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Three-charge black holes and quarter BPS states in Little String Theory

Amit Giveon,[a] **Jeffrey Harvey,**[b] **David Kutasov**[b] **and Sungjay Lee**[b]

[a] *Racah Institute of Physics, The Hebrew University, Jerusalem, 91904, Israel*
[b] *Enrico Fermi Institute and Department of Physics, The University of Chicago, 5620 S. Ellis Av., Chicago, Illinois 60637, U.S.A.*
E-mail: giveon@phys.huji.ac.il, jh25@uchicago.edu, dkutasov@uchicago.edu, sjlee79@uchicago.edu

Abstract: We show that the system of k NS5-branes wrapping $\mathbb{T}^4 \times S^1$ has non-trivial vacuum structure. Different vacua have different spectra of 1/4 BPS states that carry momentum and winding around the S^1. In one vacuum, such states are described by black holes; in another, they can be thought of as perturbative BPS states in Double Scaled Little String Theory. In general, both kinds of states are present. We compute the degeneracy of perturbative BPS states exactly, and show that it differs from that of the corresponding black holes. We comment on the implication of our results to the black hole microstate program, UV/IR mixing in Little String Theory, string thermodynamics, the string/black hole transition, and other issues.

Keywords: Black Holes in String Theory, Conformal Field Models in String Theory

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Contents

1 Introduction

In this paper we will study the system of k NS5-branes in type II string theory. We will take the fivebranes to wrap $\mathbb{R}^4 \times S^1$, and focus on states that carry momentum P and winding W around the S^1. For general (P, W), the lowest lying states with these quantum numbers preserve four of the sixteen supercharges preserved by the fivebranes. Thus, they can be thought of as quarter BPS states in the fivebrane theory.

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The states in question have the same quantum numbers as the three-charge black holes that were studied extensively in the last twenty years in the context of providing a microscopic interpretation of black hole entropy, starting with the work of [1]; for a recent review see [2]. They also figure prominently in the fuzzball program that attempts to describe these microstates by horizonless geometries [3, 4]. In these cases, one needs to replace the $\mathbb{R}^4$ that the fivebranes are wrapping by a compact manifold, such as $\mathbb{T}^4$, and we will discuss this case as well.

Our main interest will be in the dependence of the spectrum of the above states on the positions of the fivebranes. We will see that it is qualitatively different when the fivebranes are separated by any finite distance, and when they are coincident. The two cases are separated by a string-black hole transition. This may seem surprising, since separating the fivebranes corresponds in the low energy theory to Higgsing a non-abelian gauge group, and one would expect that if the W-boson mass scale is low, the physics of high mass states, such as the ones we will study, should not be affected. We will discuss why it nevertheless happens, and comment on some implications.

In the context of black hole physics, the above system is usually discussed in the full, asymptotically flat space transverse to the fivebranes. However, one can also study it in the theory obtained by restricting to the near-horizon geometry of the fivebranes. This theory is known as Little String Theory (LST). As we review in section 2, it can be alternatively defined by taking a certain scaling limit of the full string theory.

In the near-horizon geometry of the coincident fivebranes, the three-charge black holes are described by certain BPS black hole solutions in an asymptotically linear dilaton spacetime, which carry the charges (P, W); see e.g. [5, 6] and references therein. The entropy of these black holes is given by the familiar result (see section 6)

$$S_{\rm BH} = 2\pi\sqrt{kPW}\,. \tag{1.1}$$

On the other hand, if one separates the fivebranes in the transverse $\mathbb{R}^4$, one can study these states as conventional perturbative string states in a spacetime of the form $\mathbb{T}^4 \times S^1 \times \mathcal{M}_4$, where $\mathcal{M}_4$ is the background associated with directions transverse to the fivebranes. It includes a non-compact direction associated with the radial direction away from the fivebranes, and some compact directions associated with the angular part of the geometry. The precise background depends on the positions of the fivebranes.

If the separations of the fivebranes are sufficiently large, the string coupling in this background is small everywhere (unlike the case for coincident fivebranes, where it diverges as one approaches the fivebranes), and the description of these states as perturbative string states mentioned above is valid. Thus, one can use standard techniques to count them.

A convenient object for this purpose is the elliptic genus of the worldsheet CFT corresponding to $\mathcal{M}_4$. We review the definition of this object and study its properties in our case in sections 3 and 4. As we discuss there, it can be written as a power series in a parameter q. The coefficient of q^N is the (graded) number of BPS states with charges (P, W) satisfying $PW = N$. These states are standard perturbative BPS states [7], for which the right-movers on the worldsheet are in the ground state while the left-movers are in a general excited state. Thus, they satisfy

$$
\begin{aligned}
N_R &= 0\,,\ \ N_L = N = PW\,,\\
M &= \left|\frac{P}{R} + \frac{WR}{\alpha'}\right|\,,
\end{aligned}
\tag{1.2}
$$

where N_L and N_R are the excitation levels for left and right-movers on the worldsheet, R is the radius of the circle the fivebranes wrap, and M is the mass of the BPS state.

We use the elliptic genus to calculate the entropy of perturbative string states with the same quantum numbers as the black holes mentioned above, and find the result (for large PW)

$$
S_{\text{string}} = 2\pi\sqrt{\left(2 - \frac{1}{k}\right) PW}\,. \tag{1.3}
$$

This does not agree with the entropy of black holes with the same quantum numbers (1.1).[1] We argue that the system exhibits a phase transition: when the fivebranes are coincident, quarter BPS states with the quantum numbers (P, W) correspond to black holes, while when they are separated they correspond to fundamental strings. This phenomenon is an example of UV-IR mixing in LST — turning on a small IR scale (the masses of W-bosons corresponding to the separations between the fivebranes) has a large effect on the spectrum of massive states (the quarter BPS states discussed above). This UV-IR mixing is possible due to the fact that LST is not a local QFT.

As mentioned above, the black hole point of view requires us to compactify the worldvolume of the NS5-branes from $\mathbb{R}^4 \times S^1$ to, say $\mathbb{T}^4 \times S^1$. In the compact case, the theory on NS5-branes becomes $(0+1)$ dimensional (i.e. it becomes quantum mechanics). In this case, the positions of NS5-branes are no longer well defined; instead, the ground state corresponds to a wavefunction on the classical moduli space. The transition mentioned above has a slightly different flavor in this case — the quantum theory of NS5-branes has non-trivial vacuum structure. In one vacuum, the fivebranes are coindicent and the entropy of BPS states is given by S_{BH}, while in another they are separated and the entropy is given by S_{string}. The UV-IR mixing manifests itself in this theory as the fact that although the vacuum wavefunction in the string phase has support in the region where the fivebranes are arbitrarily close to each other, this phase is nevertheless distinct from the black hole phase, in which the fivebranes are all coincident. The two phases differ in their UV behavior.

We next turn to the development of the picture presented above in more detail, starting from the definition and properties of LST and in particular its holographic description.

[1] Although the result S_{string} is derived when the separations between the fivebranes are sufficiently large that the string coupling is small everywhere, we argue in section 5 that it is actually valid whenever the fivebranes are separated by any finite amount.

2 Little String Theory and its scaling limit

2.1 Little String Theory

The dynamical degrees of freedom localized on NS fivebranes can be decoupled from bulk degrees of freedom by taking a limit in which the string coupling $g_s \to 0$, with the energy scale E held fixed relative to the string scale, $E \sim m_s$. The resulting theory, known as Little String Theory, is an interacting six-dimensional theory which does not include gravity, but otherwise shares many similarities to string theory in asymptotically flat spacetime, including a Hagedorn density of states[2] and T-duality [8–10]. For reviews see [11, 12].

LST has a holographic description in terms of string theory in the near-horizon geometry of the fivebranes [13]. For k coincident fivebranes, the near-horizon metric is given by [14, 15]

$$ds^2 = dx^\mu dx_\mu + d\phi^2 + 2k d\Omega_3^2\,, \tag{2.1}$$

where x^μ ($\mu = 0, 1, .., 5$) parametrize the flat worldvolume of the fivebranes, ϕ is a (function of the) radial coordinate in the directions transverse to the fivebranes, and $d\Omega_3^2$ is the line element on the corresponding angular three-sphere. The dilaton and H-flux in the near-horizon geometry can be written in the form

$$\Phi = -\frac{Q}{2}\phi, \qquad Q = \sqrt{\frac{2}{k}}\,, \tag{2.2}$$

and

$$H = 2k\mathrm{Vol}_{S^3}\,, \tag{2.3}$$

where we took $\alpha' = 2$. The four-dimensional space transverse to the NS5-branes is described by an exactly solvable conformal field theory [14, 15], the Callan-Harvey-Strominger (CHS) CFT,

$$\mathbb{R}_\phi \times \mathrm{SU}(2)_k\,, \tag{2.4}$$

that contains a linear dilaton direction with background charge Q, a bosonic SU(2) Wess-Zumino-Witten (WZW) model at level $k-2$ with currents $\tilde{j}^a$ ($a = 1, 2, 3$), and four fermions $\psi^I = (\psi_a, \psi_\phi)$. The central charge is

$$c = c_\phi + c_{\mathrm{SU}(2)} + c_{\mathrm{f}} = \left(1 + 3Q^2\right) + \left(3 - \frac{6}{k}\right) + 2 = 6\,, \tag{2.5}$$

as expected. The total SU(2) currents of the supersymmetric SU(2) WZW model at level k are given by

$$\tilde{J}^a = \tilde{j}^a - \frac{i}{2}\epsilon^{abc}\psi_b\psi_c\,. \tag{2.6}$$

[2] String theory in asymptotically flat spacetime exhibits a Hagedorn density of states in an intermediate energy regime, while at asymptotically high energies the entropy grows faster with the energy. In LST, the Hagedorn behavior persists up to arbitrarily high energies.

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For later convenience we define

$$\psi^{\pm} = \frac{1}{\sqrt{2}}\left(\psi_\phi \pm i\psi_3\right) , \qquad \tilde{\psi}^{\pm} = \frac{1}{\sqrt{2}}\left(\psi_1 \pm i\psi_2\right) , \tag{2.7}$$

which can be bosonized as

$$\psi^{\pm} = e^{\pm iH} , \qquad \tilde{\psi}^{\pm} = e^{\pm i\tilde{H}} . \tag{2.8}$$

It will be useful in our later analysis to note that one can also describe the supersymmetric $\mathrm{SU}(2)_k$ WZW model as a $\mathbb{Z}_k$ orbifold of the tensor product of the supersymmetric $\mathrm{U}(1)_k$ WZW model and a coset CFT, $\mathrm{SU}(2)_k/\mathrm{U}(1)$,

$$\mathrm{SU}(2)_k \simeq \left(\mathrm{U}(1)_k \times \frac{\mathrm{SU}(2)_k}{\mathrm{U}(1)}\right)/\mathbb{Z}_k . \tag{2.9}$$

In this description, the compact boson Y of the $\mathrm{U}(1)_k$ WZW model is related to the current $\tilde{J}^3$ via

$$\tilde{J}^3 = \tilde{j}^3 + \psi^+\psi^- = i\sqrt{\frac{k}{2}}\partial Y . \tag{2.10}$$

The $\mathrm{SU}(2)_k/\mathrm{U}(1)$ coset is equivalent to an $N = 2$ minimal model whose central charge is $c = 3 - \frac{6}{k}$.

The CHS conformal field theory has $\mathcal{N} = 4$ superconformal symmetry with the superconformal generators

$$G_I = i\left(\psi^I\partial\phi + Q\partial\psi^I\right) - Q\eta^a_{IJ}j^a\psi^J + \frac{i}{6}Q\epsilon_{IJKL}\psi^J\psi^K\psi^L , \tag{2.11}$$

and the $\mathrm{SU}(2)_R$ currents at level one

$$J^a_{\mathrm{R}} = -\frac{i}{2}\bar{\eta}^a_{IJ}\psi^I\psi^J . \tag{2.12}$$

Here the 't Hooft tensors $\eta^a_{\mu\nu}$ and $\bar{\eta}^a_{\mu\nu}$ are antisymmetric in (μ,ν) and construct the Lie algebra of SU(2) from self-dual and anti-self-dual combinations of SO(4) generators. They are defined explicitly as

$$\begin{aligned} \eta^a_{bc} &= \epsilon_{abc} , \qquad \eta^a_{b4} = \delta^a_b , \\ \bar{\eta}^a_{bc} &= \epsilon_{abc} , \qquad \bar{\eta}^a_{b4} = -\delta^a_b . \end{aligned} \tag{2.13}$$

In particular, the current J^3_{R} is given by

$$2J^3_{\mathrm{R}} = \psi^+\psi^- + \tilde{\psi}^+\tilde{\psi}^- = i\partial H + i\partial\tilde{H} . \tag{2.14}$$

The normalizable primary vertex operators of the CHS CFT can be expressed as follows:

$$\mathcal{O} = e^{-Q(j+1)\phi}e^{i(\alpha H + \bar{\alpha}\bar{H})}e^{i(\beta\tilde{H} + \bar{\beta}\bar{\tilde{H}})}\Phi^{\mathrm{su}}_{\tilde{j},\tilde{m},\bar{\tilde{m}}} , \tag{2.15}$$

where $\Phi^{\rm su}_{\tilde{j};\tilde{m},\bar{\tilde{m}}}$ are primary operators of the bosonic $\mathrm{SU}(2)_{k-2}$ current algebra. The conformal dimensions and R-charges of $\mathcal{O}$ (2.15) are given by

$$\begin{aligned} h &= -\frac{j(j+1)}{k} + \frac{\tilde{j}(\tilde{j}+1)}{k} + \frac{\alpha^2+\beta^2}{2}\,, \\ r &= \alpha+\beta\,. \end{aligned} \tag{2.16}$$

In terms of the decomposition $\left(\mathrm{U}(1)_k \times \frac{\mathrm{SU}(2)_k}{\mathrm{U}(1)}\right)/\mathbb{Z}_k$, one can rewrite the contribution of the supersymmetric $\mathrm{SU}(2)_k$ WZW model to (2.15) as

$$e^{i\beta\tilde{H}} e^{i\bar{\beta}\bar{\tilde{H}}} \Phi^{\rm su}_{\tilde{j};\tilde{m},\bar{\tilde{m}}} = e^{i\sqrt{\frac{2}{k}}\left[(\tilde{m}+\beta)Y+(\bar{\tilde{m}}+\bar{\beta})\bar{Y}\right]}\,\tilde{V}^{\rm susy}_{\tilde{j};\tilde{m},\bar{\tilde{m}}}(\beta,\bar{\beta})\,, \tag{2.17}$$

where $\tilde{V}^{\rm susy}_{\tilde{j};\tilde{m},\bar{\tilde{m}}}(\beta,\bar{\beta})$ denotes primary operators of the $\mathcal{N}=2$ minimal model. Their conformal weights are

$$h = \frac{\tilde{j}(\tilde{j}+1)-(\tilde{m}+\beta)^2}{k} + \frac{\beta^2}{2}\,. \tag{2.18}$$

As explained in appendix A, the two parameters β and $\bar{\beta}$ can be understood as spectral flow parameters in the supersymmetric minimal model.

In the CHS background (2.4), the string coupling varies with the distance from the fivebranes as follows:

$$g_s^2 \simeq e^{-Q\phi}\,. \tag{2.19}$$

Thus $g_s \to 0$ at large distance ($\phi \to \infty$). This is the boundary of the near-horizon geometry, analogous to the boundary of AdS for gauge/gravity duality. At the same time, as one approaches the fivebranes ($\phi \to -\infty$), the string coupling diverges. Hence, the exact background (2.4) is not useful for worldsheet calculations, which rely on weak coupling. To make it useful, we need to do something about the strong coupling region. One way to deal with it is described in the next subsection.

2.2 Double scaling limit

From the discussion in the previous subsection, it is clear that the CHS geometry is applicable only for $k \geq 2$ (coincident) fivebranes, since otherwise the bosonic SU(2) WZW model at level $k-2$ does not make sense. Thus, fundamental strings propagating in the vicinity of a single NS5-brane do not see a CHS throat geometry (2.4). As a consequence, in the near-horizon geometry of k separated fivebranes, the string coupling is bounded from above [16, 17]. The maximal value of g_s depends on the separations of the fivebranes.

One can arrange the separations such that the coupling is small everywhere. This amounts to demanding that the masses of D-strings stretched between different NS5-branes in type IIB string theory,[3] denoted collectively by M_W, are much larger than the string scale,

$$M_W \gg m_s\,. \tag{2.20}$$

[3] The IIA case is very similar.

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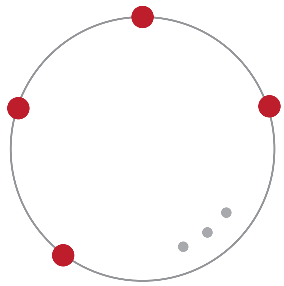

Figure 1. NS5-branes on a circle.

As we review below, the resulting theory can be studied using perturbative string techniques where m_s/M_W plays the role of the string coupling. This theory is known as Double Scaled Little String Theory (DSLST) [16, 17].

