac{\mathrm{SU}(2|1)\times\mathrm{SU}(2)}{\mathrm{U}(1)\times\mathrm{SU}(2)}\,. \tag{2.1}$$

Formally, the SU(2) factors cancel against each other. Hence, we can obtain the same superspace by considering a simpler coset

$$\frac{\mathrm{SU}(2|1)}{\mathrm{U}(1)}\,. \tag{2.2}$$

[1]For the construction of quantum mechanical models on different cosets of SU(2|1) see e.g. [17, 18] and references therein.

The only price for this is that not all the SU(2) × SU(2) isometries of S^3 are explicit in this case. However, the second SU(2) symmetry is realized as the group of external automorphisms of the $su(2|1)$ algebra and, hence, can be easily included in the construction.

The $su(2|1)$ (anti)commutation relations are

$$
\begin{aligned}
&[M_a, M_b] = \frac{2i}{r}\varepsilon_{abc}M_c\,,\\
&[M_a, Q_\alpha] = -\frac{1}{r}(\gamma_a)_\alpha^\beta Q_\beta\,, \quad [M_a, \bar{Q}_\alpha] = -\frac{1}{r}(\gamma_a)_\alpha^\beta \bar{Q}_\beta\,,\\
&\{Q_\alpha, \bar{Q}_\beta\} = \gamma^a_{\alpha\beta}M_a + \frac{1}{r}\varepsilon_{\alpha\beta}R\,, \quad [R, Q_\alpha] = -Q_\alpha\,, \quad [R, \bar{Q}_\alpha] = \bar{Q}_\alpha
\end{aligned}
\tag{2.3}
$$

(all other (anti)commutators vanish.) Here M_a, $(a = 1, 2, 3)$ are three generators of the SU(2) subgroup, while R is the U(1) R-symmetry generator and Q_α and $\bar{Q}_\alpha$, $(\alpha = 1, 2)$ are the Grassmann-odd supersymmetry generators. The parameter r is the radius of the sphere and $(\gamma^a)_\alpha^\beta$ are the Pauli matrices. For the details on our notation and conventions see appendix A.

The group of the external SU(2) automorphisms of the $su(2)$ algebra is generated by an independent set of three generators L_a

$$
[L_a, L_b] = 2i\varepsilon_{abc}L_c\,, \tag{2.4}
$$

whose commutation relations with the SU(2|1) generators are

$$
[L_a, M_b] = 2i\varepsilon_{abc}M_c\,, \quad [L_a, Q_\alpha] = -(\gamma_a)_\alpha^\beta Q_\beta\,, \quad [L_a, \bar{Q}_\alpha] = -(\gamma_a)_\alpha^\beta \bar{Q}_\beta \quad [L_a, R] = 0\,. \tag{2.5}
$$

The generators M_a and L_a form the SO(4) $\sim$ SU(2) × SU(2) isometry of S^3 with the two SU(2)'s being generated by M_a and $(L_a - \sqrt{r}M_a)$, respectively. Note that the latter commute with the whole SU(2|1).

In the limit $r \to \infty$ the algebra (2.3), (2.4) and (2.5) reduces to the standard three-dimensional Euclidean "Poincaré" superalgebra in which M_a play the role of commuting momenta operators and L_a generate the SO(3) $\sim$ SU(2) rotations in flat $3d$ space.

The superalgebra (2.3) is invariant under the following Hermitian conjugation of the generators

$$
(M_a)^\dagger = M_a\,, \qquad R^\dagger = R\,, \qquad (Q_\alpha)^\dagger = \bar{Q}^\alpha\,. \tag{2.6}
$$

Note that the spinor index changes its position under the conjugation since the spinor group is SU(2).

In the rest of this section we will derive, using the superalgebra (2.3), an explicit form of supersymmetric vielbeins, connections, torsion, curvature and covariant derivatives on the supercoset SU(2|1)/U(1) with the aim of using them afterwards for the construction of superfield actions.

2.2 Supervielbein

Let $z^M = (x^m, \theta^\mu, \bar{\theta}^\mu)$ be local coordinates parametrizing the supercoset SU(2|1)/U(1). In principle, the coordinates θ^μ and $\bar{\theta}^\mu$ can be related to each other by complex conjugation,

$(\theta^\mu)^* = \bar\theta_\mu$, in accordance with the conjugation rules (2.6) of the operators Q_α and $\bar Q_\alpha$. However, in a $d = 3$ superspace with the metric of Lorentzian signature the spinor group is $\mathrm{SL}(2,\mathbb{R})$ and the spinor index does not change its position under conjugation. We wish to consider superfield models on SU(2|1)/U(1) which are related by Wick rotation to the corresponding models in the AdS_3 superspace, considered, e.g., in [9–12]. Clearly, such Wick-rotated models are not necessary real under the conjugation (2.6). Therefore, in what follows we will treat the complex coordinates θ^μ and $\bar\theta^\mu$ as independent ones, i.e. not related to each other by the complex conjugation.

The SU(2|1)/U(1) supervielbein is given by the set of one-forms,

$$E^A = dz^M E_M{}^A(z)\,, \qquad E^A = (E^a, E^\alpha, \bar E^\alpha)\,. \tag{2.7}$$

They are components of the SU(2|1) Cartan form

$$G^{-1}dG = iE^a M_a + iE^\alpha Q_\alpha + i\bar E^\alpha \bar Q_\alpha + i\Omega_{(R)} R \equiv \omega\,, \tag{2.8}$$

where $G(z^M)$ is a representative of the supercoset SU(2|1)/U(1) and $\Omega_{(R)}$ is the U(1)-connection. In particular, one can consider the following coset representative

$$G = b(x) f(\theta,\bar\theta)\,, \quad b(x) = e^{ix^m M_m}\,, \quad f(\theta,\bar\theta) = e^{i\theta^\alpha Q_\alpha} e^{i\bar\theta^\beta \bar Q_\beta}\,, \tag{2.9}$$

such that

$$G^{-1}dG = f^{-1}(d + ie^a(x)M_a)f\,, \tag{2.10}$$

where $e^a(x) = dx^m e^a_m(x)$ is the bosonic vielbein on $S^3 \sim \mathrm{SU}(2)$. Applying the algebra (2.3) we find the components of the supervielbein in the decomposition (2.8) explicitly,[2]

$$\begin{aligned} E^\alpha &= \mathbf{d}\theta^\alpha\,, \\ \bar E^\alpha &= \mathbf{d}\bar\theta^\alpha - \frac{1}{r}\mathbf{d}\theta^\alpha\,\bar\theta^2\,, \\ E^a &= e^a - i\mathbf{d}\theta^\alpha \gamma^a_{\alpha\beta}\bar\theta^\beta\,, \end{aligned} \tag{2.11}$$

where $\mathbf{d}$ is the Killing-spinor covariant differential,

$$\mathbf{d}\theta^\alpha = d\theta^\alpha - \frac{i}{r} e^a (\gamma_a)^\alpha_\beta \theta^\beta\,, \qquad \mathbf{d}^2 = 0\,. \tag{2.12}$$

The U(1)-connection of the R-symmetry has also very simple form,

$$\Omega_{(R)} = -\frac{i}{r}\mathbf{d}\theta^\alpha\bar\theta_\alpha = -\frac{i}{r}E^\alpha\bar\theta_\alpha\,. \tag{2.13}$$

It is easy to see that the SU(2|1)/U(1) supergeometry constructed in this way has a smooth flat limit at $r \to \infty$. Note that the components E^α and $\bar E^\alpha$ enter in (2.11) asymmetrically. Therefore we refer to the basis defined by the coset representative (2.9) as the chiral basis.

[2] We use the following conventions for the contractions of spinor indices: $\theta^2 = \theta^\alpha\theta_\alpha$, $\bar\theta^2 = \bar\theta^\alpha\bar\theta_\alpha$. The spinor indices are raised and lowered by the rules $\theta_\alpha = \varepsilon_{\alpha\beta}\theta^\beta$, $\theta^\alpha = \varepsilon^{\alpha\beta}\theta_\beta$, $\varepsilon_{12} = -\varepsilon^{12} = 1$, see appendix A.

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Consider now the inverse supervielbein, i.e. the differential operator of the form

$$E_A = E_A{}^M \partial_M \,, \qquad E_A{}^M E_M{}^B = \delta_A^B \,. \tag{2.14}$$

In the chiral coordinates corresponding to the choice of the coset representative (2.9) its components have the following explicit form

$$\begin{aligned}
E_a &= \partial_a + \frac{i}{r}(\gamma_a)^\alpha_\beta \theta^\beta \partial_\alpha + \frac{i}{r}(\gamma_a)^\alpha_\beta \bar\theta^\beta \bar\partial_\alpha \,, \\
E_\alpha &= \partial_\alpha + i\gamma^a_{\alpha\beta}\bar\theta^\beta \partial_a - \frac{1}{r}\theta^\beta \bar\theta_\beta \partial_\alpha + \frac{1}{r}\theta_\alpha \bar\theta^\beta \partial_\beta - \frac{1}{2r}\bar\theta^2 \bar\partial_\alpha \,, \\
\bar E_\alpha &= \bar\partial_\alpha \,.
\end{aligned} \tag{2.15}$$

Here $\partial_a = e^m_a(x)\partial_m$ is the differential operator on S^3 with the commutation relations $[\partial_a, \partial_b] = -\frac{2}{r}\varepsilon_{abc}\partial_c$. The differential operators (2.15) obey the following algebra

$$\begin{aligned}
\{E_\alpha, \bar E_\beta\} &= i\gamma^a_{\alpha\beta}E_a + \frac{1}{r}\bar\theta_\alpha \bar E_\beta \,, \qquad \{E_\alpha, E_\beta\} = -\frac{2}{r}\bar\theta_{(\alpha}E_{\beta)} \,, \\
[E_a, \bar E_\alpha] &= -\frac{i}{r}(\gamma_a)^\beta_\alpha \bar E_\beta \,, \qquad [E_a, E_\alpha] = -\frac{i}{r}(\gamma_a)^\beta_\alpha E_\beta \,, \\
[E_a, E_b] &= -\frac{2}{r}\varepsilon_{abc}E_c \,.
\end{aligned} \tag{2.16}$$

It is interesting to note that the Berezinian of the supervielbein is independent of the Grassmann variables,

$$E \equiv \mathrm{Ber} E_M{}^A = \det e^a_m(x) = \sqrt{h(x)} \,, \tag{2.17}$$

where $h(x) = \det h_{mn}(x)$ and $h_{mn}(x)$ is a purely bosonic metric on S^3. The expression (2.17) is obtained for a particular choice of the coset representative (2.9), *i.e.* it corresponds to the chiral coordinates on SU(2|1)/U(1). However, the coordinate-independent consequence of (2.17) is the fact that the supervolume of the supercoset SU(2|1)/U(1) vanishes

$$\int d^3x d^2\theta d^2\bar\theta \, E = 0 \,. \tag{2.18}$$

In section 2.5 this property will also be checked in a different (superconformally flat) basis.

2.3 Connection, torsion and curvature

By construction, the differential form ω given in (2.8) obeys the Maurer-Cartan equation,

$$d\omega + \frac{1}{2}[\omega, \omega] = 0 \,. \tag{2.19}$$

The corresponding equations for the components of the supervielbein E^A and the U(1) connection $\Omega_{(R)}$ are

$$\begin{aligned}
dE^a - \frac{1}{r}\varepsilon^{abc}E^b \wedge E^c - iE^\alpha \wedge \bar E^\beta \gamma^a_{\alpha\beta} &= 0 \,, \\
dE^\alpha - i\Omega_{(R)} \wedge E^\alpha - \frac{i}{r}E^a \wedge E^\beta (\gamma_a)^\alpha_\beta &= 0 \,, \\
d\bar E^\alpha + i\Omega_{(R)} \wedge \bar E^\alpha - \frac{i}{r}E^a \wedge E^\beta (\gamma_a)^\alpha_\beta &= 0 \,, \\
d\Omega_{(R)} - \frac{i}{r}\varepsilon_{\alpha\beta}E^\alpha \wedge \bar E^\beta &= 0 \,.
\end{aligned} \tag{2.20}$$

Let us introduce the superconnection Ω^{AB} with the following non-vanishing components Ω^{ab}, Ω^{α}_{β} and $\bar{\Omega}^{\alpha}_{\beta}$:

$$\begin{aligned}\Omega^{ab} &= \frac{1}{r}\varepsilon^{abc}E^{c}\,,\\ \Omega^{\alpha}_{\beta} &= -\frac{i}{2r}(\gamma^{a})^{\alpha}_{\beta}E^{a} - i\delta^{\alpha}_{\beta}\Omega_{(R)}\,,\\ \bar{\Omega}^{\alpha}_{\beta} &= -\frac{i}{2r}(\gamma^{a})^{\alpha}_{\beta}E^{a} + i\delta^{\alpha}_{\beta}\Omega_{(R)}\,.\end{aligned} \tag{2.21}$$

The superconnection Ω appears in the covariant differential,

$$\mathcal{D} = d + \Omega\,. \tag{2.22}$$

In particular, the equations (2.20) take the form

$$\mathcal{D}E^{A} = dE^{A} + \Omega^{AB}\wedge E^{B} = T^{A}\,, \tag{2.23}$$

where the supertorsion T^{A} has the following components

$$\begin{aligned}T^{a} &= i\gamma^{a}_{\alpha\beta}E^{\alpha}\wedge\bar{E}^{\beta}\,,\\ T^{\alpha} &= \frac{i}{2r}(\gamma_{a})^{\alpha}_{\beta}E^{a}\wedge E^{\beta}\,,\\ \bar{T}^{\alpha} &= \frac{i}{2r}(\gamma_{a})^{\alpha}_{\beta}E^{a}\wedge\bar{E}^{\beta}\,.\end{aligned} \tag{2.24}$$

Given the superconnection Ω^{AB} we construct the supercurvature,

$$\mathcal{R}^{AB} = d\Omega^{AB} + \Omega^{AC}\wedge\Omega^{CB}\,, \tag{2.25}$$

or, explicitly,

$$\begin{aligned}\mathcal{R}^{ab} &= d\Omega^{ab} + \Omega^{ac}\wedge\Omega^{cb} = \frac{1}{r^{2}}E^{a}\wedge E^{b} + \frac{i}{r}\varepsilon^{abc}\gamma^{c}_{\alpha\beta}E^{\alpha}\wedge\bar{E}^{\beta}\,,\\ \mathcal{R}^{\alpha}_{\beta} &= d\Omega^{\alpha}_{\beta} + \Omega^{\alpha}_{\gamma}\wedge\Omega^{\gamma}_{\beta}\\ &= -\frac{i}{4r^{2}}\varepsilon^{abc}(\gamma^{c})^{\alpha}_{\beta}E^{a}\wedge E^{b} - \frac{1}{2r}(\delta^{\alpha}_{\rho}\varepsilon_{\beta\sigma} + \delta^{\alpha}_{\sigma}\varepsilon_{\beta\rho} - 2\delta^{\alpha}_{\beta}\varepsilon_{\rho\sigma})E^{\rho}\wedge\bar{E}^{\sigma}\,,\\ \bar{\mathcal{R}}^{\alpha}_{\beta} &= d\bar{\Omega}^{\alpha}_{\beta} + \bar{\Omega}^{\alpha}_{\gamma}\wedge\bar{\Omega}^{\gamma}_{\beta}\\ &= -\frac{i}{4r^{2}}\varepsilon^{abc}(\gamma^{c})^{\alpha}_{\beta}E^{a}\wedge E^{b} - \frac{1}{2r}(\delta^{\alpha}_{\rho}\varepsilon_{\beta\sigma} + \delta^{\alpha}_{\sigma}\varepsilon_{\beta\rho} + 2\delta^{\alpha}_{\beta}\varepsilon_{\rho\sigma})E^{\rho}\wedge\bar{E}^{\sigma}\,.\end{aligned} \tag{2.26}$$

These equations can be rewritten in one line,

$$\mathcal{R} = -\frac{i}{4r}M^{ab}E^{a}\wedge E^{b} + \left(\frac{1}{2}M^{a}\gamma^{a}_{\alpha\beta} - R\varepsilon_{\alpha\beta}\right)E^{\alpha}\wedge\bar{E}^{\beta}\,, \tag{2.27}$$

where we assume that the momentum operator M^{ab} acts on the tangent space vectors v^{a} and spinors ψ^{α} by the rule

$$M^{a}v^{b} = \frac{2i}{r}\varepsilon^{abc}v^{c}\,,\qquad M^{a}\psi^{\alpha} = \frac{1}{r}(\gamma^{a})^{\alpha}_{\beta}\psi^{\beta}\,. \tag{2.28}$$

The R-symmetry generator acts on a complex superfield Φ as follows

$$R\Phi = -q\Phi\,,\qquad R\bar{\Phi} = q\bar{\Phi}\,, \tag{2.29}$$

where q is the R-charge of the field.

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2.4 Covariant derivatives

Consider the covariant derivatives on the supercoset SU(2|1)/U(1),

$$\mathcal{D}_A = E_A + \Omega_A = (\mathcal{D}_a, \mathcal{D}_\alpha, \bar{\mathcal{D}}_\alpha)\,. \tag{2.30}$$

They appear in the decomposition of the covariant differential (2.22) in the tangent-space basis formed by the supervielbein,

$$\mathcal{D} = d + \Omega = E^A \mathcal{D}_A = E^a \mathcal{D}_a + E^\alpha \mathcal{D}_\alpha + \bar{E}^\alpha \bar{\mathcal{D}}_\alpha\,. \tag{2.31}$$

To find the algebra of the covariant derivatives we use the fact that the covariant differential squares to the curvature, $\mathcal{D}^2 = \mathcal{R}$. This implies that

$$T^A \mathcal{D}_A - E^A \wedge E^B \mathcal{D}_B \mathcal{D}_A = \mathcal{R}\,. \tag{2.32}$$

We plug the explicit expressions for the supercurvature (2.27) and supertorsion (2.24) into this equation and obtain the SU(2|1) (anti)commutation relations,

$$\begin{aligned}
&[\mathcal{D}_a, \mathcal{D}_b] = -\frac{i}{2r} M_{ab}\,, \quad [\mathcal{D}_a, \mathcal{D}_\alpha] = -\frac{i}{2r} (\gamma_a)_\alpha^\beta \mathcal{D}_\beta\,, \quad [\mathcal{D}_a, \bar{\mathcal{D}}_\alpha] = -\frac{i}{2r} (\gamma_a)_\alpha^\beta \bar{\mathcal{D}}_\beta\,,\\
&\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_\beta\} = i\gamma^a_{\alpha\beta} \mathcal{D}_a - \frac{1}{2} \gamma^a_{\alpha\beta} M_a + \frac{1}{r} \varepsilon_{\alpha\beta} R\,,\\
&\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = \{\bar{\mathcal{D}}_\alpha, \bar{\mathcal{D}}_\beta\} = 0\,.
\end{aligned} \tag{2.33}$$

The generators M_{ab} and R have the following commutators with $\mathcal{D}_A$

$$\begin{aligned}
&[M_{ab}, \mathcal{D}_c] = \frac{2i}{r} (\delta_{ac} \mathcal{D}_b - \delta_{bc} \mathcal{D}_a)\,,\\
&[M_{ab}, \mathcal{D}_\alpha] = -\frac{1}{r} \varepsilon_{abc} (\gamma^c)_\alpha^\beta \mathcal{D}_\beta\,, \quad [M_{ab}, \bar{\mathcal{D}}_\alpha] = -\frac{1}{r} \varepsilon_{abc} (\gamma^c)_\alpha^\beta \bar{\mathcal{D}}_\beta\,,\\
&[R, \mathcal{D}_\alpha] = \mathcal{D}_\alpha\,, \qquad [R, \bar{\mathcal{D}}_\alpha] = -\bar{\mathcal{D}}_\alpha\,.
\end{aligned} \tag{2.34}$$

The covariant derivatives $\mathcal{D}_A = E_A + \Omega_A$ can be written explicitly in the chiral coordinates corresponding to the coset representative (2.9). To this end, we need to find the form of the superconnection Ω_A in these coordinates,

$$\Omega_A = i\Omega_{(R)A} R + \frac{i}{2} \Omega_{ab\,A} M_{ab}\,, \tag{2.35}$$

where the components of $\Omega_{(R)A}$ and $\Omega_{ab\,A}$ read

$$\begin{aligned}
&\Omega_{(R)a} = 0\,, \qquad \Omega_{(R)\alpha} = -\frac{i}{r} \bar{\theta}_\alpha\,, \qquad \bar{\Omega}_{(R)\alpha} = 0\,,\\
&\Omega_{ab\,c} = -\frac{1}{2} \varepsilon_{abc}\,, \quad \Omega_{ab\,\alpha} = \bar{\Omega}_{ab\,\alpha} = 0\,.
\end{aligned} \tag{2.36}$$

Now recall that the supervielbein in these coordinates is given in (2.15), so combining the above expressions with (2.15) we get

$$\begin{aligned}
\mathcal{D}_a &= \partial_a - \frac{i}{2} M_a + \frac{i}{r} (\gamma_a)_\beta^\alpha \theta^\beta \partial_\alpha + \frac{i}{r} (\gamma_a)_\beta^\alpha \bar{\theta}^\beta \bar{\partial}_\alpha\,,\\
\mathcal{D}_\alpha &= \partial_\alpha + i\gamma^a_{\alpha\beta} \bar{\theta}^\beta \partial_a - \frac{1}{r} \theta^\beta \bar{\theta}_\beta \partial_\alpha + \frac{1}{r} \theta_\alpha \bar{\theta}^\beta \partial_\beta - \frac{1}{2r} \bar{\theta}^2 \bar{\partial}_\alpha + \frac{1}{r} \bar{\theta}_\alpha R\,,\\
\bar{\mathcal{D}}_\alpha &= \bar{\partial}_\alpha\,.
\end{aligned} \tag{2.37}$$

One can check that these differential operators obey the algebra (2.33) and (2.34). Note that the covariant derivative $\bar{\mathcal{D}}_\alpha$ is short as it should be in the chiral coordinate basis.

2.5 Superconformal flatness

On general grounds [19], it is natural to expect that the supercoset SU(2|1)/U(1) should be superconformally flat, since SU(2|1)/U(1) is an Euclidean counterpart of the AdS_3 superspace $\frac{\mathrm{OSp}(2|2)\times \mathrm{Sp}(2)}{\mathrm{SO}(2)\times \mathrm{Sp}(2)}$ which was demonstrated to be superconformally flat in [10, 11]. Here we prove this explicitly by showing that the covariant derivatives on SU(2|1)/U(1) are related to flat superspace derivatives by means of a super Weyl transformation.

Let $z^m = (x^m, \theta^\alpha, \bar\theta^\alpha)$ be the coordinates on the flat Euclidian $\mathcal{N}=2$, $d=3$ superspace. In the flat case there is no difference between the indices of the local coordinates x^m and tangent space, x^a, i.e., $\partial_a = \partial_m = \frac{\partial}{\partial x^m}$. The flat covariant spinor derivatives in the chiral basis are given by $D_M = (\partial_m, D_\alpha, \bar D_\alpha)$,

$$D_\alpha = \partial_\alpha + i\gamma^a_{\alpha\beta}\bar\theta^\beta\partial_a\,, \quad \bar D_\alpha = \bar\partial_\alpha\,, \quad \{D_\alpha, \bar D_\beta\} = i\gamma^a_{\alpha\beta}\partial_a\,. \tag{2.38}$$

Following [10, 19], we construct the operators

$$\begin{aligned}
\mathcal{D}_\alpha &= e^{\frac12\rho}(D_\alpha + \frac{r}{2}(D^\beta\rho)\gamma^a_{\alpha\beta}M_a - (D_\alpha\rho)R)\,,\\
\bar{\mathcal{D}}_\alpha &= e^{\frac12\rho}(\bar D_\alpha + \frac{r}{2}(\bar D^\beta\rho)\gamma^a_{\alpha\beta}M_a + (\bar D_\alpha\rho)R)\,,\\
\mathcal{D}_a &= e^\rho\Big(\partial_a + i\gamma_a^{\alpha\beta}(D_\alpha\rho)\bar D_\beta + i\gamma_a^{\alpha\beta}(\bar D_\alpha\rho)D_\beta\\
&\quad + \frac{ir}{2}(D^\alpha\rho)(\bar D_\alpha\rho)M_a + \frac{ir}{2}\varepsilon_{abc}\partial^b\rho M^c + i\gamma_a^{\alpha\beta}(D_{(\alpha}\rho)(\bar D_{\beta)}\rho)R\Big),
\end{aligned} \tag{2.39}$$

with $\rho(x,\theta,\bar\theta)$ being a scalar superfield. These operators happen to obey the algebra (2.33) of covariant derivatives of the supercoset SU(2|1)/U(1) under the condition that the superfield ρ solves for the following equations

$$D^2 e^{-\rho} = \bar D^2 e^{-\rho} = 0\,, \tag{2.40}$$
$$[D_{(\alpha}, \bar D_{\beta)}]e^\rho = 0\,, \tag{2.41}$$
$$e^\rho D^\alpha \bar D_\alpha \rho = \frac{1}{r}\,. \tag{2.42}$$

The equation (2.40) is nothing but the linearity condition for the superfield $e^{-\rho}$. Note that eq. (2.40) is not independent but appears as a differential consequence of (2.42).

The equations (2.39) allow us to expand the differential operator (2.14) in the basis of the covariant derivatives D_M (2.38),

$$E_A = (E_a, E_\alpha, \bar E_\alpha) = E_A{}^M\partial_M = \tilde E_A{}^M D_M\,. \tag{2.43}$$

The supermatrix $\tilde E_A{}^M$ has the following explicit form

$$\tilde E_A{}^M D_M = \begin{pmatrix} e^\rho\delta^m_a & ie^\rho\gamma_a^{\alpha'\beta}(\bar D_\beta\rho) & ie^\rho\gamma_a^{\alpha'\beta}(D_\beta\rho) \\ 0 & \delta^{\alpha'}_\alpha e^{\frac12\rho} & 0 \\ 0 & 0 & \delta^{\alpha'}_\alpha e^{\frac12\rho}\end{pmatrix}\begin{pmatrix}\partial_m \\ D_{\alpha'} \\ \bar D_{\alpha'}\end{pmatrix}. \tag{2.44}$$

The Berezinian of the inverse of this matrix reads

$$E = \mathrm{Ber} E_M{}^A = \mathrm{Ber} \tilde{E}_M{}^A = e^{-\rho} \,. \tag{2.45}$$

An important consequence of this equation is the vanishing of the volume of the SU(2|1)/U(1) superspace (already observed in the chiral basis in section 2.2)

$$\int d^3x d^2\theta d^2\bar\theta \, E = \int d^3x d^2\theta d^2\bar\theta \, e^{-\rho} = -\frac{1}{4}\int d^3x d^2\theta \, \bar D^2 e^{-\rho} = 0 \,. \tag{2.46}$$

The integral is zero owing to the linearity of $e^{-\rho}$, eq. (2.40).

