

*Research Article*

# Almost Automorphic Solutions to Abstract Fractional Differential Equations

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A new and general existence and uniqueness theorem of almost automorphic solutions is obtained for the semilinear fractional differential equation  $D_t^\alpha u(t) = Au(t) + D_t^{\alpha-1} f(t, u(t))$  ( $1 < \alpha < 2$ ), in complex Banach spaces, with *Stepanov-like almost automorphic coefficients*. Moreover, an application to a fractional relaxation-oscillation equation is given.

## 1. Introduction

In this paper, we investigate the existence and uniqueness of almost automorphic solutions to the following semilinear abstract fractional differential equation:

$$D_t^\alpha u(t) = Au(t) + D_t^{\alpha-1} f(t, u(t)), \quad t \in \mathbb{R}, \quad (1.1)$$

where  $1 < \alpha < 2$ ,  $A : \mathfrak{D}(A) \subset X \rightarrow X$  is a sectorial operator of type  $\omega$  in a Banach space  $X$ , and  $f : \mathbb{R} \times X \rightarrow X$  is Stepanov-like almost automorphic in  $t \in \mathbb{R}$  satisfying some kind of Lipschitz conditions in  $x \in X$ . In addition, the fractional derivative is understood in the Riemann-Liouville's sense.

Recently, fractional differential equations have attracted more and more attentions (cf. [1–8] and references therein). On the other hand, the Stepanov-like almost automorphic problems have been studied by many authors (cf., e.g., [9, 10] and references therein). Stimulated by these works, in this paper, we study the almost automorphy of solutions to the fractional differential equation (1.1) with Stepanov-like almost automorphic coefficients.

A new and general existence and uniqueness theorem of almost automorphic solutions to the equation is established. Moreover, an application to fractional relaxation-oscillation equation is given to illustrate the abstract result.

Throughout this paper, we denote by  $\mathbb{N}$  the set of positive integers, by  $\mathbb{R}$  the set of real numbers, and by  $X$  a complex Banach space. In addition, we assume  $1 \leq p < +\infty$  if there is no special statement. Next, let us recall some definitions of almost automorphic functions and Stepanov-like almost automorphic functions (for more details, see, e.g., [9–11]).

*Definition 1.1.* A continuous function  $f : \mathbb{R} \rightarrow X$  is called almost automorphic if for every real sequence  $(s_m)$ , there exists a subsequence  $(s_n)$  such that

$$g(t) := \lim_{n \rightarrow \infty} f(t + s_n) \quad (1.2)$$

is well defined for each  $t \in \mathbb{R}$  and

$$\lim_{n \rightarrow \infty} g(t - s_n) = f(t) \quad (1.3)$$

for each  $t \in \mathbb{R}$ . Denote by  $AA(X)$  the set of all such functions.

*Definition 1.2.* The Bochner transform  $f^b(t, s)$ ,  $t \in \mathbb{R}$ ,  $s \in [0, 1]$ , of a function  $f(t)$  on  $\mathbb{R}$ , with values in  $X$ , is defined by

$$f^b(t, s) := f(t + s). \quad (1.4)$$

*Definition 1.3.* The space  $BS^p(X)$  of all Stepanov bounded functions, with the exponent  $p$ , consists of all measurable functions  $f$  on  $\mathbb{R}$  with values in  $X$  such that

$$\|f\|_{S^p} := \sup_{t \in \mathbb{R}} \left( \int_t^{t+1} \|f(\tau)\|^p d\tau \right)^{1/p} < +\infty. \quad (1.5)$$

It is obvious that  $L^p(\mathbb{R}; X) \subset BS^p(X) \subset L^p_{\text{loc}}(\mathbb{R}; X)$  and  $BS^p(X) \subset BS^q(X)$  whenever  $p \geq q \geq 1$ .

*Definition 1.4.* The space  $AS^p(X)$  of  $S^p$ -almost automorphic functions ( $S^p$ -a.a. for short) consists of all  $f \in BS^p(X)$  such that  $f^b \in AA(L^p(0, 1; X))$ . In other words, a function  $f \in L^p_{\text{loc}}(\mathbb{R}; X)$  is said to be  $S^p$ -almost automorphic if its Bochner transform  $f^b : \mathbb{R} \rightarrow L^p(0, 1; X)$  is almost automorphic in the sense that for every sequence of real numbers  $(s'_n)$ , there exist

a subsequence  $(s_n)$  and a function  $g \in L^p_{\text{loc}}(\mathbb{R}; X)$  such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \left( \int_0^1 \|f(t + s_n + s) - g(t + s)\|^p ds \right)^{1/p} &= 0, \\ \lim_{n \rightarrow \infty} \left( \int_0^1 \|g(t - s_n + s) - f(t + s)\|^p ds \right)^{1/p} &= 0, \end{aligned} \tag{1.6}$$

for each  $t \in \mathbb{R}$ .