For example, consider a configuration of fivebranes arranged equidistantely on a circle of radius R_0 in $\mathbb{R}^4$, depicted in figure 1. In this configuration, the SU(k) gauge symmetry on the fivebranes is broken to $\mathrm{U}(1)^{k-1}$ at the scale $M_W \sim R_0/g_s l_s^2$. To study the dynamics of the fivebranes in this case, we can take the double scaling limit

$$g_s \to 0\,, \ \frac{R_0}{l_s} \to 0 \ \text{ with } \ \frac{R_0}{g_s l_s} \text{ fixed.}$$

Keeping the dimensionless constant $\frac{R_0}{g_s l_s}$ fixed means keeping the masses of W-bosons fixed in string units. If these masses are large relative to m_s, the theory is weakly coupled and can be studied using worldsheet techniques; it is a special case of the DSLST construction mentioned above.

While the DSLST construction is general, the configuration of figure 1 is special in that the corresponding worldsheet CFT is solvable, which is not the case for generic points in the moduli space of DSLST. The CHS background (2.4) is replaced in this case by

$$\mathbb{R}_\phi \times \left(\mathrm{U}(1)_k \times \frac{\mathrm{SU}(2)_k}{\mathrm{U}(1)}\right)/\mathbb{Z}_k \ \to \ \left(\frac{\mathrm{SL}(2,\mathbb{R})_k}{\mathrm{U}(1)} \times \frac{\mathrm{SU}(2)_k}{\mathrm{U}(1)}\right)/\mathbb{Z}_k\,. \tag{2.21}$$

The coset $\frac{\mathrm{SL}(2,\mathbb{R})_k}{\mathrm{U}(1)}$ describes a σ-model on a cigar with asymptotically linear dilaton. The string coupling grows towards the tip of the cigar, and attains its largest value at the tip,

$$g_s(\text{tip}) \simeq \frac{m_s}{M_W} \simeq \frac{g_s l_s}{R_0} \tag{2.22}$$

One way to understand this relation is to note that D-strings stretched between the fivebranes correspond in the deformed theory (2.21) to D0-branes located at the tip of the cigar (and particular boundary states in the $\mathcal{N}=2$ minimal model). Note that (2.22) has the property that as R_0 increases, the maximal value of the string coupling decreases. On the other hand, as $R_0 \to 0$ the string coupling at the tip of the cigar grows without bound. These properties are in agreement with the expectations mentioned above — the length of the fivebrane throat increases with decreasing R_0 and vice versa.

The relation between the CHS background (2.4) corresponding to coincident fivebranes and the background corresponding to fivebranes on a circle (the r.h.s. of (2.21)) can be understood by looking at the region far from the tip of the cigar. We will refer to it below as the CHS region. In that region, the cigar geometry reduces to a cylinder $\mathbb{R}_\phi \times S^1$, and the CFT on the r.h.s. of (2.21) reduces to that on the l.h.s. which, as explained above, is equivalent to (2.4). This agrees with the intuition that far from the fivebranes one does not notice that they are separated, and the background of separated fivebranes should reduce to that of coincident ones.

Both the $SL(2,\mathbb{R})/U(1)$ and the $SU(2)/U(1)$ CFT's have $\mathcal{N}=2$ superconformal symmetry. We will denote their $U(1)_R$ currents by $J_{\rm R}^{\rm sl}$ and $J_{\rm R}^{\rm su}$, respectively. Some basic properties of these two coset models are summarized in appendix A. While the tensor product of cigar and minimal models preserves $\mathcal{N}=2$ superconformal symmetry, one can show that the $\mathbb{Z}_k$ orbifold in (2.21) enhances the superconformal symmetry to $\mathcal{N}=4$, in agreement with the fact that the background (2.21) can be thought of as describing a near-singular non-compact $K3$ surface.

The CSA of the $SU(2)_R$ current that belongs to the $\mathcal{N}=4$ algebra can be taken to be

$$2J_{\rm R}^3 = J_{\rm R}^{\rm sl} + J_{\rm R}^{\rm su}\,. \tag{2.23}$$

The factor of two on the l.h.s. is due to the fact that $\mathcal{N}=4$ supercharges transform as a doublet under $SU(2)_R$, and hence naturally have $U(1)_R$ charge $\pm 1/2$, while the $U(1)_R$ current in the $\mathcal{N}=2$ algebra is usually normalized so that the supercharges carry charge ± 1.

Finally we discuss the normalizable vertex operators in the CFT (2.21). The primary operators of the cigar CFT, $V^{\rm susy}_{j;m,\bar m}(\alpha,\bar\alpha)$, are defined and discussed in appendix A. They have conformal weight and $U(1)_R$ charge

$$\begin{aligned} h &= \frac{(m+\alpha)^2 - j(j+1)}{k} + \frac{\alpha^2}{2}\,,\\ r &= \frac{2(m+\alpha)}{k} + \alpha\,. \end{aligned} \tag{2.24}$$

In the CHS region, the vertex operator $V^{\rm susy}_{j;m,\bar m}(\alpha,\bar\alpha)$ behaves like

$$V^{\rm susy}_{j;m,\bar m}(\alpha,\bar\alpha) \;\simeq\; e^{-Q(j+1)\phi} e^{i\alpha H} e^{i\bar\alpha \bar H} e^{i\sqrt{\frac{2}{k}}\left[(m+\alpha)Y + (\bar m+\bar\alpha)\bar Y\right]}\,. \tag{2.25}$$

Note that the parameters α and $\bar\alpha$ can be identified as spectral flow parameters.

The primary operators of the minimal model $\tilde V^{\rm susy}_{\tilde j;\tilde m,\bar{\tilde m}}(\beta,\bar\beta)$, are similarly defined and discussed in appendix A. Their conformal weight and $U(1)_R$ charge are

$$\begin{aligned} h &= \frac{\tilde j(\tilde j+1) - (\tilde m+\beta)^2}{k} + \frac{\beta^2}{2}\,,\\ r &= -\frac{2(\tilde m+\beta)}{k} + \beta \end{aligned} \tag{2.26}$$

In the CHS region one has

$$e^{i\sqrt{\frac{2}{k}}\left[(\tilde m+\beta)Y + (\bar{\tilde m}+\bar\beta)\bar Y\right]} \tilde V^{\rm susy}_{\tilde j;\tilde m,\bar{\tilde m}}(\beta,\bar\beta) \simeq e^{i\beta \tilde H} e^{i\bar\beta \bar{\tilde H}} \Phi^{\rm su}_{\tilde j;\tilde m,\bar{\tilde m}}\,. \tag{2.27}$$

This can be used to construct a normalizable vertex operator in the full DSLST background (2.21), that is asymptotic to the vertex operator (2.15) in the CHS region,

$$V^{\rm susy}_{j;m,\bar{m}}(\alpha,\bar{\alpha})\cdot\tilde{V}^{\rm susy}_{\tilde{j};\tilde{m},\bar{\tilde{m}}}(\beta,\bar{\beta}) \;\to\; e^{-Q(j+1)\phi}e^{i(\alpha H+\bar{\alpha}\bar{H})}e^{i(\beta\tilde{H}+\bar{\beta}\bar{\tilde{H}})}\Phi^{\rm su}_{\tilde{j},\tilde{m},\bar{\tilde{m}}} \tag{2.28}$$

with

$$\begin{aligned} m+\alpha &= \tilde{m}+\beta\,, \\ \bar{m}+\bar{\alpha} &= \bar{\tilde{m}}+\bar{\beta}\,. \end{aligned} \tag{2.29}$$

3 Elliptic genus of DSLST

The holographic worldsheet description of DSLST at a generic point in its moduli space involves the background $\mathcal{M}^{5,1}\times\mathcal{M}_4$, where $\mathcal{M}_4$ ($\mathcal{M}^{5,1}$) is a CFT associated with the directions transverse to (along) the fivebranes. A natural quantity to consider is the elliptic genus of the CFT on $\mathcal{M}_4$, defined as

$$\mathcal{E}_{\rm DSLST} = {\rm Tr}_{\mathcal{H}_{\rm RR}}\left[(-1)^F q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}e^{2\pi i z(2J^3_{\rm R})_0}\right], \tag{3.1}$$

where $q=e^{2\pi i\tau}$, $c=\bar{c}=6$ are the left and right-moving central charges, and $J^3_{\rm R}$ is a Cartan generator of the left-moving ${\rm SU}(2)_R$ symmetry. The trace is taken over states in the Ramond-Ramond sector of the SCFT. We discussed the spacetime significance of this quantity in the introduction. In this section, we will calculate it using worldsheet techniques. In particular the calculation will be performed at the point in moduli space corresponding to figure 1, where

$$\mathcal{M}_4 = \left(\frac{{\rm SL}(2,\mathbb{R})_k}{{\rm U}(1)}\times\frac{{\rm SU}(2)_k}{{\rm U}(1)}\right)/\mathbb{Z}_k \tag{3.2}$$

but, as we will discuss later, the result is independent of the moduli.

We will see that the elliptic genus is not holomorphic in q unlike the situation for compact SCFT's. This is because both discrete and continuum states can contribute to the elliptic genus [18–21]. We will discuss a physical way to separate the elliptic genus into two contributions, corresponding to the discrete and continuum states, respectively.

Some properties of the elliptic genus of DSLST will be discussed in the next section.

3.1 Cigar CFT

We can describe the cigar CFT as a two dimensional $\mathcal{N}=(2,2)$ non-linear σ-model whose Lagrangian takes the form

$$\mathcal{L} = -g_{i\bar{j}}\partial^\mu\phi^i\partial_\mu\bar{\phi}^{\bar{j}} + ig_{i\bar{j}}\bar{\psi}^{\bar{j}}_-D_+\psi^i_- + ig_{i\bar{j}}\bar{\psi}^{\bar{j}}_+D_-\psi^i_+ + R_{i\bar{j}k\bar{l}}\psi^i_+\psi^k_-\bar{\psi}^{\bar{j}}_-\bar{\psi}^{\bar{l}}_+\,, \tag{3.3}$$

where the target space metric is

$$ds^2 = k\left(dr^2+\tanh^2 r d\theta^2\right)\,. \tag{3.4}$$

The non-linear σ model also includes a non-trivial dilaton profile, which will not play a role in what follows. The four global supercharges are given by

$$Q_{\pm} = \int d\sigma\ 2g_{i\bar{j}}\partial_{\pm}\bar{\phi}^{\bar{j}}\psi^{i}_{\pm}\,, \qquad \bar{Q}_{\pm} = \int d\sigma\ 2g_{i\bar{j}}\bar{\psi}^{\bar{j}}_{\pm}\partial_{\pm}\phi^{i}\,. \tag{3.5}$$

Using supersymmetric localization, one can reduce the path-integral of the cigar CFT to a finite dimensional integral over the holonomy-torus [22, 23],

$$\mathcal{E}_{\text{cig}}(\tau,z) = k\int_0^1 ds_1 \int_0^1 ds_2 \sum_{(n,w)\in\mathbb{Z}^2} \frac{\vartheta_1\left(\tau, s_1\tau + s_2 + z(1+1/k)\right)}{\vartheta_1\left(\tau, s_1\tau+s_2+z\right)} \\ \times e^{-2\pi i z w} e^{-\frac{\pi k}{\tau_2}\left|s_1\tau+s_2+n+\tau w\right|^2}\,, \tag{3.6}$$

where $\vartheta_1(\tau,z)$ denotes the odd Jacobi theta function given by

$$\vartheta_1(\tau,z) = -iq^{1/8}e^{\pi i z}\prod_{n=1}^{\infty}(1-q^n)(1-e^{2\pi i z}q^n)(1-e^{-2\pi i z}q^{n-1})\,. \tag{3.7}$$

Using the Poisson resummation formula, one can rewrite the elliptic genus as

$$\mathcal{E}_{\text{cig}}(\tau,z) = \sqrt{k\tau_2}\int_0^1 ds_1 \int_0^1 ds_2 \sum_{(p,w)\in\mathbb{Z}^2} \frac{\vartheta_1\left(\tau, s_1\tau + s_2 + z(1+1/k)\right)}{\vartheta_1\left(\tau, s_1\tau+s_2+z\right)} \\ \times e^{-2\pi i z w} e^{-2\pi i s_2 p} q^{l_0}\bar{q}^{\bar{l}_0}\,, \tag{3.8}$$

where

$$l_0 = \frac{1}{4k}\left(p - k(w+u_2)\right)^2\,, \qquad \bar{l}_0 = \frac{1}{4k}\left(p + k(w+u_2)\right)^2\,. \tag{3.9}$$

The integers (p,w) are the momentum and winding around the cigar.

Following the treatment in [18, 19, 22, 23], one can show that

$$\mathcal{E}_{\text{cig}} = \mathcal{E}_{\text{d}} + \mathcal{E}_{\text{c}}\,, \tag{3.10}$$

where the contribution from the discrete spectrum is

$$\begin{aligned} \mathcal{E}_d &= +\frac{i\vartheta_1(\tau,z)}{\eta(q)^3}\sum_{\alpha=0}^{k-1}\sum_{w\in\mathbb{Z}}\frac{q^{(-\alpha+kw)w}e^{2\pi i z(-\alpha/k+2w)}}{1-e^{2\pi i z}q^{-\alpha+kw}} \\ &= +\frac{i\vartheta_1(\tau,z)}{k\eta(q)^3}\sum_{\beta,\gamma=1}^{k} e^{2\pi i\frac{\beta\gamma}{k}}\sum_{w\in\mathbb{Z}}\frac{q^{\frac{(kw+\beta)^2}{k}}\left(e^{2\pi i z}\right)^{2\frac{kw+\beta}{k}}}{1-\left(e^{2\pi i z}\right)^{\frac{1}{k}}q^{\frac{kw+\beta}{k}}e^{2\pi i\frac{\gamma}{k}}} \end{aligned} \tag{3.11}$$

and that of the scattering states is

$$\begin{aligned} \mathcal{E}_{\text{c}} &= -\frac{\vartheta_1(\tau,z)}{2\eta(q)^3}\sum_{p,w\in\mathbb{Z}}\int_{-\infty}^{\infty}dK\ \frac{1}{\pi}\left[\frac{1}{K+i(p+kw)}\right](q\bar{q})^{\frac{K^2}{4k}}q^{\frac{1}{4k}(p-kw)^2}\bar{q}^{\frac{1}{4k}(p+kw)^2}e^{2\pi i z(\frac{p}{k}-w)} \\ &= -\frac{\vartheta_1(\tau,z)}{2\eta(q)^3}\sum_{p,w\in\mathbb{Z}}\int_0^{\infty}dK\ \frac{1}{\pi}\left[\frac{1}{K+i(p+kw)}+\frac{1}{-K+i(p+kw)}\right](q\bar{q})^{\frac{K^2}{4k}} \\ &\quad\times q^{\frac{1}{4k}(p-kw)^2}\bar{q}^{\frac{1}{4k}(p+kw)^2}e^{2\pi i z(\frac{p}{k}-w)} \end{aligned} \tag{3.12}$$

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When $p + kw = 0$, we choose the integration contour in (3.12) slightly above the real axis. The discrete contribution to the elliptic genus is holomorphic in q but not modular, while the contribution from the scattering states restores modularity at the cost of a loss of holomorphy. Note that the scattering state contribution to the elliptic genus at $z = 0$, i.e. to the Witten index, vanishes.

Below we discuss a more physical way to compute the contribution of the scattering states to the elliptic genus. This method should be applicable to any non-compact CFT, and in particular to DSLST at a generic point in its moduli space.[4] The reader not interested in the details can skip section 3.2 and proceed directly to section 3.3.

3.2 Spectral asymmetry and non-holomorphic contributions

The contribution of the continuum of δ-function normalizable states to the elliptic genus is related to the difference between the densities of bosonic and fermionic states. These densities can be computed from the corresponding scattering phase shifts. The individual phase shifts for bosons and fermions are non-trivial, but the difference between them can be computed exactly using only asymptotic data. The general idea goes back to calculations of the Witten index in supersymmetric quantum mechanics with non-compact target space [25–27].

In order to perform this computation, we first consider the Scherk-Schwarz reduction of the cigar σ-model in a sector with winding number w to quantum mechanics, i.e. we take

$$\partial_1 r = 0\,, \qquad \partial_1 \theta = w\,. \tag{3.13}$$

The bosonic part of the resulting Lagrangian takes the form

$$\mathcal{L}^B_{\mathrm{QM}} = \frac{k}{2}\left(\frac{dr}{dt}\right)^2 + \frac{k}{2}\tanh^2 r \left(\frac{d\theta}{dt}\right)^2 - \frac{k}{2} w^2 \tanh^2 r\,. \tag{3.14}$$

This Lagrangian describes the center of mass motion of a string winding the cigar w times. The attractive potential, the last term in (3.14), is due to the fact that the string can decrease its energy by moving towards the tip of the cigar.