2.6 Killing supervector

Let us consider how the SU(2|1) transformations act on the superfields. These are generated by the Killing supervector defined as follows.

Let us take a local supervector $\xi^A(z) = (\xi^a, \xi^\alpha, \bar\xi^\alpha)$, and construct an operator

$$\mathbb{K} = \xi^a \mathcal{D}_a + \xi^\alpha \mathcal{D}_\alpha + \bar\xi^\alpha \bar{\mathcal{D}}_\alpha - iL^{ab}(z) M_{ab} - il(z) R \,, \tag{2.47}$$

where $L^{ab}(z)$ and $l(z)$ are local SU(2) $\times$ U(1) parameters. ξ^A is said to be a Killing supervector if the operator $\mathbb{K}$ associated with ξ^A commutes with all the covariant derivatives [7]

$$[\mathbb{K}, \mathcal{D}_A] = 0 \,. \tag{2.48}$$

This equation defines the components of the supervector $\xi^A(z)$ as well as the superfunctions $L^{ab}(z)$ and $l(z)$ in (2.47).

The Killing supervector generates the isometries of the superspace and the corresponding symmetries of a dynamical system. In the case under consideration, it is responsible for the supersymmetries generated by Q_α and $\bar Q_\alpha$, the SU(2)-rotations M_a and the R-symmetry of the SU(2|1) algebra (2.3). The SU(2|1) variation of a given superfield Φ on SU(2|1)/U(1) is

$$\delta\Phi = \mathbb{K}\Phi \,. \tag{2.49}$$

It is worth noticing that the sphere S^3 has isometry SU(2) $\times$ SU(2), but only one of these SU(2)'s is taken into account by $\mathbb{K}$. To manifestly represent the full isometry group of S^3 one should start with the supercoset (2.1) rather than (2.2).

The equation (2.48) leads to a number of differential equations for the components of ξ^A, $L^{ab}(z)$ and $l(z)$

$[\mathcal{D}_a, \mathbb{K}] = 0 \Rightarrow$

$$\mathcal{D}_{(a}\xi_{b)} = 0 \,, \qquad \mathcal{D}_{[a}\xi_{b]} = \frac{4}{r} L_{ab} \,, \tag{2.50a}$$

$$\mathcal{D}_a L_{bc} = \frac{1}{4r}(\delta_{ac}\xi_b - \delta_{ab}\xi_c) \,, \tag{2.50b}$$

$$\mathcal{D}_a \xi^\alpha = \frac{i}{2r}(\gamma_a)^\alpha_\beta \xi^\beta \,, \qquad \mathcal{D}_a \bar\xi^\alpha = \frac{i}{2r}(\gamma_a)^\alpha_\beta \bar\xi^\beta \,, \tag{2.50c}$$

$$\mathcal{D}_a l = 0 \,; \tag{2.50d}$$

$[\mathcal{D}_\alpha, \mathbb{K}] = 0 \Rightarrow$

$$\mathcal{D}_\alpha \xi^a = i\bar{\xi}^\beta \gamma^a_{\alpha\beta}\,, \qquad \mathcal{D}_\alpha L^{ab} = -\frac{i}{4}\varepsilon^{abc}(\gamma^c)_{\alpha\beta}\bar{\xi}^\beta\,, \tag{2.51a}$$

$$\mathcal{D}_\alpha \xi^\alpha = -2il\,, \qquad \mathcal{D}_\alpha l = \frac{i}{r}\bar{\xi}_\alpha\,, \tag{2.51b}$$

$$\mathcal{D}_{(\alpha}\xi_{\beta)} = -\frac{i}{2r}\xi^a(\gamma_a)_{\alpha\beta} + \frac{i}{r}L^{ab}\varepsilon_{abc}(\gamma^c)_{\alpha\beta}\,, \tag{2.51c}$$

$$\mathcal{D}_\alpha \bar{\xi}^\beta = 0\,; \tag{2.51d}$$

$[\bar{\mathcal{D}}_\alpha, \mathbb{K}] = 0 \Rightarrow$

$$\bar{\mathcal{D}}_\alpha \xi^a = i\xi^\beta \gamma^a_{\alpha\beta}\,, \qquad \bar{\mathcal{D}}_\alpha L_{ab} = -\frac{i}{4}\varepsilon_{abc}\gamma^c_{\alpha\beta}\xi^\beta\,, \tag{2.52a}$$

$$\bar{\mathcal{D}}_\alpha \bar{\xi}^\alpha = 2il\,, \qquad \bar{\mathcal{D}}_\alpha l = -\frac{i}{r}\xi_\alpha\,, \tag{2.52b}$$

$$\bar{\mathcal{D}}_{(\alpha}\bar{\xi}_{\beta)} = -\frac{i}{2r}\xi^a(\gamma_a)_{\alpha\beta} + \frac{i}{r}L^{ab}\varepsilon_{abc}\gamma^c_{\alpha\beta}\,, \tag{2.52c}$$

$$\bar{\mathcal{D}}_\alpha \xi^\beta = 0\,. \tag{2.52d}$$

The relations (2.50)–(2.52) are analogous to those for the (2,0) AdS_3 superspace derived in [10–12].

The equations (2.51d) and (2.52d) show that $\xi^\alpha(z)$ is chiral while $\bar{\xi}^\alpha(z)$ is antichiral. All the other paramters are linear as a consequence of (2.51a), (2.51b), (2.52a) and (2.52b),

$$\mathcal{D}^2\xi^a = \bar{\mathcal{D}}^2\xi^a = 0\,, \quad \mathcal{D}^2 L^{ab} = \bar{\mathcal{D}}^2 L^{ab} = 0\,, \quad \mathcal{D}^2 l = \bar{\mathcal{D}}^2 l = 0\,. \tag{2.53}$$

The parameters L^{ab} and l are not independent as they can be expressed in terms of components of ξ^A. Indeed, from (2.51b) and (2.52b) we have

$$l = \frac{i}{2}\mathcal{D}_\alpha\xi^\alpha = -\frac{i}{2}\bar{\mathcal{D}}_\alpha\bar{\xi}^\alpha\,. \tag{2.54}$$

The second equation in (2.50a) implies

$$L^{ab} = -\frac{1}{4}\varepsilon^{abc}\xi^c\,. \tag{2.55}$$

Hence, the operator (2.47) is completely specified by the components of the supervector ξ^A which obey (2.50)–(2.52).

The general solution of (2.50)–(2.52) is

$$\begin{aligned}
\xi^\alpha &= \bar{\mathcal{D}}^2\mathcal{D}^\alpha\zeta\,, \qquad \bar{\xi}^\alpha = -\mathcal{D}^2\bar{\mathcal{D}}^\alpha\zeta\,,\\
\xi^a &= -2i\gamma^a_{\alpha\beta}\bar{\mathcal{D}}^\alpha\mathcal{D}^\beta\zeta\,,\\
L_{ab} &= \frac{i}{2}\varepsilon_{abc}\gamma^c_{\alpha\beta}\bar{\mathcal{D}}^\alpha\mathcal{D}^\beta\zeta\,, \qquad l = \frac{2i}{r}\bar{\mathcal{D}}^\alpha\mathcal{D}_\alpha\zeta\,,
\end{aligned} \tag{2.56}$$

where ζ is a covariantly constant superparameter with zero R-charge defined modulo gauge transformations,

$$\mathcal{D}_a\zeta = 0\,, \quad R\zeta = 0\,, \quad \zeta \sim \zeta - i\Lambda + i\bar{\Lambda}\,, \tag{2.57}$$

with Λ being a chiral and covariantly constant superfunction, $\bar{\mathcal{D}}_\alpha\Lambda = 0$, $\mathcal{D}_a\Lambda = 0$. In particular, with the use of $\zeta(z)$ the transformation of a chiral scalar superfield Φ, $\bar{\mathcal{D}}_\alpha\Phi = 0$, can be written as

$$\delta\Phi = \bar{\mathcal{D}}^2[(\mathcal{D}^\alpha\zeta)(\mathcal{D}_\alpha\Phi)] . \tag{2.58}$$

Indeed, using the algebra of the covariant derivatives (2.33) the variation (2.58) can be rewritten in the form (2.49) in which the components of the Killing supervector are given by (2.56). The Killing vector in the form (2.56) and the corresponding superfield transformations (2.58) derived above will be applied in section 4 where superfield models with extended supersymmetry are considered.

In conclusion of this section we present an explicit expression for the operator $\mathbb{K}$ in chiral coordinates in which the covariant derivatives have the form (2.37):

$$\mathbb{K} = b^a\mathbb{M}_a + \epsilon^\alpha\mathbb{Q}_\alpha + \bar\epsilon^\alpha\bar{\mathbb{Q}}_\alpha + t\mathbb{R} , \tag{2.59}$$

where

$$\begin{aligned}
\mathbb{M}_a &= -i\Lambda_{ab}\partial_b ,\\
\mathbb{Q}_\alpha &= -i\Lambda_\alpha{}^\beta\partial_\beta ,\\
\bar{\mathbb{Q}}_\alpha &= -i\Lambda_\alpha{}^\beta[\bar\partial_\beta - i\gamma^a_{\beta\gamma}\theta^\gamma\partial_a + \frac{1}{2r}\theta^2\partial_\beta - \frac{1}{r}\theta_\beta\bar\theta^\gamma\bar\partial_\gamma + \frac{1}{r}\theta_\beta R] ,\\
\mathbb{R} &= \bar\theta^\alpha\bar\partial_\alpha - \theta^\alpha\partial_\alpha - R .
\end{aligned} \tag{2.60}$$

Here $\Lambda_a{}^b$ and $\Lambda_\alpha{}^\beta$ are purely bosonic local SO(3) $\sim$ SU(2) matrices which obey the relations

$$\partial_d\Lambda_{ab}(x) = \frac{2}{r}\varepsilon_{dcb}\Lambda_{ac}(x) , \qquad \partial_a\Lambda_\alpha{}^\beta = \frac{i}{r}\Lambda_\alpha{}^\gamma(\gamma_a)_\gamma^\beta , \qquad \Lambda_\alpha{}^\delta\gamma^b_{\delta\rho}\Lambda_\beta{}^\rho\Lambda_b{}^a = \gamma^a_{\alpha\beta} . \tag{2.61}$$

Using these properties one can check that each of the operators (2.60) independently obeys (2.48). The operator $\mathbb{M}_a$ corresponds to the SU(2) rotations on the sphere, $\mathbb{Q}_\alpha$ and $\bar{\mathbb{Q}}_\alpha$ are the generators of supersymmetries, and $\mathbb{R}$ is the R-symmetry generator. The expression (2.59) is just a linear combination of these operators with the corresponding constant parameters b^a, ϵ^α, $\bar\epsilon^\alpha$ and t.

3 Superfield actions

The supergeometry of the SU(2|1)/U(1) supercoset elaborated in the previous section is characterized by torsion and curvature that satisfy eqs. (2.24) and (2.27). Comparing these equations with the supergeometry constraints to be satisfied by the (Euclidean version of) $\mathcal{N} = 2$, $d = 3$ dynamical supergravity (see e.g. [9–11, 20]), one can see that SU(2|1)/U(1) geometry is a particular (vacuum) solution of the supergravity constraints. As such, we can bypass the step of coupling the matter superfields to off-shell supergravity and construct classical superfield actions directly on SU(2|1)/U(1) as easy as in flat superspace.

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3.1 Gauge supermultiplet

Let us take the covariant derivatives $\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha, \bar{\mathcal{D}}_\alpha)$ on SU(2|1)/U(1) and extend them with a gauge superfield connection V_A

$$\nabla_A = \mathcal{D}_A + V_A \,, \qquad V_A = (V_a, V_\alpha, \bar{V}_\alpha) \,. \tag{3.1}$$

V_A take values in the Lie algebra of a gauge group. Gauge superfield constraints are imposed by requiring that the gauge-covariant derivatives obey the (anti)commutation relations (2.33) deformed by gauge superfield strengths,

$$\begin{aligned}
\{\nabla_\alpha, \nabla_\beta\} &= \{\bar\nabla_\alpha, \bar\nabla_\beta\} = 0 \,,\\
\{\nabla_\alpha, \bar\nabla_\beta\} &= i\gamma^a_{\alpha\beta}\nabla_a - \frac{1}{2}\gamma^a_{\alpha\beta}M_a + \frac{1}{r}\varepsilon_{\alpha\beta}R + i\varepsilon_{\alpha\beta}G \,,\\
[\nabla_a, \nabla_b] &= -\frac{i}{2r}M_{ab} + i\mathbf{F}_{ab} \,,\\
[\nabla_a, \nabla_\alpha] &= -\frac{i}{2r}(\gamma_a)^\beta_\alpha\nabla_\beta - (\gamma_a)^\beta_\alpha\bar{W}_\beta \,, \quad [\nabla_a, \bar\nabla_\alpha] = -\frac{i}{2r}(\gamma_a)^\beta_\alpha\bar\nabla_\beta + (\gamma_a)^\beta_\alpha W_\beta \,,\\
[R, \nabla_\alpha] &= \nabla_\alpha \,, \quad [R, \bar\nabla_\alpha] = -\bar\nabla_\alpha \,,\\
[M_{ab}, \nabla_\alpha] &= -\frac{1}{r}\varepsilon_{abc}(\gamma^c)^\beta_\alpha\nabla_\beta \,, \quad [M_{ab}, \bar\nabla_\alpha] = -\frac{1}{r}\varepsilon_{abc}(\gamma^c)^\beta_\alpha\bar\nabla_\beta \,.
\end{aligned} \tag{3.2}$$

Here G, W_α, $\bar{W}_\alpha$ and $\mathbf{F}_{ab}$ are gauge superfield strengths subject to the Bianchi identities. In particular, W_α is covariantly chiral and $\bar{W}_\alpha$ is covariantly antichiral,

$$\bar\nabla_\alpha W_\beta = 0 \,, \qquad \nabla_\alpha \bar{W}_\beta = 0 \,. \tag{3.3}$$

These superfields obey 'standard' Bianchi identity

$$\nabla^\alpha W_\alpha = \bar\nabla^\alpha \bar{W}_\alpha \,. \tag{3.4}$$

The spinorial superfield strengths W_α and $\bar{W}_\alpha$ are expressed in terms of the scalar superfield G as follows

$$\bar{W}_\alpha = \nabla_\alpha G \,, \qquad W_\alpha = \bar\nabla_\alpha G \,. \tag{3.5}$$

The latter is covariantly linear,

$$\nabla^2 G = \bar\nabla^2 G = 0 \,. \tag{3.6}$$

The gauge connections V_A in (3.1) can be expressed in terms of a single gauge prepotential V. In particular, in the so-called chiral representation [7, 8] the covariant spinor derivatives ∇_α and $\bar\nabla_\alpha$ are given by

$$\nabla_\alpha = e^{-V}\mathcal{D}_\alpha e^V \,, \qquad \bar\nabla_\alpha = \bar{\mathcal{D}}_\alpha \,. \tag{3.7}$$

As a consequence of the constraints (3.2), the superfield strengths are expressed in terms of the prepotential V as follows

$$G = \frac{i}{2}\bar{\mathcal{D}}^\alpha(e^{-V}\mathcal{D}_\alpha e^V) \,, \quad W_\alpha = -\frac{i}{4}\bar{\mathcal{D}}^2(e^{-V}\mathcal{D}_\alpha e^V) \,, \quad \bar{W}_\alpha = \frac{i}{2}\nabla_\alpha\bar{\mathcal{D}}^\beta(e^{-V}\mathcal{D}_\beta e^V) \,. \tag{3.8}$$

The gauge transformation of V is

$$e^V \longrightarrow e^{i\bar{\Lambda}} e^V e^{-i\Lambda}\,, \tag{3.9}$$

where Λ and $\bar{\Lambda}$ are covariantly (anti)chiral local gauge parameters

$$\bar{\mathcal{D}}_\alpha \Lambda = 0\,, \qquad \mathcal{D}_\alpha \bar{\Lambda} = 0\,. \tag{3.10}$$

The superfield strengths transform covariantly under the gauge transformations (3.9),

$$G \to e^{i\Lambda} G e^{-i\Lambda}\,, \qquad W_\alpha \to e^{i\Lambda} W_\alpha e^{-i\Lambda}\,. \tag{3.11}$$

The super Yang-Mills action can be equivalently written either in the full $\mathcal{N}=2$ superspace or in the chiral subspace,

$$S_{\rm SYM} = -\frac{4}{g^2}{\rm tr}\int d^3x d^2\theta d^2\bar{\theta}\, E\, G^2 = \frac{2}{g^2}{\rm tr}\int d^3x d^2\theta\, \mathcal{E}\, W^\alpha W_\alpha\,, \tag{3.12}$$

where g^2 is the gauge coupling constant of mass dimension $[g]=1/2$ and $\mathcal{E}$ is a chiral density. The variation of the SYM action reads

$$\delta S_{\rm SYM} = -\frac{4i}{g^2}{\rm tr}\int d^3x d^2\theta d^2\bar{\theta}\, E\, \Delta V \nabla^\alpha \bar{\nabla}_\alpha G\,, \tag{3.13}$$

where ΔV is the gauge-covariant variation,

$$\Delta V = e^{-V}\delta e^V = \delta V + \frac{1}{2}[\delta V, V] + \ldots \tag{3.14}$$

Hence, the SYM equation of motion is

$$0 = \frac{\delta S_{\rm SYM}}{\Delta V} = -\frac{4i}{g^2}\nabla^\alpha \bar{\nabla}_\alpha G = -\frac{4i}{g^2}\nabla^\alpha W_\alpha\,. \tag{3.15}$$

The Abelian Chern-Simons action is known to be

$$-\frac{k}{\pi}\int d^3x d^2\theta d^2\bar{\theta}\, E\, VG\,, \tag{3.16}$$

where k is an integer. The non-Abelian generalization of this action requires the introduction of an auxiliary parameter t [21],

$$S_{\rm CS} = -\frac{ik}{\pi}{\rm tr}\int_0^1 dt\int d^3x d^2\theta d^2\bar{\theta}\, E\, \bar{\mathcal{D}}^\alpha(e^{-tV}\mathcal{D}_\alpha e^{tV})e^{-tV}\partial_t e^{tV}\,. \tag{3.17}$$

However, the variation of the Chern-Simons action does not contain this parameter,

$$\delta S_{\rm CS} = -\frac{2k}{\pi}{\rm tr}\int d^3x d^2\theta d^2\bar{\theta}\, E\, G\Delta V\,. \tag{3.18}$$

Finally, the Fayet-Iliopoulos term is given by

$$S_{\rm FI} = -4i\xi\int d^3x d^2\theta d^2\bar{\theta}\, E\, V\,, \tag{3.19}$$

where ξ is the coupling of mass dimension $+1$.

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3.1.1 Component structure

The vector supermultiplet consists of one scalar field $\sigma(x)$ one vector $A_a(x) = -i\gamma_a^{\alpha\beta} A_{\alpha\beta}$, spinors $\lambda_\alpha(x)$ and $\bar\lambda_\alpha(x)$ and one auxiliary field $D(x)$. In the Wick-rotated (Euclidean) SYM theory under consideration $\lambda_\alpha(x)$ and $\bar\lambda_\alpha(x)$ are regarded as independent fields, not related to each other by complex conjugation, and also the bosonic fields σ and A_a are assumed to be complex.

To derive the component structure in supersymmetric gauge theories it is convenient to impose the Wess-Zumino gauge,

$$V| = 0\,, \quad \mathcal{D}_\alpha V| = \bar{\mathcal{D}}_\alpha V| = 0\,, \quad \mathcal{D}^2 V| = \bar{\mathcal{D}}^2 V| = 0\,, \tag{3.20}$$

where $|$ denotes the component value of the superfields at $\theta = \bar\theta = 0$. The component fields appear in the following derivatives of the gauge superfield

$$\begin{aligned}
\frac{1}{2}[\mathcal{D}_\alpha, \bar{\mathcal{D}}_\beta]V| &= 2iA_{\alpha\beta} - \varepsilon_{\alpha\beta} i\sigma\,,\\
\frac{1}{2}\bar{\mathcal{D}}^2\mathcal{D}_\alpha V| &= i\lambda_\alpha\,, \quad \frac{1}{2}\mathcal{D}^2\bar{\mathcal{D}}_\alpha V| = i\bar\lambda_\alpha\,,\\
\frac{1}{8}\{\mathcal{D}^2, \bar{\mathcal{D}}^2\}V| &= iD\,.
\end{aligned} \tag{3.21}$$

Using the algebra of the covariant derivatives (2.33) we find the components of the superfield strengths (3.8) and their derivatives to be

$$\begin{aligned}
G| &= \sigma\,,\\
W_\alpha| &= \frac{1}{2}\lambda_\alpha\,, \qquad \bar W_\alpha| = \frac{1}{2}\bar\lambda_\alpha\,,\\
\mathcal{D}^\alpha W_\alpha| &= D + \frac{2\sigma}{r}\,,\\
\mathcal{D}_{(\alpha}W_{\beta)}| &= -\frac{i}{4}\varepsilon^{abc}\gamma^c_{\alpha\beta}F_{ab} + \frac{i}{2}\gamma^a_{\alpha\beta}\hat\nabla_a\sigma\,,\\
\mathcal{D}^2 W_\alpha| &= i\gamma^a_{\alpha\beta}\hat\nabla_a\bar\lambda^\beta + i[\sigma, \bar\lambda_\alpha] - \frac{1}{2r}\bar\lambda_\alpha\,,
\end{aligned} \tag{3.22}$$

where

$$\begin{aligned}
\hat\nabla_a\bar\lambda^\beta &= \hat{\mathcal{D}}_a\bar\lambda^\beta + i[A_a, \bar\lambda^\beta]\,,\\
\hat\nabla_a\sigma &= \hat{\mathcal{D}}_a\sigma + i[A_a, \sigma]\,, \qquad (\hat\nabla_{\alpha\beta} = -\frac{i}{2}\gamma^a_{\alpha\beta}\hat\nabla_a)\,,\\
F_{ab} &= \hat{\mathcal{D}}_a A_b - \hat{\mathcal{D}}_b A_a + i[A_a, A_b]
\end{aligned} \tag{3.23}$$

and $\hat{\mathcal{D}}_a = \partial_a + \omega_a(x)$ is a covariant derivative on S^3.

Consider now the SYM action (3.12) and replace the integration over $d^2\theta$ by corresponding spinor covariant derivatives

$$\begin{aligned}
S_{\rm SYM} &= \frac{2}{g^2}{\rm tr}\int d^3x d^2\theta\, \mathcal{E}\, W^\alpha W_\alpha\\
&= -\frac{1}{g^2}{\rm tr}\int d^3x\sqrt{h}\left(W^\alpha\mathcal{D}^2W_\alpha - \frac{1}{2}\mathcal{D}^\alpha W_\alpha\mathcal{D}^\beta W_\beta - \mathcal{D}_{(\alpha}W_{\beta)}\mathcal{D}^{(\alpha}W^{\beta)}\right)\Bigg|\,.
\end{aligned} \tag{3.24}$$

Substituting (3.22) into (3.24), we find the component structure of the classical SYM action

$$\begin{aligned}S_{\rm SYM} = &\frac{1}{g^2}{\rm tr}\int d^3x\sqrt{h}\bigg[\frac{1}{4}F^{ab}F_{ab}+\frac{1}{2}\hat{\nabla}^a\sigma\hat{\nabla}_a\sigma+\frac{1}{2}\left(D+\frac{2\sigma}{r}\right)^2\\&+\frac{i}{2}\lambda^\alpha(\gamma^a)_\alpha^\beta\hat{\nabla}_a\bar\lambda_\beta-\frac{i}{2}\lambda^\alpha[\sigma,\bar\lambda_\alpha]+\frac{1}{4r}\lambda^\alpha\bar\lambda_\alpha\bigg].\end{aligned} \tag{3.25}$$

Note that the terms containing the inverse radius of the three-sphere $1/r$ automatically appear in this procedure and this action is $\mathcal{N}=2$ supersymmetric by construction.

In a similar way one recovers the component structure of the Chern-Simons (3.17) and Fayet-Iliopoulos (3.19) superfield actions,

$$S_{\rm CS} = \frac{ik}{4\pi}{\rm tr}\int d^3x\sqrt{h}\left[\varepsilon^{abc}(A_a\hat{\mathcal{D}}_bA_c+\frac{2i}{3}A_aA_bA_c)-\bar\lambda^\alpha\lambda_\alpha-2\sigma D-\frac{2\sigma^2}{r}\right], \tag{3.26}$$

$$S_{\rm FI} = -4i\xi\int d^3xd^2\theta d^2\bar\theta\, E\, V = \xi\int d^3x\sqrt{h}\, D\,. \tag{3.27}$$

The last term in the Chern-Simons action (3.26) can be eliminated by the shift of the auxiliary field, $D\to D' = D+\frac{\sigma}{r}$. After such a shift the Chern-Simons action takes the canonical form.

3.2 Chiral matter

Let us now consider a covariantly chiral superfield Φ and an anti-chiral superfield $\bar\Phi$ i.e. the superfields that obey the constraints

$$\bar{\mathcal{D}}_\alpha\Phi=0\,,\qquad \mathcal{D}_\alpha\bar\Phi=0\,. \tag{3.28}$$

Again, as for the vector supermultiplet, we do not assume that Φ and $\bar\Phi$ are related by the complex conjugation.

A general action for the chiral superfields interacting with the background gauge superfield V is

$$S = 4\int d^3xd^2\theta d^2\bar\theta\, E\,\bar\Phi e^V\Phi + 2\int d^3xd^2\theta\,\mathcal{E}\,W(\Phi)+2\int d^3xd^2\bar\theta\,\bar{\mathcal{E}}\,\bar W(\bar\Phi)\,, \tag{3.29}$$

where $W(\Phi)$ is a superpotential. Here we assume that Φ transforms under the fundamental representation of the gauge gorup. In the case of the adjoint representation the kinetic term for Φ includes the trace of the matrix indices

$$4\,{\rm tr}\int d^3xd^2\theta d^2\bar\theta\, E\, e^{-V}\bar\Phi e^V\Phi\,. \tag{3.30}$$

The (anti)chiral superfield may carry an R-charge q, i.e.

$$R\Phi=-q\Phi\,,\qquad R\bar\Phi=q\bar\Phi\,. \tag{3.31}$$

In principle, the R-charge of the chiral superfield can be arbitrary although its canonical value for the chiral matter is $q=1/2$. Note also that the R-charge of the superpotential

$W(\Phi)$ should be -2 since the chiral measure $d^2\theta$ has the R-charge $+2$. The latter follows form the fact that $d\theta_\alpha \propto \mathcal{D}_\alpha$ and from the commutation relations (2.34).