*Remark 1.5.* It is clear that if  $1 \leq p < q < \infty$  and  $f \in L^q_{\text{loc}}(\mathbb{R}; X)$  is  $S^q$ -almost automorphic, then  $f$  is  $S^p$ -almost automorphic. Also if  $f \in AA(X)$ , then  $f$  is  $S^p$ -almost automorphic for any  $1 \leq p < \infty$ .

*Definition 1.6.* A function  $f : \mathbb{R} \times X \rightarrow X$ ,  $(t, u) \mapsto f(t, u)$  with  $f(\cdot, u) \in L^p_{\text{loc}}(\mathbb{R}, X)$  for each  $u \in X$  is said to be  $S^p$ -almost automorphic in  $t \in \mathbb{R}$  uniformly for  $u \in X$ , if for every sequence of real numbers  $(s'_n)$ , there exists a subsequence  $(s_n)$  and a function  $g : \mathbb{R} \times X \rightarrow X$  with  $g(\cdot, u) \in L^p_{\text{loc}}(\mathbb{R}, X)$  such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \left( \int_0^1 \|f(t + s_n + s, u) - g(t + s, u)\|^p ds \right)^{1/p} &= 0, \\ \lim_{n \rightarrow \infty} \left( \int_0^1 \|g(t - s_n + s, u) - f(t + s, u)\|^p ds \right)^{1/p} &= 0, \end{aligned} \tag{1.7}$$

for each  $t \in \mathbb{R}$  and for each  $u \in X$ . We denote by  $AS^p(\mathbb{R} \times X, X)$  the set of all such functions.

## 2. Almost Automorphic Solution

First, let us recall that a closed and densely defined linear operator  $A$  is called sectorial if there exist  $0 < \theta < \pi/2$ ,  $M > 0$ , and  $\omega \in \mathbb{R}$  such that its resolvent exists outside the sector

$$\begin{aligned} \omega + S_\theta &:= \{\omega + \lambda : \lambda \in \mathbb{C}, |\arg(-\lambda)| < \theta\}, \\ \|(\lambda I - A)^{-1}\| &\leq \frac{M}{|\lambda - \omega|}, \quad \lambda \notin \omega + S_\theta. \end{aligned} \tag{2.1}$$

Recently, in [3], Cuesta proved that if  $A$  is sectorial operator for some  $0 < \theta < \pi(1 - \alpha/2)$  ( $1 < \alpha < 2$ ),  $M > 0$ , and  $\omega < 0$ , then there exists  $C > 0$  such that

$$\|E_\alpha(t)\| \leq \frac{CM}{1 + |\omega|t^\alpha}, \quad t \geq 0, \tag{2.2}$$

where

$$E_\alpha(t) := \frac{1}{2\pi i} \int_\gamma e^{\lambda t} \lambda^{\alpha-1} (\lambda^\alpha - A)^{-1} d\lambda, \quad (2.3)$$

where  $\gamma$  is a suitable path lying outside the sector  $\omega + S_\theta$ .

In addition, by [2], we have the following definition.

*Definition 2.1.* A function  $u : \mathbb{R} \rightarrow X$  is called a mild solution of (1.1) if  $s \rightarrow E_\alpha(t-s)f(s, u(s))$  is integrable on  $(-\infty, t)$  for each  $t \in \mathbb{R}$  and

$$u(t) = \int_{-\infty}^t E_\alpha(t-s)f(s, u(s))ds, \quad t \in \mathbb{R}. \quad (2.4)$$

**Lemma 2.2.** Let  $\{S(t)\}_{t \geq 0} \subset \mathcal{B}(X)$  be a strongly continuous family of bounded and linear operators such that

$$\|S(t)\| \leq \phi(t), \quad t \in \mathbb{R}^+, \quad (2.5)$$

where  $\phi \in L^1(\mathbb{R}^+)$  is nonincreasing. Then, for each  $f \in AS^1(X)$ ,

$$\int_{-\infty}^t S(t-s)f(s)ds \in AA(X). \quad (2.6)$$

*Proof.* For each  $n \in \mathbb{N}$ , let

$$f_n(t) := \int_{t-n}^{t-n+1} S(t-s)f(s)ds = \int_{n-1}^n S(s)f(t-s)ds, \quad t \in \mathbb{R}. \quad (2.7)$$

