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Banach lattices of L-weakly and M-weakly compact operators

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Abstract. We give conditions for the linear span of the positive L-weakly compact (resp. M-weakly compact) operators to be a Banach lattice under the regular norm, for that Banach lattice to have an order continuous norm, to be an AL-space or an AM-space.

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1. Introduction. We use [6] as our standard reference about Banach lattices and operators on them but, for the convenience of the reader, let us recall the definitions of the operators that this work involves. An operator $T : E \to Y$, where E is a Banach lattice and Y a Banach space, is called *M-weakly compact* if whenever (x_n) is a norm bounded disjoint sequence in E, we have $\|Tx_n\| \to 0$. Dually, if $T : X \to F$, where X is a Banach space and F a Banach lattice, then T is called *L-weakly compact* if for every disjoint sequence (y_n) in the solid hull of $\{Tx : x \in X, \|x\| \le 1\}$, we have $\|y_n\| \to 0$. In this paper we write $\mathcal{W}_L(E,F)$ [resp. $\mathcal{W}_M(E,F)$] for the L-weakly compact (resp. M-weakly compact) operators from E into F, where we will only be considering operators between two Banach lattices. The linear span of the positive operators in each class will be denoted by $\mathcal{W}_L^r(E,F)$ and $\mathcal{W}_M^r(E,F)$, respectively. The reader should not confuse, for example, $\mathcal{W}_L^r(E,F)$ with $\mathcal{W}_L(E,F) \cap \mathcal{L}^r(E,F)$ which is in general much larger. There is a considerable literature concerning the relationship between L-weakly compact operators and M-weakly compact operators, on the one hand, with weakly compact operators and a myriad of

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other classes, on the other hand. Missing from the literature seems to be any discussion of the nature of these spaces as ordered Banach spaces. Prompted by known results about the spaces of all regular operators, and the linear span of the positive compact or weakly compact operators, natural questions to ask are: When does a domination property hold? When are our spaces Banach lattices? When is the norm in a Banach lattice of operators particularly nice? E.g., when is it order continuous, a KB-norm, an AL-norm, or an AM-norm? We present at least partial answers to all of these questions in this note, apart from answering when they are KB-norms. We have results in this direction, but they are as yet too partial to be worthy of publication.

2. Banach lattices of operators. For many special classes of operators between Banach lattices, e.g., compact operators, weakly compact operators, or Dunford-Pettis operators, the so-called domination problem is both important and non-trivial. I.e., what conditions on the domain and/or range together with the relationship $0 \leq S \leq T$ forces S to be in the same class of operators as T. By way of contrast, no extra conditions are needed when we deal with L-weakly compact or M-weakly compact operators. We include a proof of this for completeness, although it is certainly well known.

Proposition 2.1. *If E and F are any Banach lattices, $S, T \in \mathcal{L}(E,F)$, and $0 \leq S \leq T$, then:*

1. *If $T \in \mathcal{W}_L(E,F)$, then $S \in \mathcal{W}_L(E,F)$.*
2. *If $T \in \mathcal{W}_M(E,F)$, then $S \in \mathcal{W}_M(E,F)$.*

Proof. Suppose that $T \in \mathcal{W}_L(E,F)$ and that (y_n) is a disjoint sequence in $\operatorname{sol}\big(S(B_E)\big)$, then there is a sequence (x_n) in B_E such that $|y_n| \leq |Sx_n|$. As $|y_n| \leq |Sx_n| \leq S|x_n| \leq T|x_n|$, (y_n) is a disjoint sequence in $\operatorname{sol}\big(T(B_E)\big)$ and T is L-weakly compact so that $\|y_n\| \to 0$ and therefore $S \in \mathcal{W}_L(E,F)$.

If $T \in \mathcal{W}_M(E,F)$ and (x_n) is a disjoint sequence in B_E, then so is $(|x_n|)$ and therefore $\big\|T|x_n|\big\| \to 0$. As $|Sx_n| \leq S|x_n| \leq T|x_n|$, we have $\|Sx_n\| = \big\||Sx_n|\big\| \leq \big\|T|x_n|\big\|$ so that $\|Sx_n\| \to 0$ and $S \in \mathcal{W}_M(E,F)$. □

This does not, however, mean that the spaces $\mathcal{W}_L(E,F)$ and $\mathcal{W}_M(E,F)$ have a nice order theoretic structure. Theorem 2.2 of [4] gives an example of a regular operator that is both L-weakly compact and M-weakly compact but does not have a modulus, whilst Theorem 2.3 of [4] provides an operator which is L-weakly compact, M-weakly compact, and has a modulus but that modulus is neither L-weakly compact nor M-weakly compact. These examples make it clear that in order to have any hope of a well-behaved space of operators, we should be working in the spaces $\mathcal{W}_L^r(E,F) = \{T_1 - T_2 : T_1, T_2 \in \mathcal{W}_L(E,F)_+\}$ and $\mathcal{W}_M^r(E,F) = \{T_1 - T_2 : T_1, T_2 \in \mathcal{W}_M(E,F)_+\}$.