Quantizing the system, the fermion operators have to satisfy the canonical anticommutation relations

$$\{\bar\psi_+, \psi_+\} = \{\bar\psi_-, \psi_-\} = \frac{1}{k}\,, \tag{3.15}$$

where $\psi_\pm = e_m \psi^m_\pm$ $(m = 1, 2)$ with an orthonormal frame e_m. They can be represented by four-dimensional Dirac gamma matrices

$$\begin{aligned} \sqrt{k}\bar\psi_+ &= \frac{\sigma^1 + i\sigma^2}{2} \otimes \mathbf{1}_2\,, \qquad & \sqrt{k}\psi_+ &= \frac{\sigma^1 - i\sigma^2}{2} \otimes \mathbf{1}_2\,, \\ \sqrt{k}\bar\psi_- &= \sigma^3 \otimes \frac{\sigma^1 + i\sigma^2}{2}\,, \qquad & \sqrt{k}\psi_- &= \sigma^3 \otimes \frac{\sigma^1 - i\sigma^2}{2}\,, \end{aligned} \tag{3.16}$$

[4]The universality of this contribution played an important role in [24].

where $\vec{\sigma}$ represent the Pauli matrices. The fermion number operator then becomes

$$F = \frac{1-\sigma^3}{2} \otimes \mathbf{1}_2 + \mathbf{1}_2 \otimes \frac{1-\sigma^3}{2} \,. \tag{3.17}$$

One can regard the wavefunctions as 4-component spinors, two of which are bosonic and the others are fermionic, i.e.,

$$\begin{aligned}
\langle x|B_1\rangle &= f_1(r,\theta)|++\rangle \,, \\
\langle x|F_1\rangle &= g_1(r,\theta)|-+\rangle \,, \\
\langle x|B_2\rangle &= f_2(r,\theta)|--\rangle \,, \\
\langle x|F_2\rangle &= g_2(r,\theta)|+-\rangle \,.
\end{aligned} \tag{3.18}$$

Their left and right-moving $\mathrm{U}(1)_R$ charges denoted by $\mathrm{U}(1)_l$ and $\mathrm{U}(1)_r$ respectively are summarized, up to overall constants, in the table below

	$\lvert B_1\rangle$	$\lvert F_1\rangle$	$\lvert B_2\rangle$	$\lvert F_2\rangle$
$\mathrm{U}(1)_l$	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$
$\mathrm{U}(1)_r$	$-\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$

.

For the scattering states, the asymptotic behavior of $f_{1,2}(r,\theta)$ and $g_{1,2}(r,\theta)$ in (3.18) can be described as follows

$$\begin{aligned}
f_1(r,\theta) &\to \left(e^{-iKr} + e^{2i\delta_B^1(K)}e^{+iKr}\right)\cdot e^{ip\theta} \,, \\
g_1(r,\theta) &\to \left(e^{-iKr} + e^{2i\delta_F^1(K)}e^{+iKr}\right)\cdot e^{ip\theta} \,, \\
f_2(r,\theta) &\to \left(e^{-iKr} + e^{2i\delta_B^2(K)}e^{+iKr}\right)\cdot e^{ip\theta} \,, \\
g_2(r,\theta) &\to \left(e^{-iKr} + e^{2i\delta_F^2(K)}e^{+iKr}\right)\cdot e^{ip\theta} \,.
\end{aligned} \tag{3.19}$$

The boson and fermion scattering states are paired by the supercharges Q_+ and $\bar{Q}_+$. More precisely,

$$Q_+|B_1\rangle \sim |F_1\rangle \,, \qquad Q_+|F_2\rangle \sim |B_2\rangle \,, \tag{3.20}$$

which implies that, in the limit $r\to\infty$,

$$\begin{aligned}
\left[-\frac{i}{\sqrt{k}}\frac{\partial}{\partial r} - \frac{1}{\sqrt{k}}\frac{\partial}{\partial\theta} - i\sqrt{k}w\right] f_1(r,\theta) &\propto g_1(r,\theta) \,, \\
\left[-\frac{i}{\sqrt{k}}\frac{\partial}{\partial r} - \frac{1}{\sqrt{k}}\frac{\partial}{\partial\theta} - i\sqrt{k}w\right] g_2(r,\theta) &\propto f_2(r,\theta) \,.
\end{aligned} \tag{3.21}$$

These relations provide very strong constraints between phase-shift factors,

$$\begin{aligned}
e^{2i\left(\delta_B^1(K)-\delta_F^1(K)\right)} &= -\frac{K+i(p+kw)}{K-i(p+kw)} \,, \\
e^{2i\left(\delta_B^2(K)-\delta_F^2(K)\right)} &= -\frac{K-i(p+kw)}{K+i(p+kw)} \,.
\end{aligned} \tag{3.22}$$

This result can be verified directly by using the exact results for the bosonic and fermionic phase shifts [20, 28]. Note that the individual phase shifts receive non-trivial stringy corrections that play an important role in the discussion of [29, 30]. However, these stringy corrections cancel in the difference of phase shifts, which is given exactly by the quantum mechanical result.

Using the standard relation between the spectral density and the phase shift in quantum mechanics,

$$\rho_B(E) - \rho_F(E) = \frac{1}{\pi}\frac{\partial}{\partial E}\left(\delta_B(E) - \delta_F(E)\right) , \tag{3.23}$$

one obtains the difference in the density of states

$$\rho^1_B(K)-\rho^1_F(K) = -\rho^2_B(K)+\rho^2_F(K) = \frac{1}{2\pi i}\left[\frac{1}{K+i(p+kw)}+\frac{1}{-K+i(p+kw)}\right] . \tag{3.24}$$

This result implies that, unless one turns on a chemical potential z for the R-charge, there is no spectral asymmetry. It explains why the scattering state contribution to the Witten index vanish, $\mathcal{E}_c(\tau, z=0) = 0$.

In a sector with winding w around the cigar, the contribution of scattering states to the elliptic genus with chemical potential z can be expressed as

$$\begin{aligned}\mathcal{E}^{QM}_c(w) = \sum_{p\in\mathbb{Z}}\int_0^\infty dK\, &\left[\left(\rho^1_B(K)-\rho^1_F(K)\right)e^{-\pi i z} + \left(\rho^2_B(K)-\rho^2_F(K)\right)e^{+\pi i z}\right] \\ &\times \left(e^{-2\pi\tau_2}\right)^{E(K,p,w)}\left(e^{2\pi i\tau_1}\right)^{P(p,w)} e^{2\pi i z J(p,w)} ,\end{aligned} \tag{3.25}$$

where

$$\begin{aligned} E(K,p,w) &= \frac{K^2}{2k} + \frac{p^2}{2k} + \frac{kw^2}{2} , \\ P(p,w) &= -pw , \\ J(p,w) &= \frac{p}{k} - w . \end{aligned} \tag{3.26}$$

Using the result (3.24), one can rewrite the index $\mathcal{E}^{QM}_c$ in the following form

$$\begin{aligned}\mathcal{E}^{QM}_c(w) = -\sin\pi z\sum_p\int_0^\infty dK\,\frac{1}{\pi}&\left[\frac{1}{K+i(p+kw)}+\frac{1}{-K+i(p+kw)}\right](q\bar{q})^{\frac{K^2}{4k}} \\ &\times q^{\frac{1}{4k}(p-kw)^2}\bar{q}^{\frac{1}{4k}(p+kw)^2}e^{2\pi i z(\frac{p}{k}-w)} .\end{aligned} \tag{3.27}$$

Since the difference in the spectral densities is not affected by the oscillator modes of the string, the continuum part of the elliptic genus can be written as

$$\mathcal{E}_c = \mathcal{E}^{\rm osc}_c \times \mathcal{E}^{\rm com}_c , \qquad \mathcal{E}^{\rm com}_c = \sum_w \mathcal{E}^{\rm QM}_c(w) , \tag{3.28}$$

where the contribution from the oscillator modes is

$$\mathcal{E}^{\rm osc}_c = \prod_{n=1}^\infty\left[\frac{\left(1-q^n e^{2\pi i z}\right)\left(1-q^n e^{-2\pi i z}\right)}{\left(1-q^n\right)^2}\right] . \tag{3.29}$$

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Using the definition of the Jacobi theta function $\vartheta_1(\tau, z)$ and the Dedekind eta function $\eta(q)$, one can rewrite $\mathcal{E}_c$ in the form

$$\mathcal{E}_{\rm c} = -\frac{\vartheta_1(\tau,z)}{2\eta(q)^3} \sum_{p,w\in\mathbb{Z}} \int_0^\infty dK \, \frac{1}{\pi} \left[\frac{1}{K+i(p+kw)} + \frac{1}{-K+i(p+kw)}\right] (q\bar{q})^{\frac{K^2}{4k}} \\ \times q^{\frac{1}{4k}(p-kw)^2} \bar{q}^{\frac{1}{4k}(p+kw)^2} e^{2\pi i z(\frac{p}{k}-w)} \,, \tag{3.30}$$

which agrees with (3.12).

3.3 Mock modularity

There is a close relationship between mock modular forms and the elliptic genera of non-compact CFT's [18, 19]. In fact, one can show that the discrete part (3.11) can be written as follows

$$\mathcal{E}_d = +\frac{i\vartheta_1(\tau,z)}{k\eta(q)^3} \sum_{\beta,\gamma=1}^{k} e^{2\pi i\frac{\beta\gamma}{k}} q^{\frac{\beta^2}{k}} \left(e^{2\pi i z}\right)^{\frac{2\beta}{k}} \mathcal{A}_{1,k}\left(\tau, \frac{z+\beta\tau+\gamma}{k}\right) \,, \tag{3.31}$$

where the Appell-Lerch sum is a well-known mock modular form defined by [19]

$$\mathcal{A}_{1,k}(\tau,z) = \sum_{t\in\mathbb{Z}} \frac{q^{kt^2}\left(e^{2\pi i z}\right)^{2kt}}{1-\left(e^{2\pi i z}\right) q^t} \,. \tag{3.32}$$

On the other hand, from an integration formula,

$$\int_{\mathbb{R}\mp i\epsilon} dp \, \frac{1}{p-i\lambda} e^{-\alpha p^2} = i\pi \mathrm{sgn}(\lambda\pm\epsilon)\mathrm{Erfc}\left(\sqrt{\alpha}|\lambda|\right) e^{\alpha\lambda^2} \quad (\alpha,\epsilon>0 \text{ and } \lambda\in\mathbb{R}) \,, \tag{3.33}$$

one can express the continuum part (3.12) as

$$\mathcal{E}_{\rm c} = \frac{i\vartheta_1(\tau,z)}{2\eta(q)^3} \sum_{l=0}^{2k-1} R^-_{k,l}(\tau)\vartheta_{k,l}\left(\tau, \frac{z}{k}\right) \,, \tag{3.34}$$

where the non-holomorphic Eichler integrals $R^\pm_{k,l}(\tau)$ are defined as

$$R^\pm_{k,l}(\tau) = \sum_{\lambda=l+2k\mathbb{Z}} \mathrm{sgn}(\lambda\pm\epsilon)\mathrm{Erfc}\left(\sqrt{\frac{\pi\tau_2}{k}}|\lambda|\right) q^{-\frac{\lambda^2}{4k}} \,, \tag{3.35}$$

and $\vartheta_{k,l}(\tau,z)$ denote Jacobi theta functions at level k

$$\vartheta_{k,l}(\tau,z) = \sum_{\lambda=l+2k\mathbb{Z}} q^{\lambda^2/4k} e^{2\pi i z\lambda} \,. \tag{3.36}$$

Since the Eichler-Zagier involution maps a Jacobi theta function at level k to a different Jacobi theta function

$$\frac{1}{k}\sum_{\beta,\gamma=1}^{k} e^{2\pi i\frac{\beta\gamma}{k}} q^{\frac{\beta^2}{k}} \left(e^{2\pi i z}\right)^{2\beta} \vartheta_{k,l}\left(\tau, z+\frac{\beta\tau+\gamma}{k}\right) = \vartheta_{k,2k-l}\left(\tau,z\right) \quad \text{for} \quad l\in\mathbb{Z}_{2k} \,, \tag{3.37}$$

(3.34) can be written in the form

$$\mathcal{E}_{\rm c} = \frac{i\vartheta_1(\tau,z)}{\eta(q)^3}\cdot\frac{1}{k}\sum_{\beta,\gamma=1}^{k} e^{2\pi i\frac{\beta\gamma}{k}} q^{\frac{\beta^2}{k}} \left(e^{2\pi i\frac{z}{k}}\right)^{2\beta}\cdot\left(-\frac{1}{2}\sum_{l=1}^{2k} R^+_{k,l}(\tau)\vartheta_{k,l}\left(\tau,\frac{z+\beta\tau+\gamma}{k}\right)\right). \tag{3.38}$$

Collecting all the results, one can finally observe that the full elliptic genus can be expressed in terms of the non-holomorphic modular completion of the Appell-Lerch sum $\mathcal{A}_{1,k}(\tau,z)$,

$$\mathcal{E}_{\rm cig} = \mathcal{E}_d + \mathcal{E}_c = +\frac{i\vartheta_1(\tau,z)}{k\eta(q)^3}\sum_{\beta,\gamma=1}^{k} e^{2\pi i\frac{\beta\gamma}{k}} q^{\frac{\beta^2}{k}} \left(e^{2\pi iz}\right)^{\frac{2\beta}{k}} \hat{\mathcal{A}}_{1,k}\left(\tau,\frac{z+\beta\tau+\gamma}{k}\right), \tag{3.39}$$

where

$$\hat{\mathcal{A}}_{1,k}(\tau,z) = \mathcal{A}_{1,k}(\tau,z) - \frac{1}{2}\sum_{l=1}^{2k} R^+_{k,l}(\tau)\vartheta_{k,l}(\tau,z)\,. \tag{3.40}$$

Thus, the elliptic genus of the cigar CFT is expressed as the Eichler-Zagier involution [31] of the modular completion of Appell-Lerch sum, $\hat{\mathcal{A}}_{1,k}(\tau,z)$.

3.4 Character decomposition

We now discuss the expansion of the elliptic genus $\mathcal{E}_{\rm cig}$ (3.11) in terms of $\mathcal{N}=2$ superconformal characters in the Ramond sector with an insertion of $(-1)^F$.

Let us first introduce the $\mathcal{N}=2$ character formula with $c_{\rm cig} = 3\left(1+\frac{2}{k}\right) > 3$ of discrete representations [32, 33]

$$\mathrm{Ch}^{\rm cig}_{l,n}(\tau,z) = q^{\frac{n^2-(l-1)^2}{4k}} \left(e^{2\pi iz}\right)^{\frac{n}{k}} \frac{1}{1-e^{2\pi iz}q^{\frac{n-l+1}{2}}}\cdot\frac{i\vartheta_1(\tau,z)}{\eta(q)^3}\,, \tag{3.41}$$

where $1\le l\le k+1$. The conformal weight h and the $\mathrm{U}(1)_R$ charge r of an $\mathcal{N}=2$ primary corresponding to each character are

- $\boldsymbol{n-l+1\ge 2}$:

$$\begin{aligned} h_{l,n} - \frac{c_{\rm cig}}{24} &= \frac{n^2-(l-1)^2}{4k}\,,\\ r_{l,n} &= \frac{n}{k}-\frac{1}{2}\,. \end{aligned} \tag{3.42}$$

- $\boldsymbol{n-l+1\le -2}$:

$$\begin{aligned} h_{l,n} - \frac{c_{\rm cig}}{24} &= \frac{(k-n)^2-(k-l+1)^2}{4k}\,,\\ r_{l,n} &= -\frac{k-n}{k}+\frac{1}{2}\,. \end{aligned} \tag{3.43}$$

- $\boldsymbol{n-l+1 = 0}$:

$$\begin{aligned} h_{l,n} - \frac{c_{\rm cig}}{24} &= 0\,,\\ r_{l,n} &= \frac{n}{k}-\frac{1}{2}\,. \end{aligned} \tag{3.44}$$

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The $\mathcal{N}=2$ characters enjoy a $\mathbb{Z}_2$ reflection symmetry

$$\mathrm{Ch}^{\mathrm{cig}}_{l,n}(\tau,z)=\mathrm{Ch}^{\mathrm{cig}}_{(k+2)-l,k-n}(\tau,-z)\,, \tag{3.45}$$

and transform under the spectral flow by α units as

$$q^{\frac{c_{\mathrm{cig}}}{6}\alpha^2}\left(e^{2\pi iz}\right)^{\frac{c_{\mathrm{cig}}}{3}\alpha}\mathrm{Ch}^{\mathrm{cig}}_{l,n}(\tau,z+\tau\alpha)=(-1)^{\alpha}\mathrm{Ch}^{\mathrm{cig}}_{l,n+2\alpha}(\tau,z)\,. \tag{3.46}$$

It is straightforward to show that the discrete part of the elliptic genus of the $\frac{\mathrm{SL}(2)_k}{\mathrm{U}(1)}$ CFT can be expanded as

$$\mathcal{E}_d=\sum_{\tilde{l}=0}^{k-1}\sum_{w\in\mathbb{Z}}\mathrm{Ch}^{\mathrm{cig}}_{\tilde{l}+1,-\tilde{l}+2kw}(\tau,z)\,. \tag{3.47}$$

3.5 Minimal model

The Landau-Ginzburg theory with a superpotential

$$W=X^{k+2}+Y^2+Z^2\,, \tag{3.48}$$

is well-known to flow in the infrared to the level k SU(2)/U(1) Kazama-Suzuki model, whose central charge is given by $c_{\mathrm{min}}=3\left(1-\frac{2}{k}\right)<3$.

