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Ghost-free massive $f(R)$ theories modeled as effective Einstein spaces and cosmic acceleration

Sergiu I. Vacaru[1,2,a]

[1] Theory Division, CERN, 1211, Geneva 23, Switzerland
[2] Rector's Office, Alexandru Ioan Cuza University, Alexandru Lapuşneanu street, nr. 14, UAIC, Corpus R, office 323, 700057 Iasi, Romania

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Abstract We study how massive ghost-free gravity $f(R)$-modified theories, MGFTs, can be encoded into generic off-diagonal Einstein spaces. Using "auxiliary" connections completely defined by the metric fields and adapted to nonholonomic frames with associated nonlinear connection structure, we decouple and integrate in certain general forms the field equations in MGFT. Imposing additional nonholonomic constraints, we can generate Levi-Civita, LC, configurations and mimic MGFT effects via off-diagonal interactions of effective Einstein and/or Einstein–Cartan gravity with nonholonomically induced torsion. We show that imposing nonholonomic constraints it is possible reproduce very specific models of massive $f(R)$ gravity studied in Cai et al. (arXiv:1307.7150, 2013), Klusoňet al. (Phys Lett B 726:918, 2013), Nojiri and Odintov (Phys Lett B 716:377, 2012) and Nojiri et al. (JCAP 1305:020, 2013). The cosmological evolution of ghost-free off-diagonal Einstein spaces is investigated. Certain compatibility of MGFT cosmology to small off-diagonal deformations of ΛCDM models is established.

1 Introduction

In [1–4], two models of nonlinear massive gravitational theories including $f(R)$ modifications were elaborated. Such theories contain the benefits of the dRGT model [5,6] and are free of ghost modes [7–10]. Advantages are that by tuning the $f(R)$ functional (on such modifications, see the reviews of [11–13]), we can stabilize cosmological backgrounds, and we can elaborate various types cosmological evolution scenarios, unified description of inflation and late-time acceleration, etc. The main goal of [1] is to perform a general analysis for arbitrary $f(R)$ theory but Refs. [2–4] provide solutions for explicit cosmological problems of such theories. From general theoretical considerations, the $f(R)$ paradigm attempts to explain the universe's acceleration and dark energy/matter problems through infra-red (IR) modifications of general relativity (GR) theory and understanding possible physical implications of the massive spin-2 theory. In this paper, we generate a very specific model of massive $f(R)$ gravity constraining nonholonomically the corresponding system of modified gravitational equations. We shall analyze possible cosmological implications for such special cases containing small off-diagonal corrections. On the other hand, ultra-violet (UV) corrections are expected to be of quantum origin (see Refs. [14,15] for possible effective actions). Cosmological implications of massive gravity were also analyzed in the framework of modified gravity theories, MGT, [16–23], and also cosmological models related to bi-metric gravity [24–27].

It is the point of this paper to apply in MGFT the so-called anholonomic frame deformation method, AFDM, [28–34] for constructing generic off-diagonal exact solutions. Such a method provides geometric techniques, which allows us to integrate systems of partial differential equations, PDEs, with functional and parametric dependencies for the Levi-Civita (zero torsion) and nontrivial torsion configurations.

2 The geometric setup

We shall work on a pseudo-Riemannian manifold V, dim $V = 4$, where a Whitney sum $\mathbf{N}$ is defined for its tangent space TV, $\mathbf{N}: \ TV = hTV \oplus vTV$. Such a decomposition defines a nonholonomic (equivalently, non-integrable, or anholonomic) horizontal (h) and vertical (v) splitting, i.e. a nonlinear connection (*N-connection*) structure; see details in [28–34]. The local coefficients $\{N_i^a(u)\}$, where $\mathbf{N} = N_i^a(x,y)dx^i \otimes \partial/\partial y^a$ for certain local coordinates

S. I. Vacaru is a associated visiting researcher.

[a] e-mail: sergiu.vacaru@uaic.ro

$u=(x,y)$, or $u^{\alpha}=(x^i,y^a)$, with h-indices $i,j,=1,2$ and v-indices $a,b,\ldots=3,4,$[1] define naturally N-adapted frame and, respectively, dual frame structures, $\mathbf{e}_\nu=(\mathbf{e}_i,e_a)$ and $\mathbf{e}^\mu=(e^i,\mathbf{e}^a)$, where

$$\mathbf{e}_i=\partial/\partial x^i-\ N_i^a(u)\partial/\partial y^a,\ e_a=\partial_a=\partial/\partial y^a,\\ \text{and } e^i=\mathrm{d}x^i,\ \mathbf{e}^a=dy^a+\ N_i^a(u)\mathrm{d}x^i. \qquad (1)$$