In addition, for each  $n \in \mathbb{N}$ , by the principle of uniform boundedness,

$$M_n := \sup_{n-1 \leq s \leq n} \|S(s)\| < +\infty. \quad (2.8)$$

Fix  $n \in \mathbb{N}$  and  $t \in \mathbb{R}$ . We have

$$\begin{aligned} \|f_n(t+h) - f_n(t)\| &\leq \int_{n-1}^n \|S(s)\| \cdot \|f(t+h-s) - f(t-s)\| ds \\ &\leq M_n \cdot \int_{t-n}^{t-n+1} \|f(s+h) - f(s)\| ds. \end{aligned} \quad (2.9)$$

In view of  $f \in L^1_{\text{loc}}(\mathbb{R}; X)$ , we get

$$\lim_{h \rightarrow 0} \int_{t-n}^{t-n+1} \|f(s+h) - f(s)\| ds = 0, \quad (2.10)$$

which yields that

$$\lim_{h \rightarrow 0} \|f_n(t+h) - f_n(t)\| = 0. \quad (2.11)$$

This means that  $f_n(t)$  is continuous.

Fix  $n \in \mathbb{N}$ . By the definition of  $AS^1(X)$ , for every sequence of real numbers  $(s'_m)$ , there exist a subsequence  $(s_m)$  and a function  $g \in L^1_{\text{loc}}(\mathbb{R}; X)$  such that

$$\lim_{m \rightarrow \infty} \int_0^1 \|f(t+s_m+s) - g(t+s)\| ds = \lim_{m \rightarrow \infty} \int_0^1 \|g(t-s_m+s) - f(t+s)\| ds = 0, \quad (2.12)$$

for each  $t \in \mathbb{R}$ . Combining this with

$$\begin{aligned} \left\| f_n(t+s_m) - \int_{n-1}^n S(s)g(t-s)ds \right\| &\leq M_n \cdot \int_{n-1}^n \|f(t+s_m-s) - g(t-s)\| ds \\ &= M_n \cdot \int_0^1 \|f(t-n+s_m+s) - g(t-n+s)\| ds, \end{aligned} \quad (2.13)$$

we get

$$\lim_{m \rightarrow \infty} f_n(t+s_m) = \int_{n-1}^n S(s)g(t-s)ds \quad (2.14)$$

for each  $t \in \mathbb{R}$ . Similar to the above proof, one can show that

$$\lim_{m \rightarrow \infty} \int_{n-1}^n S(s)g(t-s_m-s)ds = f_n(t) \quad (2.15)$$

for each  $t \in \mathbb{R}$ . Therefore,  $f_n \in AA(X)$  for each  $n \in \mathbb{N}$ .

Noticing that

$$\begin{aligned} \|f_n(t)\| &\leq \int_{n-1}^n \phi(s) \cdot \|f(t-s)\| ds \leq \phi(n-1) \cdot \|f\|_{S^1}, \\ \sum_{n=1}^{\infty} \phi(n-1) \cdot \|f\|_{S^1} &\leq \left( \phi(0) + \sum_{n=2}^{\infty} \int_{n-2}^{n-1} \phi(t) dt \right) \cdot \|f\|_{S^1} \\ &\leq \left( \phi(0) + \|\phi\|_{L^1(\mathbb{R}^+)} \right) \cdot \|f\|_{S^1} < +\infty, \end{aligned} \quad (2.16)$$

we know that  $\sum_{n=1}^{\infty} f_n(t)$  is uniformly convergent on  $\mathbb{R}$ . Thus

$$\int_{-\infty}^t S(t-s)f(s)ds = \sum_{n=1}^{\infty} f_n(t) \in AA(X). \quad (2.17)$$

□

*Remark 2.3.* For the case of  $f \in AA(X)$ , the conclusion of Lemma 2.2 was given in [1, Lemma 3.1].

The following theorem will play a key role in the proof of our existence and uniqueness theorem.

**Theorem 2.4** (see [11]). *Assume that*

- (i)  $f \in AS^p(\mathbb{R} \times X, X)$  with  $p > 1$ ;
- (ii) there exists a nonnegative function  $L \in AS^r(\mathbb{R})$  with  $r \geq \max\{p, p/(p-1)\}$  such that for all  $u, v \in X$  and  $t \in \mathbb{R}$ ,

$$\|f(t, u) - f(t, v)\| \leq L(t)\|u - v\|; \quad (2.18)$$

- (iii)  $x \in AS^p(X)$  and  $K = \overline{\{x(t) : t \in \mathbb{R}\}}$  is compact in  $X$ .