In the first case, we have an extremely satisfactory result. Recall, [6, p. 212], that an L-weakly compact subset of a Banach lattice F is contained in F^a, the maximal ideal in F on which the norm is order continuous. It follows that if $T \in \mathcal{W}_L(E,F)$, then $T(E) \subseteq F^a$.

Theorem 2.2. *For any Banach lattices E and F, $\mathcal{W}_L^r(E,F)$, equipped with the regular norm, is a Dedekind complete Banach lattice.*

Proof. If $T \in W^r_L(E,F)$, there is $U \in W_L(E,F)_+$ with $\pm T \le U$. As both T and U take values in the Dedekind complete ideal F^a, T has a modulus in $\mathcal{L}^r(E,F^a)$ which will be its modulus in $\mathcal{L}^r(E,F)$. As $-U \le |T| \le U$, Proposition 2.1 tells us that $|T| \in \mathcal{W}^r_L(E,F)$ so that $\mathcal{W}^r_L(E,F)$ is a vector lattice. It is similarly simple to see that $\mathcal{W}^r_L(E,F)$ is Dedekind complete. To complete the proof, we need only show that $\mathcal{W}^r_L(E,F)$ is closed in $\mathcal{L}^r(E,F)$ for the regular norm. If each $T_n \in \mathcal{W}^r_L(E,F)$, $T \in \mathcal{L}^r(E,F)$, and $\|T_n - T\|_r \to 0$, then give that each T_n has a modulus in $\mathcal{L}^r(E,F)$, then Theorem 2.1 of [5] tells us that T has a modulus in $\mathcal{L}^r(E,F)$ and that $\left\||T_n| - |T|\right\|_r \to 0$. It follows that $\left\||T_n| - |T|\right\| \to 0$. As each $|T_n| \in \mathcal{W}_L(E,F)$ and $\mathcal{W}_L(E,F)$ is closed in $\mathcal{L}(E,F)$ (Theorem 18.14 (2) of [1], Theorem 5.65 (2) of [2]), we see that $|T| \in W_L(E,F)$. Again, Proposition 2.1 tells us that $T \in \mathcal{W}^r_L(E,F)$. □

As M-weakly compact operators need not take values in F^a, we need an extra assumption in order to guarantee the existence of a modulus. The proof of the following result is virtually identical with that of Theorem 2.2, using part (1) of the result used from [1] or [2]. There will, of course, be variants of this theorem which assume, for example, that E is separable and F Dedekind σ-complete or that E is atomic with an order continuous norm with appropriately weaker conclusions.

Theorem 2.3. *If E is any Banach lattice and F a Dedekind complete Banach lattice, then $\mathcal{W}^r_M(E,F)$, equipped with the regular norm, is a Dedekind complete Banach lattice.*

We are unsure what can be said about the order structure of $\mathcal{W}^r_M(E,F)$ in the absence of Dedekind completeness of F. In a work in preparation, the current authors will discuss matrix representations of L-weakly compact and M-weakly compact operators between the standard sequence spaces. As a by-product of this work, it will be seen that for E being any of the standard sequence spaces, $W^r_M(E,c)$ is a vector lattice even though c is far from being Dedekind complete. Our guess is that, as with compactness, the fact that c is an AM-space helps here and that $W^r_M(E,F)$ will not be a vector lattice in general.

We turn now to the question of what kind of Banach lattice these spaces of operators can be.

3. Order continuity of the norm. We first look at order continuity of the norm in $\mathcal{W}^r_L(E,F)$. If F^a is trivial, then so is $\mathcal{W}^r_L(E,F)$ and we can deduce nothing about E. With the proviso that F^a is non-zero, we have an extremely satisfactory result. We write $f \otimes y$ for the operator $x \mapsto f(x)y$ if $f \in E'$ and $y \in Y$. Note that if $y \in F^a$, then $f \otimes y \in \mathcal{W}^r_L(E,F)$ whilst if $f \in (E')^a$, then $f \otimes y \in \mathcal{W}^r_M(E,F)$.