In general, such local (co) bases are nonholonomic, i.e. $[\mathbf{e}_\alpha,\mathbf{e}_\beta]=\mathbf{e}_\alpha\mathbf{e}_\beta-\mathbf{e}_\beta\mathbf{e}_\alpha=W^{\gamma}_{\alpha\beta}\mathbf{e}_\gamma$ with the anholonomy coefficients $W^b_{ia}=\partial_a N^b_i$, $W^a_{ji}=\Omega^a_{ij}=\mathbf{e}_j(N^a_i)-\mathbf{e}_i(N^a_j)$, where Ω^a_{ij} is the N-connection curvature. With respect to (1), any metric tensor $\mathbf{g}$ can be expressed as a distinguished metric, a d-metric,

$$\mathbf{g}=g_\alpha(u)\mathbf{e}^\alpha\otimes\mathbf{e}^\beta\\ =g_i(x^k)\mathrm{d}x^i\otimes\mathrm{d}x^i+g_a(x^k,y^b)\mathbf{e}^a\otimes\mathbf{e}^a. \qquad (2)$$

For any prescribed N-connection and d-metric structures, we can work equivalently with two linear connections,

$$(\mathbf{g},\mathbf{N})\rightarrow\begin{cases}\nabla: & \nabla\mathbf{g}=0;\ {}^{\nabla}\mathcal{T}=0;\\ \widehat{\mathbf{D}}: \widehat{\mathbf{D}}\mathbf{g}=0; & h\widehat{\mathcal{T}}=0,v\widehat{\mathcal{T}}=0,hv\widehat{\mathcal{T}}\neq0,\end{cases}$$

where ∇ is the torsionless Levi-Civita, LC, connection and $\widehat{\mathbf{D}}=h\widehat{\mathbf{D}}+v\widehat{\mathbf{D}}$ is the so-called canonical distinguished connection, the d-connection. The value $\widehat{\mathbf{D}}$ preserves the h–v splitting under parallel transport, but ∇ does not have such a property. Nevertheless, there is a canonical distortion distinguished tensor, d-tensor, $\widehat{\mathbf{Z}}=\{\widehat{\mathbf{T}}^{\alpha}{}_{\beta\gamma}\}$, which is an algebraic combination of the coefficients of the corresponding torsion d-tensor $\widehat{\mathcal{T}}=\{\widehat{\mathbf{T}}^{\alpha}{}_{\beta\gamma}\}$. This defines a canonical distortion relation $\widehat{\mathbf{D}}=\nabla+\widehat{\mathbf{Z}}$ which is adapted to the N-splitting. The torsions, $\widehat{\mathcal{T}}$ and ${}^{\nabla}\mathcal{T}=0$, and curvatures, $\widehat{\mathcal{R}}=\{\widehat{\mathbf{R}}^{\alpha}{}_{\beta\gamma\delta}\}$ and ${}^{\nabla}\mathcal{R}=\{R^{\alpha}{}_{\beta\gamma\delta}\}$, respectively, of $\widehat{\mathbf{D}}$ and ∇ can be defined and computed in standard coordinate free and/or coefficient forms.

The Ricci tensors of $\widehat{\mathbf{D}}$ and ∇ are defined $\widehat{\mathcal{R}ic}=\{\widehat{\mathbf{R}}_{\ \beta\gamma}:=\widehat{\mathbf{R}}^{\gamma}{}_{\alpha\beta\gamma}\}$ and Ric $=\{R_{\ \beta\gamma}:=R^{\gamma}{}_{\alpha\beta\gamma}\}$. For instance, the Ricci d-tensor $\widehat{\mathcal{R}ic}$ is characterized by four subsets of h–v N-adapted coefficients,

$$\widehat{\mathbf{R}}_{\alpha\beta}=\{\widehat{R}_{ij}:=\widehat{R}^{k}{}_{ijk},\ \widehat{R}_{ia}:=-\widehat{R}^{k}{}_{ika},\\ \widehat{R}_{ai}:=\widehat{R}^{b}{}_{aib},\ \widehat{R}_{ab}:=\widehat{R}^{c}{}_{abc}\}. \qquad (3)$$