The (anti)chiral multiplet consists of the complex scalar field ϕ ($\bar\phi$), the spinor ψ_α ($\bar\psi_\alpha$) and the auxiliary field F ($\bar F$). These fields appear in the θ-decomposition of the superfields Φ and $\bar\Phi$ as follows

$$\begin{aligned} \phi(x) &= \Phi| & \bar\phi(x) &= \bar\Phi| \\ \psi_\alpha(x) &= \mathcal{D}_\alpha\Phi| & \bar\psi_\alpha(x) &= \bar{\mathcal{D}}_\alpha\bar\Phi| \\ F(x) &= -\tfrac{1}{2}\mathcal{D}^2\Phi| & \bar F(x) &= -\tfrac{1}{2}\bar{\mathcal{D}}^2\bar\Phi|\,. \end{aligned} \tag{3.32}$$

Upon integrating out the Grassmann variables we find the component structure of the action (3.29),

$$\begin{aligned} S =& \int d^3x\sqrt{h}\Big[\hat\nabla^a\bar\phi\hat\nabla_a\phi + \bar\phi\left(\sigma^2 + \frac{q(2-q)}{r^2} + \frac{2iq}{r}\sigma + iD\right)\phi + \bar F F \\ & -i\gamma^a_{\alpha\beta}\bar\psi^\alpha\hat\nabla_a\psi^\beta + \bar\psi^\alpha\left(i\sigma + \frac{1-2q}{2r}\right)\psi_\alpha + i\bar\psi^\beta\bar\lambda_\alpha\phi + i\bar\phi\lambda^\alpha\psi_\alpha\Big] \\ & + \int d^3x\sqrt{h}\left(W'(\phi)F + W'(\bar\phi)\bar F - \frac{1}{2}W''(\phi)\psi^\alpha\psi_\alpha - \frac{1}{2}W''(\bar\phi)\bar\psi^\alpha\bar\psi_\alpha\right). \end{aligned} \tag{3.33}$$

Here $\hat\nabla_a$ is the gauge covariant derivative on S^3 in the fundamental representation of the gauge group

$$\hat\nabla_a\phi = \hat{\mathcal{D}}_a\phi + iA_a(x)\phi\,,\quad \hat\nabla_a\bar\phi = \hat{\mathcal{D}}_a\bar\phi - iA_a(x)\bar\phi\,,\quad \hat\nabla_a\psi_\alpha = \hat{\mathcal{D}}_a\psi_\alpha + iA_a(x)\psi_\alpha\,. \tag{3.34}$$

The generalization to any other representation of the gauge group is straightforward.

4 Superfield models with extended supersymmetry

In the previous section we constructed the superfield gauge and matter models on S^3 with minimal ($\mathcal{N}=2$) supersymmetry.[3] This construction was very similar to the formulation of superfield theories in a general curved superspace of Lorentz signature (see, e.g. [7] for this topic in four dimensions or a series of papers [9–12, 22, 23] for relevant three-dimensional supergravity-matter models in superspace). The classical actions introduced in this section can be considered as the Wick-rotated gauge and matter superfield actions in the (2,0) AdS_3 superspace [10, 11].

In this section we will consider models with extended $\mathcal{N}>2$ supersymmetry on the three-sphere. In particular, the classical actions of $\mathcal{N}=4$ and $\mathcal{N}=8$ SYM theories, as well as the Gaiotto-Witten and ABJM models will be constructed. In principle, for these models it would be natural to introduce curved superspaces with extended ($\mathcal{N}=4,6,8$) supersymmetry and to construct the actions directly in these superspaces. However, even in the flat space the use of the extended superspaces is not always convenient because it usually employs special methods with harmonic or projective coordinates which help to achieve unconstrained superfield formulations. So, we will avoid introducing extended superspaces and continue to use the $\mathcal{N}=2$ superspace formalism.

[3]Recall that since on S^3 the spinors are complex, S^3 does not admit $\mathcal{N}=1$ supersymmetry which would correspond to a single real 2-component spinor.

The description of the models with extended supersymmetry in the $\mathcal{N}=2$ supercoset SU(2|1)/U(1) mimics the construction of the classical actions of supersymmetric gauge and matter models in the conventional component field formulation [13]. Let us recall that such a construction is carried out in two steps. First, one couples the flat actions to the S^3 background geometry and then one finds extra terms which come with inverse radius of S^3 and which are necessary for the invariance under the supersymmetry on the sphere generated by S^3 Killing spinors. Similarly, in the $\mathcal{N}=2$ superspace we will use the chiral matter and gauge superfields for constructing actions with extra supersymmetries and will reveal new terms of order $\frac{1}{r}$ required for the action to be invariant under extended supersymmetry for superfields carrying a non-zero R-charge. As will be shown, the parameters of the extra supersymmetries and their bosonic (R-symmetry) partners are encoded in $\mathcal{N}=2$ superfield parameters which include, as their components, S^3 Killing spinors corresponding to the extra supersymmetries.

We will consider in detail the construction of the classical action for an $\mathcal{N}=4$ SYM model and will shortly discuss actions for other models ($\mathcal{N}=8$ SYM, Gaiotto-Witten and ABJM models).

To have a theory on S^3-sphere with $\mathcal{N}=4$ supersymmetry we need one more copy of the Killing spinors, in addition to those which have already appeared in SU(2|1)/U(1). Recall that the Killing spinor equation reads

$$\hat{\mathcal{D}}_a \xi^\alpha(x) \pm \frac{i}{2r}(\gamma_a)^\alpha_\beta \xi^\beta(x) = 0\,, \tag{4.1}$$

where $\hat{\mathcal{D}}_a$ is purely bosonic covariant derivative on S^3. The choice of the sign in (4.1) can be arbitrary. In the $\mathcal{N}=2$ case we should have two spinors, ξ_α and $\bar{\xi}_\alpha$, of the same "chirality"[4] with respect to the sign in (4.1), which is required by the SU(2|1) supergroup structure (see eqs. (2.50c)). In the $\mathcal{N}=4$ case we need another copy of Killing spinors associated with extra $\mathcal{N}=2$ supersymmetries, say η_α and $\bar{\eta}_\alpha$ the "chirality" of which can either coincide with the one of ξ_α, or can be opposite. For instance, in the case of superfield models in the AdS_3 space [11, 24], the corresponding $\mathcal{N}=4$ superspaces are denoted as (4,0) and (2,2), respectively. In this paper we will consider mainly the models with extended supersymmetry associated with all the S^3 Killing spinors of the same "chirality" and will shortly describe the models with Killing spinors of different "chiralities" on the example of $\mathcal{N}=4$ SYM model.

4.1 $\mathcal{N}=4$ SYM with SU(2) $\times$ SU(2) R-symmetry

$\mathcal{N}=4$ gauge supermultiplet is given by a pair (V,Φ), where $V(x,\theta,\bar\theta)$ is the $\mathcal{N}=2$ gauge superfield and $\Phi(x,\theta,\bar\theta)$ is a chiral superfield in the adjoint representation of the gauge group. We start the construction by lifting the flat $d=3$, $\mathcal{N}=4$ SYM action (written in terms of the $\mathcal{N}=2$ superfields, see e.g. [25]) onto the SU(2|1)/U(1) background

$$S_0 = -\frac{4}{g^2}\mathrm{tr}\int d^3x d^2\theta d^2\bar\theta\, E(G^2 - e^{-V}\bar\Phi e^V\Phi)\,. \tag{4.2}$$

[4] We refer to the spinors obeying the equation (4.1) with different signs as the Killing spinors of different "chirality". We hope that this will not cause the confusion with the conventional notion of the chiral spinors (which do not exist on S^3).

The superfields V and G are neutral under the U(1) R-transformations associated with the manifest $\mathcal{N}=2$ SU(2|1) supersymmetry, while the R-charge of the chiral superfield Φ can be, *a priori*, arbitrary

$$R\bar{\Phi} = q\bar{\Phi}\,, \qquad R\Phi = -q\Phi\,. \tag{4.3}$$

For further convenience, it is useful to introduce the gauge-covariant chiral superfields,

$$\overline{\mathit{\Phi}} = e^{-V}\bar{\Phi}e^{V}\,, \quad \mathit{\Phi} = \Phi\,, \qquad \nabla_\alpha\overline{\mathit{\Phi}} = 0\,, \quad \bar{\nabla}_\alpha\mathit{\Phi} = 0\,. \tag{4.4}$$

In terms of these superfields the gauge transformations are given by

$$\Delta V = i\bar{\Lambda} - i\Lambda\,, \qquad \delta\mathit{\Phi} = i[\Lambda, \mathit{\Phi}]\,, \qquad \delta\overline{\mathit{\Phi}} = i[\bar{\Lambda}, \overline{\mathit{\Phi}}]\,, \tag{4.5}$$

with Λ being a covariantly chiral gauge superfield parameter, $\bar{\nabla}_\alpha\Lambda = 0$. Recall that ΔV is a gauge-covariant variation for the $\mathcal{N}=2$ gauge superfield (3.14).

The general variation of the action (4.2) reads

$$\delta S_0 = -\frac{4}{g^2}\mathrm{tr}\int d^3x d^2\theta d^2\bar{\theta}\, E(i\Delta V\bar{\nabla}^\alpha\nabla_\alpha G - \delta\overline{\mathit{\Phi}}\mathit{\Phi} - \overline{\mathit{\Phi}}\delta\mathit{\Phi} - \Delta V[\mathit{\Phi}, \overline{\mathit{\Phi}}])\,. \tag{4.6}$$

We now assume that the hidden $\mathcal{N}=2$ supersymmetry and its bosonic partners encoded in (anti)chiral superfield parameters $\Upsilon(z)$ and $\bar{\Upsilon}(z)$ transform the superfields V and Φ into each other as follows

$$\Delta_\Upsilon V = i(\Upsilon\overline{\mathit{\Phi}} - \bar{\Upsilon}\mathit{\Phi})\,, \quad \delta_\Upsilon\mathit{\Phi} = \bar{\nabla}^\alpha G\mathcal{D}_\alpha\Upsilon + \frac{q}{r}G\Upsilon\,, \qquad \delta_\Upsilon\overline{\mathit{\Phi}} = -\nabla^\alpha G\bar{\mathcal{D}}_\alpha\bar{\Upsilon} - \frac{q}{r}G\bar{\Upsilon}\,. \tag{4.7}$$

In addition to be (anti)chiral

$$\bar{\mathcal{D}}_\alpha\Upsilon = 0\,, \qquad \mathcal{D}_\alpha\bar{\Upsilon} = 0\,, \tag{4.8}$$

Υ and $\bar{\Upsilon}$ are also subject to the constraints

$$\mathcal{D}_a\Upsilon = 0\,, \qquad \mathcal{D}_a\bar{\Upsilon} = 0\,. \tag{4.9}$$

The above constraints are required for the superfields $\mathit{\Phi}$ and $\overline{\mathit{\Phi}}$ to remain (anti)chiral upon the extra supersymmetry transformations, i.e.

$$\bar{\nabla}_\alpha\delta_\Upsilon\mathit{\Phi} = 0\,, \qquad \nabla_\alpha\delta_\Upsilon\overline{\mathit{\Phi}} = 0\,. \tag{4.10}$$

Note that V and $\mathit{\Phi}$ are non-Abelian superfields in the adjoint representation of the gauge group, while Υ does not carry the gauge group indices, so $\nabla_A\Upsilon = \mathcal{D}_A\Upsilon$. Note also that $\Upsilon(z)$ should have the same R-charge as the superfield Φ (4.3), namely

$$R\bar{\Upsilon} = q\bar{\Upsilon}\,, \qquad R\Upsilon = -q\Upsilon\,. \tag{4.11}$$

In comparison with the flat case, the transformations (4.7) involve additional terms with the inverse radius of the sphere. These extra terms are necessary to preserve the covariant chirality of the variation of the chiral superfield (4.10).

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Off the mass shell, the commutator of two transformations (4.7) closes on the SU(2|1) transformations considered in section 2.6

$$
\begin{aligned}
&[\delta_{\Upsilon_2},\delta_{\Upsilon_1}]\varPhi = \bar\nabla^2[(\mathcal{D}^\alpha\zeta)(\nabla_\alpha\varPhi)]\,, \qquad [\delta_{\Upsilon_2},\delta_{\Upsilon_1}]\overline{\varPhi} = -\nabla^2[(\bar{\mathcal{D}}^\alpha\zeta)(\bar\nabla_\alpha\overline{\varPhi})]\,,\\
&[\delta_{\Upsilon_2},\delta_{\Upsilon_1}]G = -2i\gamma^a_{\alpha\beta}(\bar{\mathcal{D}}^\alpha\mathcal{D}^\beta\zeta)\nabla_a G + (\bar{\mathcal{D}}^2\mathcal{D}^\alpha\zeta)\nabla_\alpha G - (\mathcal{D}^2\bar{\mathcal{D}}^\alpha\zeta)\bar\nabla_\alpha G\,,
\end{aligned} \tag{4.12}
$$

where

$$
\zeta = \frac{1}{4}(\bar\Upsilon_1\Upsilon_2 - \bar\Upsilon_2\Upsilon_1)\,. \tag{4.13}
$$

Indeed, the transformation of the chiral superfield in (4.12) has exactly the same form as (2.58) while the transformation of G has the general form (2.49) with the parameters given by (2.56).

Let us consider the commutator of the transformations (4.7) with the SU(2|1) transformations (2.49). Using the fact that the operator $\mathbb{K}$ (2.47) commutes with the covariant derivatives (2.48) we have

$$
[\delta_\Upsilon,\delta_{\mathbb{K}}]V = \delta_{\Upsilon'}V\,, \quad [\delta_\Upsilon,\delta_{\mathbb{K}}]\varPhi = \delta_{\Upsilon'}\varPhi\,, \quad [\delta_\Upsilon,\delta_{\mathbb{K}}]\overline{\varPhi} = \delta_{\Upsilon'}\overline{\varPhi}\,, \tag{4.14}
$$

where

$$
\Upsilon' = \mathbb{K}\Upsilon\,. \tag{4.15}
$$

Thus the commutator of (4.7) and (2.49) is again of the form (4.7). Therefore, the SU(2|1) transformations and the extra $\mathcal{N}=2$ supertransformations (4.7) form an $\mathcal{N}=4$ superalgebra. Though we do not have a clear understanding of algebraic stricture of the transformations (4.7) for generic values of q, for $q=0$ the form of the supersymmetry transformations suggests that this superalgebra is $su(2|2)\times su(2)$.

To show this, consider the component structure of the chiral superfield parameter Υ. In the chiral superspace coordinates its θ-decomposition is

$$
\Upsilon = a + \theta^\alpha\eta_\alpha + \theta^2 b\,. \tag{4.16}
$$

Using the explicit form of the superspace derivatives (2.37) one can easily check that the equation (4.9) implies that the components a and b are constant

$$
a = const\,, \qquad b = const\,, \tag{4.17}
$$

while η^α, associated with the extra $\mathcal{N}=2$ supersymmetry, obeys the Killing spinor equation similar to (2.50c) satisfied by the supersymmetry parameters of the manifest SU(2|1) supersymmetry

$$
\hat{\mathcal{D}}_a\eta^\alpha - \frac{i}{2r}(\gamma_a)^\alpha_\beta\eta^\beta = 0\,. \tag{4.18}
$$

The lowest component a in Υ, and its conjugate $\bar a$ appearing in $\bar\Upsilon$, are the parameters of the coset elements $\mathrm{SU(2)_R/U(1)}$, where $\mathrm{SU(2)_R}$ is part of the R-symmetry group in the $\mathcal{N}=4$ SYM theory. This indicates that the transformations (4.7) together with the SU(2|1) symmetry generate the supergroup SU(2|2). One can also verify that the highest component b in Υ, and its conjugate $\bar b$ in $\bar\Upsilon$, are the parameters of another SU(2) which rotates a triplet of auxiliary fields in the $\mathcal{N}=4$ gauge supermultiplet.

One-line computations show that the naive action (4.6) is not invariant under the Υ-transformations (4.7) for $q \neq 0$,

$$\delta_\Upsilon S_0 = -\frac{4q}{rg^2}\mathrm{tr}\int d^3x d^2\theta d^2\bar\theta\, E\, G(\bar\Upsilon\varPhi - \Upsilon\bar{\varPhi})\,. \tag{4.19}$$

Surprisingly, the non-invariance of S_0 cancels against the variation of the following Chern-Simons term

$$S_{\rm CS} = -\frac{2q}{rg^2}\mathrm{tr}\int_0^1 dt\int d^3x d^2\theta d^2\bar\theta\, E\,\bar{\mathcal{D}}^\alpha(e^{-tV}\mathcal{D}_\alpha e^{tV})e^{-tV}\partial_t e^{tV}\,. \tag{4.20}$$

This action differs from (3.17) only by the overall *real* coefficient in front of the superspace integral. Indeed, using (3.18) it is easy to find the variation of (4.20) under (4.7),

$$\delta_\Upsilon S_{\rm CS} = \frac{4iq}{rg^2}\mathrm{tr}\int d^3x d^2\theta d^2\bar\theta\, E\, G\Delta_\Upsilon V = \frac{4q}{rg^2}\mathrm{tr}\int d^3x d^2\theta d^2\bar\theta\, E\, G(\bar\Upsilon\varPhi - \Upsilon\bar{\varPhi})\,. \tag{4.21}$$

Thus, we conclude that the action of the $\mathcal{N}=4$ SYM model on the three-sphere is given by the sum of the action (4.2) and the Chern-Simons term (4.20),

$$S^{\mathcal{N}=4}_{\rm SYM} = -\frac{4}{g^2}\mathrm{tr}\int d^3x d^2\theta d^2\bar\theta\, E\left[G^2 - e^{-V}\bar\Phi e^V\Phi + \frac{q}{2r}\int_0^1 dt\bar{\mathcal{D}}^\alpha(e^{-tV}\mathcal{D}_\alpha e^{tV})e^{-tV}\partial_t e^{tV}\right]. \tag{4.22}$$

This action is manifestly invariant under SU(2|1) and under the hidden $\mathcal{N}=2$ transformations (4.7),

$$\delta_\Upsilon S^{\mathcal{N}=4}_{\rm SYM} = 0\,. \tag{4.23}$$

The requirement to have the Chern-Simons term (for $q \neq 0$) together with the YM term in the SYM action (4.22) to make it $\mathcal{N}=4$ supersymmetric is a somewhat unexpected feature of this model.[5] The Chern-Simons term disappears in the flat limit as it comes about with the inverse radius of the sphere. The Chern-Simons term is also absent for $q=0$, but we stress that the action (4.22) is consistent also for $q \neq 0$. However, as we will show in the next section, there is a natural bound on the values of this parameter $0 \le q \le 2$ which originates from the requirement of the absence of negative energy states in the spectrum of the model (4.22). Note that, the choice of $q=1$ is the most natural since this value of the R-charge coincides with the conformal dimension of the chiral supermultiplet which has applications in studying various aspects of dualities of three-dimensional gauge theories [14].

The term (4.20) comes with a real coefficient in front of the integral, in contrast to the Chern-Simons action (3.17) which appears with the imaginary unit factor since it was obtained by Wick rotating the Chern-Simons term in space-time of Lorentz signature. This has two consequences: (i) The non-invariance of the Chern-Simons term (4.20) under topologically non-trivial large gauge transformations will result in the large gauge non-invariance of the partition function unless q=0.[6] (ii) The term (4.20) produces a negative

[5] When gauge supermultiplets are part of supergravity supermultiplets, it is well known that the invariance of the supergravity action under supersymmetry may require the presence of Chern-Simons terms, as e.g. in the case of $D=11$ supergravity [26] or $\mathcal{N}=4$, $d=3$ supergravity [27, 28]. The necessity to add the Chern-Simons term to the SYM action coupled to the chiral supermultiplet with the R-charge $q=1$ for getting the $\mathcal{N}=4$ SYM theory on S^3 was noticed in [15].

[6] We thank referee for drawing our attention to this issue.

topological mass squared for the gauge field and can, in principle, cause the states with negative energies. To find the allowed values of q for which these states are absent we will consider the component form of the action (4.22).

4.1.1 Component form of the $\mathcal{N}=4$ SYM action on S^3

The action (4.22) consists of the pure $\mathcal{N}=2$ SYM term, the (anti)chiral superfield part and the $\mathcal{N}=2$ Chern-Simons term. Component structure of all these three terms is given by (3.25), (3.33) and (3.26), respectively. Putting these expressions together, we get

$$
\begin{aligned}
S^{\mathcal{N}=4}_{\rm SYM} = \frac{1}{g^2}{\rm tr}\int d^3x\sqrt{h}\Big[&\frac14 F^{ab}F_{ab} + \frac12\hat\nabla^a\sigma\hat\nabla_a\sigma + \frac12\left(D+\frac{2\sigma}{r}\right)^2 \\
&+\frac{i}{2}\lambda^\alpha(\gamma^a)_\alpha^\beta\hat\nabla_a\bar\lambda_\beta - \frac{i}{2}\lambda^\alpha[\sigma,\bar\lambda_\alpha] + \frac{1}{4r}\lambda^\alpha\bar\lambda_\alpha \\
&+\hat\nabla^a\bar\phi\hat\nabla_a\phi + \frac{q(2-q)}{r^2}\bar\phi\phi + \bar\phi[\sigma,[\sigma,\phi]] + \frac{2iq}{r}\bar\phi[\sigma,\phi] + i\bar\phi[D,\phi] + \bar F F \\
&+i(\gamma^a)_\alpha^\beta\bar\psi^\alpha\hat\nabla_a\psi_\beta + \frac{1-2q}{2r}\bar\psi^\alpha\psi_\alpha + i\bar\psi^\alpha[\sigma,\psi_\alpha] + i\bar\psi^\beta[\bar\lambda_\alpha,\phi] + i[\bar\phi,\lambda^\alpha]\psi_\alpha \\
&+\frac{q}{2r}\varepsilon^{abc}(A_a\hat{\mathcal{D}}_bA_c + \frac{2i}{3}A_aA_bA_c) - \frac{q}{2r}\bar\lambda^\alpha\lambda_\alpha - \frac{q}{r}\sigma D - \frac{q\sigma^2}{r^2}\Big]\,.
\end{aligned} \tag{4.24}
$$

For $q=1$ the action (4.24) coincides with that of [15] upon a suitable redefinition of the auxiliary field D, $D = F_3 + \frac{q-2}{r}\sigma - i[\phi,\bar\phi]$. The auxiliary fields

$$
F_3\,,\quad F = \frac{1}{\sqrt2}(F_1 - iF_2)\,,\quad \bar F = \frac{1}{\sqrt2}(F_1+iF_2) \tag{4.25}
$$

completely decouple from the physical sector and form an SO(3) $\sim$ SU(2) triplet contributing to the action (4.24) with the term $\frac12 F^AF_A$, where $F_A = (F_1,F_2,F_3)$.

Analogously, let us decompose the physical scalars ϕ and $\bar\phi$ into their real and imaginary parts

$$
\phi = \frac{1}{\sqrt2}(\phi_1 - i\phi_2)\,,\qquad \bar\phi = \frac{1}{\sqrt2}(\phi_1+i\phi_2)\,. \tag{4.26}
$$

The three scalars ϕ_1, ϕ_2 and σ form the triplet of another SO(3) $\sim$ SU(2)$_{\rm R}$

$$
\phi_I = (\phi_1,\phi_2,\sigma)\,,\qquad I=1,2,3\,. \tag{4.27}
$$

It is important to note that the physical scalars ϕ_I and the auxiliary fields F_A transform under different SU(2) groups which together form the SU(2) $\times$ SU(2) R-symmetry of the $\mathcal{N}=4$ SYM model.

Finally, we introduce the SU(2)$_{\rm R}$ doublets of spinors $\psi_{i\alpha}$, $\bar\psi^{i\alpha}$, $i=1,2$. These spinors are related to the ones in (4.24) as follows

$$
\bar\psi^{1\alpha} = \frac{1}{\sqrt2}\bar\lambda^\alpha\,,\quad \psi_{1\alpha} = \frac{1}{\sqrt2}\lambda_\alpha\,,\quad \psi_{2\alpha} = \bar\psi_\alpha\,,\quad \bar\psi^{2\alpha} = \psi^\alpha\,. \tag{4.28}
$$

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Eliminating the auxiliary fields and using the fields (4.27) and (4.28) we recast the action (4.24) in the manifestly $\mathrm{SU}(2)_{\mathrm{R}}$ invariant form

$$
\begin{aligned}
S_{\mathrm{SYM}}^{\mathcal{N}=4} &= \frac{1}{g^2}\mathrm{tr}\int d^3x\sqrt{h}\big(\mathcal{L}_{\mathrm{gauge}}+\mathcal{L}_{\mathrm{scalar}}+\mathcal{L}_{\mathrm{spinor}}\big)\,, && (4.29)\\
\mathcal{L}_{\mathrm{gauge}} &= \frac{1}{4}F^{ab}F_{ab}+\frac{q}{2r}\varepsilon^{abc}(A_a\hat{\mathcal{D}}_bA_c+\frac{2i}{3}A_aA_bA_c)\,, && (4.30)\\
\mathcal{L}_{\mathrm{scalar}} &= \frac{1}{2}\hat\nabla^a\phi^I\hat\nabla_a\phi_I+\frac{q(2-q)}{2r^2}\phi^I\phi_I\\
&\quad -\frac{3q-2}{6r}\varepsilon^{IJK}\phi_I[\phi_J,\phi_K]-\frac{1}{4}[\phi^I,\phi^J][\phi_I,\phi_J]\,, && (4.31)\\
\mathcal{L}_{\mathrm{spinor}} &= i(\gamma^a)_\alpha^\beta\bar\psi^{i\alpha}\hat\nabla_a\psi_{i\beta}+\frac{1-2q}{2r}\bar\psi^{i\alpha}\psi_{i\alpha}+i\bar\psi^{i\alpha}(\gamma^I)_i^j[\phi_I,\psi_{j\alpha}]\,. && (4.32)
\end{aligned}
$$

Here $(\gamma^I)_i^j$ are $\mathrm{SO}(3)\sim\mathrm{SU}(2)_{\mathrm{R}}$ gamma-matrices similar to (A.1). It is straightforward to check that (4.29) is invariant under the following $\mathcal{N}=4$ supersymmetry transformations

$$
\begin{aligned}
\delta A_a &= i(\gamma_a)_\alpha^\beta(\bar\eta^{i\alpha}\psi_{i\beta}+\eta_{i\beta}\bar\psi^{i\alpha})\,,\\
\delta\phi^I &= (\gamma^I)_j^i(\bar\eta^{j\alpha}\psi_{i\alpha}+\eta_{i\alpha}\bar\psi^{j\alpha})\,,\\
\delta\bar\psi^{i\alpha} &= \frac{i}{2}\varepsilon^{abc}(\gamma_c)_\beta^\alpha\bar\eta^{i\beta}F_{ab}+i(\gamma^a)_\beta^\alpha(\gamma^I)_j^i\hat\nabla_a\phi_I\bar\eta^{j\beta}\\
&\quad -\frac{q}{r}\phi^I(\gamma_I)_j^i\bar\eta^{j\alpha}-\frac{1}{2}\varepsilon^{IJK}(\gamma_K)_j^i[\phi_I,\phi_J]\bar\eta^{j\alpha}\,,\\
\delta\psi_{i\alpha} &= -\frac{i}{2}\varepsilon^{abc}(\gamma_c)_\alpha^\beta\eta_{i\beta}F_{ab}+i(\gamma^a)_\alpha^\beta(\gamma^I)_i^j\hat\nabla_a\phi^I\eta_{j\beta}\\
&\quad +\frac{q}{r}\phi^I(\gamma_I)_i^j\eta_{j\alpha}+\frac{1}{2}\varepsilon^{IJK}(\gamma_K)_i^j[\phi_I,\phi_J]\eta_{j\alpha}\,, && (4.33)
\end{aligned}
$$

where $\eta_{i\alpha}$ and $\bar\eta_\alpha^i$ are SU(2)-doublets of Killing spinors obeying standard equation (4.18). For $q=0$ these transformations close according to the (anti)commutation relations in the $su(2|2)$ superalgebra. The algebraic properties of these transformations for generic values of q should still be understood.[7]

Let us consider the gauge field equations of motion which follow from the Lagrangian (4.30), for simplicity, in the Abelian case

$$
\frac{\delta}{\delta A_a}\int d^3x\sqrt{h}\,\mathcal{L}_{\mathrm{gauge}}=0\quad\Rightarrow\quad\hat{\mathcal{D}}^bF_{ab}+\frac{q}{2r}\varepsilon_{abc}F^{bc}=0\,. \tag{4.34}
$$

It is convenient to introduce the dual field strength

$$
\tilde F_a=\frac{1}{2}\varepsilon_{abc}F^{bc}\,, \tag{4.35}
$$

which obeys the Bianchi identity

$$
\hat{\mathcal{D}}^a\tilde F_a=0\,. \tag{4.36}
$$

[7]Note that an $\mathcal{N}=4$ superfield description of a similar model in an AdS_3 superspace with $\mathrm{OSp}(4|2)\times\mathrm{SL}(2,\mathbb{R})$ as its symmetry group was developed in a recent paper [24].