Theorem 3.1. *If E and F are Banach lattices with $F^a \neq \{0\}$, then the regular norm on $\mathcal{W}^r_L(E,F)$ is order continuous if and only if E' has an order continuous norm.*

Proof. Assume that E' does have an order continuous norm. If $T \in \mathcal{W}_L^r(E,F)_+$, then the order intervals $[0,T]$ in $\mathcal{W}_L^r(E,F)$ and in $\mathcal{L}^r(E,F)$ are the same by Proposition 2.1. If $T \in \mathcal{W}_L^r(E,F)_+$, then $T' \in W_M(F',E')$ by Theorem 3.6.11 of [6], so that $T' \in \mathcal{W}_L(F',E')$ by Theorem 3.6.14 of [6] (as E' has an order continuous norm) and hence $T \in \mathcal{W}_M^r(E,F)$, using Theorem 3.6.11 of [6] again. As T takes values in F^a, Theorem 3.6.19 of [6] now tells us that the norm on $[0,T]$ is order continuous.

Suppose now that the norm on $\mathcal{W}_L^r(E,F)$ is order continuous and that $F^a \neq \{0\}$. Let $0 \neq y \in F_+^a$ and consider any net (f_α) in E'_+ which decreases to 0. Clearly, $f_\alpha \otimes y \downarrow 0$ in $\mathcal{W}_L^r(E,F)$, so that $\|f_\alpha\|\|y\| = \|f_\alpha \otimes y\| \downarrow 0$ and the norm in E' is indeed order continuous. □

If $F^a = \{0\}$, then $\mathcal{W}_L^r(E,F) = \{0\}$, so certainly has an order continuous norm whatever E may be.

Even though $\mathcal{W}_M^r(E,F)$ need not be a lattice, we can still talk about order continuity of the norm in the sense that any net in the positive cone which is downward directed to 0 must converge in norm to 0. Of course, we will not have the usual characterisations of this that apply for Banach lattices.

Theorem 3.2. *If E and F are Banach lattices with $(E')^a \neq \{0\}$, then the regular norm on $\mathcal{W}_M^r(E,F)$ is order continuous if and only if F has an order continuous norm.*

Proof. The proof of "if" is simple in this case using Theorem 3.6.19 of [6] and Proposition 2.1, whilst the proof of the converse is very similar to that in the preceding proof. □

Corollary 3.3. *If E and F are Banach lattices with $(E')^a \neq \{0\}$ and $F^a \neq \{0\}$, then:*

1. *$\mathcal{W}_L^r(E,F)$ has an order continuous norm if and only if $\mathcal{W}_M^r(F',E')$ has an order continuous norm.*
2. *If $\mathcal{W}_L^r(F',E')$ has an order continuous norm, then so has $\mathcal{W}_M^r(E,F)$.*
3. *$\mathcal{W}_L^r(F',E')$ has an order continuous norm if and only if $\mathcal{W}_M^r(E,F'')$ has an order continuous norm.*

The converse of Corollary 3.3 (2) is false.

Example 3.4. Take $E = F = c_0$. As F has an order continuous norm, $\mathcal{W}_M^r(E,F)$ has an order continuous norm. However, because $E' = \ell_1$ has an order continuous norm, so that $E'^a \neq \{0\}$, and $F'' = \ell_\infty$ does not have an order continuous norm, Theorem 3.1 tells us that $W_L^r(F',E')$ does not have an order continuous norm.

4. AL-spaces and AM-spaces of operators. Our results in this section will not be unexpected, but we do need to take some care in their formulation. We deal first with the rather simpler question of when these spaces are AL-spaces.

Theorem 4.1. *If E and F are Banach lattices, then $\mathcal{W}_L^r(E,F)$ is an AL-space under the regular norm if and only if one the following conditions holds:*

1. *$F^a = \{0\}$.*
2. *E is an AM-space and F^a is an AL-space.*

Proof. Suppose that $\mathcal{W}_L^r(E,F)$ is an AL-space and that $F^a \neq \{0\}$. To see that (2) holds, repeat the corresponding part of the proof of Theorem 2.1 in [7], but requiring that all y's lie in F^a.