The elliptic genus of this minimal model can also be computed by supersymmetric localization with the result [34]

$$\mathcal{E}_{\mathrm{min}}(\tau,z)=\frac{\vartheta_1\left(\tau,\left(1-\frac{1}{k}\right)z\right)}{\vartheta_1\left(\tau,\frac{1}{k}z\right)}\,. \tag{3.49}$$

The $\mathcal{N}=2$ superconformal character formulae of the SU(2)/U(1) Kazama-Suzuki model at level k are

$$\mathrm{Ch}^{\mathrm{min}}_{l,n}(\tau,z)=\chi^l_{n,1}(\tau,z)-\chi^l_{n,3}(\tau,z)\,, \tag{3.50}$$

where the branching functions $\chi^l_{m,s}(\tau,z)$ are defined by [35]

$$\chi^{\widehat{\mathfrak{su}}(2)_{k-2}}_l(\tau,w)\cdot\chi^{\widehat{\mathfrak{u}}(1)_2}_s(\tau,w-z)=\sum_{n=0}^{2k-1}\chi^{\widehat{\mathfrak{u}}(1)_k}_n(\tau,w-\frac{2z}{k+2})\cdot\chi^l_{n,s}(\tau,z)\,, \tag{3.51}$$

where $\chi^{\widehat{\mathfrak{su}}(2)_k}_l(\tau,z)$ and $\chi^{\widehat{\mathfrak{u}}(1)_k}_n(\tau,z)$ denote the $\widehat{\mathfrak{su}}(2)_k$ and $\widehat{\mathfrak{u}}(1)_k$ characters given by

$$\begin{aligned}\chi^{\widehat{\mathfrak{su}}(2)_k}_l(\tau,z)&=\frac{\vartheta_{k+2,l+1}(\tau,z/2)-\vartheta_{k+2,-l-1}(\tau,z/2)}{\vartheta_{2,1}(\tau,z/2)-\vartheta_{2,-1}(\tau,z/2)}\qquad(l=0,1,..,k)\,,\\ \chi^{\widehat{\mathfrak{u}}(1)_k}_n(\tau,z)&=\frac{\vartheta_{k,n}(\tau,z/2)}{\eta(q)}\qquad(n=0,1,..,2k-1)\,.\end{aligned} \tag{3.52}$$

The minimal model characters are periodic in m with period $2k$,

$$\mathrm{Ch}^{\mathrm{min}}_{l,n}(\tau,z)=\mathrm{Ch}^{\mathrm{min}}_{l,n+2k}(\tau,z)\,, \tag{3.53}$$

and also enjoy a $\mathbb{Z}_2$ reflection symmetry

$$\mathrm{Ch}^{\min}_{l,n}(\tau,z) = -\mathrm{Ch}^{\min}_{(k-2)-l,k+n}(\tau,z)\,. \tag{3.54}$$

Using these two properties, one can always choose (l,n) to satisfy the constraint

$$0 \le |n \mp 1| \le l\,. \tag{3.55}$$

Then, the conformal weight h and r charge of the highest weight representation corresponding to $\mathrm{Ch}^{\min}_{l,n}(\tau,z)$ are

$$\begin{aligned} h_{l,n} - \frac{c_{\min}}{24} &= \frac{(l+1)^2 - n^2}{4k}\,, \\ r_{l,n} &= -\frac{n}{k} \pm \frac{1}{2}\,. \end{aligned} \tag{3.56}$$

Under spectral flow by α units, the $\mathcal{N}=2$ characters $\mathrm{Ch}^{\min}_{l,n}$ transform as

$$q^{\frac{c_{\min}}{6}\alpha^2}\left(e^{2\pi i z}\right)^{\frac{c_{\min}}{3}\alpha}\mathrm{Ch}^{\min}_{l,n}(\tau,z+\alpha\tau) = (-1)^{\alpha}\mathrm{Ch}^{\min}_{l,n+2\alpha}(\tau,z)\,. \tag{3.57}$$

We can then express the elliptic genus of the $\mathcal{N}=2$ $\frac{\mathrm{SU}(2)}{\mathrm{U}(1)}$ minimal as

$$\mathcal{E}_{\min}(\tau,z) = \sum_{l=1}^{k-1}\mathrm{Ch}^{\min}_{l-1,l}(\tau,z)\,, \tag{3.58}$$

where the $\mathcal{N}=2$ characters can be written as

$$\mathrm{Ch}^{\min}_{l-1,l}(\tau,z) = \frac{i\vartheta_1(\tau,z)}{\eta(\tau)^3}e^{-2\pi i z\frac{l}{k}}\sum_{w\in\mathbb{Z}}q^{kw^2+lw}\left[\frac{1}{1-q^{l+kw}e^{2\pi i z}}+\frac{1}{1-q^{kw}e^{-2\pi i z}}-1\right]. \tag{3.59}$$

Note that the $\mathcal{N}=2$ characters $\mathrm{Ch}^{\min}_{l-1,l}(\tau,z)$ correspond to primary vertex operators $\tilde{V}^{\mathrm{susy}}_{\frac{l-1}{2};\frac{l-1}{2},\frac{l-1}{2}}\left(\frac{1}{2},\frac{1}{2}\right)$. This implies that the elliptic genus $\mathcal{E}_{\min}$ receives contributions only from the characters associated with the Ramond ground states.

3.6 DSLST

We saw earlier that the holographic dual of DSLST at the particular point in the moduli space corresponding to the brane configuration of figure 1 contains the $\mathbb{Z}_k$ orbifold of the product of an $\mathcal{N}=2$ cigar SCFT and an $\mathcal{N}=2$ minimal model:

$$\left(\frac{\mathrm{SL}(2)_k}{\mathrm{U}(1)}\times\frac{\mathrm{SU}(2)_k}{\mathrm{U}(1)}\right)/\mathbb{Z}_k\,. \tag{3.60}$$

The $\mathbb{Z}_k$ orbifold action, that is generated by $e^{2\pi i(2J^3_{\mathrm{R}})}$ with

$$2J^3_{\mathrm{R}} = J^{\mathrm{cig}}_{\mathrm{R}} + J^{\min}_{\mathrm{R}}\,, \tag{3.61}$$

is necessary for space-time supersymmetry.[5]

[5] And worldsheet $\mathcal{N}=4$ superconformal symmetry.

In the case of this particular class of orbifold theories, we can use the results of [36] to obtain (see e.g. [37–39])

$$\mathcal{E}_{\text{DSLST}}(\tau, z) = \frac{1}{k} \sum_{\alpha,\beta=0}^{k-1} q^{\frac{\hat{c}}{2}\alpha^2} \left(e^{2\pi i z}\right)^{\hat{c}\alpha} \mathcal{E}_{\text{cig}}(\tau, z + \alpha\tau + \beta)\mathcal{E}_{\text{min}}(\tau, z + \alpha\tau + \beta)\,, \tag{3.62}$$

where the elliptic genera of the two coset models are given by (3.39) and (3.49), and the central charge is

$$\hat{c} = \frac{c_{\text{cig}}}{3} + \frac{c_{\text{min}}}{3} = 2\,. \tag{3.63}$$

Clearly we obtain a non-holomorphic elliptic genus since the cigar elliptic genus is not holomorphic. The contribution from the discrete states of DSLST can be read off from (3.31),

$$\mathcal{E}^d_{\text{DSLST}}(\tau, z) = \frac{1}{k} \sum_{\alpha,\beta=0}^{k-1} q^{\frac{\hat{c}}{2}\alpha^2} \left(e^{2\pi i z}\right)^{\hat{c}\alpha} \mathcal{E}^d_{\text{cig}}(\tau, z + \alpha\tau + \beta)\mathcal{E}_{\text{min}}(\tau, z + \alpha\tau + \beta)\,. \tag{3.64}$$

Using (3.58) and (3.47), it is also useful to rewrite the discrete part of the elliptic genus in terms of $\mathcal{N} = 2$ superconformal characters as

$$\mathcal{E}^d_{\text{DSLST}} = \sum_{\alpha=0}^{k-1} \sum_{l=1}^{k-1} \sum_{\tilde{l}=0}^{k-1} \sum_{w\in\mathbb{Z}} \delta(l + \tilde{l} - k) \cdot \text{Ch}^{\text{cig}}_{\tilde{l}+1,-\tilde{l}+2\alpha+2kw}(\tau, z) \cdot \text{Ch}^{\text{min}}_{l-1,l+2\alpha}(\tau, z)\,. \tag{3.65}$$

The $\mathbb{Z}_k$ projection gives rise to the Kronecker delta in the above expression.

In the next section we discuss various features of the discrete contribution to the elliptic genus and their physical implications.

4 Properties of the Elliptic Genus

4.1 $\mathcal{N} = 4$ Character Decomposition

The superconformal field theory appearing in the holographic description of DSLST has an $\mathcal{N} = 4$ superconformal algebra with $c = 6$. It must therefore be possible to decompose the discrete contribution to the elliptic genus into a (in general infinite) sum of $\mathcal{N} = 4$ characters. The irreducible highest weight representations $V^{(m)}_{h,j}$ of the $\mathcal{N} = 4$ superconformal algebra with $c = 6(m-1)$ are labelled by h and j, the eigenvalues of L_0 and $\left(2J^3_{\text{R}}\right)_0$. We define the Ramond sector characters as

$$\text{ch}^{(m)}_{h,j}(\tau, z) = \text{Tr}_{V^{(m)}_{h,j}} \left[(-1)^F e^{2\pi i z (2J^3_{\text{R}})_0} q^{L_0 - c/24}\right]. \tag{4.1}$$

These characters are given by [40]

$$\text{ch}^{(m)}_{\frac{m-1}{4},j}(\tau, z) = i\mu^{(m)}_j(\tau, z)\frac{(\vartheta_1(\tau, z))^2}{\vartheta_1(\tau, 2z) \cdot \eta(\tau)^3} \tag{4.2}$$

for the massless or BPS characters with $h = \frac{m-1}{4}$ and $j \in \{0, 1, \cdots m-1\}$, and by

$$\mathrm{ch}^{(m)}_{h,j} = i(-1)^j q^{h-\frac{m-1}{4}-\frac{j^2}{4m}} \left(\vartheta_{m,j}(\tau,z) - \vartheta_{m,-j}(\tau,z)\right) \frac{(\vartheta_1(\tau,z))^2}{\vartheta_1(\tau,2z)\cdot\eta(\tau)^3} \tag{4.3}$$

for the massive or non-BPS characters with $h > \frac{m-1}{4}$ and $j \in \{1, 2, \cdots m-1\}$. Here the function

$$\mu^{(m)}_j(\tau,z) = (-1)^{j+1} \sum_{k\in\mathbb{Z}} q^{mk^2} \left(e^{2\pi i z}\right)^{2mk} \sum_{a=-j}^{j+1} \frac{\left(e^{2\pi i z} q^k\right)^a}{1 - e^{2\pi i z} q^k}\,, \tag{4.4}$$

is a generalized Appell-Lerch sum and for $m = 2$ is closely related to the Appell-Lerch sum $\mu(\tau, z)$ that plays a prominent role in Zwegers influential work on mock theta functions [41]. One can show that the second Taylor coefficients of $\mathcal{N} = 4$ massless and massive characters are given by

$$\begin{aligned}
\left(\frac{1}{2\pi i}\right)^2 \frac{d^2}{dz^2} \mathrm{ch}^{(m)}_{\frac{m-1}{4},0}(\tau,z)\bigg|_{z=0} &= 4\sum_{n=1}^{\infty} \left[\frac{q^n\left(1-q^{mn^2}\right)}{(1-q^n)^2} - mn\frac{q^{mn^2}(1+q^n)}{1-q^n}\right],\\
\left(\frac{1}{2\pi i}\right)^2 \frac{d^2}{dz^2} \mathrm{ch}^{(m)}_{h,1}(\tau,z)\bigg|_{z=0} &= -2q^{h-\frac{m-1}{4}} \vartheta^{(1)}_{m,1}(\tau)\,.
\end{aligned} \tag{4.5}$$

It is not hard to see that the decomposition into $\mathcal{N} = 4$ characters involves the massless character $\mathrm{ch}^{(2)}_{\frac{1}{4},0}$ with multplicity $(k-1)$. As discussed in section 4.4, these correspond under spectral flow to the chiral operators in the NS sector that can be understood as the relative translation modes of the fivebranes. In terms of world-volume fields on IIB fivebranes they belong to the same supermultiplet as the $k-1$ massless gauge bosons in the Cartan subalgebra of SU(k). Denoting the multiplicities of massive characters by a_n we thus have the decomposition

$$\mathcal{E}^d_{\mathrm{DSLST}}(\tau,z) = (k-1)\mathrm{ch}^{(2)}_{\frac{1}{4},0}(\tau,z) + \sum_{n=1}^{\infty} a_n \mathrm{ch}^{(2)}_{\frac{1}{4}+n,1}(\tau,z)\,. \tag{4.6}$$

Based on non-trivial numerical experimentation, we believe that the second Taylor coefficient of $\mathcal{E}^d_{\mathrm{DSLST}}(\tau,z)$ for arbitrary k is

$$\left(\frac{1}{2\pi i}\right)^2 \frac{d^2}{dz^2} \mathcal{E}^d_{\mathrm{DSLST}}(\tau,z)\bigg|_{z=0} = 4\mathcal{F}^{k,1}_2\,, \tag{4.7}$$

and thus the coefficients a_n satisfy the relation

$$-\frac{1}{2}\vartheta^{(1)}_{2,1}(\tau)\sum_{n=1} a_n q^{n-\frac{1}{8}} = \mathcal{F}^{k,1}_2(q) - (k-1)\mathcal{F}^{2,1}_2(q) \tag{4.8}$$

with

$$\mathcal{F}^{k,1}_2 = \left[\sum_{\substack{r,s\in\mathbb{Z}\\0<s<kr}} - k\sum_{\substack{r,s\in\mathbb{Z}\\0<ks<r}}\right] s q^{rs} = \sum_{n=1}^{\infty}\left[\frac{q^n\left(1-q^{kn^2}\right)}{(1-q^n)^2} - kn\frac{q^{kn^2}(1+q^n)}{1-q^n}\right]. \tag{4.9}$$

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Here $\vartheta^{(1)}_{2,1}(\tau)$ denotes the first Taylor coefficient of a Jacobi theta function with level 2, $\vartheta_{2,1}(\tau,z)$,

$$\vartheta^{(1)}_{2,1}(\tau) = \sum_{n\in\mathbb{Z}}(1+4n)q^{\frac{(4n+1)^2}{8}} = \eta(q)^3\,. \tag{4.10}$$

The $\mathcal{F}_2^{k,1}$ are mixed mock modular forms of weight two that played an important role in the analysis of [42]. It is natural to expect a relation between the second derivative of $\mathcal{E}^d_{\rm DSLST}(\tau,z)$ at $z=0$ and the spacetime BPS index computed in [42] since they are both weight two (mixed) mock modular forms computed in the SCFT describing the holographic background of DSLST.

For later convenience, we present the first few coefficients a_n $(k>2)$ below

$$a_1 = 2k-4\ , \qquad a_2 = 8k-20\,, \qquad a_3 = \begin{cases} 6 & \text{if } \ k=3 \\ 22k-66 & \text{if } \ k>3 \end{cases}\,, \tag{4.11}$$

and so on. At $k=2$, all the coefficients a_n vanish and the elliptic genus is simply given by the $\mathcal{N}=4$ massless character with $j=0$

$$\mathcal{E}^d_{\rm DSLST}(\tau,z) = {\rm ch}^{(2)}_{\frac14,0}(\tau,z) \ \text{ at } \ k=2\,. \tag{4.12}$$

4.2 Comments on $k=2$

Note that the $\mathcal{N}=2$ minimal model contribution to the elliptic genus is not present at $k=2$. It is therefore natural to ask how our result at $k=2$ is related to the elliptic genus of the $\mathbb{Z}_2$ orbifold of the cigar theory at $k=2$ studied in [18–20].

Using the results of [36], the elliptic genus of the $\mathbb{Z}_k$ orbifold of the coset CFT takes the general form

$$\mathcal{E}^D_{\rm orb}(\tau,z) = \frac{1}{k}\sum_{\alpha,\beta=0}^{k-1}(-1)^{D(\alpha+\beta+\alpha\beta)}e^{2\pi i\frac{\hat c}{2}\alpha\beta}q^{\frac{\hat c}{2}\alpha^2}\left(e^{2\pi iz}\right)^{\hat c\alpha}\mathcal{E}_{\rm cig}(\tau,z+\alpha\tau+\beta)\,, \tag{4.13}$$

where D is an integer satisfying

$$Dk = \hat c k \ \text{mod}\ 2\,. \tag{4.14}$$

For a generic k, we can choose $D=1$ satisfying the relation (4.14),

$$k = \left(1+\frac{2}{k}\right)k \ \text{mod}\ 2\,. \tag{4.15}$$

and the elliptic genus then becomes

$$\mathcal{E}^{D=1}_{\rm orb}(\tau,z) = \frac{1}{k}\sum_{\alpha,\beta=0}^{k-1}(-1)^{(\alpha+\beta)}e^{2\pi i\frac{1}{k}\alpha\beta}q^{\frac{\hat c}{2}\alpha^2}\left(e^{2\pi iz}\right)^{\hat c\alpha}\mathcal{E}_{\rm cig}(\tau,z+\alpha\tau+\beta)\,, \tag{4.16}$$

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which agrees with the results in [18–20]. When $k=2$, one can show that

$$\begin{aligned}\mathcal{E}^{D=1}_{\text{orb}}(\tau,z) &= \frac{i\vartheta_1(\tau,z)}{\eta(q)^3}\cdot\sum_{m\in\mathbb{Z}}\frac{q^{2m^2}\xi^{2m}}{1-\xi^{\frac{1}{2}}q^m}\\ &= \left(1+\frac{1}{\sqrt{\xi}}\right)+\frac{\left(\sqrt{\xi}+1\right)^3\left(\sqrt{\xi}-1\right)^2}{\xi^{\frac{3}{2}}}q+\mathcal{O}(q^2)\,,\end{aligned} \tag{4.17}$$

where $\xi=e^{2\pi i z}$. However, the above result cannot be decomposed into $\mathcal{N}=4$ superconformal characters. This implies that $\mathcal{N}=2$ supersymmetry can not be enhanced to $\mathcal{N}=4$ supersymmetry when $k=2$ and $D=1$. Furthermore, there are states in (4.17) that carry fractional U(1) R-charges indicating that the choice $D=1$ leads to a theory which is not compatible with spacetime supersymmetry of DSLST.