From equation (4.34) it follows that the dual field strength satisfies the massive "Klein-Gordon" equation

$$(-\hat{\mathcal{D}}^a\hat{\mathcal{D}}_a + \frac{2}{r^2})\tilde{F}_b - \frac{q^2}{r^2}\tilde{F}_b = 0\,. \tag{4.37}$$

Note that the Laplacian operator acting in the space of divergenceless vector fields on S^3 is given by

$$\Delta = -\hat{\mathcal{D}}^a\hat{\mathcal{D}}_a + \frac{2}{r^2}\,. \tag{4.38}$$

Its spectrum is given in (B.7). In particular, its lowest eigenvalue is $\frac{4}{r^2}$. Hence, to avoid negative energy states in the solution of eq. (4.37) we should impose the bound $q \leq 2$. Similarly, the absence of negative energy states for the scalar field in (4.31) requires $q \geq 0$. Hence, the allowed values of the parameter q are

$$0 \leq q \leq 2\,. \tag{4.39}$$

The situation here is analogous to the Breitelohner-Freedman bound [29] on the negative mass square of fields in AdS.

4.2 $\mathcal{N} = 4$ SYM with $\mathrm{U}(1) \times \mathrm{U}(1)$ R-supersymmetry

The authors of [14] considered an $\mathcal{N} = 4$ SYM model which consists of one $\mathcal{N} = 2$ gauge multiplet and one chiral multiplet with the unite R-charge $q = 1$. In contrast with (4.22) the classical action of this model is given simply by (4.2) with no extra Chern-Simons term. In this section we demonstrate that this model is invariant under the $\mathcal{N} = 4$ supersymmetry which has two Killing spinors with positive "chirality" and other two with the negative one.

Recall that the isometries of the $\mathrm{SU}(2|1)/\mathrm{U}(1)$ supercoset are generated by the operator $\mathbb{K}$ given by (2.47) which includes the Killing spinors of the positive "chirality", see eq. (2.50c). Consider now the Killing spinors on S^3 of the opposite "chirality",

$$\hat{\mathcal{D}}_a\eta^\alpha + \frac{i}{2r}(\gamma_a)^\alpha_\beta\eta^\beta = 0\,. \tag{4.40}$$

On S^3 one can choose such a gauge for the Lorentz connection in which the covariant derivative acts on spinors as

$$\hat{\mathcal{D}}_a\eta^\alpha = \partial_a\eta^\alpha - \frac{i}{2r}(\gamma_a)^\alpha_\beta\eta^\beta\,, \tag{4.41}$$

where $\partial_a = e^m_a(x)\partial_m$ is purely bosonic. Hence, in this gauge the Killing spinor equation (4.40) is simply

$$\partial_a\eta^\alpha = 0\,. \tag{4.42}$$

Moreover, we require that η^α is neutral under the action of the U(1) R-symmetry of the manifest SU(2|1) supersymmetry

$$R\eta^\alpha = 0\,. \tag{4.43}$$

In this case, using (2.37), it is straightforward to check that η^α is annihilated by the covariant spinor derivatives,

$$\mathcal{D}_\alpha\eta^\beta = \bar{\mathcal{D}}_\alpha\eta^\beta = 0\,. \tag{4.44}$$

Given a pair of Killing spinors η^α and $\bar\eta^\alpha$ with the properties described above one can construct an analog of superfield transformations (4.7)

$$\begin{aligned}\Delta_\eta V &= \Theta^\alpha \eta_\alpha \overline{\Phi} - \bar\theta^\alpha \bar\eta_\alpha \Phi\,,\\ \delta_\eta \Phi &= -i\eta^\alpha \bar\nabla_\alpha G\,, \qquad \delta_\eta \overline{\Phi} = i\bar\eta^\alpha \nabla_\alpha G\,,\end{aligned} \tag{4.45}$$

where we have introduced the object

$$\Theta^\alpha = \theta^\alpha - \frac{1}{r}\theta^2 \bar\theta^\alpha\,, \tag{4.46}$$

which has an important property

$$\mathcal{D}_\alpha \Theta^\beta = \delta_\alpha^\beta\,. \tag{4.47}$$

Using this property one can also find the transformation of the superfield strength G,

$$\delta_\eta G = \frac{i}{2}\eta^\alpha \bar\nabla_\alpha \overline{\Phi} - \frac{i}{2}\bar\eta^\alpha \nabla_\alpha \Phi - \bar\theta^\alpha \bar\eta_\alpha [G, \Phi]\,. \tag{4.48}$$

Note that, because of (4.43), the R-charge of the chiral superfield is fixed as

$$R\Phi = -\Phi\,, \qquad R\overline{\Phi} = \overline{\Phi}\,. \tag{4.49}$$

The general variation of the action (4.2) is given by (4.6). It is a simple exercise to check that this variation vanishes for the transformations of the fields (4.45), $\delta_\eta S_0 = 0$. So, this action is manifestly invariant under the $\mathcal{N} = 2$ supersymmetry and respects also the hidden $\mathcal{N} = 2$ supersymmetries (4.45). By construction, these supersymmetries are generated by the Killing spinors of opposite "chiralities".

It is instructive to find the closure of the transformations (4.45). For instance, with the use of (4.48) one can easily find the commutator of two transformations (4.45) for the chiral superfield

$$[\delta_{\eta_2}, \delta_{\eta_1}]\Phi = i\zeta^{\alpha\beta}\gamma^a_{\alpha\beta}\nabla_a \Phi + \frac{\zeta}{r}\Phi + i\zeta[G, \Phi] + 2i\zeta^{\alpha\beta}\bar\theta_\beta [W_\alpha, \Phi]\,, \tag{4.50}$$

where

$$\zeta^{\alpha\beta} = \frac{1}{2}(\eta_1^\alpha \bar\eta_2^\beta - \eta_2^\alpha \bar\eta_1^\beta)\,, \qquad \zeta = \zeta_\alpha^\alpha\,. \tag{4.51}$$

The first term in the r.h.s. of (4.50) is the bosonic translation while the second one is a U(1) transformation. The terms with commutators in (4.50) provide the covariant chirality of the superfield expression in the r.h.s. These terms are required in the non-Abelian case only.

The relation (4.50) suggests that the transformations (4.45) close according to the commutation relations of the super Lie algebra of the group SU(2|1). However, this SU(2|1) group is different from the one generated by the Killing supervector considered in section 2.6. Indeed, using the equations (4.42)–(4.44) one can verify that the Killing spinor η^α (as well as $\bar\eta^\alpha$) defined by (4.40) is annihilated by the operator (2.47),

$$\mathbb{K}\eta^\alpha = \mathbb{K}\bar\eta^\alpha = 0\,. \tag{4.52}$$

As a consequence, the transformations (4.45) commute with the SU(2|1) ones, up to a field dependent gauge transformation,

$$[\delta_\eta, \delta_{\mathbb{K}}]\mathit{\Phi} = 0\,, \quad [\delta_\eta, \delta_{\mathbb{K}}]\overline{\mathit{\Phi}} = 0\,, \quad [\delta_\eta, \delta_{\mathbb{K}}]G = [(\mathbb{K}\bar\theta^\alpha\bar\eta_\alpha)\mathit{\Phi}, G]\,. \tag{4.53}$$

Note that the expression $(\mathbb{K}\bar\theta^\alpha\bar\eta_\alpha)$ in the last commutator is chiral, $\bar{\mathcal{D}}_\alpha(\mathbb{K}\bar\theta^\alpha\bar\eta_\alpha) = 0$. This can be verified, e.g. using the explicit form of the operator $\mathbb{K}$ in the chiral coordinates given in (2.59) and (2.60). In the Abelian case the superfield strength is gauge invariant and the last commutator (4.53) vanishes identically. So, the full symmetry group of the model (4.2) is SU(2|1) × SU(2|1). A similar model in the AdS_3 space was considered recently [24].

For completeness, in this section we present the component structure of the action (4.2),

$$\begin{aligned} S_0 = \frac{1}{g^2}\mathrm{tr}\int d^3x\sqrt{h}\Big[&\frac14 F^{ab}F_{ab} + \frac12\hat\nabla^a\sigma\hat\nabla_a\sigma + \hat\nabla^a\bar\phi\hat\nabla_a\phi + \frac{1}{r^2}\bar\phi\phi \\ &-[\sigma,\phi][\sigma,\bar\phi] + \frac12[\phi,\bar\phi]^2 + \bar F F + \frac12\left(D + \frac{2\sigma}{r} + i[\phi,\bar\phi]\right)^2 \\ &+\frac{i}{2}\lambda^\alpha(\gamma^a)^\beta_\alpha\hat\nabla_a\bar\lambda_\beta - \frac{i}{2}\lambda^\alpha[\sigma,\bar\lambda_\alpha] + \frac{1}{4r}\lambda^\alpha\bar\lambda_\alpha \\ &+i(\gamma^a)^\beta_\alpha\bar\psi^\alpha\hat\nabla_a\psi_\beta - \frac{1}{2r}\bar\psi^\alpha\psi_\alpha + i\bar\psi^\alpha[\sigma,\psi_\alpha] + i\bar\psi^\beta[\bar\lambda_\alpha,\phi] + i[\bar\phi,\lambda^\alpha]\psi_\alpha\Big]\,. \end{aligned} \tag{4.54}$$

In contrast with (4.29), the scalars ϕ, $\bar\phi$ and σ have different masses and do not form an SU(2) triplet. The full R-symmetry of this model is U(1) × U(1) because the complex scalars ϕ, $\bar\phi$ and the auxiliary fields F, $\bar F$ transform independently under two different U(1) groups.

4.3 $\mathcal{N} = 8$ SYM

In the $\mathcal{N} = 2$ superfield description of $\mathcal{N} = 8$, $d = 3$ SYM theory its multiplet consists of the gauge superfield V and an SU(3)-triplet of chiral superfields Φ_i, $i = 1, 2, 3$ in the adjoint representation. The generalization of the flat $\mathcal{N} = 8$, $d = 3$ SYM classical action [30] to the supercoset SU(2|1) is

$$S^{\mathcal{N}=8}_{\rm SYM} = S_{\rm YM} + S_{\rm CS} + S_{\rm pot}\,, \tag{4.55}$$

$$S_{\rm YM} = -\frac{4}{g^2}\mathrm{tr}\int d^3x d^2\theta d^2\bar\theta\, E(G^2 - e^{-V}\bar\Phi^i e^V\Phi_i)\,, \tag{4.56}$$

$$S_{\rm CS} = -\frac{4}{3rg^2}\mathrm{tr}\int_0^1 dt\int d^3x d^2\theta d^2\bar\theta\, E\,\bar{\mathcal{D}}^\alpha(e^{-tV}\mathcal{D}_\alpha e^{tV})e^{-tV}\partial_t e^{tV}\,, \tag{4.57}$$

$$S_{\rm pot} = -\frac{i\sqrt2}{3g^2}\mathrm{tr}\int d^3x d^2\theta\,\mathcal{E}\,\varepsilon^{ijk}\Phi_i[\Phi_j,\Phi_k] + \frac{i\sqrt2}{3g^2}\mathrm{tr}\int d^3x d^2\bar\theta\,\bar{\mathcal{E}}\,\varepsilon_{ijk}\bar\Phi^i[\bar\Phi^j,\bar\Phi^k]. \tag{4.58}$$

This action is invariant under the following transformations which include hidden $\mathcal{N} = 6$ supersymmetry,

$$\begin{aligned} \delta_\Upsilon V &= i\Upsilon_i\overline{\mathit{\Phi}}^i - i\bar\Upsilon^i\mathit{\Phi}_i\,, \\ \delta_\Upsilon\mathit{\Phi}_i &= \bar\nabla^\alpha G\mathcal{D}_\alpha\Upsilon_i + \frac{2}{3r}G\Upsilon_i + \frac{1}{2\sqrt2}\varepsilon_{ijk}\bar\nabla^2(\bar\Upsilon^j\overline{\mathit{\Phi}}^k)\,, \\ \delta_\Upsilon\overline{\mathit{\Phi}}^i &= -\nabla^\alpha G\bar{\mathcal{D}}_\alpha\bar\Upsilon^i - \frac{2}{3r}G\bar\Upsilon^i - \frac{1}{2\sqrt2}\varepsilon^{ijk}\nabla^2(\Upsilon_j\mathit{\Phi}_k)\,, \end{aligned} \tag{4.59}$$

where Υ_i is a triplet of chiral superfield parameters, $\bar{\mathcal{D}}_\alpha \Upsilon_i = 0$, subject to

$$\mathcal{D}_a \Upsilon_i = 0 \,. \tag{4.60}$$

Similarly to (4.16), the superparameters Υ_i contain three Killing spinors $\eta_{i\alpha}$, each of which obeys (4.18). In (4.59) we use covariantly chiral superfields $\mathit{\Phi}_i$, $\overline{\mathit{\Phi}}^i$ defined as in (4.4).

The form of the superpotential (4.58) fixes the R-charges of the chiral superfields to be

$$R\Phi_i = -\frac{2}{3}\Phi_i \,, \qquad R\bar{\Phi}^i = \frac{2}{3}\bar{\Phi}^i \,. \tag{4.61}$$

This R-charge differs from the scaling dimension of the chiral superfields. As a consequence, the localization methods cannot be directly applied to the $\mathcal{N}=8$ SYM theory, see [14] for a discussion of this issue.

Clearly, the superparameters Υ_i should have the same charges as Φ_i

$$R\Upsilon_i = -\frac{2}{3}\Upsilon_i \,, \qquad R\bar{\Upsilon}^i = \frac{2}{3}\bar{\Upsilon}^i \,. \tag{4.62}$$

Taking these values of the R-charges into account, it is straightforward to check that the transformations of the (anti)chiral superfields in (4.59) preserve the chirality

$$\bar{\nabla}_\alpha \delta_\Upsilon \mathit{\Phi}_i = 0 \,, \qquad \nabla_\alpha \delta_\Upsilon \overline{\mathit{\Phi}}^i = 0 \,. \tag{4.63}$$

It is also rather straightforward but a bit lengthy to check, using the identities

$$\bar{\nabla}^2 \nabla_\alpha \mathit{\Phi}_i = 4i[W_\alpha, \mathit{\Phi}_i] \,, \qquad \nabla^2 \bar{\nabla}_\alpha \overline{\mathit{\Phi}}^i = -4i[\bar{W}_\alpha, \overline{\mathit{\Phi}}^i] \,, \tag{4.64}$$

that (4.55) is invariant under (4.59), $\delta_\Upsilon S^{\mathcal{N}=8}_{\rm SYM} = 0$. In this procedure, the cancelation of some terms becomes evident only after passing to the (anti)chiral subspace.

The action (4.55) contains the real Chern-Simons term similar to that in the $\mathcal{N}=4$ SYM model (4.22). However, in contrast to the $\mathcal{N}=4$ case the value of the R-charge of the chiral superfields is now fixed (4.61) by the presence of the superpotential. This value is within the bound (4.39), hence, although the Chern-Simons term in (4.55) gives a negative topological mass squared, there are no negative energy states in the theory.

In this section we considered the $\mathcal{N}=8$ SYM model with Killing spinors of the same "chirality" (obeying the equation (4.1) with minus sign). Similar to section 4.2, it is straightforward to construct an $\mathcal{N}=8$ SYM action invariant under supersymmetry with Killing spinors of different "chiralities". For instance, one can check that the action (4.55) with vanishing Chern-Simons term is still invariant under hidden $\mathcal{N}=6$ supersymmetry, but which is associated with six Killing spinors obeying (4.40) rather than (4.18). The transformations of these hidden supersymmetries are a simple generalization of (4.45). There is also a possibility of constructing an $\mathcal{N}=8$ SYM model with four Killing spinors of positive "chirality" and four extra ones of the negative "chirality". It would be of interest to study all these cases in detail and determine corresponding underlying supergroup structures.

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4.4 Gaiotto-Witten theory

In this section we construct a classical action of the Gaiotto-Witten model [31] on S^3. This is a superconformal Chern-Simons-matter model with $\mathcal{N}=4$ supersymmetry which consists of two $\mathcal{N}=2$ gauge superfields V and $\tilde{V}$ corresponding to two different gauge groups and two chiral superfields (a hypermultiplet), X_+ and X_-, in the bi-fundamental representation. We find the classical action of this model in the form

$$\begin{aligned} S_{\rm GW} &= S_{\rm CS}[V] - S_{\rm CS}[\tilde{V}] + S_X\,, \qquad (4.65)\\ S_X &= 4\,{\rm tr}\int d^3x d^2\theta d^2\bar\theta\, E(\bar X_+ e^V X_+ e^{-\tilde V} + X_- e^{-V}\bar X_- e^{\tilde V})\,, \qquad (4.66)\end{aligned}$$

where $S_{\rm CS}[V]$ and $S_{\rm CS}[\tilde V]$ are two Chern-Simons terms for left and right gauge superfields each of which has the form (3.17). The action S_X is the standard action for the chiral superfields minimally interacting with gauge superfields in the bi-fundamental representation and carrying R-charge q. It is straightforward to check that the action (4.65) is invariant under the following superfield transformation

$$\begin{aligned} \Delta V &= \bar\Sigma \mathcal{X}_+\mathcal{X}_- + \Sigma\bar{\mathcal{X}}_-\bar{\mathcal{X}}_+\,, & \Delta\tilde V &= \bar\Sigma\mathcal{X}_-\mathcal{X}_+ + \Sigma\bar{\mathcal{X}}_+\bar{\mathcal{X}}_-\,,\\ \delta\mathcal{X}_\pm &= \pm\bar\nabla^2(\bar\Upsilon\bar{\mathcal{X}}_\mp)\,, & \delta\bar{\mathcal{X}}_\pm &= \pm\nabla^2(\Upsilon\mathcal{X}_\mp)\,. \end{aligned} \qquad (4.67)$$

Here $\mathcal{X}_\pm$ and $\bar{\mathcal{X}}_\pm$ are covariantly (anti)chiral superfields,

$$\bar{\mathcal{X}}_+ = e^{-\tilde V}\bar X_+ e^V\,,\quad \mathcal{X}_+ = X_+\,,\quad \bar{\mathcal{X}}_- = e^{-V}\bar X_- e^{\tilde V}\,,\quad \mathcal{X}_- = X_-\,, \qquad (4.68)$$

and Υ is a chiral superfield parameter subject to the constraint (4.9). As is shown in (4.16), in components it contains the Killing spinor η_α and a parameter of the ${\rm SU}(2)_{\rm R}$ symmetry group. Hence, the variations (4.67) include transformations of the hidden $\mathcal{N}=2$ supersymmetry as well as part of the ${\rm SU}(2)_{\rm R}$ R-symmetry.

The superfields Υ and Σ possess the following U(1) R-charges associated with the manifest $\mathcal{N}=2$ SU(2|1) supersymmetry

$$R\Upsilon = 2(q-1)\Upsilon\,,\quad R\Sigma = -2q\Sigma\,,\quad R\bar\Upsilon = 2(1-q)\bar\Upsilon\,,\quad R\bar\Sigma = 2q\bar\Sigma\,. \qquad (4.69)$$

We stress that the (anti)chiral superfields Σ and $\bar\Sigma$ in (4.67) are not independent. They are related to Υ and $\bar\Upsilon$ as follows

$$\mathcal{D}_\alpha\Sigma = -\frac{8i\pi}{k}\bar{\mathcal{D}}_\alpha\bar\Upsilon\,,\qquad \bar{\mathcal{D}}_\alpha\bar\Sigma = -\frac{8i\pi}{k}\mathcal{D}_\alpha\Upsilon\,. \qquad (4.70)$$

These equations define Σ and $\bar\Sigma$ in terms of Υ and $\bar\Upsilon$ uniquely. For instance, for the chiral superfield parameter Υ in the form (4.16) we find the following component field decomposition of $\bar\Sigma$ in the chiral basis

$$\bar\Sigma = -\frac{8i\pi}{k}\left(\frac{q-1}{r}\bar\theta^2 b + \bar\theta^\alpha\eta_\alpha + \frac{q-1}{r}\bar\theta^2\theta^\alpha\eta_\alpha + 2\theta^\alpha\bar\theta_\alpha a + \frac{q-1}{r}\theta^2\bar\theta^2 a - \frac{r}{q}a\right)\,. \qquad (4.71)$$

4.4.1 Component form of the Gaiotto-Witten action on S^3

Let us denote the components of the $\mathcal{N}=2$ superfieds in the Gaiotto-Witten model as follows

$$V:\{\sigma,A_a,\lambda_\alpha,\bar\lambda_\alpha,D\}\,,\quad \tilde V:\{\tilde\sigma,\tilde A_a,\tilde\lambda_\alpha,\tilde{\bar\lambda}_\alpha,\tilde D\}\,,\quad X_\pm:\{\phi_\pm,\psi^\alpha_\pm,F_\pm\}\,. \tag{4.72}$$

These components are defined in accordance with the rules (3.21) and (3.32).