Alternatively to the LC-scalar curvature, $R:=\mathbf{g}^{\alpha\beta}R_{\alpha\beta}$, we can introduce the scalar of canonical d-curvature, $\widehat{\mathbf{R}}:=\mathbf{g}^{\alpha\beta}\widehat{\mathbf{R}}_{\alpha\beta}=g^{ij}\widehat{R}_{ij}+g^{ab}\widehat{R}_{ab}$.[2]

3 Field equations in MGFT and N-adapted variables

We follow the model elaborated in [2–4] and reformulate it on a nonholonomic manifold $\mathbf{V}$ enabled with N-connection structure $\mathbf{N}$ and two d-metrics where $\mathbf{g}=\{\mathbf{g}_{\alpha\beta}\}$ is the dynamical d-metric and $\mathbf{q}=\{\mathbf{q}_{\alpha\beta}\}$ is the so-called non-dynamical reference metric. In our approach, we work with $\widehat{\mathbf{D}}$ instead of ∇ and $\widehat{\mathbf{R}}$ is computed for $\mathbf{g}$, the nonzero graviton mass is denoted by μ, M_P is the Planck mass.[3]

Let us consider the d-tensor $\left(\sqrt{\mathbf{g}^{-1}\mathbf{q}}\right)^{\mu}_{\ \nu}$ computed as the square root of $\mathbf{g}^{\mu\rho}\mathbf{q}_{\rho\nu}$, where

$$\left(\sqrt{\mathbf{g}^{-1}\mathbf{q}}\right)^{\mu}_{\ \rho}\left(\sqrt{\mathbf{g}^{-1}\mathbf{q}}\right)^{\rho}_{\ \nu}=\mathbf{g}^{\mu\rho}\mathbf{q}_{\rho\nu},\quad\text{and}\\ \sum_{k=0}^{4}{}^{k}\beta e_k\left(\sqrt{\mathbf{g}^{-1}\mathbf{q}}\right)=3-\mathrm{tr}\sqrt{\mathbf{g}^{-1}\mathbf{q}}-\det\sqrt{\mathbf{g}^{-1}\mathbf{q}},$$

for some coefficients ${}^{k}\beta$. The values $e_k(\mathbf{X})$ can be defined for any d-tensor $\mathbf{X}^{\mu}_{\rho}$ and trace $X=[X]:=\mathrm{tr}(\mathbf{X})=\mathbf{X}^{\mu}_{\mu}$, where

$$e_0(X)=1,e_1(X)=X,2e_2(X)=X^2-[X^2],\\ 6e_3(X)=X^3-3X[X^2]+2[X^3],\\ 24e_4(X)=X^4-6X^2[X^2]+3[X^2]^2+8X[X^3]-6[X^4];\\ e_k(X)=0\ \text{ for }\ k>4.$$

We shall use also the mass-deformed scalar curvature $\tilde{\mathbf{R}}:=\widehat{\mathbf{R}}+2\,\mu^2(3-\mathrm{tr}\sqrt{\mathbf{g}^{-1}\mathbf{q}}-\det\sqrt{\mathbf{g}^{-1}\mathbf{q}})$.

The action $\mathcal{S}$ for MGFT is postulated in the form

$$\mathcal{S}=M_P^2\int \mathrm{d}^4u\sqrt{|\mathbf{g}|}[\mathbf{f}(\tilde{\mathbf{R}})+\ {}^{m}\mathcal{L}], \qquad (4)$$

where ${}^{m}\mathcal{L}(\mathbf{g},\mathbf{N})$ is the Lagrange density for the matter fields.[4] The energy-momentum d-tensor can be computed via N-adapted variational calculus,

[1] We shall use the Einstein rule on summation on "upper-lower" cross indices. Boldface letters are written in order to emphasize that a N-connection spitting is considered on a manifold $\mathbf{V}=(V,\mathbf{N})$.