It is interesting to understand where the discrepancy between the two results (4.12) and (4.17) at $k=2$ comes from. In fact, for $k=2$ we find that there is another solution to (4.14), namely $D=2$ since

$$D\cdot 2=\left(1+\frac{2}{2}\right)\cdot 2\,. \tag{4.18}$$

The corresponding elliptic genus $\mathcal{E}^{D=2}_{\text{orb}}(\tau,z)$ turns out to coincide with a single $\mathcal{N}=4$ massless character with $j=0$, i.e.,

$$\begin{aligned}\mathcal{E}^{D=2}_{\text{orb}}(\tau,z) &= \frac{1}{2}\sum_{\alpha,\beta=0}^{1} q^{\frac{\hat{c}}{2}\alpha^2}\left(e^{2\pi i z}\right)^{\hat{c}\alpha}\mathcal{E}_{\text{cig}}(\tau,z+\alpha\tau+\beta)\\ &= \text{ch}^{(2)}_{\frac{1}{4},0}(\tau,z)\,,\end{aligned} \tag{4.19}$$

which now in turn agrees perfectly with the elliptic genus of double-scaled little string theory (DSLST) at $k=2$ (4.12).

4.3 Large k limit

Consider the discrete contribution to the elliptic genus of DSLST for k fivebranes in the limit $k\to\infty$. We might expect that it becomes easier to identify vertex operators for various states in this limit, which will be discussed in section 4.4, since the algebraic structure simplifies.

It is not hard to check that

$$\lim_{k\to\infty}\mathcal{F}^{k,1}_2(q)=\frac{1-E_2(\tau)}{24} \tag{4.20}$$

with $E_2(\tau)$ the quasi modular Eisenstein series of weight 2. In particular it is independent of k at large k. We therefore have, pulling out an overall factor of k,

$$\lim_{k\to\infty}\mathcal{E}^{d}_{\text{DSLST}}(\tau,z)=k\left(\text{ch}^{(2)}_{\frac{1}{4},0}(\tau,z)+\sum_{n=1}^{\infty}a_n\text{ch}^{(2)}_{\frac{1}{4}+n,1}(\tau,z)\right) \tag{4.21}$$

where

$$\sum_{n=1} a_n q^{n-1/8} = \frac{2}{\eta(\tau)^3}\mathcal{F}_2^{2,1}(q) \tag{4.22}$$

It is perhaps interesting to rewrite this further using the fact [43] that

$$\mathcal{F}_2^{2,1}(q) = \frac{\eta(\tau)^3 H^{(2)}(\tau)}{48} + \frac{E_2(\tau)}{24} \tag{4.23}$$

where

$$H^{(2)}(\tau) = 2q^{-1/8}\left(-1 + 45q + 231q^2 + 770q^3 + \cdots\right) \tag{4.24}$$

is the weight 1/2 mock modular form connected to Mathieu Moonshine [44] that appears in the decomposition of the the elliptic genus of $K3$ into $\mathcal{N}=4$ characters. We thus have

$$\sum_{n=1} a_n q^{n-1/8} = \frac{H^{(2)}(\tau)}{24} + \frac{E_2(\tau)}{12\eta(\tau)^3} \tag{4.25}$$

The elliptic genus of $K3$ has a decomposition into characters of the $\mathcal{N}=4$ SCA given by

$$\mathcal{E}_{K3}(\tau, z) = 24\mathrm{ch}^{(2)}_{\frac{1}{4},0}(\tau,z) + \sum_{n=0}^{\infty} c_n \mathrm{ch}^{(2)}_{\frac{1}{4}+n,1}(\tau,z) \tag{4.26}$$

with

$$\sum_{n=0}^{\infty} c_n q^{n-1/8} = H^{(2)}(\tau)\,. \tag{4.27}$$

We thus have

$$\lim_{k\to\infty} \mathcal{E}^d_{\mathrm{DSLST}}(\tau,z) = k\left(\frac{\mathcal{E}_{K3}(\tau,z)}{24} + Z_{\mathrm{quasi}}(\tau,z)\right) \tag{4.28}$$

where

$$Z_{\mathrm{quasi}}(\tau,z) = \sum_{n=0}^{\infty} b_n \mathrm{ch}^{(2)}_{\frac{1}{4}+n,1}(\tau,z) \tag{4.29}$$

and

$$\sum_{n=0}^{\infty} b_n q^{n-1/8} = \frac{E_2(\tau)}{12\eta(\tau)^3} \tag{4.30}$$

This shows that the large k limit of the DSLST elliptic genus is not modular since E_2 is only quasi modular. It would be interesting to develop a physical interpretation of the above decomposition of the large k limit of the DSLST elliptic genus into a modular part, proportional to the elliptic genus of $K3$, and a quasi-modular part.

4.4 Vertex operators and null states

The elliptic genus of DSLST, $\mathcal{E}_{\mathrm{DSLST}}$, is independent of the position of the fivebranes (see section 5). Thus, if we make the radius of the circle in figure 1, R_0, large, the naive expectation is that the fivebranes do not interact with each other and the elliptic genus should be proportional to k.

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However we can see from (4.6) and (4.11) that the elliptic genus of DSLST exhibit a more complicated dependence on k,

$$\begin{aligned}\mathcal{E}^{d}_{\mathrm{DSLST}}(\tau,z) &= (k-1)\mathrm{ch}^{(2)}_{\frac{1}{4},0}(\tau,z) + 2(k-2)\mathrm{ch}^{(2)}_{\frac{1}{4}+1,1}(\tau,z) + \cdots \\ &= (k-1)\left[1 + \left(e^{4\pi iz} - 2e^{2\pi iz} + 2 - 2e^{-2\pi iz} + e^{-4\pi iz}\right) q + \mathcal{O}(q^2)\right], \\ &\quad + 2(k-2)\left[\left(-e^{2\pi iz} + 2 - e^{-2\pi iz}\right) q + \mathcal{O}(q^2)\right] + \mathcal{O}(q^2)\,. \end{aligned} \tag{4.31}$$

We will explain in section 5 that this result does not contradict the fact that the elliptic genus is independent of the positions of the fivebranes. Here we will try to identify the vertex operators that correspond to the first few terms in (4.31).

To find vertex operators contributing to the elliptic genus, the expression (3.65) in terms of $\mathcal{N}=2$ superconformal characters is very useful. The terms in (3.65) corresponding to the primary operators that contribute the $\mathcal{N}=4$ massless character $\mathrm{ch}^{(0)}_{\frac{1}{4},0}$ are

$$\mathrm{Ch}^{\mathrm{cig}}_{l+1,l}(\tau,z)\mathrm{Ch}^{\mathrm{min}}_{k-l-1,k+l}(\tau,z) = 1 + 2\left(2 - e^{2\pi iz} - e^{-2\pi iz}\right) q + \mathcal{O}(q^2) \tag{4.32}$$

where $l = 2, .., k-2$ and

$$\begin{aligned}\mathrm{Ch}^{\mathrm{cig}}_{2,1}(\tau,z)\mathrm{Ch}^{\mathrm{min}}_{k-2,k+1}(\tau,z) &= 1 + \left(3 - e^{2\pi iz} - 2e^{-2\pi iz}\right) q + \mathcal{O}(q^2)\,, \\ \mathrm{Ch}^{\mathrm{cig}}_{k,k-1}(\tau,z)\mathrm{Ch}^{\mathrm{min}}_{0,2k-1}(\tau,z) &= 1 + \left(3 - 2e^{2\pi iz} - e^{-2\pi iz}\right) q + \mathcal{O}(q^2)\,. \end{aligned} \tag{4.33}$$

From these expressions, we can identify the vertex operators of the lowest conformal weight $h - \frac{c}{24} = 0$ as

$$\mathcal{O}^{(0)}_{j,0} \equiv V^{\mathrm{susy}}_{j;j+1,j+1}\left(-\frac{1}{2},-\frac{1}{2}\right)\cdot \tilde{V}^{\mathrm{susy}}_{j;j,j}\left(+\frac{1}{2},+\frac{1}{2}\right)\,, \tag{4.34}$$

where $j = 0, \frac{1}{2}, .., \frac{k-2}{2}$. Note that the operators $\mathcal{O}^{(0)}_{j,0}$ are related to the translational modes of the fivebranes via spectral flow. This explains why the $\mathcal{N}=4$ massless character contributions are proportional to $(k-1)$ rather than k. It is due to the fact that, as will be discussed in details in section 5, we need to exclude a non-normalizable translational mode corresponding to the center of mass of the system.

The other vertex operators in (4.32) and (4.33) of higher conformal weights can be obtained by acting with $\mathcal{N}=2$ superconformal currents of cigar and minimal CFTs on $\mathcal{O}^{(0)}_{j,0}$. For instance, we can show from the OPEs in appendix A that there are $(2k-3)$ vertex operators of conformal weight $h - \frac{c}{24} = 1$ and $\mathrm{U}(1)_R$ charge $r = +1$

$$\left[\left(G^{+}_{\mathrm{cig}}\right)_{-1}, \mathcal{O}^{(0)}_{j,0}\right] \propto \mathcal{O}^{(0)}_{j,1} \equiv V^{\mathrm{susy}}_{j;j,j+1}\left(\frac{1}{2},-\frac{1}{2}\right)\cdot \tilde{V}^{\mathrm{susy}}_{j;j,j}\left(+\frac{1}{2},+\frac{1}{2}\right)\,, \tag{4.35}$$

where $j = 0, \frac{1}{2}, .., \frac{k-2}{2}$, and

$$\left[\left(G^{+}_{\mathrm{min}}\right)_{-1}, \mathcal{O}^{(0)}_{j,0}\right] \propto \tilde{\mathcal{O}}^{(0)}_{j,1} \equiv V^{\mathrm{susy}}_{j;j+1,j+1}\left(-\frac{1}{2},-\frac{1}{2}\right)\cdot \tilde{V}^{\mathrm{susy}}_{j;j-1,j}\left(+\frac{3}{2},+\frac{1}{2}\right)\,, \tag{4.36}$$

where $j = \frac{1}{2}, 1, .., \frac{k-2}{2}$. Note that superficially the number of states of the form (4.35), (4.36) should be proportional to $k-1$, like that of the states (4.34). However, this is not the case due to the presence of null states, in this case associated with the action of $\left(G^+_{\rm min}\right)_{-1}$ on $\mathcal{O}^{(0)}_{j=0,0}$.

The other terms in (3.65) relevant to find the vertex operators of conformal weight $h - \frac{c}{24} = 1$ and positive $\mathrm{U}(1)_R$ charge $r > 0$ are

$$\mathrm{Ch}^{\rm cig}_{l+1,l+2}(\tau,z)\mathrm{Ch}^{\rm min}_{k-l-1,k+l+2} = \left(e^{4\pi iz} - 2e^{2\pi iz} + 1\right) q + \mathcal{O}(q^2)\,, \tag{4.37}$$

where $l = 1, 2, .., k-2$, and

$$\mathrm{Ch}^{\rm cig}_{k,k+1}(\tau,z)\mathrm{Ch}^{\rm min}_{0,1}(\tau,z) = \left(e^{4\pi iz} - e^{2\pi iz}\right) q + \mathcal{O}(q^2)\,. \tag{4.38}$$

From these $\mathcal{N} = 2$ characters, it is straightforward to identify the $(k-1)$ vertex operators of conformal weight $h = 1 + \frac{1}{4}$ and $\mathrm{U}(1)_R$ charge $r = 2$,

$$\mathcal{O}^{(1)}_{j,2} \equiv V^{\rm susy}_{j;j+1,j+1}\left(+\frac{1}{2}, -\frac{1}{2}\right) \cdot \tilde{V}^{\rm susy}_{j;j,j}\left(+\frac{3}{2}, +\frac{1}{2}\right)\,, \tag{4.39}$$

where $j = 0, \frac{1}{2}, .., \frac{k-2}{2}$. These operators are in fact $\mathcal{N} = 4$ descendants in the massless characters

$$\left[\left(J^{++}_{\rm R}\right)_{-1}, \mathcal{O}^{(0)}_{j,0}\right] = \mathcal{O}^{(1)}_{j,2}\,, \tag{4.40}$$

which explains why their contributions are proportional to $(k-1)$. Here $J^{++}_{\rm R} = J^1_{\rm R} + iJ^2_{\rm R}$. On the other hands, the $(2k-3)$ vertex operators in (4.37) and (4.38) of conformal weight $h - \frac{c}{24} = 1$ and R-charge $r = 1$ can be obtained from acting with $G^-_{\rm cig}$ and $G^-_{\rm min}$ on $\mathcal{O}^{(2)}_{j,2}$,

$$\left[\left(G^-_{\rm cig}\right)_0, \mathcal{O}^{(1)}_{j,2}\right] \propto \mathcal{O}^{(1)}_{j,1} \equiv V^{\rm susy}_{j;j+2,j+1}\left(-\frac{1}{2}, -\frac{1}{2}\right) \cdot \tilde{V}^{\rm susy}_{j;j,j}\left(+\frac{3}{2}, +\frac{1}{2}\right)\,, \tag{4.41}$$

where $j = 0, \frac{1}{2}, .., \frac{k-2}{2}$, and

$$\left[\left(G^-_{\rm min}\right)_0, \mathcal{O}^{(1)}_{j,2}\right] \propto \tilde{\mathcal{O}}^{(1)}_{j,1} \equiv V^{\rm susy}_{j;j+1,j+1}\left(+\frac{1}{2}, -\frac{1}{2}\right) \cdot \tilde{V}^{\rm susy}_{j;j+1,j}\left(+\frac{1}{2}, +\frac{1}{2}\right)\,, \tag{4.42}$$

where $j = 0, 1, .., \frac{k-3}{2}$. In this case there is a null state associated with the action of $\left(G^-_{\rm min}\right)_0$ on $\mathcal{O}^{(0)}_{j=\frac{k-2}{2},0}$.

To summarize, we constructed $(4k-6)$ vertex operators of conformal weight $h - \frac{c}{24} = 1$ and R-charge $r = 1$ that contribute to the elliptic genus. Among them, one can show that certain linear combinations of $\mathcal{O}^{(0)}_{j,1}$ and $\tilde{\mathcal{O}}^{(0)}_{j,1}$, and those of $\mathcal{O}^{(1)}_{j,1}$ and $\tilde{\mathcal{O}}^{(1)}_{j,1}$ for $j = 0, \frac{1}{2}, .., \frac{k-2}{2}$ are in fact $\mathcal{N} = 4$ descendants of the ground state, and thus belong to the massless character. The remaining $2(k-2)$ linear combinations of these operators, orthogonal to the above $\mathcal{N} = 4$ descendants, belong to the massive character $\mathrm{ch}^{(2)}_{1+\frac{1}{4},0}(\tau,z)$.

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4.5 Density of states at large level

In preparation for a discussion of the black hole/string transition in section 6 we now turn to an estimate of the entropy of states contributing to the elliptic genus. The entropy formula can be read off from the asymptotic behavior of the level density for highly excited perturbative string BPS states. In other words, we would like to determine the large level N behavior of $D(N,z)$ defined by

$$D(N,z) = \oint \frac{dq}{2\pi i} \frac{\mathcal{E}^d_{\mathrm{DSLST}}(\tau,z)}{q^{N+1}}\,, \tag{4.43}$$

where a small circle around the origin is chosen as a contour.

To evaluate the above contour integral for large N, we first need to know how the discrete part of the elliptic genus $\mathcal{E}^d_{\mathrm{DSLST}}(\tau,z)$ behaves as $q \to 1^-$. It is straightforward to estimate crudely that $\mathcal{E}^d_{\mathrm{DSLST}}$ is asymptotic to

$$\mathcal{E}^d_{\mathrm{DSLST}} \sim \mathrm{Exp}\left[\frac{C(z)}{1-q}\right] \quad \text{as } q \to 1^- \tag{4.44}$$

with

$$C(z) = \left(\mathrm{Li}_2(1) - \mathrm{Li}_2(\xi) + \mathrm{Li}_2(\xi^{1/k}) - \mathrm{Li}_2(\xi^{1-1/k})\right) + \text{ c.c.}\,, \tag{4.45}$$

where $\xi = e^{2\pi i z}$ and $\mathrm{Li}_2(x) = \sum_{m=1} \frac{x^m}{m^2}$.