The action (4.65) contains Chern-Simons terms for the gauge superfields V and $\tilde V$ each of which has the component structure (3.26) as well as the matter superfields part S_X the component structure of which can be read from (3.33). We thus get

$$S_{\rm GW}={\rm tr}\int d^3x\,\sqrt{h}(\mathcal{L}_{\rm CS}+\mathcal{L}_{X_+}+\mathcal{L}_{X_-})\,, \tag{4.73}$$

$$\begin{aligned}\mathcal{L}_{\rm CS}=&\frac{ik}{4\pi}\left[\varepsilon^{abc}(A_a\hat{\mathcal{D}}_bA_c+\frac{2i}{3}A_aA_bA_c)-\bar\lambda^\alpha\lambda_\alpha-2\sigma D-\frac{2\sigma^2}{r}\right]\\&-\frac{ik}{4\pi}\left[\varepsilon^{abc}(\tilde A_a\hat{\mathcal{D}}_b\tilde A_c+\frac{2i}{3}\tilde A_a\tilde A_b\tilde A_c)-\tilde{\bar\lambda}^\alpha\tilde\lambda_\alpha-2\tilde\sigma\tilde D-\frac{2\tilde\sigma^2}{r}\right]\,,\end{aligned} \tag{4.74}$$

$$\begin{aligned}\mathcal{L}_{X_+}=&\Big[\hat\nabla^a\bar\phi_+\hat\nabla_a\phi_++\frac{q(2-q)}{r^2}\bar\phi_+\phi_+-(\sigma\phi_+-\phi_+\tilde\sigma)(\tilde\sigma\bar\phi_+-\bar\phi_+\sigma)\\&+\frac{2iq}{r}\bar\phi_+(\sigma\phi_+-\phi_+\tilde\sigma)+i\bar\phi_+(D\phi_+-\phi_+\tilde D)+F_+\bar F_+\\&+i(\gamma^a)_\alpha^\beta\bar\psi^\alpha_+\hat\nabla_a\psi_{+\beta}+\frac{1-2q}{2r}\bar\psi^\alpha_+\psi_{+\alpha}+i\bar\psi^\alpha_+(\sigma\psi_{+\alpha}-\psi_{+\alpha}\tilde\sigma)\\&+i\bar\psi^\alpha_+(\bar\lambda_\alpha\phi_+-\phi_+\tilde{\bar\lambda}_\alpha)-i(\tilde\lambda^\alpha\bar\phi_+-\bar\phi_+\lambda^\alpha)\psi_{+\alpha}\Big]\,,\end{aligned} \tag{4.75}$$

$$\begin{aligned}\mathcal{L}_{X_-}=&\Big[\hat\nabla^a\bar\phi_-\hat\nabla_a\phi_-+\frac{q(2-q)}{r^2}\bar\phi_-\phi_--(\tilde\sigma\phi_--\phi_-\sigma)(\sigma\bar\phi_--\bar\phi_-\tilde\sigma)\\&+\frac{2iq}{r}\bar\phi_-(\tilde\sigma\phi_--\phi_-\tilde\sigma)+i\bar\phi_-(\tilde D\phi_--\phi_-D)+F_-\bar F_-\\&+i(\gamma^a)_\alpha^\beta\bar\psi^\alpha_-\hat\nabla_a\psi_{-\beta}+\frac{1-2q}{2r}\bar\psi^\alpha_-\psi_{-\alpha}+i\bar\psi^\alpha_-(\tilde\sigma\psi_{-\alpha}-\psi_{-\alpha}\sigma)\\&+i\bar\psi^\alpha_-(\tilde{\bar\lambda}_\alpha\phi_--\phi_-\bar\lambda_\alpha)-i(\lambda^\alpha\bar\phi_--\bar\phi_-\tilde\lambda^\alpha)\psi_{-\alpha}\Big]\,.\end{aligned} \tag{4.76}$$

The component fields $F_\pm$, $\bar F_\pm$, D, $\tilde D$, σ, $\tilde\sigma$, λ_α, $\tilde\lambda_\alpha$, $\bar\lambda_\alpha$, $\tilde{\bar\lambda}_\alpha$ enter the action $S_{\rm GW}$ algebraically. They can be eliminated using their equations of motion,

$$\begin{aligned}&F_\pm=\bar F_\pm=0\,,\\&\sigma=\frac{2\pi}{k}(\phi_+\bar\phi_+-\bar\phi_-\phi_-)\,,&&\tilde\sigma=\frac{2\pi}{k}(\bar\phi_+\phi_+-\phi_-\bar\phi_-)\,,\\&\lambda_\alpha=\frac{4\pi}{k}(\phi_+\bar\psi_{+\alpha}-\bar\psi_{-\alpha}\phi_-)\,,&&\bar\lambda_\alpha=\frac{4\pi}{k}(\psi_{+\alpha}\bar\phi_+-\bar\phi_-\psi_{-\alpha})\,,\\&\tilde\lambda_\alpha=\frac{4\pi}{k}(\bar\psi_{+\alpha}\phi_+-\phi_-\bar\psi_{-\alpha})\,,&&\tilde{\bar\lambda}_\alpha=\frac{4\pi}{k}(\bar\phi_+\psi_{+\alpha}-\psi_{-\alpha}\bar\phi_-)\,.\end{aligned} \tag{4.77}$$

Next, we combine the scalar and spinor fields into SU(2) doublets as follows

$$\begin{aligned}&\phi^i=(\phi^1,\phi^2)=(\phi_+,\bar\phi_-)\,,&&\bar\phi_i=(\bar\phi_1,\bar\phi_2)=(\bar\phi_+,\phi_-)\,,\\&\psi^i_\alpha=(\psi^1_\alpha,\psi^2_\alpha)=(\psi_{+\alpha},\bar\psi_{-\alpha})\,,&&\bar\psi_{i\alpha}=(\bar\psi_{1\alpha},\bar\psi_{2\alpha})=(\bar\psi_{+\alpha},\psi_{-\alpha})\,.\end{aligned} \tag{4.78}$$

As a result, we get the component form of the Gaiotto-Witten action on S^3 in the form

$$
\begin{aligned}
S_{\rm GW} &= S_{\rm CS} + S_2 + S_{\rm int}\,, && (4.79)\\
S_{\rm CS} &= \frac{ik}{4\pi}{\rm tr}\int d^3x\sqrt{h}\,\varepsilon^{abc}(A_a\hat{\mathcal{D}}_bA_c + \frac{2i}{3}A_aA_bA_c - \tilde{A}_a\hat{\mathcal{D}}_b\tilde{A}_c - \frac{2i}{3}\tilde{A}_a\tilde{A}_b\tilde{A}_c)\,, && (4.80)\\
S_2 &= {\rm tr}\int d^3x\sqrt{h}\Big[\hat\nabla^a\phi^i\hat\nabla_a\bar\phi_i + \frac{q(2-q)}{r^2}\phi^i\bar\phi_i + i(\gamma^a)^\beta_\alpha\bar\psi^\alpha_i\hat\nabla_a\psi^i_\beta + \frac{1-2q}{2r}\bar\psi^\alpha_i\psi^i_\alpha\Big] && (4.81)\\
S_{\rm int} &= \frac{2\pi}{k}{\rm tr}\int d^3x\sqrt{h}\Big[\frac{i}{r}(1-2q)\phi^i\bar\phi^j\phi_i\bar\phi_j + \frac{2\pi}{k}(\phi^i\bar\phi_i\phi^j\bar\phi^k\phi_j\bar\phi_k + \bar\phi_i\phi^i\bar\phi^j\phi^k\bar\phi_j\phi_k)\\
&\quad -i\psi^{i\alpha}\bar\psi_{i\alpha}\phi^j\bar\phi_j + i\bar\psi^\alpha_i\psi^i_\alpha\bar\phi_j\phi^j + i\phi^i\bar\psi^\alpha_j\phi_i\bar\psi^j_\alpha - i\bar\phi^i\psi^\alpha_j\bar\phi_i\psi^j_\alpha\Big]\,. && (4.82)
\end{aligned}
$$

Here the gauge-covariant derivative $\hat\nabla$ acts on the matter fields in the bi-fundamental representation by the rule $\hat\nabla\phi^i = \hat{\mathcal{D}}_a\phi^i + iA_a\phi^i - i\phi^i\tilde{A}_a$.

Although the canonical value of the $\mathcal{N}=2$ supersymmetry R-charge of the chiral matter is $q=\frac{1}{2}$, the action (4.79) is explicitly SU(2) invariant for arbitrary value of the R-charge, thus manifesting the presence of the extended $\mathcal{N}=4$ supersymmetry. The natural bound for this parameter q is (4.39) for which the mass square of the scalar fields is positive.

4.5 ABJ(M) model

Finally, let us construct the classical action of the ABJ(M) theory [32–34] on S^3. This model can be considered as an $\mathcal{N}=6$ supersymmetric generalization of the Gaoitto-Witten theory [31] which involves two hypermultiplets, (X_{+i}, X^i_-), $i=1,2$, where X_{+i} and X^i_- are chiral superfields in the bi-fundamental representation of the gauge group. Each of these two chiral superfields can be rotated independently by its own SU(2) group which make part of the full SU(4) R-symmetry group of the ABJM model. The action of the ABJM model involves a superpotential which is consistent with this symmetry. We find the following generalization of this action on S^3:

$$
\begin{aligned}
S_{\rm ABJM} &= S_{\rm CS}[V] - S_{\rm CS}[\tilde V] + S_X + S_{\rm pot}\,, && (4.83)\\
S_X &= 4{\rm tr}\int d^3xd^2\theta d^2\bar\theta\,E\left(\bar X^i_+ e^V X_{+i}e^{-\tilde V} + X^i_- e^{-V}\bar X_{-i}e^{\tilde V}\right)\,, && (4.84)\\
S_{\rm pot} &= -\frac{4\pi i}{k}{\rm tr}\int d^3xd^2\theta\,\mathcal{E}\left(X_{+i}X^i_-X_{+j}X^j_- - X^i_-X_{+i}X^j_-X_{+j}\right)\\
&\quad -\frac{4\pi i}{k}{\rm tr}\int d^3xd^2\bar\theta\,\bar{\mathcal{E}}\left(\bar X_{-i}\bar X^i_+\bar X_{-j}\bar X^j_+ - \bar X^i_+\bar X_{-i}\bar X^j_+\bar X_{-j}\right)\,. && (4.85)
\end{aligned}
$$

Similar to the Gaiotto-Witten model (4.65), this action has two Chern-Simons terms $S_{\rm CS}[V]$ and $S_{\rm CS}[\tilde V]$ for the two gauge superfields and the standard kinetic term S_X for the chiral superfields minimally interacting with the gauge superfields. The superpotential $S_{\rm pot}$ has the standard ABJM form which is fixed by the requirement that the action (4.83) be invariant under the following superfield transformations

$$\begin{aligned}\Delta V &= -\frac{8i\pi}{k}(\bar\Upsilon^i{}_j\mathcal{X}_{+i}\mathcal{X}_-^j+\Upsilon_i{}^j\bar{\mathcal{X}}_{-j}\bar{\mathcal{X}}_+^i)\,,\\ \Delta\tilde V &= -\frac{8i\pi}{k}(\bar\Upsilon^j{}_i\mathcal{X}_-^i\mathcal{X}_{+j}+\Upsilon_i{}^j\bar{\mathcal{X}}_+^i\bar{\mathcal{X}}_{-j})\,, \end{aligned}\tag{4.86}$$

$$\delta\mathcal{X}_{+i} = \bar\nabla^2(\bar\Upsilon_i{}^j\bar{\mathcal{X}}_{-j})\,,\qquad \delta\mathcal{X}_-^j = -\bar\nabla^2(\bar\Upsilon_i{}^j\bar{\mathcal{X}}_+^i)\,,\tag{4.87}$$

$$\delta\bar{\mathcal{X}}_+^i = \nabla^2(\Upsilon^i{}_j\mathcal{X}_-^j)\,,\qquad \delta\bar{\mathcal{X}}_{-j} = -\nabla^2(\Upsilon^i{}_j\mathcal{X}_{+i})\,.\tag{4.88}$$

Here $\mathcal{X}_{\pm i}$ and $\bar{\mathcal{X}}_{\pm i}$ are covariantly (anti)chiral superfields defined similarly to (4.68) and $\Upsilon^i{}_j$ is a quartet of chiral superfield parameters each of which is constrained by (4.9). In components, it involves four Killing spinors $(\eta^i{}_j)_\alpha$ (their conjugate are present in $\bar\Upsilon_i{}^j$) which, together with the manifest supersymmetry, form the $\mathcal{N}=6$ supersymmetry of the ABJ(M) model.

To summarize, in this section we have constructed $\mathcal{N}=2$ superfield actions for the models with extended supersymmetry, namely, for $\mathcal{N}=4$ and $\mathcal{N}=8$ SYM, Gaiotto-Witten and ABJ(M) theories. For these models we have derived the transformations of $\mathcal{N}=2$ superfields under the hidden supersymmetries. Although these transformations are the generalization to SU(2|1)/U(1) superspace of the corresponding flat-space supersymmetries, to the best of our knowledge, their explicit form has not been given in the literature before. The extended $\mathcal{N}=4$ and $\mathcal{N}=8$ supersymmetry, associated with the S^3 Killing spinors of the same "chirality", requires the extension of the SYM actions on S^3 with the Chern-Simons terms. It would be of interest to understand the nature of these terms from the point of view of $\mathcal{N}=4$ superfield formulations of these theories and coupling these models to the extended three-dimensional supergravities considered e.g. in [9, 11, 23, 24].

5 One-loop partition functions

We will now compute one-loop effective actions and corresponding partition functions for superfield theories on SU(2|1)/U(1) discussed in the previous section.

5.1 Chiral superfield on the gauge superfield background

Let us consider a pair of chiral superfields Φ and $\widetilde\Phi$ interacting with an Abelian external background gauge superfield V

$$S = 4\int d^3x d^2\theta d^2\bar\theta\, E\,(\bar\Phi e^V\Phi + \bar{\widetilde\Phi}e^{-V}\widetilde\Phi)\,.\tag{5.1}$$

A reason why we consider the pair of the chiral fields is because they carry opposite charges with respect to the U(1) gauge group. Hence, there is no parity anomaly and the Chern-Simons term is not generated at one loop [35–38].

The problem of computing the partition function of the chiral supermultiplet on S^3 with an arbitrary R-charge was considered in [13–15, 39, 40] using component field calculations. Here we will derive similar results using superfield methods. Note also that the problem of low-energy effective action of the model (5.1) in flat space-time was considered in [43].

As we have already done in the previous section, it is convenient to introduce gauge-covariant (anti)chiral superfields

$$\overline{\mathit{\Phi}} = \overline{\Phi} e^{V} , \quad \mathit{\Phi} = \Phi , \quad \overline{\widetilde{\mathit{\Phi}}} = \overline{\widetilde{\Phi}} e^{-V} , \quad \widetilde{\mathit{\Phi}} = \widetilde{\Phi} , \tag{5.2}$$

such that $\nabla_\alpha \overline{\mathit{\Phi}} = 0$ and $\bar{\nabla}_\alpha \mathit{\Phi} = 0$. In terms of these superfields the classical action (5.1) is simply

$$S = 4 \int d^3x d^2\theta d^2\bar{\theta}\, E\, (\overline{\mathit{\Phi}} \mathit{\Phi} + \overline{\widetilde{\mathit{\Phi}}} \widetilde{\mathit{\Phi}}) . \tag{5.3}$$

Since the background gauge field is non-propagating, the effective action in this model is one-loop exact,

$$\Gamma = -\frac{1}{2} \mathrm{Tr} \ln H - \frac{1}{2} \mathrm{Tr} \ln \widetilde{H} , \tag{5.4}$$

where H and $\widetilde{H}$ are the operators acting in the space of the superfields $(\mathit{\Phi}, \overline{\mathit{\Phi}})$ and $(\widetilde{\mathit{\Phi}}, \overline{\widetilde{\mathit{\Phi}}})$, respectively, i.e.

$$H = \begin{pmatrix} 0 & -\bar{\nabla}^2 \\ -\nabla^2 & 0 \end{pmatrix} . \tag{5.5}$$

The operator $\widetilde{H}$ differs from H only in the sign of the background gauge superfield V due to the opposite U(1) charges of $\mathit{\Phi}$ and $\widetilde{\mathit{\Phi}}$. The standard procedure of computing the effective action in the chiral superfield model is based on squaring the operators H and $\widetilde{H}$ [7] and rewriting (5.4) as follows

$$\Gamma = -\frac{1}{4} \mathrm{Tr} \ln H^2 - \frac{1}{4} \mathrm{Tr} \ln \widetilde{H}^2 . \tag{5.6}$$

However, one should be careful with this squaring because some part of the effective action can be lost.[8] Therefore, we will avoid naive squaring like (5.6) and consider instead the variation of the effective action with respect to the background gauge superfield V,

$$\delta\Gamma = \int d^3x d^2\theta d^2\bar{\theta}\, E\, \delta V \langle J \rangle , \tag{5.7}$$

where $\langle J \rangle$ is an effective current which is expressed in terms of the Green's functions of the chiral superfields as follows

$$\langle J \rangle = \langle \frac{\delta S}{\delta V} \rangle = 4 \langle \overline{\mathit{\Phi}} \mathit{\Phi} \rangle - 4 \langle \overline{\widetilde{\mathit{\Phi}}} \widetilde{\mathit{\Phi}} \rangle . \tag{5.8}$$

Once the variation (5.7) is computed, its integration will give us the value of the effective action.

To compute (5.8), consider the Green's function $\langle \overline{\mathit{\Phi}}(z) \mathit{\Phi}(z') \rangle \equiv \mathrm{G}_{-+}(z, z')$ which obeys the equation

$$\bar{\nabla}^2 \mathrm{G}_{-+}(z, z') = \delta_+(z, z') , \tag{5.9}$$

where $\delta_+(z, z')$ is a chiral delta-function $(\bar{\nabla}_\alpha \delta_+(z, z') = 0)$,

$$\delta_+(z, z') = -\frac{1}{4} \bar{\nabla}^2 \delta^7(z, z') , \qquad \delta^7(z, z') = \frac{1}{E} \delta^3(x - x') \delta^2(\theta - \theta') \delta^2(\bar{\theta} - \bar{\theta}') . \tag{5.10}$$

---

[8] This is similar to the case of the Dirac operator on S^3 which has both positive and negative eigenvalues. So, if one naively takes its square, the negative eigenvalues will not be counted.

As a result, to obtain the variation of the effective action (5.7) we should find the Green's function G_{-+} at coincident superspace points.

As in the flat superspace [7], the Green's function G_{-+} is related to the covariantly chiral Green's function G_{+},

$$\mathrm{G}_{-+}(z,z') = -\frac{1}{4}\nabla^2 \mathrm{G}_{+}(z,z')\,, \tag{5.11}$$

where G_{+} obeys

$$\Box_{+}\mathrm{G}_{+}(z,z') = -\delta_{+}(z,z')\,, \qquad \Box_{+} \equiv \frac{1}{4}\bar{\nabla}^2\nabla^2\,. \tag{5.12}$$

Using the algebra (3.2), the operator $\Box_{+}$ can be represented as

$$\Box_{+} = -\nabla^a\,\nabla_a + (G - \frac{i}{r}R)^2 + i(\bar{\nabla}^\alpha \bar{W}_\alpha) + 2iW^\alpha\nabla_\alpha + \frac{1}{r}[\nabla^\alpha, \bar{\nabla}_\alpha]\,. \tag{5.13}$$

Let us take a very particular background gauge superfield $V = V_0$ such that its superfield strength $G = G_0$ is constant,

$$G_0 = \frac{i}{2}\bar{\mathcal{D}}^\alpha\mathcal{D}_\alpha V_0 = \sigma_0 = cosnt\,, \qquad W_{0\,\alpha} = W_{0\,\alpha} = 0\,. \tag{5.14}$$

As will be discussed in the next section, exactly the background of this kind is interesting from the point of view of the localization technique.

In the chiral coordinates, the background gauge superfield V_0 corresponding to (5.14) is

$$V_0 = i\sigma_0(\theta\bar{\theta} - \frac{1}{2r}\theta^2\bar{\theta}^2)\,, \tag{5.15}$$

and from (3.22) we see that the background values of the component fields are

$$\sigma = \sigma_0\,, \quad D = -\frac{2\sigma_0}{r}\,, \quad F_{ab} = 0\,, \quad \lambda_\alpha = \bar{\lambda}_\alpha = 0\,. \tag{5.16}$$

For this background the spinorial components of the superfield strengths vanish (5.14), and the form of the operator (5.13) simplifies to

$$\Box_{+} = -\nabla^a\nabla_a + m^2\,, \qquad m^2 \equiv G_0^2 + \frac{2i}{r}G_0(q-1) + \frac{q(2-q)}{r^2}\,, \tag{5.17}$$

where m is the effective mass. Here we have assumed that $\Box_{+}$ acts on the covariantly chiral scalar superfields of R-charge q.

For the gauge superfield background described above the chiral Green's function G_{+} (5.12) can be written as[9]

$$\mathrm{G}_{+}(z,z') = -\frac{1}{4}\bar{\nabla}^2\mathrm{G}_{\mathrm{o}}(z,z') = -\frac{1}{4}\bar{\nabla}'^2\mathrm{G}_{\mathrm{o}}(z,z')\,, \tag{5.18}$$

where $\bar{\nabla}'$ acts on z' and $\mathrm{G}_{\mathrm{o}}(z,z')$ solves for

$$\Box_{\mathrm{o}}\mathrm{G}_{\mathrm{o}}(z,z') = -\delta^7(z,z')\,, \qquad \Box_{\mathrm{o}} = -\nabla^a\nabla_a + m^2\,. \tag{5.19}$$

[9]Four-dimensional analogs of the relations (5.18) and (5.21) were first derived in [41, 42].

The operator $\Box_o$ has the same expression as $\Box_+$, eq. (5.17), but it acts on the superfields defined in the full superspace rather than on the chiral superfields. To check that (5.18) obeys (5.12) one should use the identities

$$[\nabla^2, \Box_o] = [\bar{\nabla}^2, \Box_o] = 0\,, \tag{5.20}$$

which hold for the considered gauge superfield background.

Combining (5.11) with (5.18) we find

$$\mathrm{G}_{-+}(z,z') = \frac{1}{16}\nabla^2\bar{\nabla}'^2\mathrm{G}_o(z,z') = -\frac{1}{16}\nabla^2\bar{\nabla}'^2\frac{1}{-\nabla^a\nabla_a + m^2}\delta^7(z,z')\,. \tag{5.21}$$

Next, using (5.20) we commute the operators ∇^2 and $\bar{\nabla}'^2$ with $(-\nabla^a\nabla_a + m^2)^{-1}$ and consider the Green's function (5.21) at coincident superspace points

$$\mathrm{G}_{-+}(z,z) = -\frac{1}{-\nabla^a\nabla_a + m^2}\frac{1}{16}\nabla^2\bar{\nabla}'^2\delta^7(z,z')|_{z=z'} = -\frac{1}{\Delta_{S^3} + m^2}\delta^3(x,x')|_{x=x'}\,. \tag{5.22}$$

Note that all the fermionic components of the superspace delta-function $\delta^7(z,z')$ should be differentiated out by the operators ∇^2 and $\bar{\nabla}'^2$ to get the non-vanishing result. The remaining expression is nothing but the trace of the inverse of the purely bosonic Laplace-Beltrami operator Δ_{S^3} acting on scalar fields on the S^3-sphere

$$-\operatorname{tr}\frac{1}{\Delta_{S^3} + m^2} \propto -\sum_{j=0}^{\infty}\frac{d_j}{\lambda_j + m^2}\,, \tag{5.23}$$

where λ_j are the eigenvalues of the Laplace-Beltrami operator and d_j are their degeneracies

$$\lambda_j = \frac{1}{r^2}j\,(j+2)\,, \qquad d_j = (j+1)^2\,, \quad j = 0,1,2,\ldots \tag{5.24}$$

The sum (5.23) is divergent. Regularizing it in a standard way, $\sum 1 = \zeta(0) = -\frac{1}{2}$, we find

$$\begin{aligned}\mathrm{G}_{-+}(z,z) &= \frac{c\pi r^2}{2}\sqrt{1-m^2r^2}\cot\left(\pi\sqrt{1-m^2r^2}\right)\\ &= \frac{c\pi r^2}{2}(irG_0 + 1 - q)\cot\left(\pi\,(irG_0 + 1 - q)\right).\end{aligned} \tag{5.25}$$

Here we used the explicit expression for the effective mass m^2 given in (5.17). The constant c can be fixed from the flat space limit which was studied in [43], namely

$$\lim_{r\to\infty}\mathrm{G}_{-+} = \frac{1}{4\pi}G_0 \quad \Rightarrow \quad c = \frac{1}{2\pi^2r^3}\,. \tag{5.26}$$

The formula (5.25) is valid for the arbitrary value of the R-charge q. Let us consider several particular values of q. $q = \frac{1}{2}$ corresponds to the chiral matter fields with canonical R-charge, $q = 0$ and $q = 2$ are carried by ghost superfields in the SYM theory (see next subsection), and $q = 1$ is the value of R-charge of the adjoint chiral multiplet in the $\mathcal{N} = 4$

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SYM action (4.22) which is singled out by its equality with the scale dimension of the chiral superfield. For these particular cases the formula (5.25) reduces to

$$\begin{aligned}
\mathrm{G}_{-+}|_{q=\frac{1}{2}} &= \frac{1}{4\pi}G_0 \tanh \pi r G_0 - \frac{i}{8\pi r} \tanh \pi r G_0 \,, && (5.27)\\
\mathrm{G}_{-+}|_{q=0} &= \frac{1}{4\pi}G_0 \coth \pi r G_0 - \frac{i}{4\pi r} \coth \pi r G_0 \,, && (5.28)\\
\mathrm{G}_{-+}|_{q=1} &= \frac{1}{4\pi}G_0 \coth \pi r G_0 \,, && (5.29)\\
\mathrm{G}_{-+}|_{q=2} &= \frac{1}{4\pi}G_0 \coth \pi r G_0 + \frac{i}{4\pi r} \coth \pi r G_0 \,. && (5.30)
\end{aligned}$$

Let us now consider in detail the computation of the effective action for the chiral superfield with the R-charge $q = \frac{1}{2}$. Recall that the Green's function $\langle \widetilde{\bar{\Phi}} \widetilde{\Phi} \rangle$ is obtained from $\langle \bar{\Phi} \Phi \rangle$ by changing the sign of the gauge superfield

$$\langle \bar{\Phi} \Phi \rangle \xrightarrow{G \to -G} \langle \widetilde{\bar{\Phi}} \widetilde{\Phi} \rangle \,. \tag{5.31}$$

As a result, the real part of the Green's functions (5.27) cancel in the effective current (5.8)

$$\langle J \rangle = -\frac{i}{\pi r} \tanh \pi r G_0 = -\frac{i}{\pi r} \tanh \pi r \sigma_0 \,. \tag{5.32}$$

Now, we substitute this expression for the effective current into (5.7) and compute the superspace integral similarly to the Fayet-Iliopoulos term (3.27),

$$\delta\Gamma = -\frac{i}{\pi r} \tanh(\pi r \sigma_0) \int d^3x d^2\theta d^2\bar{\theta}\, E\, \delta V = \frac{1}{4\pi r} \tanh(\pi r \sigma_0) \int d^3x \sqrt{h}\, \delta D \,. \tag{5.33}$$

Recall that for the considered background (5.16) the auxiliary field D is proportional to the scalar σ, $\delta D = -\frac{2}{r}\delta\sigma_0$. Taking into account that σ_0 is a constant parameter, we obtain

$$\delta\Gamma = -\frac{1}{2\pi r^2}\delta\sigma_0 \tanh(\pi r \sigma_0) \mathrm{Vol} S^3 = -\pi r \delta\sigma_0 \tanh(\pi r \sigma_0) \,. \tag{5.34}$$

Hence,

$$\Gamma = -\ln\left(c_1 \cosh(\pi r \sigma_0)\right) \,, \tag{5.35}$$

where c_1 is an integration constant. The corresponding partition function is

$$Z = e^{\Gamma} = \frac{1}{c_1 \cosh(\pi r \sigma_0)} \,. \tag{5.36}$$

For $\sigma_0 = 0$ the expression (5.36) should reproduce the partition function of a free chiral supermultiplet on S^3, [44]. This fixes the value of the integration constant c_1,

$$c_1 = 2 \,. \tag{5.37}$$

Using the Green's functions (5.28)–(5.30) in a similar way we find that the partition functions of the chiral superfields with R-charges $q = 0$, $q = 1$ and $q = 2$ have the following form

$$\begin{aligned}
q = 0: &\quad Z = \frac{1}{(2\sinh \pi r \sigma_0)^2} \,, && (5.38)\\
q = 1: &\quad Z = 1 \,, && (5.39)\\
q = 2: &\quad Z = (2\sinh \pi r \sigma_0)^2 \,. && (5.40)
\end{aligned}$$

The partition function of the chiral superfield with the R-charge $q = 1$ is equal to one because the propagator (5.29) has no imaginary part which could contribute to the effective current (5.8). The fact that this partition function is trivial was first noticed in [14].

5.2 $\mathcal{N} = 2$ super Yang-Mills partition function

Let us now consider the $\mathcal{N} = 2$ super Yang-Mills theory (3.12) with the gauge group SU(N). We are interested in the one-loop partition function Z which is related to the one-loop effective action Γ as

$$Z^{\mathcal{N}=2}_{\text{SYM}} = e^{\Gamma[V]} \,. \tag{5.41}$$

To derive the effective action Γ we perform the standard background-quantum splitting [8] $V \to (V_0, v)$ such that

$$e^V = e^{\Omega^\dagger} e^{gv} e^{\Omega} \,, \tag{5.42}$$

where v is the Hermitian *quantum* gauge superfield and Ω is a complex unconstrained prepotential which defines the Hermitian *background* gauge superfield V_0 as follows

$$e^{V_0} = e^{\Omega^\dagger} e^{\Omega} \,. \tag{5.43}$$

With this splitting we acquire extra gauge symmetry which leaves eqs. (5.42) and (5.43) invariant

$$e^{\Omega} \to e^{i\tau} e^{\Omega} \,, \qquad e^{gv} \to e^{i\tau} e^{gv} e^{-i\tau} \,, \tag{5.44}$$

where $\tau(z)$ is a real (Hermitian) superfield parameter. These transformations are called the 'background' gauge transformations.