For the converse, if $F^a = \{0\}$, then $\mathcal{W}_L^r(E,F) = \{0\}$, so is certainly an AL-space. Otherwise, Theorem 2.1 of [7] tells that $\mathcal{L}^r(E,F^a)$ is an AL-space and so is its sublattice $\mathcal{W}_L^r(E,F^a)$. But there is an obvious identification of $\mathcal{W}_L^r(E,F^a)$ with $\mathcal{W}_L^r(E,F)$ so the claim is clear. □

Similar ideas give us:

Theorem 4.2. *If E and F are Banach lattices, then $\mathcal{W}_M^r(E,F)$ is an AL-space under the regular norm if and only if one of the following conditions holds:*
1. *$(E')^a = \{0\}$.*
2. *$(E')^a$ and F are AL-spaces.*

Corollary 4.3. *Let E and F be Banach lattices.*
1. *If F has an order continuous norm, then $\mathcal{W}_L^r(E,F)$ is an AL-space if and only if E is an AM-space and F an AL-space.*
2. *If E' has an order continuous norm, then $\mathcal{W}_M^r(E,F)$ is an AL-space if and only if E is an AM-space and F an AL-space.*

The criteria for the L-weakly compact operators to form an AM-space are rather simpler than we might have expected as they take values in the order continuous part of F and the Fatou property is automatic.

Theorem 4.4. *If E and F are Banach lattices, then $\mathcal{W}_L^r(E,F)$ is an AM-space under the regular norm if and only if one of the following conditions holds:*
1. *$F^a = \{0\}$.*
2. *E is an AL-space and F^a is an AM-space*

Proof. Again, the proof that if $\mathcal{W}_L^r(E,F)$ is an AM-space, then either (1) or (2) holds is modelled on part of the proof of Theorem 2.2 of [7]. If F^a, and therefore $\mathcal{W}_L^r(E,F)$, is trivial, the conclusion is clear. If (2) holds, then $\mathcal{L}^r(E,F^a)$ is an AM-space by Theorem 2.3 of [7] as order continuous norms are Fatou. As $\mathcal{W}_L^r(E,F)$ is, for all intents and purposes, a closed sublattice of $\mathcal{L}^r(E,F)$ it also is an AM-space. □

Notice that if F^a is an AM-space, then it is isomorphic to a space, $C_0(I)$, of continuous functions vanishing at infinity on a discrete topological space. Therefore the kind of AM-spaces that can arise as $\mathcal{W}_L^r(E,F)$ are rather limited in variety.

Matters become somewhat more difficult when we consider the M-weakly compact operators. If these form an AM-space, then standard methods only tell us that $(E')^a$ is an AM-space (and therefore again of the form $C_0(I)$) which does not tell us much about E itself. Also, even to know that the space of M-weakly compact operators is a Banach lattice, we must assume, for example, that F is Dedekind complete. The following result, therefore, is somewhat disappointing in its completeness. We do, however, manage to avoid the Fatou condition on F.

Theorem 4.5. *If E is an AL-space and F a Dedekind complete Banach lattice, then $\mathcal{W}_M^r(E,F)$ is an AM-space under the regular norm if and only if F is an AM-space.*

Proof. It is pointed out in the proof of Theorem 2.6 in [4] that an M-weakly compact operator on an AL-space must be zero on the non-atomic part. Hence we may assume that E is an atomic AL-space. From the preceding theorem, $\mathcal{W}_L^r(F',E') = \mathcal{W}_L^r(F',(E')^a)$ is an AM-space. If $S,T \in \mathcal{W}_M^r(E,F)_+$, then using Theorem 4.1 in [3], we have

$$\begin{aligned} \|S \vee T\|_r &= \|S \vee T\| = \|(S \vee T)'\| = \|S' \vee T'\| \\ &= \|S'\| \vee \|T'\| = \|S\| \vee \|T\| = \|S\|_r \vee \|T\|_r, \end{aligned}$$

so that $\mathcal{W}_M^r(E,F)$ is an AM-space. The proof of the converse is routine. □

There will be variants of this results similar to those mentioned before Theorem 2.3. Given the prevalence of AM-spaces with an order continuous norm in our discussions, the following result may be worthy of note.

Theorem 4.6. *If E and F are Banach lattices, then $\mathcal{W}_M^r(E,F)$ under the regular norm is an AM-space with order continuous norm if and only if one of the following conditions holds.*

1. $(E')^a = \{0\}$.
2. $(E')^a$ *is an AM-space and F is an AM-space with order continuous norm.*

Proof. If $W_M^r(E,F)$ is an AM-space, then it is routine to check that $(E')^a$ and F must be AM-spaces whilst the order continuity of the norm in F is also immediate from that of $\mathcal{W}_M^r(E,F)$ if $(E')^a \neq \{0\}$ using operators of the form $f \otimes y$ for $0 \neq f \in (E')^a_+$.

The converse is trivial if $(E')^a$, and therefore $\mathcal{W}_M^r(E,F)$, is trivial. In case (2), repeat the proof from Theorem 4.5 using Synnatzschke's theorem, Proposition 1.4.17 of [6], given that F has an order continuous norm, in place of Theorem 4.1 of [3]. □

There are, of course, also isomorphic versions of all the theorems in this section.

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