[2] Any (pseudo) Riemannian geometry can be equivalently described by both geometric data $(\mathbf{g},\nabla)$ and $(\mathbf{g},\mathbf{N},\widehat{\mathbf{D}})$, where the canonical distortion relations $\widehat{\mathcal{R}}={}^{\nabla}\mathcal{R}+{}^{\nabla}\mathcal{Z}$ and $\widehat{\mathcal{R}ic}=\mathrm{Ric}+\widehat{\mathcal{Z}ic}$, with respective

Footnote 2 continued
distortion d-tensors ${}^{\nabla}\mathcal{Z}$ and $\widehat{\mathcal{Z}ic}$, are computed for $\widehat{\mathbf{D}}=\nabla+\widehat{\mathbf{Z}}$. To prove the decoupling of fundamental gravitational equations in general relativity, GR, and various MGFTs is possible for d-metrics and the canonical d-connection working with respect to N-adapted frames. LC-configurations can be extracted from certain classes of solutions of (modified) gravitational field equations if additional conditions are imposed, resulting in zero values for the canonical d-torsion, $\widehat{\mathcal{T}}=0$.

[3] Our system of "N-adapted notations" is similar to that considered in [35,36].

[4] For simplicity, we consider matter actions ${}^{m}\mathcal{S}=\int\mathrm{d}^4u\sqrt{|\mathbf{g}|}{}^{m}\mathcal{L}$ which only depend on the coefficients of a metric field and not on their derivatives. Here we note that the geometric constructions in this paper can also be performed in similar form for cosmological models [2–4] but must be supplemented by a number of formulas that would contain nonholonomic constraints for additional physical assumptions. To work with the action (4) is a more convenient choice for emphasizing in an "economic" way all priorities of our geometric approach.

$$
\begin{aligned}
{}^{m}\mathbf{T}_{\alpha\beta} &:= -\frac{2}{\sqrt{|\mathbf{g}_{\mu\nu}|}}\frac{\delta(\sqrt{|\mathbf{g}_{\mu\nu}|}\ {}^{m}\mathcal{L})}{\delta\mathbf{g}^{\alpha\beta}} \\
&= {}^{m}\mathcal{L}\mathbf{g}^{\alpha\beta} + 2\frac{\delta({}^{m}\mathcal{L})}{\delta\mathbf{g}_{\alpha\beta}}.
\end{aligned} \tag{5}
$$

Applying such a calculus to $\mathcal{S}$ (4), with ${}^{1}\mathbf{f}(\tilde{\mathbf{R}}) := d\mathbf{f}(\tilde{\mathbf{R}})/d\tilde{\mathbf{R}}$, see details in [28–34], we obtain the field equations

$$
\widehat{\mathbf{R}}_{\mu\nu} = \Upsilon_{\mu\nu}, \tag{6}
$$

where $\Upsilon_{\mu\nu} = {}^{m}\Upsilon_{\mu\nu} + {}^{f}\Upsilon_{\mu\nu} + {}^{\mu}\Upsilon_{\mu\nu}$, for

$$
\begin{aligned}
{}^{m}\Upsilon_{\mu\nu} &= \frac{1}{2M_P^2}\ {}^{m}\mathbf{T}_{\alpha\beta}, \\
{}^{f}\Upsilon_{\mu\nu} &= \left(\frac{\mathbf{f}}{2\ {}^{1}\mathbf{f}} - \frac{\widehat{\mathbf{D}}^2\ {}^{1}\mathbf{f}}{{}^{1}\mathbf{f}}\right)\mathbf{g}_{\mu\nu} + \frac{\widehat{\mathbf{D}}_\mu\widehat{\mathbf{D}}_\nu\ {}^{1}\mathbf{f}}{{}^{1}\mathbf{f}}, \\
{}^{\mu}\Upsilon_{\mu\nu} &= -2\mu^2\Bigg[\left(3 - \mathrm{tr}\sqrt{\mathbf{g}^{-1}\mathbf{q}} - \det\sqrt{\mathbf{g}^{-1}\mathbf{q}}\right) \\
&\quad -\frac{1}{2}\det\left(\sqrt{\mathbf{g}^{-1}\mathbf{q}}\right)\Bigg]\mathbf{g}_{\mu\nu} \\
&\quad +\frac{\mu^2}{2}\left\{\mathbf{q}_{\mu\rho}\left[\left(\sqrt{\mathbf{g}^{-1}\mathbf{q}}\right)^{-1}\right]^{\rho}_{\nu}\right. \\
&\quad \left. +\mathbf{q}_{\nu\rho}\left[\left(\sqrt{\mathbf{g}^{-1}\mathbf{q}}\right)^{-1}\right]^{\rho}_{\mu}\right\}.
\end{aligned} \tag{7}
$$