The so-called 'quantum' form of the original gauge transformation (3.9) is

$$e^{\Omega} \to e^{i\lambda} e^{\Omega} e^{-i\lambda} \,, \qquad e^{gv} \to e^{i\bar{\lambda}} e^{gv} e^{-i\lambda} \,, \tag{5.45}$$

where $\lambda(z)$ is a chiral superfield parameter. The basic idea of the background field method is to fix the gauge symmetry corresponding to the parameter λ such that the effective action remains invariant under the background gauge transformations (5.44) with arbitrary τ.

In general, it is a difficult problem to find the effective action $\Gamma[V_0]$ for an arbitrary unconstrained background gauge superfield V_0. To simplify the problem, we restrict ourself to the consideration of the low-energy effective action for V_0 taking vales in the Cartan subalgebra of $su(N)$,

$$V_0 = \text{diag}(V_1, V_2, \ldots V_N) \,, \qquad \sum_{I=1}^{N} V_I = 0 \,. \tag{5.46}$$

Moreover, we assume that each of the superfields V_I in (5.46) has a constant superfield strength, $G_I = \frac{i}{2}\mathcal{D}^\alpha \mathcal{D}_\alpha V_I = \sigma_I = const$, $I = 1, \ldots, N$. In components, such a background is given in (5.16). Although these restrictions may look too strong, as we will show in the next section, they will allow us to compute the $\mathcal{N} = 2$ Chern-Simons partition function with the localization method applied to the superfield action.

One-loop partition function is defined by quadratic fluctuations of the quantum superfield v around the classical gauge superfield background V_0,[10]

$$S_2 = -\frac{1}{2}\mathrm{tr}\int d^7z\, E\, v(\nabla^\alpha\bar\nabla^2\nabla_\alpha - 4iW^\alpha\nabla_\alpha)v\,, \tag{5.47}$$

where the superfield strength W_α and gauge-covariant derivatives ∇_α and $\bar\nabla_\alpha$ are constructed with the use of the background gauge superfield V_0 by the rules (3.7) and (3.8). These derivatives obey the (anti)commutation relations similar to (3.2). Note that V_0 in (5.46) has a constant superfield strength G_0. Hence, the superfield W_α vanishes, $W_\alpha = 0$, and the action for the quadratic fluctuations simplifies to

$$S_2 = -\frac{1}{2}\mathrm{tr}\int d^7z\, E\, v\nabla^\alpha\bar\nabla^2\nabla_\alpha v\,. \tag{5.48}$$

The operator $\nabla^\alpha\bar\nabla^2\nabla_\alpha$ in (5.48) is degenerate and requires gauge fixing. Following the conventional background field method in the $\mathcal{N}=2$, $d=3$ superspace [30, 45], we fix the gauge freedom for the quantum transformations (5.45) by imposing the conditions

$$i\bar\nabla^2 v = f\,, \qquad i\nabla^2 v = \bar f\,, \tag{5.49}$$

where f is a fixed covariantly chiral superfunction, $\bar\nabla_\alpha f = 0$. This gauge is manifestly supersymmetric.

The corresponding ghost superfield action has the form

$$S_{\rm FP} = \mathrm{tr}\int d^7z\, E(b+\bar b)L_{gv}[c+\bar c+\coth(L_{gv})(c-\bar c)] = \mathrm{tr}\int d^7z\, E(\bar b c - b\bar c) + O(g)\,, \tag{5.50}$$

where b and c are two covariantly chiral anticommuting ghost superfields and $L_{gv}X$ denotes the commutator, $L_{gv}X = [g\,v, X]$. As a result, the one-loop partition function in the SYM theory is given by the following functional integral

$$Z^{\mathcal{N}=2}_{\rm SYM} = \int \mathcal{D}v\mathcal{D}b\mathcal{D}c\,\delta(f - i\bar\nabla^2 v)\delta(\bar f - i\nabla^2 v)e^{-S_2 - S_{\rm FP}}\,. \tag{5.51}$$

To represent the delta-functions in (5.51) in the Gaussian form, we average this functional integral with the weight

$$1 = \int \mathcal{D}f\mathcal{D}\varphi e^{\alpha \mathrm{tr}\int d^7z\, E[\bar f f + \bar\varphi\varphi]}\,, \tag{5.52}$$

where α is a real parameter and φ is the Grassmann-odd Nielsen-Kallosh ghost. This yields the following gauge-fixing and Nielsen-Kallosh ghost actions

$$S_{\rm gf} = -\frac{\alpha}{2}\mathrm{tr}\int d^7z\, E\, v\{\nabla^2, \bar\nabla^2\}v\,, \qquad S_\varphi = \alpha\,\mathrm{tr}\int d^7z\, E\,\bar\varphi\varphi\,. \tag{5.53}$$

For $\alpha = 1/2$ we have

$$S_2 + S_{\rm gf} = -\mathrm{tr}\int d^7z\, E\, v\Box_{\rm v} v\,, \tag{5.54}$$

[10] The details of the background-quantum expansion of the SYM action in $\mathcal{N}=1$, $d=4$ superspace can be found in [8]. This procedure is also directly applied to the $\mathcal{N}=2$, $d=3$ SYM model under consideration.

where

$$\Box_{\rm v} = \frac{1}{4}\{\nabla^2, \bar\nabla^2\} - \frac{1}{2}\nabla^\alpha\bar\nabla^2\nabla_\alpha + 2iW^\alpha\nabla_\alpha \tag{5.55}$$

is a covariant d'Alembertian operator in the space of real superfields v. With the use of the algebra of the covariant derivatives (3.2), this operator can be represented as

$$\begin{aligned}\Box_{\rm v} =& -\nabla^a\nabla_a + (G_0 - \frac{i}{r}R)^2 + \frac{1}{r}[\nabla^\alpha, \bar\nabla_\alpha] \\ &+2iW^\alpha\nabla_\alpha - 2i\bar W^\alpha\bar\nabla_\alpha - i(\nabla^\alpha W_\alpha)\,. \end{aligned}\tag{5.56}$$

Since we consider the constant gauge superfield background $G_0 = const$ for which $W_\alpha = 0$ and the gauge superfield v has vanishing R-charge, the form of the operator (5.56) gets simplified to

$$\Box_{\rm v} = -\nabla^a\nabla_a + G_0^2 + \frac{1}{r}[\nabla^\alpha, \bar\nabla_\alpha]\,. \tag{5.57}$$

In the one-loop approximation the functional integrals in (5.51) for the gauge and ghost superfields factorize and the partition function takes the form

$$Z^{\mathcal{N}=2}_{\rm SYM} = {\rm Det}^{-1/2}\Box_{\rm v}\cdot Z_\varphi\cdot Z_{b,c}\,. \tag{5.58}$$

Here Z_φ and $Z_{b,c}$ are one-loop partition functions corresponding to the chiral ghost superfields φ and (b,c), repsectively. It is important to note that, as is seen from the action (5.50), the b,c ghosts have vanishing R-charge while the Nielsen-Kallosh ghost φ has R-charge $+2$ as a consequence of the gauge-fixing (5.49),

$$q_{(b,c)} = 0\,,\qquad q_{(\varphi)} = 2\,. \tag{5.59}$$

Let us consider the operator $\Box_{\rm v}$ in (5.58). In general, as a consequence of the gauge invariance of the effective action, the trace of the logarithm of this operator is given by a functional of the gauge superfield strength G

$$-\frac{1}{2}{\rm Tr}\ln\Box_{\rm v} = \int d^7z\, E\,\mathcal{L}(G_0)\,, \tag{5.60}$$

with some effective Lagrangian $\mathcal{L}(G_0)$. We stress that $\mathcal{L}$ explicitly depends on the superfield strength G_0, but not on the gauge field potential V_0, since the Chern-Simons like terms can be produced by chiral field loops only. So, since we consider the constant superfield background, $G_0 = const$, $\mathcal{L}(G_0)$ is also a constant. Therefore, the full superspace integral over this effective Lagrangian vanishes owing to (2.18). We conclude that[11]

$${\rm Det}^{-1/2}\Box_{\rm v} = 1\,, \tag{5.61}$$

i.e. there are no contributions from the quantum superfield v to the partition function (5.58).

At first glance the result (5.61) might look strange, because the component field computations of the $\mathcal{N} = 2$ SYM partition function [13] show that the fields of the gauge multiplet contribute non-trivially. In our case, the $\mathcal{N} = 2$ SYM partition function is entirely due to the chiral ghost superfields, while the gauge multiplet itself brings only trivial

[11] A direct proof of (5.61) based on the analysis of the spectrum of the operator $\Box_{\rm v}$ is given in appendix B.2.

contribution (5.61). In fact, this mismatch is not so surprising, since we use the supersymmetric gauge (5.49) while in the component field computation [13] one imposes the standard Lorentz gauge which is obviously non-supersymmetric. In different gauges the modes giving non-trivial contributions to the partition function can be distributed differently among the gauge multiplet and ghosts, however the final result should be the same, since the partition function is a gauge invariant object.

Consider now the contributions to the partition function (5.58) of the chiral ghost superfields. For simplicity, let us look at the Nielsen-Kallosh ghost φ, the contributions from b, c-ghosts can be analyzed in a similar way. Recall that φ is a covariantly chiral superfield, $\bar{\nabla}_\alpha \varphi = 0$, with the action

$$S_\varphi = \int d^3x d^2\theta d^2\bar{\theta}\, E\, \bar{\varphi}\varphi\,. \tag{5.62}$$

These superfields are in the adjoint representation of SU(N). They can be expanded in the basis elements e_{IJ}[12]

$$\varphi = \sum_{I\neq J}^{N} e_{IJ}\varphi_{IJ}\,, \qquad \bar{\varphi} = \sum_{I\neq J}^{N} e_{JI}\bar{\varphi}_{IJ}\,. \tag{5.63}$$

where e_{IJ} are $N \times N$ matrices in $gl(N)$ with the following matrix elements

$$(e_{IJ})_{KL} = \delta_{IK}\delta_{JL}\,. \tag{5.64}$$

Thus, the action (5.62) is given by the sum of actions for covariantly chiral superfields φ_{IJ} which do not interact with each other

$$S_\varphi = \sum_{I\neq J}^{N} \int d^3x d^2\theta d^2\bar{\theta}\, E\, \bar{\varphi}_{IJ}\varphi_{IJ}\,. \tag{5.65}$$

Each of the superfields $\bar{\varphi}_{IJ}$ is covariantly antichiral,

$$e^{-V_{IJ}}\mathcal{D}_\alpha e^{V_{IJ}}\bar{\varphi}_{IJ} = 0 \text{ for } I < J\,, \qquad e^{V_{IJ}}\mathcal{D}_\alpha e^{-V_{IJ}}\bar{\varphi}_{IJ} = 0 \text{ for } I > J\,, \tag{5.66}$$

where

$$V_{IJ} = V_I - V_J\,. \tag{5.67}$$

The equations (5.66) show that the superfields φ_{IJ} appear in the action (5.65) in pairs in which the two fields have opposite charges associated with the gauge superfield V_{IJ}. Hence, each of the terms in the sum (5.65) is equivalent to the chiral superfield action (5.3) for which the partition function was given in (5.40). There are $N(N-1)/2$ pairs of the superfields φ_{IJ}, hence

$$Z_\varphi = \prod_{I<J}^{N} \frac{1}{(2\sinh \pi r\sigma_{IJ})^2}\,. \tag{5.68}$$

Note that the ghost φ has Grassmann-odd statistics and contributes as in (5.40), but in the inverse power.

[12] Here we exclude the diagonal (Cartan) elements form the sum, $\sum_{I=1}^{N} e_{II}\varphi_{II}$, as they do not interact with the background gauge superfield (5.46).

The ghost superfields b and c can be considered analogously, keeping in mind that they have vanishing R-charges. So one uses the expression given in eq. (5.38), but in the inverse power since the ghosts are Grassmann-odd,

$$Z_{b,c} = \prod_{I<J} (2\sinh \pi r\sigma_{IJ})^4 \,. \tag{5.69}$$

We plug the equations (5.68) and (5.69) into (5.58) and obtain the one-loop partition function of the SYM theory,

$$Z_{\rm SYM}^{\mathcal{N}=2} = \prod_{I<J} 4\sinh^2(\pi r\sigma_{IJ}) \,. \tag{5.70}$$

This partition function differs from the one computed in [13] by the factor $\prod_{I<J}(\sigma_I - \sigma_J)^2$. As we prove in appendix C, this mismatch is due to the fact that in (5.51) we perform the functional integration over the unconstrained superfield v while in the calculations of [13] the zero modes of the scalar field σ in the $\mathcal{N}=2$, $d=3$ gauge multiplet are effectively removed from the corresponding functional integration. In section 6 we will demonstrate that the partition function (5.70) gives the correct result for the $\mathcal{N}=2$ Chern-Simons partition function calculated with the use of the superfield version of the localization method.

5.3 $\mathcal{N}=4$ SYM partition function

In comparison to the $\mathcal{N}=2$ case (3.12), the classical action of $\mathcal{N}=4$ SYM theory on S^3 (4.22) has one extra chiral superfield and a Chern-Simons term which comes about with a real parameter q. It is natural, from the point of view of the SU(2|2) group structure of $\mathcal{N}=4$ supersymmetry, to consider two cases, $q=0$ and $q=1$, both of which are within the bound (4.39). For $q=0$ the action (4.22) has no Chern-Simons term and resembles the $\mathcal{N}=4$ SYM action in flat space. The value $q=1$ is interesting from the point of view of applications of localization methods [14] because it coincides with the scaling dimension of the chiral superfield Φ which constitutes part of the $\mathcal{N}=4$ gauge multiplet. Consider one-loop partition functions in the model (4.22) for these two values of q separately.

For $q=0$ the one-loop partition function in the $\mathcal{N}=4$ SYM model can be represented as

$$Z_{\rm SYM}^{\mathcal{N}=4} = {\rm Det}^{-1/2}(\Box_{\rm v}) \cdot Z_\varphi \cdot Z_{b,c} \cdot Z_\Phi = Z_{\rm SYM}^{\mathcal{N}=2} \cdot Z_\Phi \,, \tag{5.71}$$

where ${\rm Det}^{-1/2}(\Box_{\rm v})$ corresponds to the one-loop determinant for the gauge superfield, $Z_{b,c}$ and Z_φ are contributions from the ghost superfields which are the same as for the $\mathcal{N}=2$ SYM while Z_Φ takes into account the contribution from the chiral superfield Φ. For $q=0$ the latter was computed in (5.38), namely

$$Z_\Phi = \prod_{I<J} \frac{1}{4\sinh^2(\pi r\sigma_{IJ})} \,. \tag{5.72}$$

This expression is the inverse for (5.70). Thus, we conclude that for $q=0$ the $\mathcal{N}=4$ SYM one-loop partition function is

$$Z_{\rm SYM}^{\mathcal{N}=4} = 1 \,. \tag{5.73}$$

Consider now the $\mathcal{N}=4$ SYM partition function for $q=1$,

$$Z_{\rm SYM}^{\mathcal{N}=4} = {\rm Det}^{-1/2}(\Box_{\rm v} - \frac{1}{2r}[\nabla^\alpha, \bar\nabla_\alpha]) \cdot Z_\varphi \cdot Z_{b,c} \cdot Z_\Phi \,. \tag{5.74}$$

In contrast to the previous case, the quadratic operator for the quantum gauge superfield $\Box_{\rm v}$ gets shifted by the term $-\frac{1}{2r}[\nabla^\alpha, \bar\nabla_\alpha]$ which originates from the second variational derivative of the Chern-Simons term in the $\mathcal{N}=4$ SYM action (4.22). The same arguments as in eqs. (5.60) and (5.61) can be employed to show that

$${\rm Det}^{-1/2}(\Box_{\rm v} - \frac{1}{2r}[\nabla^\alpha, \bar\nabla_\alpha]) = 1 \,. \tag{5.75}$$

One can check this identity by analyzing the spectrum of this operator by the methods of appendix C and to verify that this operator has equal numbers of bosonic and fermionic states with the same eigenvalue. Note that for $q=1$ the one-loop partition function of the chiral superfield is trivial, (5.39).

The identities (5.39) and (5.75) show that the $\mathcal{N}=4$ SYM partition function (5.74) receives non-trivial contributions only from the ghost superfields which have the same structure as in the $\mathcal{N}=2$ SYM. Thus, we conclude that the partition functions of the $\mathcal{N}=4$ and $\mathcal{N}=2$ SYM theories coincide

$$Z_{\rm SYM}^{\mathcal{N}=4} = Z_{\rm SYM}^{\mathcal{N}=2} = \prod_{I<J} 4\sinh^2(\pi r \sigma_{IJ}) \,. \tag{5.76}$$

This fact was first noticed in [14].

Naively, it is straightforward to extend the formula (5.74) to the case of the $\mathcal{N}=8$ SYM model (4.55), just by taking the factor Z_Φ in eq. (5.74) three times, since there are three chiral superfields in the game. However, in contrast to the $\mathcal{N}=4$ case, each of these factors becomes non-trivial as soon as the R-charges of the chiral superfields are fractional, eq. (4.61), and do not coincide with the scaling dimensions of these fields. As is argued in [14], the naive computation of the partition function with the chiral superfields having the fractional R-charge (4.61) does not give the partition function corresponding to an infrared fixed point of the $\mathcal{N}=8$ supersymmetric gauge theory. The authors of [14] showed that to get the relevant partition function one should consider a 'mirror' version of the $\mathcal{N}=8$ SYM theory which consists of $\mathcal{N}=4$ SYM action supplemented with one adjoint and one fundamental hypermultiplet. The partition function in the latter model describes the $\mathcal{N}=8$ SYM theory in the infrared regime and agrees with the partition function in the ABJM theory.

6 On localization in $\mathcal{N}=2$ Chern-Simons theory

Before gauge fixing, the path integral for the $\mathcal{N}=2$ Chern-Simons partition function is given by

$$Z_{\rm CS} = \int \mathcal{D}V e^{-S_{\rm CS}} \,. \tag{6.1}$$

According to the localization method [2], one deforms this partition function by an operator X,

$$Z_{\rm CS}(t) = \int \mathcal{D}V e^{-S_{\rm CS}-tX}\,, \tag{6.2}$$

which should be Q-exact, $X = QY$, with respect to a supersymmetry generator Q. This guarantees that the partition function does not depend on the deformation parameter t

$$\frac{dZ(t)}{dt} = 0\,. \tag{6.3}$$

The quantity X should obey some reasonable constraints. Namely, it should be given by a local gauge-invariant functional of the gauge superfield V with a 'good' kinetic term. The conventional choice of this operator is just the SYM action [13]

$$X = S_{\rm SYM}\,. \tag{6.4}$$

The Lagrangian of the $\mathcal{N} = 2$ SYM action is known to be Q-exact [13].

In the functional integral (6.2) one performs the background-quantum splitting similar to eq. (5.42), but with the parameter $1/\sqrt{t}$ instead of the gauge coupling constant g

$$e^V = e^{\Omega^\dagger} e^{\frac{1}{\sqrt{t}}v'} e^{\Omega}\,, \qquad e^{V_0} = e^{\Omega^\dagger} e^{\Omega}\,. \tag{6.5}$$

Note that at this stage the background gauge field V_0 is not restricted to be constant yet.

This generic background-quantum splitting should satisfy the following natural property

$$\{V\} = \{V_0\} \oplus \{v'\}\,, \tag{6.6}$$

i.e. the space of all the fields (trajectories) $\{V\}$ is a direct sum of the spaces of the fields $\{V_0\}$ and $\{v'\}$. Then, the integration measure factorizes

$$\mathcal{D}V = \mathcal{D}V_0 \mathcal{D}v'\,. \tag{6.7}$$

For instance, when all the fields are represented as series in spherical harmonics on S^3 (modes) the decomposition (6.7) assumes that some of these modes (in particular zero modes) are in $\mathcal{D}V_0$ and the others are accounted by $\mathcal{D}v'$. For different choices of V_0 the corresponding redistributions of the modes between the background and the quantum fields are different. This will be important for the comparison of the superfield computations with the component field ones.

The functional integration in (6.2) requires gauge fixing. We use the same gauge fixing procedure as in section 5.2, by taking the gauge-fixing functions (5.49) and inserting them into the functional integral in a standard way

$$Z_{\rm CS}(t) = \int \mathcal{D}V_0 \mathcal{D}v' \mathcal{D}b \mathcal{D}c\, \delta(f - i\bar\nabla^2 v')\delta(\bar f - i\nabla^2 v')\, e^{-S_{\rm CS}[V_0, \frac{1}{\sqrt{t}}v'] - tS_{\rm SYM}[V_0,\frac{1}{\sqrt{t}}v'] - S_{\rm FP}}\,. \tag{6.8}$$

The Faddeev-Popov ghost action $S_{\rm FP}$ has the form of eq. (5.50) but with the gauge coupling constant g replaced with $\frac{1}{\sqrt{t}}$.

The main idea of the localization method is to compute the functional integral (6.2) at $t \to \infty$. In this limit the contribution to the functional integral (6.2) is dominated by quadratic fluctuations around the so-called critical points, i.e. the points for which $X = 0$. In the case under consideration these are the values of the gauge superfield V for which the classical SYM action vanishes

$$S_{\rm SYM}[V_0] = 0\,. \tag{6.9}$$

According to the equation (3.24), the SYM action is equal to zero for the vanishing superfield strength W_α,

$$S_{\rm SYM} = 0 \quad \Leftrightarrow \quad W_\alpha = 0\,, \quad G = G_0 = \text{constant matrix}\,. \tag{6.10}$$

So, the functional integral over $\mathcal{D}V_0$ in (6.2) is localized to such gauge superfield configurations V_0 which have a constant gauge superfield strength G_0. Recall that the lowest component of G is the scalar $\sigma(x)$ which takes its values in the Lie algebra $\mathfrak{g}$ of the gauge group, hence,

$$G_0 = \sigma_0 \in \mathfrak{g}\,, \qquad V_0 = i\sigma_0(\theta\bar\theta - \frac{1}{2r}\theta^2\bar\theta^2)\,. \tag{6.11}$$

As a result, the integration measure $\mathcal{D}V_0$ exactly corresponds to the integration over the zero modes of the Lie-algebra-valued scalar $\sigma(x)$. Therefore, according to (6.7), these zero modes should be removed from the measure $\mathcal{D}v'$.

The actions $S_{\rm CS}[V_0, \frac{1}{\sqrt{t}}v']$ and $S_{\rm SYM}[V_0, \frac{1}{\sqrt{t}}v']$ in (6.8) should be expanded in series with respect to v' around the background field V_0. It is easy to see that for large t only classical part in the Chern-Simons action remains

$$S_{\rm CS}[V_0, g\, v'/\sqrt{t}] = S_{\rm CS}[V_0] + O(1/\sqrt{t})\,, \tag{6.12}$$

while in the SYM action only the quadratic fluctuations survive,

$$-\, t S_{\rm SYM}[V_0, g\, v'/\sqrt{t}] = -S_2[V_0, v'] + O(1/\sqrt{t})\,, \tag{6.13}$$

where $S_2[V_0, v']$ is given by (5.48). Thus, the path integral defining the partition function in the Chern-Simons theory takes the following form

$$Z_{\rm CS} = \int \mathcal{D}V_0\, e^{-S_{\rm CS}[V_0]} \cdot Z'_{\rm SYM}[V_0]\,, \tag{6.14}$$

where

$$Z'_{\rm SYM}[V_0] = \int' \mathcal{D}v'\mathcal{D}b\mathcal{D}c\,\delta(f - i\bar\nabla^2 v')\delta(\bar f - i\nabla^2 v')\, e^{-S_2[V_0,v'] - S_{\rm FP}} \tag{6.15}$$

is the functional integral which has one important difference from the $\mathcal{N} = 2$ SYM partition function (5.51). In (6.15) the integration is over the fields v' excluding their zero modes because they are already taken into account by the measure $\mathcal{D}V_0$ in (6.14) while in (5.51) there are no restrictions on the field v. The reason for this is that in (5.51) we computed the partition function for the particular background in which the gauge field has vacuum expectation values only in the Cartan subalgebra of the gauge group.

Within the superfield methods considered in the previous section the computation of the functional integral (6.15) in a generic Lie-algebra valued background V_0 is much more subtle as compared with (5.51) because it requires the separation of zero modes from non-zero ones *within* superfields. Fortunately, it is possible to rearrange the integration measures in (6.14) such that the $\mathcal{N}=2$ SYM one-loop partition function (5.51) can be used instead of (6.15). To this end, let us separate the Cartan subalgebra directions of the Lie-algebra-valued V_0 from the rest,

$$V_0 = V_0^{\mathfrak{h}} + V_0^{\mathfrak{r}}\,, \qquad V_0^{\mathfrak{h}} \in \mathfrak{h}\,, \quad V_0^{\mathfrak{r}} \in \mathfrak{r}\,. \tag{6.16}$$

Here $\mathfrak{h}$ stands for the Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{r}$ labels the root space directions, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{r}$. The integration measure $\mathcal{D}V_0$ decomposes as

$$\mathcal{D}V_0 = \mathcal{D}V_0^{\mathfrak{h}} \mathcal{D}V_0^{\mathfrak{r}}\,. \tag{6.17}$$

Now, let us combine the measure $\mathcal{D}V_0^{\mathfrak{r}}$ with $\mathcal{D}v'$

$$\mathcal{D}v = \mathcal{D}V_0^{\mathfrak{r}} \mathcal{D}v'\,. \tag{6.18}$$

This new integration measure $\mathcal{D}v$ includes zero modes (as well as all the non-zero ones) which were missing in (6.15). As a result, we get

$$Z_{\rm CS} = \int \mathcal{D}V_0^{\mathfrak{h}}\, e^{-S_{\rm CS}[V_0^{\mathfrak{h}}]} \cdot Z_{\rm SYM}[V_0^{\mathfrak{h}}]\,, \tag{6.19}$$

where $Z_{\rm SYM}[V_0^{\mathfrak{h}}]$ is exactly the $\mathcal{N}=2$ SYM partition function (5.51).