Then there exists  $q \in [1, p)$  such that  $f(\cdot, x(\cdot)) \in AS^q(X)$ .

Now, we are ready to present the existence and uniqueness theorem of almost automorphic solutions to (1.1).

**Theorem 2.5.** *Assume that  $A$  is sectorial operator for some  $0 < \theta < \pi(1 - \alpha/2)$ ,  $M > 0$  and  $\omega < 0$ ; and the assumptions (i) and (ii) of Theorem 2.4 hold. Then (1.1) has a unique almost automorphic mild solution provided that*

$$\|L\|_{S^1} < \frac{\alpha \sin(\pi/\alpha)}{CM(\alpha \sin(\pi/\alpha) + |\omega|^{-1/\alpha} \pi)}. \quad (2.19)$$

*Proof.* For each  $\varphi \in AA(X)$ , let

$$\mathfrak{F}(\varphi)(t) := \int_{-\infty}^t E_\alpha(t-s)f(s, \varphi(s))ds, \quad t \in \mathbb{R}. \quad (2.20)$$

In view of  $\overline{\{\varphi(t) : t \in \mathbb{R}\}}$  which is compact in  $X$ , by Theorem 2.4, there exists  $q \in [1, p)$  such that  $f(\cdot, \varphi(\cdot)) \in AS^q(X)$ . On the other hand, by (2.2), we have

$$\|E_\alpha(t)\| \leq \frac{CM}{1 + |\omega|t^\alpha}, \quad t \geq 0. \quad (2.21)$$

Since  $1 < \alpha < 2$ ,  $CM/(1 + |\omega|t^\alpha) \in L^1(\mathbb{R}^+)$  and is nonincreasing. So Lemma 2.2 yields that  $\mathfrak{F}(\varphi) \in AA(X)$ . This means that  $\mathfrak{F}$  maps  $AA(X)$  into  $AA(X)$ .

For each  $\varphi, \psi \in AA(X)$  and  $t \in \mathbb{R}$ , we have

$$\begin{aligned}
 \|\mathfrak{F}(\varphi)(t) - \mathfrak{F}(\psi)(t)\| &\leq \int_{-\infty}^t \|E_\alpha(t-s)\| \cdot \|f(s, \varphi(s)) - f(s, \psi(s))\| ds \\
 &\leq \int_{-\infty}^t \frac{CM}{1 + |\omega|(t-s)^\alpha} L(s) ds \cdot \|\varphi - \psi\| \\
 &\leq \int_0^{+\infty} \frac{CM}{1 + |\omega|s^\alpha} L(t-s) ds \cdot \|\varphi - \psi\| \\
 &= \sum_{k=0}^{\infty} \int_k^{k+1} \frac{CM}{1 + |\omega|s^\alpha} L(t-s) ds \cdot \|\varphi - \psi\| \\
 &\leq \sum_{k=0}^{\infty} \frac{CM}{1 + |\omega|k^\alpha} \int_k^{k+1} L(t-s) ds \cdot \|\varphi - \psi\| \tag{2.22} \\
 &\leq \sum_{k=0}^{\infty} \frac{CM}{1 + |\omega|k^\alpha} \cdot \|L\|_{S^1} \cdot \|\varphi - \psi\| \\
 &\leq \left( CM + \sum_{k=1}^{\infty} \int_{k-1}^k \frac{CM}{1 + |\omega|s^\alpha} ds \right) \cdot \|L\|_{S^1} \cdot \|\varphi - \psi\| \\
 &= \left( CM + \int_0^{+\infty} \frac{CM}{1 + |\omega|s^\alpha} ds \right) \cdot \|L\|_{S^1} \cdot \|\varphi - \psi\| \\
 &= CM \left( 1 + \frac{|\omega|^{-1/\alpha} \pi}{\alpha \sin(\pi/\alpha)} \right) \cdot \|L\|_{S^1} \cdot \|\varphi - \psi\|,
 \end{aligned}$$

which gives

$$\|\mathfrak{F}(\varphi) - \mathfrak{F}(\psi)\| \leq CM \left( 1 + \frac{|\omega|^{-1/\alpha} \pi}{\alpha \sin(\pi/\alpha)} \right) \cdot \|L\|_{S^1} \cdot \|\varphi - \psi\|. \tag{2.23}$$

In view of (2.19),  $\mathfrak{F}$  is a contraction mapping. On the other hand, it is well known that  $AA(X)$  is a Banach space under the supremum norm. Thus,  $\mathfrak{F}$  has a unique fixed point  $u \in AA(X)$ , which satisfies

$$u(t) = \int_{-\infty}^t E_\alpha(t-s) f(s, u(s)) ds, \tag{2.24}$$

for all  $t \in \mathbb{R}$ . Thus (1.1) has a unique almost automorphic mild solution. □

In the case of  $L(t) \equiv L$ , by following the proof of Theorem 2.5 and using the standard contraction principle, one can get the following conclusion.