While we could estimate the asymptotic behavior at any value of z, there is cancellation between fermion and boson states at $z = 0$ while at $z = \frac{1}{2}$ the elliptic genus is essentially the partition function in the Ramond-Ramond sector with boson and fermion states contributing with equal signs. Since physically we are interested in the density of the total number of states we are most interested in the asymptotic behavior at $z = \frac{1}{2}$.

Mathematically we note that by using the identity

$$\mathrm{Li}_2(e^{2\pi i x}) + \mathrm{Li}_2(e^{-2\pi i x}) = 2\pi^2\left(x^2 - x + \frac{1}{6}\right)\,, \tag{4.46}$$

we can show that $C(z)$ has a maximum at $z = \frac{1}{2}$ and

$$C\left(z = \frac{1}{2}\right) = 2\pi^2\left(\frac{1}{2} - \frac{1}{2k}\right)\,. \tag{4.47}$$

We now continue with the saddle point approximation to evaluate the above contour integral (4.43) at $z = \frac{1}{2}$. One finds the saddle point for q near 1. Indeed the integrand at $q = 1 - \epsilon$

$$\mathrm{Exp}\left[\frac{C(\frac{1}{2})}{1-q} - (N+1)\log q\right] \simeq \mathrm{Exp}\left[\frac{C(\frac{1}{2})}{\epsilon} + (N+1)\epsilon\right]\,, \tag{4.48}$$

becomes stationary at

$$\epsilon \simeq \sqrt{\frac{C(\frac{1}{2})}{N+1}} \text{ as } N \to \infty\,. \tag{4.49}$$

Therefore, the leading behavior of the degeneracy at high level N is

$$D\left(N,\frac{1}{2}\right)\simeq e^{2\pi\sqrt{(1-\frac{1}{k})N}}\,. \tag{4.50}$$

After including the additional contribution from $\mathbb{R}^{1,4}\times S^1$, we can determine the entropy of Dabholkar-Harvey states at high energy in DSLST as

$$S_{\text{string}}=2\pi\sqrt{\left(1+1-\frac{1}{k}\right)N}\,. \tag{4.51}$$

This agrees with a naive application of the Cardy formula for a theory with $c_{\text{eff}}=6(2-\frac{1}{k})$, as in [5].

5 (In)dependence of moduli

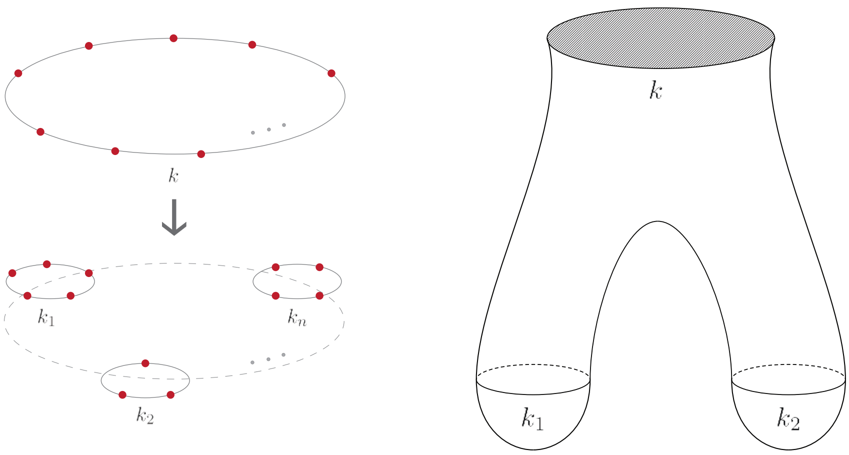
In our discussion above we focused on the elliptic genus of Little String Theory at a particular point in its moduli space at which the fivebranes are placed equidistantly on a circle in the transverse $\mathbb{R}^4$. More general points in the moduli space correspond to other distributions of fivebranes in $\mathbb{R}^4$ and it is natural to ask how the answer depends on the positions of the fivebranes.

Superficially, we expect the elliptic genus to be independent of the moduli since, as we explained before, it encodes the number of spacetime 1/4 BPS states with particular momentum and winding (P,W) on a longitudinal S^1. The mass of these states (1.2) is independent of the position moduli, and their degeneracy is an integer that cannot depend on continuous parameters such as positions of fivebranes.[6]

This leaves the possibility of jumps in the number of such states at some specific values of the moduli, that is a wall-crossing phenomenon that is known to occur for some BPS states in field and string theory. In particular, our analysis above is directly applicable when the string coupling of DSLST in small, i.e. when the mass of a $D1$-brane stretched between any two NS5-branes is much larger than m_s. As mentioned above, in that regime the states in question are perturbative string states, quite analogous to the perturbative BPS states studied in [7].

One might wonder whether there are possible jumps in the spectrum when the DSLST coupling is of order one, and the perturbative analysis may receive order one corrections. We do not expect such jumps when the fivebranes are separated. In general, the jumps are due to the fact that the supersymmetric central charges carried by the BPS states depend on the moduli; they occur when the central charge vectors of different BPS states align. In our case, the charges carried by the states in question are independent of the position moduli. As we will see below, the spectrum of BPS states however does exhibit jumps at points in moduli space where fivebranes collide. At such points, the spectrum of BPS states goes from that of strings to that of black holes.

[6]A related point is that one can think of the fivebrane background (3.2) as a non-compact K3, and it is well known that for compact K3's the elliptic genus is independent of the moduli.



(a) (b)

Figure 2. (a) k fivebranes on a single circle are separated into groups of $(k_1, k_2, .., k_n)$ fivebranes on n circles, (b) a single throat of k fivebranes is divided into smaller throats of k_1 and k_2 fivebranes where $k = k_1 + k_2$.

In this section we will stay in the realm of weakly coupled DSLST, where the spectrum of BPS states is expected to be independent of the positions of the fivebranes. This independence may seem surprising from the spacetime point of view. Consider, for example, a deformation that takes the original configuration of k fivebranes on a circle to one where they are separated into groups of $(k_1, k_2, \cdots, k_n)$ fivebranes that live on n well separated circles, depicted in figure 2 (a). The discrete part of the elliptic genus receives contributions from normalizable states that live in the fivebrane geometry. Let us denote the number of such states with given (P, W) in the single circle configuration of figure 1 by $F_{P,W}(k)$. If we separate the fivebranes into n circles, as in figure 2 (a), and assume that the states that contribute to the elliptic genus are localized in the vicinity of the individual circles, then the number of states in the second configuration should be $\sum_{i=1}^{n} F_{P,W}(k_i)$. Since the degeneracy of states with given (P, W) must be independent of the positions of the fivebranes, we conclude that

$$F_{P,W}(k) = \sum_{i=1}^{n} F_{P,W}(k_i) \tag{5.1}$$

for all k_i satisfying $\sum_{i=1}^{n} k_i = k$. However, we saw in previous sections that the degeneracies computed from the elliptic genus do not actually satisfy this relation. For instance, see equation (4.31). In this section we will discuss the origin of this discrepancy.

While the states we are interested in are 1/4 BPS, it is useful first to recall the situation with 1/2 BPS states. These are the modes that correspond to the positions of NS5-branes in $\mathbb{R}^4$, and their partners under spacetime supersymmetry. The translational modes can be viewed as deformations of the harmonic function

$$H = l_s^2 \sum_{j=1}^{k} \frac{1}{|\vec{x} - \vec{x}_j|^2}\,, \tag{5.2}$$

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which determines the metric, dilaton and NS two-form B-field of fivebranes located at $\vec{x} = \vec{x}_j$, $j = 1, 2, \cdots, k$,

$$\begin{aligned} ds^2 &= dx_\mu dx^\mu + H(\vec{x}) d\vec{x} \cdot d\vec{x}\,, \\ e^{2(\Phi-\Phi_0)} &= H(\vec{x})\,, \\ H_{mnp} &= -\epsilon^q_{mnp} \partial_q \Phi\,. \end{aligned} \tag{5.3}$$

Thus, they can be thought of as gravitons with wave functions obtained by replacing $\vec{x}_j \to \vec{x}_j + \vec{\delta}_j$ in (5.2), and expanding in $\vec{\delta}_j$.

The term in $H(\vec{x})$ that goes like δ^s (or, more precisely, $\delta_{j_1}\delta_{j_2}\cdots\delta_{j_s}$ with the vector indices on δ_j suppressed), behaves at large $|\vec{x}|$ like $1/|\vec{x}|^{s+2}$, i.e. like $1/|\vec{x}|^s$ relative to the leading $1/|\vec{x}|^2$ term. Expanding the gravitational action, taking into account the factor of $\exp(-2\Phi)$ in front of the Einstein term, we see that the behavior of the norm at large $|\vec{x}|$ is given by

$$\int \frac{d|\vec{x}|}{|\vec{x}|} |\vec{x}|^{2-2s}\,. \tag{5.4}$$

Therefore, the $s = 1$ perturbation is non-normalizable. Looking back at (5.2), we see that this perturbation corresponds to displacing the center of mass of the fivebrane system in $\mathbb{R}^4$. The fact that it is non-normalizable in the near horizon geometry of the fivebranes was found in the original work [14], who showed that the wavefunction of this mode is centered in the transition region between the near-horizon geometry and the asymptotically flat space far from the branes. In terms of LST, this implies that the low energy theory on k NS5-branes in type IIB string theory has gauge group SU(k) rather than U(k).

On the other hand, the modes with $s > 1$ are normalizable. Their wavefunctions can be obtained by performing the expansion described above. We will not need the details of this expansion, except for the fact that the corresponding wavefunctions are centered in the region near the fivebranes. Far from the fivebranes, the wavefunctions decay exponentially in the natural variable $\ln|\vec{x}|$.

Using this picture, we can now revisit the question of the (in)dependence of the spectrum of BPS states on the positions of the fivebranes. The number of translational modes of the fivebranes and their superpartners is clearly independent of the positions of the fivebranes. With the center of mass excluded, it is given by $4(k-1)$. Superficially, this is inconsistent with the discussion around the equation (5.1), but now we can resolve the discrepancy.

Consider the multi-circle configuration of fivebranes depicted in figure 2 (a). Following the analysis above, we know that each cluster of k_i fivebranes gives rise to $4(k_i - 1)$ translational modes, with the center of mass degrees of freedom excluded. This gives rise to $4\sum_i (k_i - 1) = 4(k - n)$ modes, that are localized near the respective circles. The missing $4(n-1)$ modes are easy to identify — they correspond to modes that preserve the center of mass of the whole fivebrane configuration, but not the centers of mass of the separate groups of k_i fivebranes. As in [14], their wave functions are not localized near the individual circles in figure 2 (a). Such a configuration can be thought of as a single throat of k

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fivebranes at large $|\vec{x}|$ that splits into smaller throats of the individual groups of fivebranes as $|\vec{x}|$ decreases, as depicted in figure 2 (b). The wave functions of the $4(n-1)$ missing multiplets are supported in the transition regions between the large throat and the smaller ones, and lead to a violation of the logic that led to (5.1).

So far we discussed the behavior of the 1/2 BPS states, which can be identified as Ramond-Ramond ground states leading to the constant contribution to the elliptic genus given by (4.31). In particular, we explained why their contribution $(k-1)$ is independent of the positions of the fivebranes, and yet is not proportional to k. A key point was that the wave functions of these states do not satisfy decoupling, i.e., if we split the fivebranes into arbitrarily well separated groups, these groups remain entangled via their centers of mass.

As discussed in previous sections, the full elliptic genus can be decomposed into the contributions of different representations of the $\mathcal{N}=4$ superconformal algebra, which is present everywhere in the moduli space of the multi-fivebrane CFT. The massless $\mathcal{N}=4$ character contribution in (4.31) corresponds to states that can be thought of as the 1/2 BPS states discussed above acted on by left-moving $\mathcal{N}=4$ superconformal generators. Thus, the wave functions of these states are the same as those of the 1/2 BPS states, and our discussion of the latter applies directly to them.

Massive $\mathcal{N}=4$ character contributions are in general more complicated, as can be seen from (4.31). At the special point in the moduli space described by figure 1, we demonstrated in section 4.4 that these states can be obtained by acting on the 1/2 BPS states with left-moving $\mathcal{N}=2$ generators of the $SL(2,\mathbb{R})/U(1)$ and $SU(2)/U(1)$ factors. We also found that null states sometimes can be generated by acting on the Ramond-Ramond ground states with such $\mathcal{N}=2$ superconformal generators, which explains why their contribution is not even proportional to $(k-1)$.

However, at generic points in moduli space, the chiral algebra of the model is just the $\mathcal{N}=4$ superconformal algebra, and such a description is not available. Nevertheless, we expect the wave functions of these states to have the same qualitative structure as that of states in the massless $\mathcal{N}=4$ characters, for the following reason. All states contributing to the elliptic genus are Ramond ground states in the right-moving worldsheet sector. At large k, we can think of them as zero modes of the Dirac equation in the fivebrane background. Squaring this equation gives the massless Klein-Gordon equation, whose solution is the harmonic function (5.2). Thus, the properties of these states as we change the moduli should be the same as those in the massless $\mathcal{N}=4$ representations.

6 Black holes versus perturbative string states

In the previous section we saw that the contribution of normalizable LST states to the elliptic genus is independent of the positions of the fivebranes in $\mathbb{R}^4$. In weakly coupled DSLST these states can be thought of as perturbative string states living in the fivebrane background, but the corresponding spectrum can be extended to regions in moduli space where the DSLST coupling is of order one. As mentioned in the previous section, this picture is expected to be valid for separated fivebranes, but it receives important modifications when fivebranes are allowed to coincide.

Consider, for example, the configuration of fivebranes on a circle of radius R_0 in the transverse space $\mathbb{R}^4$ depicted in figure 1. For $R_0 > 0$ we expect the analysis of the previous sections to be valid. However, for $R_0 = 0$ there is another competing contribution to the elliptic genus from a black hole with the same quantum numbers as the perturbative string states described above. To construct this black hole, we start with the coincident fivebrane background [14], $\mathbb{R}^{4,1} \times S^1 \times \mathbb{R}_\phi \times \mathrm{SU}(2)_k$, and look for solutions that carry the two charges P (momentum) and W (winding) along the S^1 of radius R that the fivebranes wrap. In string theory, it is convenient to label these charges in terms of left and right moving momenta,

$$(P_L, P_R) = \left(\frac{P}{R} - \frac{WR}{\alpha'}, \frac{P}{R} + \frac{WR}{\alpha'}\right) . \tag{6.1}$$

For $P_L = P_R = 0$, the black hole solution takes the form $\mathrm{SL}(2,\mathbb{R})_k/\mathrm{U}(1) \times \mathbb{R}^4 \times S^1 \times \mathrm{SU}(2)_k$. It describes the background of k non-extremal fivebranes, with the value of the dilaton at the horizon labeling the energy density above extremality. For general (P_L, P_R), one can find the black hole solution by reduction of the three-dimensional rotating, charged black string background obtained from the uncharged black hole solution by a sequence of boosts and T dualities. Algebraically, this leads to a CFT in which the $\mathrm{SL}(2,\mathbb{R})_k/\mathrm{U}(1) \times S^1$ factor above is replaced by $\frac{\mathrm{SL}(2,\mathbb{R})_k \times \mathrm{U}(1)}{\mathrm{U}(1)}$, where the embedding of the gauged $\mathrm{U}(1)$ into $\mathrm{U}(1) \times \mathrm{U}(1) \subset \mathrm{SL}(2,\mathbb{R}) \times \mathrm{U}(1)$ is determined by the charges (P_L, P_R). For a review of this construction, as well as for the precise sigma-model background fields in the general two-charge case, see e.g. [5, 45].

A tractable special case, which has all the essential ingredients is $P_L = 0$. The corresponding charged black hole has metric, dilaton and gauge field,

$$\begin{aligned} ds^2 &= -f dt^2 + \frac{k\alpha'}{4}\frac{dr^2}{r^2 f} , \\ \Phi &= -\frac{1}{2}\ln\left(\sqrt{\frac{k}{\alpha'}}\, r\right) , \\ A_t &= \frac{\alpha'}{2r} P_R , \end{aligned} \tag{6.2}$$

where the function $f(r)$ is

$$f = \left(1 - \frac{r_-}{r}\right)\left(1 - \frac{r_+}{r}\right) , \tag{6.3}$$

and the inner and outer horizons of the black hole are at

$$r_\pm = \frac{\alpha'}{2}\left(M_{\mathrm{BH}} \pm \sqrt{M_{\mathrm{BH}}^2 - P_R^2}\right) . \tag{6.4}$$

The entropy of this black hole, and its generalization to generic $(P_L, P_R, M_{\mathrm{BH}})$ is given by [5]

$$S_{\mathrm{BH}} = \pi l_s \sqrt{k}\left(\sqrt{M_{\mathrm{BH}}^2 - P_L^2} + \sqrt{M_{\mathrm{BH}}^2 - P_R^2}\right) . \tag{6.5}$$

For the extremal case $M_{\mathrm{BH}} = |P_R|$, this takes the form

$$S_{\mathrm{BH}} = 2\pi\sqrt{kPW} , \tag{6.6}$$

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familiar from studies of three-charge black hole. The above entropy formula (6.6) can be derived microscopically by computing the high energy density of states of the CFT on the system of coincident NS5-branes and fundamental strings [1]. Looking back at the analogous expression for perturbative strings (4.51) of the same charges (P_L, P_R),

$$S_{\text{string}} = 2\pi\sqrt{\left(2-\frac{1}{k}\right)PW}\,, \tag{6.7}$$

we see that the two are qualitatively similar, but the factor $(2-\frac{1}{k})$ in the fundamental string entropy is replaced by k for black holes. Thus, the black hole entropy is always larger for $k \geq 2$.