Let us consider now the gauge group U(N) with the Lie algebra $\mathfrak{g} = u(N)$. In this case $V_0^{\mathfrak{h}}$ is a diagonal matrix

$$V_0^{\mathfrak{h}} = {\rm diag}(V_1, V_2, \ldots V_N)\,, \tag{6.20}$$

where each of V_I is as in (5.15), $V_I = i\sigma_I(\theta\bar\theta - \frac{1}{2r}\theta^2\bar\theta^2)$, $\sigma_I = const$. Hence, the integration measure $\mathcal{D}V_0^{\mathfrak{h}}$ reduces to

$$\mathcal{D}V_0^{\mathfrak{h}} = \prod_{I=1}^{N} d\sigma_I\,. \tag{6.21}$$

It is easy to compute the value of the Chern-Simons action (3.26) for the constant gauge superfield background (5.16). One gets

$$S_{\rm CS}[V_0^{\mathfrak{h}}] = i\pi k r^2\, {\rm tr}\, \sigma_0^2 = i\pi k r^2 \sum_{L=1}^{N} \sigma_L^2\,. \tag{6.22}$$

Finally, we substitute (5.51), (6.22) and (6.21) into (6.19) and arrive at the well-known expression for the partition function of the Chern-Simons theory [13],

$$Z_{\rm CS} = \int \prod_{L=1}^{N} d\sigma_L\, e^{-i\pi k r^2 \sigma_L^2} \prod_{I<J}^{N} (2\sinh \pi r(\sigma_I - \sigma_J))^2\,. \tag{6.23}$$

We point out that the expression (6.23) of the partition function is exactly the same as in [13], but the procedure of arriving at this result is different. Let us discuss this difference in more detail.

The authors of [13] notice that the functional integral (6.14) has a residual symmetry (5.44) which can be used to reduce the integration over the Lie-algebra-valued field V_0 to the integration over its Cartan subalgebra values

$$\mathcal{D}V_0 \to \prod_{I=1}^{N} dV_I \prod_{K<L} (V_K - V_L)^2 = \prod_{I=1}^{N} d\sigma_I \prod_{K<L} (\sigma_K - \sigma_L)^2 \,, \tag{6.24}$$

where V_I parametrize the Cartan subalgebra as in (6.20). Here $\prod_{K<L}(V_K - V_L)^2$, which appears in the reduced measure, is the so-called Vandermonde (or Weyl) determinant see, e.g. [1]. Next, one evaluates the factor $Z'_{\rm SYM}[V_0]$ in (6.14) by computing one-loop determinants for all the component fields in the $\mathcal{N} = 2$ gauge multiplet with the following outcome

$$Z'_{\rm SYM}[V_0] = \prod_{I<J}^{N} \left(\frac{2 \sinh \pi r(\sigma_I - \sigma_J)}{\sigma_I - \sigma_J} \right)^2 . \tag{6.25}$$

The denominator in (6.25) exactly cancels the Vandermonde factor in (6.24) and one gets the same result for the partition functions as the one obtained by the superfield computations, i.e. eq. (6.23).

In the superfield approach for computing the partition function we have effectively imposed an additional constraint that the critical points around which the theory is localized are not generic constant scalars valued in the Lie algebra (6.11), but take values only in the Cartan subalgebra. In this case there is no residual symmetry and the Vandermonde factor does not appear. The 'non-Cartan' degrees of freedom are taken into account in the factor $Z_{\rm SYM}[V_0^{\mathfrak{h}}]$ in (6.19).

Comparing (6.25) with (5.70) one can see that the one-loop partition function in the $\mathcal{N} = 2$ SYM differs from the one computed in [13] by the Vandermonde factor

$$Z_{\rm SYM} = Z'_{\rm SYM} \cdot \prod_{I<J} (\sigma_I - \sigma_J)^2 \,. \tag{6.26}$$

This identity is proved explicitly in appendix C by comparing one-loop determinants contributing to $Z_{\rm SYM}$ and $Z'_{\rm SYM}$ within the component field approach. In appendix C we show that the factor $\prod_{I<J}(\sigma_I - \sigma_J)^2$ in (6.26) appears due to the zero modes of the scalar field which were systematically removed from the SYM partition function considered in [13].

To summarize, these two ways of computing the partition function are equivalent since they differ only in the place where the zero modes of the scalars σ are accommodated, i.e. either in the measure $\mathcal{D}V_0$ or in $\mathcal{D}v$. The latter option has turned out to be more convenient in the superfield approach because it is easier to compute the one-loop superfield partition function $Z_{\rm SYM}[V_0^{\mathfrak{h}}]$ with no restrictions on the integration measure (i.e. without separating the zero modes).

In this section we considered the Coulomb branch localization formula only for the pure $\mathcal{N} = 2$ Chern-Simons theory.[13] It is straightforward to generalize this procedure to models

[13] Higgs branch localization of various $\mathcal{N} = 2$ gauge theories on S^3 have been considered recently in [46, 47].

of major interest, such as the Gaiotto-Witten or ABJM theories. To this end one should include into the consideration additional chiral matter fields taking values in appropriate representations of the gauge group. Then the localization formula (6.19) just acquires extra factors with one-loop partition functions of these additional matter superfields. In the component field formulation many such examples were studied in [13, 14].

7 Discussion

In this paper, we have constructed the $\mathcal{N}=2$ superfield formulations of gauge and matter field theories with rigid $\mathcal{N}=2$ supersymmetry on three-sphere S^3. Our construction is based on the supercoset SU(2|1)/U(1) which has S^3 as its bosonic body. For this coset we have derived an explicit form of the supervielbein, covariant derivatives and curvature and used these objects to construct superfield actions for gauge and matter $\mathcal{N}=2$ supermultiplets. Upon the integration over the Grassmann-odd coordinates these actions reduce to the known component field actions which contain terms with S^3 curvature [13]. The $\mathcal{N}=2$ superfield actions on $\frac{\mathrm{SU}(2|1)}{\mathrm{U}(1)} \sim \frac{\mathrm{SU}(2|1)_{\mathrm{L}}\times \mathrm{SU}(2)_{\mathrm{R}}}{\mathrm{U}(1)\times\mathrm{SU}(2)}$ are Euclidean counterparts of actions on the AdS_3 supercoset $\frac{\mathrm{OSp}(2|2)_{\mathrm{L}}\times\mathrm{Sp}(2)_{\mathrm{R}}}{\mathrm{SO}(2)\times\mathrm{Sp}(2)}$ constructed in [9–11] within the study of three-dimensional superfield supergravities.

Using $\mathcal{N}=2$ superfields on SU(2|1)/U(1) we have also constructed superfield actions with extended supersymmetry for $\mathcal{N}=4$ SYM and Gaiotto-Witten theories, $\mathcal{N}=8$ SYM, and $\mathcal{N}=6$ ABJM theory. An interesting new feature of the $\mathcal{N}=4$ SYM action is that it respects the $\mathcal{N}=4$ supersymmetry and SU(2) R-symmetry for arbitrary value of the charge q of the chiral superfield Φ under the $\mathrm{U}(1)_{\mathrm{R}}$ subgroup of SU(2|1). This parameter q appears explicitly both in the action and in the supersymmetry transformations. The value $q=1$ corresponds to the canonical scaling dimension of this superfield. To understand the nature of generic values of q from the point of view of $\mathcal{N}=4$ superalgebra it would be interesting to develop an $\mathcal{N}=4$ superfield formulation of this model. Analogously, the extended supersymmetry does not impose constraints on the values of the $\mathrm{U}(1)_{\mathrm{R}}$ charge of the chiral superfields in the Gaiotto-Witten theory and they may be, in principle, different from the canonical one $q=\frac{1}{2}$.

As a further extension and application of the superfield methods it will be interesting to consider superfield theories on the supercoset SU(2|1)/[U(1) × U(1)] which contains the sphere S^2 as its bosonic body. Gauge and matter multiplets on S^2 were considered in components in [48, 49] where their partition functions were studied with the localization technique. It would be also of interest to develop a superfield formulation for five-dimensional gauge theories on curved backgrounds considered, e.g. in [50–53].

The localization method in supersymmetric field theories effectively reduces the computation of the full partition functions to the calculation of one-loop partition functions for quadratic fluctuations around critical points [1, 16]. As a rule, in the process of the computation of these one-loop determinants many cancelations happen among bosonic and fermionic eigenvalues due to supersymmetry. In superspace, these cancelations occur automatically in the supersymmetric gauge in which the operators of the quadratic fluctuations of the superfields in gauge theory are manifestly supersymmetric. In particular, in

$\mathcal{N} = 2$, $d = 3$ superspace the SYM partition function is represented as a product of one-loop determinants for the gauge superfield v itself and the ghost superfield contributions. Simple superspace arguments allowed us to conclude that the one-loop determinant of the Laplace-like operator $\Box_{\rm v}$ for the superfield v is equal to one and only the ghost superfields contribute to the SYM partition function. The cancelation of the bosonic and fermionic eigenvalues of this operator is verified also by explicit computations of its spectrum given in appendix B.

In superspace, the problem of computing the one-loop partition functions of chiral superfields reduces to finding the chiral superfield propagator at coincident superspace points. We have obtained the result by analyzing the formal superspace expression for this propagator and reducing the problem to the eigenvalue problem of usual bosonic Laplace operator acting on scalar fields. However, it will be useful to derive exact expressions for the chiral and gauge superfield propagators on curved supersymmetric backgrounds such as AdS space or a sphere. Having at hand exact superfield propagators one could compute one-loop partition functions in supersymmetric theories without appealing to the eigenvalue problem for the component fields. Note that propagators of some superfields on $AdS_5 \times S^5$ superspace were studied in [54]. It would be useful to extend these results to chiral and gauge superfields considered in the present paper.

The one-loop partition function in $\mathcal{N} = 2$, $d = 3$ SYM theory computed in section 5.2 differs from the one obtained in [13] by the factor $\prod_{I<J}(\sigma_I - \sigma_J)^2$, where σ_I are vacuum expectation values of the scalar $\sigma(x)$ in the $\mathcal{N} = 2$ gauge supermultiplet. This mismatch is due to the fact that when computing the one-loop SYM partition function in the superfield formulation we performed functional integration over unconstrained superfields, while in [13] the zero modes of component fields are effectively removed from the functional integrals. Such a partition function with removed zero modes appeared in the localization formula for the Chern-Simons partition function. In section 6 we have shown that the $\mathcal{N} = 2$ SYM partition function (5.70) which includes contributions of all the modes is equally good for the localization formula of the Chern-Simons partition function. To this end, one should take care that the scalar zero modes are not counted twice. With the use of superfields, it is more natural to exclude the scalar zero modes from the measure in the localization formula for the Chern-Simons partition function rather than from the one-loop SYM partition function.

To conclude, we have demonstrated that the superfield methods not only simplify the problem of the construction of classical actions for supersymmetric field theories on curved backgrounds, but are also useful for studying their quantum aspects with the use of the localization method. Although we have restricted ourselves to three-dimensional gauge and matter theories, it is straightforward to extend these results to models in other space-time dimensions in which superspace description is applicable.

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A Euclidian $d = 3$ gamma-matrices

The three-dimensional gamma-matrices, taken to be those of Pauli

$$(\gamma_1)_\alpha{}^\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (\gamma_2)_\alpha{}^\beta = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma_3 = -i\gamma_1\gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.1}$$

obey the Clifford algebra

$$\{\gamma_a, \gamma_b\} = 2\delta_{ab}\,, \qquad a, b = 1, 2, 3\,, \tag{A.2}$$

and generate the spinor representation of SU(2)

$$[\gamma_a, \gamma_b] = i\varepsilon_{abc}\gamma_c\,. \tag{A.3}$$

Basic gamma-matrix relations are

$$(\gamma_a)_{\alpha\beta}(\gamma_a)^{\gamma\delta} = -(\delta_\alpha^\gamma\delta_\beta^\delta + \delta_\alpha^\delta\delta_\beta^\gamma)\,, \qquad \mathrm{tr}\,\gamma_a\gamma_b = 2\delta_{ab}\,, \tag{A.4}$$

where the spinorial indices are raised and lowered with the antisymmetric tensors $\varepsilon^{\alpha\beta}$ and $\varepsilon_{\alpha\beta}$ $\varepsilon_{12} = -\varepsilon^{12} = 1$. Useful formulae for products of gamma-matrices:

$$\gamma^a\gamma^b = i\varepsilon^{abc}\gamma^c + \delta^{ab}\mathbf{1}\,, \qquad \gamma^a\gamma^b\gamma^c = i\varepsilon^{abc}\mathbf{1} + \delta^{ab}\gamma^c + \delta^{bc}\gamma^a - \delta^{ac}\gamma^b\,. \tag{A.5}$$

An antisymmetric tensor ω^{ab} can be converted to a vector ω^c and vice versa with the help of Levi-Civita symbol,

$$\omega^{ab} = \varepsilon^{abc}\omega_c\,, \qquad \omega_c = \frac{1}{2}\varepsilon_{abc}\omega^{ab}\,. \tag{A.6}$$

We use the following conventions for converting the vector and spinorial indices into each other,

$$\begin{aligned} \omega^a &= -i\gamma^a_{\alpha\beta}\omega^{\alpha\beta}\,, & \omega^{\alpha\beta} &= -\frac{i}{2}\gamma_a^{\alpha\beta}\omega_a\,, \\ \omega^{ab} &= -i\varepsilon^{abc}\gamma^c_{\alpha\beta}\omega^{\alpha\beta}\,, & \omega^{\alpha\beta} &= -\frac{i}{4}\varepsilon_{abc}\gamma_c^{\alpha\beta}\omega^{ab}\,. \end{aligned} \tag{A.7}$$

In particular, for the bosonic derivative we have

$$\partial_{\alpha\beta} = -\frac{i}{2}\gamma^a_{\alpha\beta}\partial_a\,, \quad \partial_a = -i\gamma_a^{\alpha\beta}\partial_{\alpha\beta}\,, \quad \partial^{\alpha\beta}\partial_{\alpha\beta} = \frac{1}{2}\partial_a\partial_a\,. \tag{A.8}$$

B Spectra of supersymmetric operators on S^3

The supersymmetric Laplacian operator on the sphere has the form (5.13) or (5.56) depending on whether it acts in the space of covariantly chiral Φ or vector superfields V. As we will show below, the supersymmetric eigenvalue problems of these operators are always reduced to the eigenvalue problems of the component fields in Φ and V. Therefore, before we start considering supersymmetric operators we summarize the result about the spectra of conventional Laplacian and Dirac operators on S^3. All these results are well known and can be found e.g. in the appendices of [16].

- Laplacian operator $-\partial^a\partial_a$ acting on scalar fields ϕ has the following eigenvales
$$-\partial^a\partial_a\phi^{(n)} = \lambda_n\phi^{(n)}\,, \quad \lambda_n = \frac{1}{r^2}n(n+2)\,, \quad d_n = (n+1)^2\,, \quad n = 0,1,2,\ldots \tag{B.1}$$
Here (and further) d_n means the degeneracy of the corresponding eigenvalue.

- Dirac operator $-i\gamma^a\hat{\mathcal{D}}_a$ on S^3 has the spectrum
$$\lambda_n^\pm = \pm\frac{1}{r}(n+\frac{1}{2})\,, \qquad d_n^\pm = n(n+1)\,, \qquad n = 1,2,3,\ldots \tag{B.2}$$

- The operator of square of the full angular momentum $\mathbf{J}^2 = -(\partial_a + \frac{i}{r}\gamma_a)^2$ acting on spinors ψ_α has the spectrum
$$-(\partial_a + \frac{i}{r}\gamma_a)^2\psi_\alpha^{(n)} = \lambda_n\psi_\alpha^{(n)}\,, \quad \lambda_n = \frac{1}{r^2}n(n+2)\,, \quad d_n = 2(n+1)^2\,, \quad n = 0,1,2,\ldots \tag{B.3}$$
This spectrum coincides with the scalar spectrum (B.1), but the number of states is doubled because the spinor ψ_α has two independent components. Indeed, the operator $\mathbf{J}$ of the total angular momentum is given by the sum of orbital and spin parts,
$$\mathbf{J} = \mathbf{L} + \mathbf{S}\,, \qquad \mathbf{L}_a = -\frac{i}{2}\partial_a\,, \qquad \mathbf{S}_a = \frac{1}{2}\gamma_a\,. \tag{B.4}$$
All these three operators $\mathbf{J}$, $\mathbf{L}$ and $\mathbf{S}$ obey the commutation relations of the $su(2)$ algebra,
$$[\mathbf{J}_a,\mathbf{J}_b] = i\varepsilon_{abc}\mathbf{J}_c\,, \qquad [\mathbf{L}_a,\mathbf{L}_b] = i\varepsilon_{abc}\mathbf{L}_c\,, \qquad [\mathbf{S}_a,\mathbf{S}_b] = i\varepsilon_{abc}\mathbf{S}_c\,. \tag{B.5}$$
Hence, the spectrum of $\mathbf{L}^2$ is $\frac{1}{r^2}n(n+2)$ and the spectrum of $\mathbf{J}^2$ is similar, but with shifted values of n as $n\to n\pm1$,
$$\lambda_n = \begin{cases} \frac{1}{r^2}(n+1)(n+3)\,, & d_n = (n+2)(n+1) \\ \frac{1}{r^2}(n-1)(n+1)\,, & d_n = n(n+1)\,, \qquad n = 0,1,2,\ldots \end{cases} \tag{B.6}$$
This spectrum is equivalent to (B.3).

- The covariant Laplacian operator $\Delta = -\hat{\mathcal{D}}^a\hat{\mathcal{D}}_a + \frac{2}{r^2}$ acting in the space of divergenceless one-forms B_a on S^3, $\partial^a B_a = 0$, has the spectrum
$$\lambda_n = \frac{1}{r^2}(n+1)^2\,, \qquad d_n = 2n(n+2)\,. \tag{B.7}$$

B.1 Chiral superfield Laplacian

Consider the eigenvalue problem for the operator H (5.5) in the case of vanishing gauge superfield background,

$$\frac{1}{2}\begin{pmatrix} 0 & -\bar{\mathcal{D}}^2 \\ -\mathcal{D}^2 & 0 \end{pmatrix}\begin{pmatrix} \Phi \\ \bar{\Phi} \end{pmatrix} = \lambda \begin{pmatrix} \Phi \\ \bar{\Phi} \end{pmatrix}. \tag{B.8}$$

Here Φ is a chiral superfield, $\bar{\mathcal{D}}_\alpha \Phi = 0$. For any $\lambda \neq 0$ this equation implies

$$\frac{1}{4}\bar{\mathcal{D}}^2\mathcal{D}^2\Phi = \lambda^2\Phi\,, \qquad \frac{1}{4}\mathcal{D}^2\bar{\mathcal{D}}^2\bar{\Phi} = \lambda^2\bar{\Phi}\,, \tag{B.9}$$

or

$$\left(-\mathcal{D}^a\mathcal{D}_a + M^2\right)\Phi = \lambda^2\Phi\,, \qquad \left(-\mathcal{D}^a\mathcal{D}_a + M^2\right)\bar{\Phi} = \lambda^2\bar{\Phi}\,, \tag{B.10}$$

where

$$M^2 = \frac{q(2-q)}{r^2} \qquad (R\Phi = -q\Phi)\,. \tag{B.11}$$

The equations (B.10) allow one to find the eigenvalues λ up to signs.

Using the explicit expression (2.37) for the derivative $\mathcal{D}_a$ we find

$$\mathcal{D}^a\mathcal{D}_a\Phi = (\partial_a + \frac{i}{r}(\gamma_a)_\beta^\alpha\theta^\beta\partial_\alpha)^2\Phi\,. \tag{B.12}$$

Recall that component field decomposition for the chiral superfield reads

$$\Phi = \varphi + \theta^\alpha\psi_\alpha + \frac{1}{2}\theta^2 F\,. \tag{B.13}$$

As a result, we get the following equations for the component fields

$$-\partial^a\partial_a\varphi + M^2\varphi = \lambda^2_{(\varphi)}\varphi\,, \qquad -\partial^a\partial_a F + M^2 F = \lambda^2_{(F)}F\,, \tag{B.14}$$

$$-(\partial_a + \frac{i}{r}\gamma_a)^2\psi + M^2\psi = \lambda^2_{(\psi)}\psi\,. \tag{B.15}$$

The bosonic spectrum for the fields φ and F in (B.14) can be found from (B.1),

$$\lambda^2_{(\varphi)n} = \lambda^2_{(F)n} = \frac{1}{r^2}n(n+2) + M^2\,, \qquad d_n = (n+1)^2\,, \qquad n = 0,1,2,\ldots \tag{B.16}$$

Owing to (B.3), the fermions spectrum for the fields ψ_α in (B.15) appears to be exactly the same,

$$\lambda^2_{(\psi)n} = \frac{1}{r^2}n(n+2) + M^2\,, \qquad d_n = 2(n+1)^2\,, \qquad n = 0,1,2,\ldots \tag{B.17}$$

Hence, these eigenvalues cancel among each other and the determinant of the operator (B.10) is equal to one,

$$\det\left(-\mathcal{D}^a\mathcal{D}_a + M^2\right) = \frac{\prod_m(\lambda^2_{(\varphi)m})^{d_m}\prod_n(\lambda^2_{(F)n})^{d_n}}{\prod_k(\lambda^2_{(\psi)k})^{d_k}} = 1\,. \tag{B.18}$$

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We point out that thr operator $-\mathcal{D}^a\mathcal{D}_a + M^2$ appears by squaring the operator H in (5.5). However, this squaring is possible for every $\lambda \neq 0$ while the zero modes require special considerations. Indeed, the zero modes obey the equations

$$\mathcal{D}^2\Phi = 0\,, \qquad \bar{\mathcal{D}}^2\bar{\Phi} = 0\,, \tag{B.19}$$

instead of (B.10). These two equations are equivalent and we consider the first of them. Using the explicit form of the covariant spinor derivatives (2.37) for the components of the chiral superfield (B.13) we find

$$F = 0\,, \tag{B.20}$$

$$-\partial^a\partial_a\varphi + \frac{q(2-q)}{r^2}\varphi = 0\,, \tag{B.21}$$

$$-i(\gamma^a)_\alpha^\beta\hat{\mathcal{D}}_a\psi_\beta + \frac{2q-1}{2r}\psi_\alpha = 0\,, \tag{B.22}$$

$$-(\partial_a + \frac{i}{r}\gamma^a)^2\psi_\alpha + \frac{q(2-q)}{r^2}\psi_\alpha = 0\,. \tag{B.23}$$

Here $\hat{\mathcal{D}}_a = \partial_a - \frac{i}{2}M_a$ is purely bosonic covariant derivative acting on the spinor field. Note that (B.23) is a differential consequence of (B.22), hence, it does not require separate treatment.

Using (B.1) and (B.2) we find the eigenvalues of the operators in the equations (B.21) and (B.22):

$$\lambda_{(\varphi)n} = \frac{1}{r^2}n(n+2) + \frac{q(2-q)}{r^2}\,, \qquad d_n = (n+1)^2\,, \qquad n = 0,1,2,\ldots \tag{B.24}$$

$$\lambda_{(\psi)n} = \pm\frac{1}{r}(n+\frac{1}{2}) + \frac{2q-1}{2r}\,, \qquad d_n^\pm = n(n+1)\,, \qquad n = 1,2,3,\ldots \tag{B.25}$$

These eigenvalues can vanish for some particular values of the charge q. In particular, the values $q = 0$ and $q = 2$ should be investigated.

For $q = 0$ the equation (B.21) has one zero mode $\varphi = const$ while the fermionic equation (B.22) has no zero modes. Hence, for $q = 0$ the operator H has two bosonic zero modes in its spectrum corresponding to $\varphi = const$ and $\bar{\varphi} = const$ (the latter appears in the antichiral superfield $\bar{\Phi}$).

For $q = 2$ the equation (B.21) has one bosonic zero mode, but there are also two fermionic zero modes in (B.22) as follows from (B.2). Hence, for $q = 2$ the operator H has two bosonic and four fermionic zero modes (the doubling is because the antichiral superfield $\bar{\Phi}$ contributes similarly as Φ).

We point out that for $q = \frac{1}{2}$ the equations (B.21) and (B.22) do not have zero modes and $\lambda = 0$ only for $\Phi = 0$. Hence, for the chiral matter superfields with canonical R-charge the operator H has no zero modes.