**Theorem 2.6.** Assume that  $A$  is sectorial operator for some  $0 < \theta < \pi(1 - \alpha/2)$ ,  $M > 0$  and  $\omega < 0$ ; and the assumptions (i) and (ii) of Theorem 2.4 hold with  $L(t) \equiv L$ , then (1.1) has a unique almost automorphic mild solution provided that

$$L < \frac{\alpha \sin(\pi/\alpha)}{CM|\omega|^{-1/\alpha}\pi}. \quad (2.25)$$

*Remark 2.7.* Theorem 2.6 is due to [2, Theroem 3.4] in the case of  $f(t, u)$  being almost automorphic in  $t$ . Thus, Theorem 2.6 is a generalization of [2, Theroem 3.4].

At last, we give an application to illustrate the abstract result.

*Example 2.8.* Let us consider the following fractional relaxation-oscillation equation given by

$$\partial_t^\alpha u(t, x) = \partial_x^2 u(t, x) - \mu u(t, x) + \partial_t^{\alpha-1} [a(t) \sin(u(t, x))], \quad t \in \mathbb{R}, x \in [0, \pi], \quad (2.26)$$

with boundary conditions

$$u(t, 0) = u(t, \pi) = 0, \quad t \in \mathbb{R}, \quad (2.27)$$

where  $1 < \alpha < 2$ ,  $\mu > 0$ , and

$$a(t) = \begin{cases} \sin \frac{1}{2 + \cos n + \cos \pi n}, & t \in (n - \varepsilon, n + \varepsilon), n \in \mathbb{Z}, \\ 0, & \text{otherwise,} \end{cases} \quad (2.28)$$

for some  $\varepsilon \in (0, 1/2)$ .

Let  $X = L^2[0, \pi]$ ,  $Au = u'' - \mu u$  with

$$\mathfrak{D}(A) = \left\{ u \in L^2[0, \pi] : u'' \in L^2[0, \pi], u(0) = u(\pi) = 0 \right\}, \quad (2.29)$$

and  $f(t, \varphi)(s) = a(t) \sin(\varphi(s))$  for  $\varphi \in X$  and  $s \in [0, \pi]$ . Then (2.26) is transformed into (1.1). It is well known that  $A$  is a sectorial operator for some  $0 < \theta < \pi/2$ ,  $M > 0$  and  $\omega < 0$ . By [10, Example 2.3],  $a(t) \in AS^2(\mathbb{R})$ . Then  $f \in AS^2(\mathbb{R} \times X, X)$ . In addition, for each  $t \in \mathbb{R}$  and  $u, v \in X$ ,

$$\|f(t, u) - f(t, v)\| = \left( \int_0^\pi |a(t) \sin(u(s)) - a(t) \sin(v(s))|^2 ds \right)^{1/2} \leq |a(t)| \cdot \|u - v\|. \quad (2.30)$$

Since

$$\|a(\cdot)\|_{S^1} = \sup_{t \in \mathbb{R}} \int_t^{t+1} |a(s)| ds \leq 2\varepsilon, \quad (2.31)$$



by Theorem 2.5, there exists a unique almost automorphic mild solution to (2.26) provided that  $1 < \alpha < 2(1 - \theta/\pi)$  and  $\varepsilon$  is sufficiently small.

*Remark 2.9.* In the above example, for any  $\varepsilon > 0$ ,  $f(t, u)$  is Lipschitz continuous about  $u$  uniformly in  $t$  with Lipschitz constant  $L \equiv 1$ , this means that  $f(t, u)$  has a better Lipschitz continuity than (2.30). However, one cannot ensure the unique existence of almost automorphic mild solution to (2.26) when

$$\frac{\alpha \sin(\pi/\alpha)}{CM|\omega|^{-1/\alpha}\pi} \leq 1, \quad (2.32)$$

by using Theorem 2.6. On the other hand, it is interesting to note that one can use Theorem 2.5 to obtain the existence in many cases under this restriction.

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