We conclude that as the fivebranes approach each other, the number of 1/4 BPS LST states jumps from (6.7) to (6.6). At first sight this is rather surprising — the positions of the fivebranes are moduli in the theory, and can be thought of as Higgsing the $\mathrm{SU}(k)$ gauge group to $\mathrm{U}(1)^{k-1}$. When the fivebranes are nearly coincident, the symmetry breaking scale, namely the mass of W-bosons corresponding to D-strings stretched between NS5-branes, becomes very low. We would not expect it to influence the physics of very massive states, such as the BPS states contributing to the elliptic genus. Indeed, in a local QFT such a phenomenon could not occur. However, LST is not a local QFT, and the states we are interested in can probe the non-locality; e.g., T-duality, which is often cited as evidence for non-locality of LST, acts non-trivially on them. We therefore believe that the jump in the spectrum of 1/4 BPS states is an example of UV/IR mixing in LST. The Higgs scale (IR) influences the spectrum of very massive BPS states (UV).

Another element of the above discussion that we need to address concerns the (non) compactness of the worldvolume of the fivebranes. Above, we took it to be $\mathbb{R}^{4,1}\times S^1$, but this leads to the following issue. We see from (6.2) that the two dimensional string coupling is determined by the mass of the (extremal) black hole,

$$e^{-2\Phi(r_\pm)} = \sqrt{\frac{k}{2}}M_{\text{BH}}\,, \tag{6.8}$$

where we set $\alpha' = 2$ for simplicity. If the four dimensional space along the fivebranes, $\mathcal{M}_4$, is non-compact, the six dimensional string coupling in the directions along the fivebranes is infinite. Hence, the coset description cannot be studied at small string coupling.

To avoid these singularities it is convenient to compactify the worldvolume $\mathbb{R}^4$ to $\mathbb{T}^4$. However, this raises another issue that needs to be addressed. When the fivebrane worldvolume is taken to be $\mathbb{T}^4\times S^1$, the LST in question lives in $1+0$ dimensions. Thus, the moduli corresponding to positions of fivebranes cannot be taken as fixed, but are rather fluctuating quantum mechanical degrees of freedom. The states of the theory are characterized by wave functions on the classical moduli space. This leads to the question what is the correct interpretation of our results above in these low dimensional vacua of LST.

Our view on this is that compactified LST has a discrete set of vacua labeled by the number of coincident fivebranes, which ranges from 0 to k (or, more generally, the numbers of coincident fivebranes $(k_1, \cdots, k_n)$ with $\sum_i k_i = k$). The vacuum with no coincident fivebranes has an elliptic genus that was computed in previous sections. It can be thought of as due to perturbative string states in the separated fivebrane background. The elliptic genus of the vacuum with k coincident fivebranes, defined formally as the object counting spacetime 1/4 BPS states, is dominated by the contribution of the black hole described in this section. For intermediate numbers of coincident fivebranes we have a combination of the two effects.

Note that the above picture is reminiscent of, but not identical to, the one discussed in [46]. There, the fivebranes were always coincident and the strings were part of the background. The issue was what is the $1+1$ dimensional low energy theory on the system of strings and fivebranes, and it was argued that it splits into Coulomb and Higgs branch CFT's with different central charges. The Coulomb branch corresponds to strings propagating in the vicinity of, but outside the fivebranes. The Higgs branch describes the theory of strings dissolved in the fivebranes as self-dual Yang-Mills instantons. An important role in their separation is played by the fivebrane throat of [14] seen by strings propagating in the fivebrane background.

In our case, the only branes in the background are the fivebranes. We are interested in the full theory rather than just its low energy limit, and the different branches of the theory correspond to different numbers of coincident fivebranes. However, the difference in the entropy of BPS states between (6.7) and (6.6) is the same as in [46]. In the Higgs branch this can be read off the Cardy formula with central charge $c = 6k$. In the Coulomb branch one has $c = 6$ but the object that governs the high energy density of states is $c_{\text{eff}} = 6(2 - \frac{1}{k})$. The fivebrane throat plays an important role in our discussion as well since in a sector of the Hilbert space of LST with $W \neq 0$, we effectively have strings propagating in the vicinity of the fivebranes, as in [46].

Another closely related phenomenon is the string-black hole transition discussed in [5]. There, it was shown that the high energy spectrum of string theory in asymptotically linear dilaton vacua (i.e. vacua of LST) is dominated for $k > 1$ by black holes, while for $k < 1$ the black holes are non-normalizable, and the spectrum is that of perturbative strings. The dependence of the entropy on the slope of the linear dilaton, Q, that can be parametrized by the number of fivebranes k in this paper via the relation $Q = \sqrt{\alpha'/k}$ for strings and black holes is precisely the same as in our analysis above. However, unlike in [5], we work at a fixed $k > 1$, and the transition between strings and black holes in our case is between different branches of the theory of k fivebranes. It would be interesting to understand the relation between the two phenomena better.

7 Non-extremal case

In the previous sections we saw that LST on $\mathbb{T}^4 \times S^1$ exhibits a non-trivial vacuum structure. Classically, the number of 1/4 BPS states carrying the charges (6.1) jumps when fivebranes coincide. Quantum mechanically, the theory splits into distinct sectors labeled by the numbers of coincident fivebranes, each with its own spectrum of BPS states.

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In this section we would like to briefly comment on the physics of near-BPS states in this theory. Consider a point in the moduli space of LST at which the fivebranes are separated, such that the string coupling is everywhere small. In that case, we can compute the entropy of near-BPS states using perturbative string techniques. It is given by

$$S_{\text{string}} = \pi l_s \sqrt{2 - \frac{1}{k}} \left(\sqrt{M_{\text{BH}}^2 - P_L^2} + \sqrt{M_{\text{BH}}^2 - P_R^2} \right). \tag{7.1}$$

In the BPS case, $M_{\text{BH}} = |P_R|$, this reduces to (6.7).

The energy above the BPS bound effectively makes the fivebranes non-extremal. Thus, their gravitational attraction exceeds the repulsion due to their $B_{\mu\nu}$ charge, and the moduli associated with their positions develop an attractive potential. Thus, the problem becomes time-dependent. However, if the system is near-BPS, the timescale associated with the motion of the fivebranes towards each other is large, and one can treat the problem in the adiabatic approximation.

In particular, there is a long time period in which the thermodynamics of the system is that of fundamental strings, and the entropy is given by (7.1). Eventually, as the fivebranes get closer, the effective string coupling becomes of order one. In this regime, the time evolution becomes rapid and the adiabatic approximation breaks down.

When $t \to \infty$, the fivebranes approach each other, and the effective string coupling in their vicinity becomes large. In this limit, one can again do thermodynamics, but this time it is governed by the black hole solutions discussed in the previous section. The corresponding entropy is given by (6.5), which has the same form as (7.1), with $2 - \frac{1}{k} \to k$. As expected, the entropy increases with time.

The discussion of the non-extremal case above is useful for understanding the jump in the BPS entropy discussed in the previous sections. In the non-extremal problem the behavior of the entropy is a smooth function of the two parameters in the problem: time, t, and the energy above extremality, ϵ. We start with the system of separated fivebranes ($M_W \neq 0$), and take the limit $\epsilon \to 0$ (the BPS limit) and $t \to \infty$. This limit can be taken in two ways that give different answers. If we first take $\epsilon \to 0$, and then $t \to \infty$, we get the fundamental string entropy (7.1). On the other hand, if we take $t \to \infty$ first and then $\epsilon \to 0$, we get the black hole answer (6.5). Slightly away from extremity, the time dependence interpolates smoothly between the two.

8 Discussion

In this paper we saw that string theory in a background with k NS5-branes wrapping $\mathbb{T}^4 \times S^1$ has non-trivial vacuum structure. We studied the spectrum of 1/4 BPS states in the different vacua and saw that when the fivebranes are coincident, they can be described as black holes carrying the relevant charges, while along the Coulomb branch they correspond to pertubrative string states in the separated fivebrane geometry.

We computed the degeneracies of the two kinds of states, and found that they do not agree. In particular, at the origin we found the entropy (6.6), while along the Coulomb branch it was (6.7). We interpreted this discrepancy as due to a non-trivial vacuum structure of the fivebrane theory.

We pointed out that this phenomenon is counterintuitive, since the origin is a finite distance away from points along the Coulomb branch, and the metric on the Coulomb branch does not receive quantum corrections. In terms of the theory of the fivebranes, known as Little String Theory, it is possibly due to the non-locality of the theory and it implies that it exhibits UV/IR mixing.[7]

Our analysis is based on the elliptic genus of DSLST at the special point in the moduli space that allows the weakly-coupled solvable CFT description (2.21). Using various properties of the elliptic genus discussed in section 4, we argued in section 5 that the elliptic genus is independent of the positions of the NS5-branes. This result is consistent with the fact that, when the LST is defined on $\mathbb{T}^4 \times S^1$, the notion of classical moduli space is not well defined and the ground state of the theory can be characterized by a wavefunction on the position moduli space. Finally, we obtained the degeneracy of such ground states (4.51) from the asymptotic behavior of the elliptic genus at large level.

Our results have implications to other issues. One is the program to describe microstates of supersymmetric black holes in terms of horizonless geometries [3, 4]. The quarter BPS black holes that figured in our analysis are nothing but the three-charge black holes whose microstates are discussed in that program. Usually, these black holes are studied in the full, asymptotically flat, spacetime of string theory. However, as discussed in the present work, one can also study them in the near-horizon geometry of the fivebranes, which is an asymptotically linear dilaton spacetime.

The main idea of the microstate program is to find geometries that look asymptotically far from the horizon like the corresponding black hole, but that deviate from it near the location of the would-be horizon, and in particular do not have a horizon themselves. The hope is that the entropy of these horizonless geometries agrees with the Bekenstein entropy of the black hole.

Our results point to a subtlety with this program. We saw that when the fivebranes are separated, even by a small distance, the BPS states can be thought of as standard fundamental string states in the smooth background of the fivebranes. One can describe these states by vertex operators in the fivebrane background, but one can also write the supergravity fields around the strings that carry momentum and winding. In flat spacetime this was done in [47, 48], and a similar construction should work in the fivebrane background.

The supergravity fields around these fundamental strings are presumably essentially the same as those describing the black hole solution with the same charges, at least at large distance from the horizon. Thus, one might be tempted to think of them as microstates of the black hole. However, the picture we were led to in this paper is different. The horizonless geometries corresponding to the fundamental string states in the separated fivebrane background and the black hole are different objects. In fact, they live in different vacua of the fivebrane theory, and their entropies are not the same. Thus, our results suggest that a horizonless geometry that approximates well the black hole geometry outside of the would be horizon can not necessarily be thought of as a microstate of the black hole.

[7]Other manifestations of UV/IR mixing in LST have been studied in [29, 30].

Our point of view is compatible with that of [49], where it was argued that horizonless geometries and black holes with the same quantum numbers correspond to different states. In that case the different descriptions were valid in different duality frames, i.e. different regions in coupling space, whereas for us the black holes and fundamental strings describe the BPS states in different vacua of the same theory. Our picture also seems to be related to that of [50], although the precise relation remains to be understood.

Our discussion in section 7 is reminiscent of other phenomena in string theory. For example, the authors of [51] discussed the transition from fundamental strings to black holes that happens as a function of the string coupling. In particular, they argued that if one considers a typical highly excited fundamental string state, and continuously raises the string coupling, at some point the Schwarzschild radius of a black hole with the same mass and charges as the fundamental string exceeds the string scale, and the fundamental string description gives way to a black hole one.

Something similar happens dynamically in our system. If we start with non-extremal fivebranes in the region where the effective LST string coupling is small, the entropy is dominated by fundamental string states. As time goes by, the fivebranes approach each other, the effective string coupling grows, and at late time the system is better described as a black hole. Thus, our system can be used to study the transition of [51] in a controlled setting.

Another related problem is that discussed in [52]. These authors studied the thermodynamics of weakly coupled string theory in asymptotically flat spacetime, and emphasized that due to the Jeans instability, thermodynamics does not really make sense in this system. Rather, at any finite density the system will develop time dependence. However, if the time variation is sufficiently slow, one can still study weakly coupled string thermodynamics, and the resulting description is valid for a long time.

The bulk description of our system (in terms of an asymptotically linear dilaton spacetime) is similar to that of [52]. Away from extremality, the system is time dependent, but if the fivebranes are sufficiently well separated and the non-extremality is sufficiently small, the time evolution is slow. Thus the fundamental string picture is valid for a long time, but it eventually breaks down when the fivebranes get close and the system makes a transition to a black hole phase. In our case we have an alternative description of the dynamics in terms of fivebrane physics (due to LST holography), and one can use it to shed additional light on the discussion of [52].

The discussion of this paper can be generalized in a number of directions. We studied the vacuum structure of six dimensional LST, but one could generalize the analysis to other dimensional vacua of LST, e.g. those studied in [53]. There are reasons to believe that the study of such vacua involves new qualitative and quantitative phenomena.

In our analysis, the degeneracy of BPS states on the Coulomb branch was obtained by studying string propagation in the fivebrane background. It is natural to ask whether the results could alternatively be obtained from the holographically dual point of view. (IIB) LST reduces in the IR to six dimensional $\mathcal{N} = (1,1)$ supersymmetric Yang-Mills theory, and it would be interesting to see how much of the structure we found can be understood in that theory, e.g. along the lines of [54].

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A Review on coset CFTs

A.1 Cigar CFT

The supersymmetric $\mathrm{SL}(2)_k$ WZW model can be described by SL(2) currents J^i and three free fermions ψ^i $(i = 1, 2, 3)$ satisfying the OPEs below

$$\begin{aligned} J^i(z)J^j(0) &\sim \frac{\frac{k}{2}\eta^{ij}}{z^2} + i\epsilon^{ijk}\frac{J_k(0)}{z} \\ J^i(z)\psi^j(0) &\sim i\epsilon^{ijk}\frac{\psi_k(0)}{z} \\ \psi^i(z)\psi^j(0) &\sim \frac{\eta^{ij}}{z}\,, \end{aligned} \tag{A.1}$$

where $\eta^{ij} = \mathrm{diag}(+1, +1, -1)$. Let us define a new $\mathrm{SL}(2,\mathbb{R})$ current j^i,

$$j^i = J^i + \frac{i}{2}\epsilon^{ijk}\psi_j\psi_k\,. \tag{A.2}$$

One can then show that the currents j^i commute with three fermions ψ^i and generate a bosonic SL(2) WZW model at level $k + 2$. Let us define $\psi^{\pm} = \frac{1}{\sqrt{2}}\left(\psi^1 + i\psi^2\right)$ for later convenience.