B.2 Vector superfield Laplacian on gauge superfield background

In this section we perform direct computation of the determinant of the vector superfield Laplacian by calculating its spectrum. We will use chiral coordinates in which the covariant derivatives are given by (2.37) and the background gauge superfield has the form (5.15). Then, the operator (5.57) can be written as

$$\Box_{\rm v} = -\mathcal{D}^a\mathcal{D}_a + \frac{1}{r}[\mathcal{D}^\alpha, \bar{\mathcal{D}}_\alpha] + \frac{2i}{r}\sigma_0\bar\theta^\alpha\bar{\mathcal{D}}_\alpha + \sigma_0^2 - \frac{2i}{r}\sigma_0\,, \tag{B.26}$$

where σ_0 is a constant. We consider the eigenvalue problem

$$\Box_{\rm v} V = \lambda V\,, \tag{B.27}$$

where V is a chargeless superfield without any further constraints. It has the following expansion over Grassmann coordinates

$$V(x,\theta,\bar\theta) = w(x,\theta) + \bar\theta^\alpha\Psi_\alpha(x,\theta) + \bar\theta^2 F(x,\theta)\,, \tag{B.28}$$

where w, Ψ_α and F are chiral superfields,

$$\begin{aligned} w &= w_0(x) + \theta^\alpha w_\alpha(x) + \theta^2\mathbf{w}(x)\,,\\ \Psi_\alpha &= \psi_\alpha(x) + \theta_\alpha\varphi(x) + \theta^\beta A_{(\alpha\beta)}(x) + \theta^2\boldsymbol{\Psi}_\alpha(x)\,,\\ F &= F_0(x) + \theta^\alpha F_\alpha(x) + \theta^2\mathbf{F}(x)\,. \end{aligned} \tag{B.29}$$

Substituting (B.28) into (B.27) we get the following eigenvalue problems for the chiral superfields w, Ψ_α and F,

$$(-\mathcal{D}^a\mathcal{D}_a + \sigma_0^2 - \frac{2i}{r}\sigma_0)w - \frac{2}{r}\partial_\alpha\Psi^\alpha = \lambda w\,, \tag{B.30}$$

$$\begin{aligned} &-(\mathcal{D}_a + \frac{i}{2r}\gamma_a)^2\Psi_\alpha + \sigma_0^2\Psi_\alpha\\ &+\frac{2i}{r}(\gamma^a)_\alpha^\beta\partial_a\Psi_\beta + \frac{2}{r^2}\theta_\alpha\partial_\beta\Psi^\beta + \frac{2}{r^2}\theta_\beta\partial_\alpha\Psi^\beta - \frac{2}{r^2}\Psi_\alpha + \frac{4}{r}\partial_\alpha F = \lambda\Psi_\alpha\,, \end{aligned} \tag{B.31}$$

$$(-\mathcal{D}_a\mathcal{D}^a + \sigma_0^2 + \frac{2i}{r}\sigma_0)F = \lambda F\,. \tag{B.32}$$

Here we used the fact that the R-charges of the chiral superfields are $R\Psi_\alpha = -\Psi_\alpha$, $RF = -2F$. Next, we expand remaining derivatives in (B.30), (B.31) and (B.32) and arrive at the following set of equations for the component fields

$$-\partial^a\partial_a w_0 - \frac{4}{r}\varphi + (\sigma_0^2 - \frac{2i}{r}\sigma_0)w_0 = \lambda w_0\,, \tag{B.33a}$$

$$-(\partial_a + \frac{i}{r}\gamma_a)^2 w_\alpha + \frac{4}{r}\boldsymbol{\Psi}_\alpha + (\sigma_0^2 - \frac{2i}{r}\sigma_0)w_\alpha = \lambda w_\alpha\,, \tag{B.33b}$$

$$-\partial^a\partial_a\mathbf{w} + (\sigma_0^2 - \frac{2i}{r}\sigma_0)\mathbf{w} = \lambda\mathbf{w}\,; \tag{B.33c}$$

$$-\partial^a\partial_a F_0 + (\sigma_0^2 + \frac{2i}{r}\sigma_0)F_0 = \lambda F_0\,, \tag{B.34a}$$

$$-\partial^a\partial_a\mathbf{F} + (\sigma_0^2 + \frac{2i}{r}\sigma_0)\mathbf{F} = \lambda\mathbf{F}\,, \tag{B.34b}$$

$$-(\partial_a + \frac{i}{r}\gamma_a)^2 F_\alpha + (\sigma_0^2 + \frac{2i}{r}\sigma_0)F_\alpha = \lambda F_\alpha\,; \tag{B.34c}$$

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$$(-\partial_a^2 + \frac{1}{r^2} + \sigma_0^2)\psi_\alpha + \frac{4}{r^2}F_\alpha = \lambda\psi_\alpha\,, \tag{B.35a}$$

$$(-\partial_a^2 + \frac{1}{r^2} + \sigma_0^2)\mathbf{\Psi}_\alpha = \lambda\mathbf{\Psi}_\alpha\,, \tag{B.35b}$$

$$(-\partial_a^2 + \frac{4}{r^2} + \sigma_0^2)\varphi + \frac{i}{r}(\gamma_a)^{\alpha\beta}\partial_a A_{\alpha\beta} + \frac{8}{r}\mathbf{F} = \lambda\varphi\,, \tag{B.35c}$$

$$(-\partial_a^2 + \frac{4}{r^2} + \sigma_0^2)A_{\alpha\beta} - \frac{2i}{r}\gamma^a_{\alpha\beta}\partial_a\varphi - \frac{2i}{r}(\gamma^a)_{(\beta}^{\gamma}\partial_a A_{\alpha)\gamma} = \lambda A_{\alpha\beta}\,. \tag{B.35d}$$

The bispinor $A_{\alpha\beta}$ is equivalent to a vector, $A_a = -i\gamma_a^{\alpha\beta}A_{\alpha\beta}$. Hence, the equation (B.35d) can be rewritten as

$$(-\hat{\mathcal{D}}^b\hat{\mathcal{D}}_b + \frac{2}{r^2} + \sigma_0^2)A_a + \frac{4}{r}\partial_a\varphi = \lambda A_a\,. \tag{B.36}$$

where we used the fact that the covariant derivative acts on the vector by the rule $\hat{\mathcal{D}}_a A_b = \partial_a A_b + \frac{1}{r}\varepsilon_{abc}A_c$. Next, we decompose this vector into the divergenceless B_a and gradient parts,

$$A_a = B_a + \partial_a b\,, \qquad \partial^a B_a = 0\,, \quad b \neq \text{const}\,. \tag{B.37}$$

The equation (B.36) leads to two independent equations for these components,

$$(-\hat{\mathcal{D}}_b^2 + \frac{2}{r^2} + \sigma_0^2)B_a = \lambda B_a\,, \tag{B.38}$$

$$(-\hat{\mathcal{D}}_b^2 + \sigma_0^2)b + \frac{4}{r}\varphi = \lambda b\,. \tag{B.39}$$

Note also that eq. (B.35c) is equivalent to

$$(-\partial_a^2 + \frac{4}{r^2} + \sigma_0^2)\varphi - \frac{1}{r}\partial_a^2 b + \frac{8}{r}\mathbf{F} = \lambda\varphi\,. \tag{B.40}$$

Our purpose now is to find the eigenvalues λ from the system of equations (B.33a)–(B.35b) and (B.38)–(B.40). Some of these equations are entangled because of the fact that we work in the chiral coordinates. We start with the case when the equations (B.34) have trivial solution, $F_0 = F_\alpha = \mathbf{F} = 0$. In this case (B.35) can be rewritten as

$$(-\partial_a^2 + \frac{1}{r^2} + \sigma_0^2)\psi_\alpha = \lambda\psi_\alpha\,, \tag{B.41}$$

$$(-\partial_a^2 + \frac{1}{r^2} + \sigma_0^2)\mathbf{\Psi}_\alpha = \lambda\mathbf{\Psi}_\alpha\,, \tag{B.42}$$

$$(-\hat{\mathcal{D}}_b^2 + \frac{2}{r^2} + \sigma_0^2)B_a = \lambda B_a\,, \tag{B.43}$$

$$(-\partial_a^2 + \sigma_0^2)b + \frac{4}{r}\varphi = \lambda b\,, \tag{B.44}$$

$$(-\partial_a^2 + \frac{4}{r^2} + \sigma_0^2)\varphi - \frac{1}{r}\partial_a^2 b = \lambda\varphi\,. \tag{B.45}$$

The equations (B.41) and (B.42) for the spinors ψ_α and $\mathbf{\Psi}_\alpha$ have the form of the bosonic equation (B.1), but with shifted value of λ. Hence, we find the spectrum,

$$\lambda_n = \frac{1}{r^2}(n+1)^2 + \sigma_0^2\,, \qquad n = 0, 1, 2, \ldots\,, \tag{B.46}$$

with altogether $d_n = 4(n+1)^2$ fermionic states on the corresponding level.

The operator $\Delta = -\hat{\mathcal{D}}_a^2 + \frac{2}{r^2}$ in (B.43) is nothing but the Laplacian operator acting in the space of divergenceless one-forms. Its spectrum is given (B.7). Thus, the equation for the vector B_a gives eigenvalues $\lambda_n = \frac{1}{r^2}(n+1)^2 + \sigma_0^2$, with degeneracies $d_n = 2n(n+2)$.

The equations (B.43) and (B.45) also have the spectrum (B.46) with $d_n = 2n^2+4n+4$ states on the corresponding level. Thus, the equations (B.41)–(B.45) have non-trivial solutions for the values of λ given by (B.46) with $4(n+1)^2$ bosonic and $4(n+1)^2$ states on the n-th level. Finally, we point out that for every non-trivial solution of these equations the system (B.33) has the unique solution of the form

$$\mathbf{w} = 0\,, \quad w_\alpha = w_\alpha(\mathbf{\Psi}_\alpha)\,, \quad w_0 = w_0(\varphi)\,. \tag{B.47}$$

with some functions $w_\alpha(\mathbf{\Psi}_\alpha)$ and $w_0(\varphi)$. Therefore, no new independent degrees of freedom appear from (B.33).

Let us turn to the case when the system (B.34) has non-trivial solutions. Equations (B.34) are similar to (B.14) and (B.15) which correspond to the supersymmetric Laplacian operator acting on the chiral superfield (B.10) in the case of vanishing R-charge q. Therefore the equations (B.34) give the spectrum

$$\lambda_n = \frac{1}{r^2}n(n+2) + \sigma_0^2 + \frac{2i}{r}\sigma_0\,, \qquad n = 0,1,2,\ldots, \tag{B.48}$$

with $d_n = 2(n+1)^2$ bosonic and $d_n = 2(n+1)^2$ fermionic states on n-th level. One can easily see that for every non-trivial solution of (B.34) it is possible to find unique solution of the remaining equations (B.33) and (B.35). Hence, these equations do not give any new degrees of freedom corresponding to the eigenvalues (B.48).

The last case to consider is when both systems (B.34) and (B.35) have trivial solutions, $F_0 = \mathbf{F} = F_\alpha = 0$, $\psi_\alpha = \mathbf{\Psi}_\alpha = A_{\alpha\beta} = \varphi = 0$. In this case the set of equations (B.33) is simply

$$-\partial^a\partial_a w_0 + (\sigma_0^2 - \frac{2i}{r}\sigma_0)w_0 = \lambda w_0\,, \tag{B.49}$$

$$-(\partial_a + \frac{i}{r}\gamma_a)^2 w_\alpha + (\sigma_0^2 - \frac{2i}{r}\sigma_0)w_\alpha = \lambda w_\alpha\,, \tag{B.50}$$

$$-\partial^a\partial_a \mathbf{w} + (\sigma_0^2 - \frac{2i}{r}\sigma_0)\mathbf{w} = \lambda \mathbf{w}\,. \tag{B.51}$$

These equations are identical to the ones (B.14), (B.15) arising from the chiral superfield eigenvalue problem. Hence, using (B.16), we can immediately write down the spectrum,

$$\lambda_n = \frac{1}{r^2}n(n+2) + \sigma_0^2 - \frac{2i}{r}\sigma_0\,, \qquad n = 0,1,2,\ldots. \tag{B.52}$$

For any given eigenvalue there are $d_n = 2(n+1)^2$ bosonic and fermionic modes.

To summarize, the system of equations (B.33)–(B.35) has the spectrum (B.46), (B.48) and (B.52). The numbers of states (degeneracies) for these eigenvalues are given in table 1. This table shows that for every eigenvalue λ_n there are equal numbers of bosonic and fermionic eigenstates. Hence, they exactly cancel against each other in the determinant of the operator $\Box_{\rm v}$,

$$\det \Box_{\rm v} = \frac{\prod \lambda_{\rm bos}}{\prod \lambda_{\rm ferm}} = 1\,. \tag{B.53}$$

This result was used in section 5.2 when computing the SYM partition function.

	$\lambda = \frac{1}{r^2}(n+1)^2 + \sigma_0^2$	$\lambda = \frac{1}{r^2}n(n+2) + \sigma_0^2 + \frac{2i}{r}\sigma_0$	$\lambda = \frac{1}{r^2}n(n+2) + \sigma_0^2 - \frac{2i}{r}\sigma_0$
w_0	0	0	$(n+1)^2$
$\mathbf{w}$	0	0	$(n+1)^2$
w_α	0	0	$2(n+1)^2$
F_0	0	$(n+1)^2$	0
$\mathbf{F}$	0	$(n+1)^2$	0
F_α	0	$2(n+1)^2$	0
ψ_α	$2(n+1)^2$	0	0
$\mathbf{\Psi}_\alpha$	$2(n+1)^2$	0	0
B_a	$2n(n+2)$	0	0
φ, b	$2n^2+4n+4$	0	0

Table 1. Degeneracies of eigenvalues of the operator $\square_{\rm v}$ acting on general superfield V.

C Component field calculation of the $\mathcal{N} = 2$ SYM one-loop partition function revisited

The one-loop partition function in the $\mathcal{N} = 2$ SYM theory was computed in [13] by considering the spectra of operators of quadratic fluctuations for bosonic and fermionic fields of the $\mathcal{N} = 2$ gauge multiplet. Here we revisit these computations with a special attention to zero modes of scalar fields. In contrast to [13] we use a modified Lorentz gauge which has no zero modes and gives a mass term to the Laplacian operators of the Faddeev-Popov ghosts and physical scalar σ making these operators invertible.

Consider the $\mathcal{N} = 2$ super Yang-Mills action in the component form (3.25) and make background-quantum splitting for the scalar field σ,

$$\sigma \to \sigma_0 + g\,\sigma\,, \quad A_a \to gA_a\,, \quad \lambda_\alpha \to g\lambda_\alpha\,, \quad D \to gD\,, \tag{C.1}$$

were g is the gauge coupling and σ_0 is a constant background field which is chosen to belong to the Cartan subalgebra of the gauge algebra. For computing the one-loop partition function it is sufficient to consider the part of the action (3.25) which describes quadratic fluctuations around this background,

$$S_2 = {\rm tr}\int d^3x\,\sqrt{h}(\mathcal{L}_{\rm bos} + \mathcal{L}_{\rm ferm})\,, \tag{C.2}$$

$$\begin{aligned}\mathcal{L}_{\rm bos} &= \frac{1}{2}\hat{\mathcal{D}}_a A_b \hat{\mathcal{D}}^a A^b - \frac{1}{2}\hat{\mathcal{D}}_a A_b \hat{\mathcal{D}}^b A^a + \frac{1}{2}\partial^a\sigma\partial_a\sigma + i\partial_a\sigma[A^a,\sigma_0] - \frac{1}{2}[A_a,\sigma_0]^2 \\ &\quad + \frac{1}{2}\left(D + \frac{2\sigma}{r}\right)^2\,,\end{aligned} \tag{C.3}$$

$$\mathcal{L}_{\rm ferm} = \frac{i}{2}\lambda^\alpha(\gamma^a)_\alpha^\beta\hat{\mathcal{D}}_a\bar\lambda_\beta - \frac{i}{2}\lambda^\alpha[\sigma_0,\bar\lambda_\alpha] + \frac{1}{4r}\lambda^\alpha\bar\lambda_\alpha\,. \tag{C.4}$$

Here $\hat{\mathcal{D}}_a$ is purely bosonic covariant derivative on S^3 with standard commutation rule, $[\hat{\mathcal{D}}_a, \hat{\mathcal{D}}_b] = -\frac{i}{4r}M_{ab}$.

The one-loop partition function

$$Z_{\rm SYM}[\sigma_0] = \int \mathcal{D}A_a\,\mathcal{D}\sigma\,\mathcal{D}\lambda_\alpha\,\mathcal{D}D\,e^{-S_2} \tag{C.5}$$

requires gauge fixing since the SYM action is gauge invariant. The standard Lorentz gauge

$$\hat{\mathcal{D}}^a A_a = 0 \tag{C.6}$$

(although admissible) is not convenient here because there is the cross-term $i\partial_a\sigma[A^a, \sigma_0]$ in (C.3). It is desirable to have a propagator in the diagonal form, without mixing of the fields A_a and σ. The simplest way to eliminate this crossing term from the action is to impose the modified Lorentz gauge,

$$f = \hat{\mathcal{D}}^a A_a + i[\sigma_0, \sigma]\,, \tag{C.7}$$

where $f(x)$ is some fixed function. In principle, one can put this function to zero, but we keep it to represent the gauge-fixing condition in the functional integral in Gaussian form. Indeed, the functional delta-function $\delta(\hat{\mathcal{D}}^a A_a + i[\sigma_0, \sigma] - f)$, after averaging over f with a suitable weight, leads to the gauge-fixing term

$$S_{\rm gf} = {\rm tr} \int d^3x\, \sqrt{h}\, \mathcal{L}_{\rm gf}\,, \qquad \mathcal{L}_{\rm gf} = \frac{1}{2} f^2\,. \tag{C.8}$$

Adding this action to (C.2) we find[14]

$$\mathcal{L}_{\rm bos} + \mathcal{L}_{\rm gf} = \frac{1}{2} A_a \Delta A^a - \frac{1}{2}[A_a, \sigma_0]^2 - \frac{1}{2}\sigma\partial^2\sigma - \frac{1}{2}[\sigma_0, \sigma]^2\,, \tag{C.9}$$

where

$$\Delta = -\hat{\mathcal{D}}^2 + \frac{2}{r^2} \tag{C.10}$$

is the covariant Laplacian operator in the space on one-forms on S^3. As a result, the gauge fixed version of the functional integral (C.5) reads

$$Z_{\rm SYM}[\sigma_0] = \int \mathcal{D}A_a\, \mathcal{D}\sigma\, \mathcal{D}\lambda_\alpha\, \Delta_{\rm FP}\, e^{-\int d^3x\,\sqrt{h}(\mathcal{L}_{\rm bos} + \mathcal{L}_{\rm ferm} + \mathcal{L}_{\rm gf})}\,, \tag{C.11}$$

where $\Delta_{\rm FP}$ is the Faddeev-Popov determinant. We stress that the functional integration $\int \mathcal{D}\sigma$ in (C.11) runs over all configurations of the scalar field σ, including its zero mode (i.e., the zero mode of the operator ∂^2).

Consider the variation of the gauge-fixing function (C.7) under gauge transformations with local gauge parameter $\lambda = \lambda(x)$,

$$\delta f = i(\partial^a\partial_a\lambda - [\sigma_0, [\sigma_0, \lambda]] + ig\partial^a[A_a, \lambda] - g[\sigma_0, [\sigma, \lambda]])\,. \tag{C.12}$$

The last two terms in (C.12) are not essential for one-loop computations as they are responsible for interactions of the ghost fields with the vector A_a and scalar σ. The quadratic term for the ghost fields corresponds to the operator

$$\mathcal{O} = -\partial^a\partial_a + [\sigma_0, [\sigma_0, \cdot]]\,. \tag{C.13}$$

[14] We omit the term $\frac{1}{2}\left(D + \frac{2\sigma}{r}\right)^2$ in (C.3) since the functional integration over the auxiliary field D gives trivial contribution to the partition function.

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Hence, the one-loop Faddeev-Popov determinant $\Delta_{\rm FP} = {\rm Det}\,\mathcal{O}$ is represented by the functional integral over anticommuting Faddeev-Popov ghosts b and c,

$$\Delta_{\rm FP} = \int \mathcal{D}b\mathcal{D}c\, e^{-S_{\rm FP}}\,, \qquad S_{\rm FP} = {\rm tr} \int d^3x\, \sqrt{h}\, b(-\partial^2 c + [\sigma_0, [\sigma_0, c]])\,. \tag{C.14}$$

Note that the functional integration in (C.14) is taken over unrestricted ghost fields b and c, including their zero modes. Indeed, the operator (C.13) is non-degenerate owing to the last term which is nothing but the mass parameter. This term can be also interpreted as the interaction of the ghost fields with the background field σ_0. This is the crucial difference of our computation from the one given in [13] where the Lorentz gauge (C.6) was imposed and the zero modes of σ did not enter the functional integral over $\mathcal{D}\sigma$ in (C.5) (we will comment on this case in the end of this section).

In what follows we concentrate on the gauge group SU(N). In this case all the fields are given by Hermitian matrices. Consider, for instance, the gauge field A_a and expand it over the basis in the Lie algebra $gl(N)$,

$$A_a = \sum_{I,J=1}^{N} e_{IJ} A_a^{IJ}\,, \quad \bar{A}_a^{IJ} = A_a^{JI}\,, \quad \sum_{I=1}^{N} A_a^{II} = 0\,, \tag{C.15}$$

where the basis elements e_{IJ} are given by the matrices

$$(e_{IJ})_{KL} = \delta_{IK}\delta_{JL} \tag{C.16}$$

with the orthogonality property

$${\rm tr}\, e_{IJ} e_{KL} = \delta_{IL}\delta_{JK}\,. \tag{C.17}$$

The field σ_0 in the Cartan subalgebra of $su(N)$ is just the diagonal matrix,

$$\sigma_0 = {\rm diag}(\sigma_1, \sigma_2, \ldots, \sigma_N)\,, \qquad \sum_{I=1}^{N} \sigma_I = 0\,. \tag{C.18}$$

Hence, we have the following properties

$$[\sigma_0, A_a] = \sum_{I\neq J}^{N} (\sigma_I - \sigma_J) e_{IJ} A_a^{IJ}\,, \qquad {\rm tr}[\sigma_0, A_a]^2 = -\sum_{I\neq J}^{N} (\sigma_I - \sigma_J)^2 A_a^{IJ} \bar{A}_a^{IJ}\,. \tag{C.19}$$

Applying these rules to all fields in the gauge multiplet we rewrite the expressions (C.4) and (C.9) as well as the Lagrangian for the ghost fields as

$${\rm tr}(\mathcal{L}_{\rm bos} + \mathcal{L}_{\rm gf}) = \frac{1}{2} \sum_{I\neq J} \left[\bar{A}_a^{IJ}(\Delta + (\sigma_I - \sigma_J)^2) A_a^{IJ} + \bar{\sigma}^{IJ}(-\partial^2 + (\sigma_I - \sigma_J)^2)\sigma^{IJ}\right], \tag{C.20}$$

$${\rm tr}\,\mathcal{L}_{\rm ferm} = \frac{1}{2} \sum_{I\neq J} \left[\lambda^{IJ}(i\gamma^a \hat{\mathcal{D}}_a + \frac{1}{2r} - i(\sigma_I - \sigma_J))\bar{\lambda}^{IJ}\right]\,, \tag{C.21}$$

$${\rm tr}\,\mathcal{L}_{\rm FP} = \sum_{I\neq J} b^{IJ}(-\partial^2 + (\sigma_I - \sigma_J)^2) c^{IJ}\,. \tag{C.22}$$

Hence, the one-loop partition function $Z_{\rm SYM}[\sigma_0]$ factorizes according to the contributions from different fields as

$$Z_{\rm SYM}[\sigma_0] = Z_A \cdot Z_\sigma \cdot Z_{\rm ferm} \cdot Z_{b,c}\,, \tag{C.23}$$

$$Z_A = {\rm Det}^{-\frac{1}{2}}(\Delta + (\sigma_I - \sigma_J)^2)\,, \tag{C.24}$$

$$Z_\sigma = {\rm Det}^{-\frac{1}{2}}(-\partial^2 + (\sigma_I - \sigma_J)^2)\,, \tag{C.25}$$

$$Z_{\rm ferm} = {\rm Det}(i\gamma^a \hat{\mathcal{D}}_a - \frac{1}{2r} + i(\sigma_I - \sigma_J))\,, \tag{C.26}$$

$$Z_{b,c} = {\rm Det}(-\partial^2 + (\sigma_I - \sigma_J)^2)\,. \tag{C.27}$$

The factor Z_A in (C.23) deserves special attention. The determinant in (C.24) is computed in the space of unconstrained one-forms A_a on S^3. This space naturally decomposes into the divergenceless one-forms B_a, $\partial^a B_a = 0$, and the one-forms given by the gradient of a scalar, $\partial_a \phi$,

$$A_a = B_a + \partial_a \phi\,, \qquad \partial^a B_a = 0\,. \tag{C.28}$$

However, the zero mode of the scalar ϕ does not contribute to A_a and, hence, it should be eliminated. Therefore Z_A decomposes as

$$Z_A = Z_B \cdot Z_\phi\,, \tag{C.29}$$

$$Z_B = {\rm Det}'^{-\frac{1}{2}}(\Delta + (\sigma_I - \sigma_J)^2)\,, \tag{C.30}$$

$$Z_\phi = {\rm Det}'^{-\frac{1}{2}}(-\partial^2 + (\sigma_I - \sigma_J)^2)\,, \tag{C.31}$$

where the determinant in Z_B is computed in the space of diverdenceless one-forms B_a and Z_ϕ is given by the determinant of the Laplacian in the space of scalar fields ϕ, with the zero mode excluded from the spectrum.

It is straightforward to compute the partition function since the spectra of all the operators in (C.23)–(C.27) in known (see appendix B). The part

$$Z_B \cdot Z_{\rm ferm} = \prod_{I>J} \left(\frac{2\sinh(\pi r(\sigma_I - \sigma_J))}{\sigma_I - \sigma_J} \right)^2 \tag{C.32}$$

of the partition function was computed in [13]. Therein, the determinants of the other fields did not contribute to the partition function because they do not interact with the background fiend σ_0 in the Lorentz gauge (C.6).

In our case the modified Lorentz gauge (C.7) effectively gives the mass term for the scalar σ and ghosts and we earn additional contribution to the partition function depending on σ_0,

$$Z_\sigma \cdot Z_{b,c} \cdot Z_\phi = \frac{{\rm Det}(-\partial^2 + (\sigma_I - \sigma_J)^2)}{{\rm Det}^{\frac{1}{2}}(-\partial^2 + (\sigma_I - \sigma_J)^2){\rm Det}'^{\frac{1}{2}}(-\partial^2 + (\sigma_I - \sigma_J)^2)}\,. \tag{C.33}$$

All these determinants correspond to the same operator $-\partial^2 + (\sigma_I - \sigma_J)^2$ acting in the space of scalar fields. Hence, all the eigenvalues in (C.33) cancel except for the zero mode because it is absent in ${\rm Det}'^{\frac{1}{2}}(-\partial^2 + (\sigma_I - \sigma_J)^2)$. Thus,

$$Z_\sigma \cdot Z_{b,c} \cdot Z_\phi = \prod_{I>J} (\sigma_I - \sigma_J)^2\,. \tag{C.34}$$

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This expression cancels the denominator in (C.32) and we get exactly the partition function (5.70) computed in section 5.2 by superfield methods,

$$Z_{\rm SYM} = \prod_{\alpha>0} 4\sinh^2(\pi r(\sigma_I - \sigma_J))\,. \tag{C.35}$$

Let us now consider the partition function $Z'_{\rm SYM}$ introduced in (6.15) and computed in [13]. In components, this partition function is represented by the same functional integral (C.11), but with one important difference, namely, the integration over $\mathcal{D}\sigma$ runs over the space of scalar fields excluding their zero modes. When the zero models of σ are dropped out, the gauge fixing function (C.7) does not have zero modes as well and, as a consequences, the zero modes are absent in the Faddeev-Popov ghost fields b and c in (C.14). As a result, the zero modes are now absent in all determinants entering (C.33),

$$Z_\sigma \cdot Z_{b,c} \cdot Z_\phi = \frac{{\rm Det}'(-\partial^2 + (\sigma_I - \sigma_J)^2)}{{\rm Det}'^{\frac{1}{2}}(-\partial^2 + (\sigma_I - \sigma_J)^2){\rm Det}'^{\frac{1}{2}}(-\partial^2 + (\sigma_I - \sigma_J)^2)} = 1\,. \tag{C.36}$$

So, only (C.32) contributes to $Z'_{\rm SYM}$,

$$Z'_{\rm SYM} = \prod_{\alpha>0} \left(\frac{2\sinh(\pi r(\sigma_I - \sigma_J))}{\sigma_I - \sigma_J}\right)^2\,. \tag{C.37}$$

Exactly this partition function was employed in [13] in the localization formula in the $\mathcal{N}=2$ Chern-Simons theory. The denominator of (C.37) gets cancelled in the final stage of calculations of [13] by the Vandermonde determinant in the integration measure of the scalar σ_0.

Comparing the formulae (C.35) and (C.37) we get the proof of the identity (6.26) used in section 6.

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