The supersymmetric SL(2)/U(1) coset model can be obtained by gauging the U(1) $\mathcal{N} = 1$ supermultiplet that contains the primary λ^3 and $J^3 = \{G_{-1/2}, \lambda_3\}$. Then the coset has an *enhanced* $\mathcal{N} = 2$ algebra generated by

$$\begin{aligned} G^{\pm}_{\mathrm{cig}} &= \sqrt{\frac{2}{k}}j^{\mp}\psi^{\pm} \\ J^{\mathrm{sl}}_R &= \frac{k+2}{k}\psi^+\psi^- + \frac{2}{k}j^3 = \lambda^+\lambda^- + \frac{2}{k}J^3\,. \end{aligned} \tag{A.3}$$

We denote by x, H, X, and X_R the bosonizations of various currents j^3, $\psi^+\psi^-$, J^3 and J_R,

$$j^3 = -\sqrt{\frac{k+2}{2}}\partial x\,,$$

$$\begin{aligned}
\psi^+\psi^- &= +i\partial H\,,\\
J^3 &= -\sqrt{\frac{k}{2}}\partial X\,,\\
J_R &= +i\sqrt{\frac{k+2}{k}}\partial X_R\,.
\end{aligned} \tag{A.4}$$

Note that two U(1) currents J^3 and J^R commute. From (A.2) and (A.3), one can show that

$$\begin{aligned}
x &= \sqrt{\frac{k+2}{k}}X + i\sqrt{\frac{2}{k}}X_R\,,\\
iH &= \sqrt{\frac{2}{k}}X + i\sqrt{\frac{k+2}{k}}X_R\,.
\end{aligned} \tag{A.5}$$

Using the non-compact parafermion fields $\pi, \pi^\dagger$, the ladder operators $j^\pm$ then can be expressed as follows

$$\begin{aligned}
j^+ &= \sqrt{k+2}\cdot\pi(z)\cdot e^{-\sqrt{\frac{2}{k+2}}x(z)}\,,\\
j^- &= \sqrt{k+2}\cdot\pi^\dagger(z)\cdot e^{+\sqrt{\frac{2}{k+2}}x(z)}\,.
\end{aligned} \tag{A.6}$$

The two supercurrent $G^\pm_{\text{cig}}$ can be then expressed as

$$\begin{aligned}
G^+_{\text{cig}} &= \sqrt{\frac{2(k+2)}{k}}\cdot\pi^\dagger(z)\cdot e^{i\sqrt{\frac{k}{k+2}}X_R(z)}\,,\\
G^-_{\text{cig}} &= \sqrt{\frac{2(k+2)}{k}}\cdot\pi(z)\cdot e^{-i\sqrt{\frac{k}{k+2}}X_R(z)}\,,
\end{aligned} \tag{A.7}$$

Vertex Operators. Let us discuss the primaries of the coset model. We start with the SL(2) vertex operator $\Phi^{\text{sl}}_{j;m,\bar m}$ of conformal weight $-\frac{j(j+1)}{k}$. One can define the SL(2)/U(1) vertex operator $V^{\text{susy}}_{j;m,\bar m}(\alpha,\bar\alpha)$ by removing the $\mathrm{U}(1)_{J^3}$ part of the operator $\Phi^{\text{sl}}_{j;m,\bar m}$,

$$e^{i\alpha H}e^{i\bar\alpha\bar H}\Phi^{\text{sl}}_{j;m,\bar m} \equiv e^{\sqrt{\frac{2}{k}}\left((m+\alpha)X+(\bar m+\bar\alpha)\bar X\right)}V^{\text{susy}}_{j;m,\bar m}(\alpha,\bar\alpha)\,. \tag{A.8}$$

The conformal weight and U(1) R-charge of $V^{\text{susy}}_{j;m,\bar m}(\alpha,\bar\alpha)$ are

$$\begin{aligned}
h &= \frac{(m+\alpha)^2 - j(j+1)}{k} + \frac{1}{2}\alpha^2\,,\\
\bar h &= \frac{(\bar m+\bar\alpha)^2 - j(j+1)}{k} + \frac{1}{2}\bar\alpha^2\,,
\end{aligned} \tag{A.9}$$

and

$$\begin{aligned}
r &= \frac{2(m+\alpha)}{k} + \alpha\,,\\
\bar r &= \frac{2(\bar m+\bar\alpha)}{k} + \bar\alpha\,.
\end{aligned} \tag{A.10}$$

From the definition of the primary operator $V^{\mathrm{B}}_{j;m,\bar{m}}$ for the bosonic SL(2)/U(1) at level $k+2$

$$\Phi^{\mathrm{sl}}_{j;m,\bar{m}} \equiv e^{\sqrt{\frac{2}{k+2}}(mx+\bar{m}\bar{x})} V^{\mathrm{B}}_{j;m,\bar{m}}\,, \tag{A.11}$$

we have

$$V^{\mathrm{susy}}_{j;m,\bar{m}}(\alpha,\bar{\alpha}) = V^{\mathrm{B}}_{j;m,\bar{m}} e^{i\frac{2}{\sqrt{k(k+2)}}(mX_R+\bar{m}\bar{X}_R)} e^{i\sqrt{\frac{k+2}{k}}(\alpha X_R+\bar{\alpha}\bar{X}_R)}\,, \tag{A.12}$$

where X_R denotes the bosonization of the U(1) R-current J^{sl}_R (A.4). From the well-known equivalence between the non-compact parafermionic primaries,

$$V^{\mathrm{B}}_{j;m,\bar{m}} = V^{\mathrm{B}}_{\frac{k-2}{2}-j;\frac{k+2}{2}+m,\frac{k+2}{2}+\bar{m}}\,, \tag{A.13}$$

we can verify an interesting property that $V^{\mathrm{susy}}_{j;m,\bar{m}}(\alpha,\bar{\alpha})$ should satisfy

$$V^{\mathrm{susy}}_{j;m,\bar{m}}(\alpha,\bar{\alpha}) = V^{\mathrm{susy}}_{\frac{k-2}{2}-j;\pm\frac{k+2}{2}+m,\pm\frac{k+2}{2}+\bar{m}}(\alpha\mp 1,\bar{\alpha}\mp 1)\,. \tag{A.14}$$

Useful OPEs. Finally, let us summarize several useful OPEs for primaries. The parafermionic primaries $V^{\mathrm{B}}_{j;m,\bar{m}}$ satisfy the following OPEs

$$\begin{aligned}
\pi(z)V^{B}_{j;m,\bar{m}}(0) &\sim \frac{m+(j+1)}{\sqrt{k+2}}\frac{1}{z^{1-\frac{2m}{k+2}}}V_{j;m+1,\bar{m}}(0)\,,\\
\pi^{\dagger}(z)V^{B}_{j;m,\bar{m}}(0) &\sim \frac{m-(j+1)}{\sqrt{k+2}}\frac{1}{z^{1+\frac{2m}{k+2}}}V_{j;m-1,\bar{m}}(0)\,.
\end{aligned} \tag{A.15}$$

Then, one can easily show that

$$\begin{aligned}
G^{+}_{\mathrm{cig}}(z)V^{\mathrm{susy}}_{j;m,\bar{m}}(\alpha,\bar{\alpha})(0) &\sim \frac{m-(j+1)}{z^{1-\alpha}}\sqrt{\frac{2}{k}}V^{\mathrm{susy}}_{j;m-1,\bar{m}}(\alpha+1,\bar{\alpha})+\cdots\,,\\
G^{-}_{\mathrm{cig}}(z)V^{\mathrm{susy}}_{j;m,\bar{m}}(\alpha,\bar{\alpha})(0) &\sim \frac{m+(j+1)}{z^{1+\alpha}}\sqrt{\frac{2}{k}}V^{\mathrm{susy}}_{j;m+1,\bar{m}}(\alpha-1,\bar{\alpha})+\cdots\,.
\end{aligned} \tag{A.16}$$

A.2 Minimal model

The supersymmetric $\mathrm{SU}(2)_k$ WZW model can be described by SU(2) currents $\tilde{J}^a$ and three free fermions ψ^a $(a=1,2,3)$ satisfying the OPEs below

$$\begin{aligned}
\tilde{J}^a(z)\tilde{J}^b(0) &\sim \frac{\frac{k}{2}\delta^{ab}}{z^2}+i\epsilon^{abc}\frac{\tilde{J}_c(0)}{z}\\
\tilde{J}^a(z)\psi^b(0) &\sim i\epsilon^{abc}\frac{\psi_c(0)}{z}\\
\psi^a(z)\psi^b(0) &\sim \frac{\delta^{ab}}{z}\,,
\end{aligned} \tag{A.17}$$

where $\delta^{ab} = \mathrm{diag}(+1,+1,+1)$. Let us define a new SU(2) current $\tilde{j}^a$,

$$\tilde{j}^a = \tilde{J}^a + \frac{i}{2}\epsilon^{abc}\psi_b\psi_c\,. \tag{A.18}$$

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One can then show that the currents $\tilde{j}^a$ commute with three fermions ψ^a and generate a bosonic SU(2) WZW model at level $k-2$.

The supersymmetric SU(2)/U(1) coset model can be obtained by gauging the U(1) $\mathcal{N}=1$ supermultiplet that contains the primary ψ^3 and $\tilde{J}^3 = \{G_{-1/2}, \psi_3\}$. Then the coset has an *enhanced* $\mathcal{N}=2$ algebra generated by

$$\begin{aligned} G^{\pm}_{\rm min} &= \sqrt{\frac{2}{k}}\tilde{j}^{\mp}\psi^{\pm} \\ J^{\rm su}_R &= \frac{k-2}{k}\psi^+\psi^- - \frac{2}{k}\tilde{j}^3 = \psi^+\psi^- - \frac{2}{k}\tilde{J}^3\,. \end{aligned} \tag{A.19}$$

For later convenience, let us denote by $\tilde{x}$, $\tilde{H}$, $\tilde{X}$, and $\tilde{X}_R$ the bosonizations of various currents $\tilde{j}^3$, $\psi^+\psi^-$, $\tilde{J}^3$ and $J^{\rm su}_R$,

$$\begin{aligned} \tilde{j}^3 &= i\sqrt{\frac{k-2}{2}}\partial\tilde{x}\,, \\ \psi^+\psi^- &= i\partial\tilde{H}\,, \\ \tilde{J}^3 &= i\sqrt{\frac{k}{2}}\partial\tilde{X}\,, \\ J^{\rm su}_R &= +i\sqrt{\frac{k-2}{k}}\partial\tilde{X}_R\,. \end{aligned} \tag{A.20}$$

Note that two U(1) currents $\tilde{J}^3$ and $J^{\rm su}_R$ commute. From (A.18) and (A.19), one can show that

$$\begin{aligned} \tilde{x} &= \sqrt{\frac{k-2}{k}}\tilde{X} - \sqrt{\frac{2}{k}}\tilde{X}_R\,, \\ \tilde{H} &= \sqrt{\frac{2}{k}}\tilde{X} + \sqrt{\frac{k-2}{k}}\tilde{X}_R\,. \end{aligned} \tag{A.21}$$

Using the parafermion fields $\tilde{\pi}, \tilde{\pi}^\dagger$, the ladder operators $\tilde{j}^{\pm}$ then can be expressed as follows

$$\begin{aligned} \tilde{j}^+(z) &= \sqrt{k-2}\cdot\tilde{\pi}(z)\cdot e^{+i\sqrt{\frac{2}{k-2}}\tilde{x}}(z)\,, \\ \tilde{j}^-(z) &= \sqrt{k-2}\cdot\tilde{\pi}^\dagger(z)\cdot e^{-i\sqrt{\frac{2}{k+2}}\tilde{x}}(z)\,. \end{aligned} \tag{A.22}$$

The two supercurrent $G^{\pm}_{\rm min}$ can be also written as

$$\begin{aligned} G^{+}_{\rm min} &= \sqrt{\frac{2(k-2)}{k}}\cdot\tilde{\pi}^\dagger(z)\cdot e^{+i\sqrt{\frac{k}{k-2}}\tilde{X}_R(z)}\,, \\ G^{-}_{\rm min} &= \sqrt{\frac{2(k-2)}{k}}\cdot\tilde{\pi}(z)\cdot e^{-i\sqrt{\frac{k}{k-2}}\tilde{X}_R(z)}\,, \end{aligned} \tag{A.23}$$

Vertex Operator. Let us then discuss the primaries of the supersymmetric SU(2)/U(1) coset model. We start with the $\mathrm{SU}(2)_{k-2}$ vertex operator $\Phi^{\rm su}_{\tilde{j};\tilde{m},\bar{\tilde{m}}}$ of conformal weight $\frac{\tilde{j}(\tilde{j}+1)}{k}$. One can obtain the SU(2)/U(1) vertex operator $\tilde{V}^{\rm susy}_{\tilde{j};\tilde{m},\bar{\tilde{m}}}(\beta,\bar{\beta})$ by removing the $\mathrm{U}(1)_{\tilde{J}^3}$ part of the operator $\Phi^{\rm su}_{\tilde{j};\tilde{m},\bar{\tilde{m}}}$,

$$e^{i\beta\tilde{H}}e^{i\bar{\beta}\bar{\tilde{H}}}\Phi^{\rm su}_{\tilde{j};\tilde{m},\bar{\tilde{m}}} \equiv e^{i\sqrt{\frac{2}{k}}\left((\tilde{m}+\beta)\tilde{X}+(\bar{\tilde{m}}+\bar{\beta})\bar{\tilde{X}}\right)}\tilde{V}^{\rm susy}_{\tilde{j};\tilde{m},\bar{\tilde{m}}}(\beta,\bar{\beta})\,. \tag{A.24}$$

The conformal weight and U(1) R-charge of $\tilde{V}^{\text{susy}}_{\tilde{j};\tilde{m},\bar{\tilde{m}}}(\beta,\bar{\beta})$ are

$$
\begin{aligned}
h &= \frac{\tilde{j}(\tilde{j}+1)-(\tilde{m}+\beta)^2}{k}+\frac{1}{2}\beta^2\,,\\
\bar{h} &= \frac{\tilde{j}(\tilde{j}+1)-(\bar{\tilde{m}}+\bar{\beta})^2}{k}+\frac{1}{2}\bar{\beta}^2\,,
\end{aligned}
\tag{A.25}
$$

and

$$
\begin{aligned}
r &= \frac{2(\tilde{m}+\beta)}{k}+\beta\,,\\
\bar{r} &= \frac{2(\bar{\tilde{m}}+\bar{\beta})}{k}+\bar{\beta}\,.
\end{aligned}
\tag{A.26}
$$

From the definition of the primary operator $\tilde{V}^{\text{B}}_{\tilde{j};\tilde{m},\bar{\tilde{m}}}$ for the bosonic SU(2)/U(1) at level $k-2$

$$
\Phi^{\text{su}}_{\tilde{j};\tilde{m},\bar{\tilde{m}}} \equiv e^{i\sqrt{\frac{2}{k-2}}(\tilde{m}\tilde{x}+\bar{\tilde{m}}\bar{\tilde{x}})}\tilde{V}^{\text{B}}_{\tilde{j};\tilde{m},\bar{\tilde{m}}}\,,
\tag{A.27}
$$

we have

$$
\tilde{V}^{\text{susy}}_{\tilde{j};\tilde{m},\bar{\tilde{m}}}(\beta,\bar{\beta}) = \tilde{V}^{\text{B}}_{\tilde{j};\tilde{m},\bar{\tilde{m}}}e^{-i\frac{2}{\sqrt{k(k-2)}}\left(\tilde{m}\tilde{X}_R+\bar{\tilde{m}}\bar{\tilde{X}}_R\right)}e^{i\sqrt{\frac{k-2}{k}}\left(\beta\tilde{X}_R+\bar{\beta}\bar{\tilde{X}}_R\right)}
\tag{A.28}
$$

where $\tilde{X}_R$ denotes the bosonization of the U(1) R-current J^{su}_R (A.20). From the well-known equivalence between the non-compact parafermionic primaries,

$$
\tilde{V}^{\text{B}}_{\tilde{j};\tilde{m},\bar{\tilde{m}}} = \tilde{V}^{\text{B}}_{\frac{k-2}{2}-\tilde{j};-\frac{k-2}{2}+\tilde{m},-\frac{k-2}{2}+\bar{\tilde{m}}}\,,
\tag{A.29}
$$

we can verify an interesting property that $\Psi^{\text{B}}_{\tilde{j};\tilde{m},\bar{\tilde{m}}}$ should satisfy

$$
\tilde{V}^{\text{susy}}_{\tilde{j};\tilde{m},\bar{\tilde{m}}}(\beta,\bar{\beta}) = \tilde{V}^{\text{susy}}_{\frac{k-2}{2}-\tilde{j};\pm\frac{k-2}{2}+\tilde{m},\pm\frac{k-2}{2}+\bar{\tilde{m}}}(\beta\pm 1,\bar{\beta}\pm 1)\,.
\tag{A.30}
$$

Useful OPEs. Finally, let us summarize several useful OPEs for primaries. The parafermionic primaries $\tilde{V}^{\text{B}}_{j;m,\bar{m}}$ satisfy the following OPEs

$$
\begin{aligned}
\tilde{\pi}(z)\tilde{V}^{B}_{\tilde{j};\tilde{m},\bar{\tilde{m}}}(0) &\sim \frac{\tilde{j}-\tilde{m}}{\sqrt{k-2}}\frac{1}{z^{1+\frac{2\tilde{m}}{k-2}}}\tilde{V}^{B}_{\tilde{j};\tilde{m}+1,\bar{\tilde{m}}}(0)\,,\\
\tilde{\pi}^{\dagger}(z)\tilde{V}^{B}_{\tilde{j};\tilde{m},\bar{\tilde{m}}}(0) &\sim \frac{\tilde{j}+\tilde{m}}{\sqrt{k-2}}\frac{1}{z^{1-\frac{2\tilde{m}}{k-2}}}\tilde{V}^{B}_{\tilde{j};\tilde{m}-1,\bar{\tilde{m}}}(0)\,.
\end{aligned}
\tag{A.31}
$$

Then, one can easily show that

$$
\begin{aligned}
G^{+}_{\text{min}}(z)\tilde{V}^{\text{susy}}_{\tilde{j};\tilde{m},\bar{\tilde{m}}}(\beta,\bar{\beta})(0) &\sim \frac{\tilde{j}+\tilde{m}}{z^{1-\beta}}\sqrt{\frac{2}{k}}\tilde{V}^{\text{susy}}_{\tilde{j};\tilde{m}-1,\bar{\tilde{m}}}(\beta+1,\bar{\beta})(0)+\cdots\,,\\
G^{-}_{\text{min}}(z)\tilde{V}^{\text{susy}}_{\tilde{j};\tilde{m},\bar{\tilde{m}}}(\beta,\bar{\beta})(0) &\sim \frac{\tilde{j}-\tilde{m}}{z^{1+\beta}}\sqrt{\frac{2}{k}}\tilde{V}^{\text{susy}}_{\tilde{j};\tilde{m}+1,\bar{\tilde{m}}}(\beta-1,\bar{\beta})(0)+\cdots\,.
\end{aligned}
\tag{A.32}
$$

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