We note that the Bianchi identities for the data $(\mathbf{g}, \mathbf{N}, \widehat{\mathbf{D}})$ are given by introducing nonholonomic deformations $\nabla = \widehat{\mathbf{D}} - \widehat{\mathbf{Z}}$ into the standard relations $\nabla^\alpha(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R) = 0$ and $\nabla^\alpha T_{\alpha\beta} = 0$. Even, in general, as $\widehat{\mathbf{D}}^\alpha\mathbf{T}_{\alpha\beta} = \mathbf{Q}_\beta \neq 0$, such a $\mathbf{Q}_\beta[\mathbf{g}, \mathbf{N}]$ is completely defined by the d-metric and the chosen N-connection structure. This is a consequence of the nonholonomic structure. A similar "problem" exists in Lagrange mechanics with non-integrable constraints when the standard conservation laws do not hold true. A new class of effective variables can be introduced using Lagrange multiples. We omit cumbersome formulas for the Bianchi densities and conservation laws with nonholonomic constraints written in the variables $(\mathbf{g}, \mathbf{N}, \widehat{\mathbf{D}})$.

4 Encoding and decoupling properties of field equations in MGFT

The generalized gravitational field equations written with respect to N-adapted frames (6) are similar to those studied in our works [28–36]. The main difference of such MTGs is determined by a corresponding source which in this work is considered in the form (7). Applying the AFDM, we can construct very general classes of generic off-diagonal solutions which encode both f-modifications and massive gravity effects with nonzero μ.

For simplicity, we shall consider nonholonomic dynamical systems in MGTs which via frame transforms and connection deformations can be transformed into certain effective off-diagonal Einstein manifolds described by d-metrics with one Killing symmetry on $\partial/\partial y^3$, i.e. the gravitational and matter fields do not depend on variable y^4.[5] This is described by the ansatz (2) with

$$
\begin{aligned}
& g_i = e^{\psi(x^i)},\ g_a = h_a(x^i, t), \\
& N_i^3 = n_i(x^k),\ N_i^4 = w_i(x^k, t).
\end{aligned} \tag{8}
$$

The effective source is chosen for a timelike coordinate $y^4 = t$, where

$$
\begin{aligned}
\Upsilon_{\mu\nu} &\to \hat{\Upsilon}_{\mu\nu} = \mathrm{diag}[\Upsilon_1 = \Upsilon_2, \Upsilon_2 = {}^{m}\tilde{\Upsilon}(x^i) \\
&\quad + {}^{f}\tilde{\Upsilon}(x^i) + {}^{\mu}\tilde{\Upsilon}(x^i), \\
\Upsilon_3 &= \Upsilon_4,\ \Upsilon_4 = {}^{m}\Upsilon(x^i, t) + {}^{f}\Upsilon(x^i, t) + {}^{\mu}\Upsilon(x^i, t)] \\
&\to \check{\Upsilon}_{\mu\nu} = ({}^{m}\check{\Lambda} + {}^{f}\check{\Lambda} + {}^{\mu}\check{\Lambda})\mathbf{g}_{\alpha\beta}.
\end{aligned} \tag{9}
$$

The assumption for the first parametrization in (9) is that the matter fields and effective sources, $\Upsilon_{\mu'\nu'} = e^{\mu}_{\ \mu'}e^{\nu}_{\ \nu'}\hat{\Upsilon}_{\mu\nu}$, are generated in N-adapted frames by two types of functions/distributions $\tilde{\Upsilon}(x^i)$ and $\Upsilon(x^i, t)$. The left labels refer to contributions in such sources by f-modifications and/or by mass μ-modifications. In general, we get four independent N-adapted coefficients of $\Upsilon_{\mu\nu} = \mathrm{diag}\{\Upsilon_\mu(x^i, t)\}$ for variations in (5) using (8). For cosmological applications, we can model sources of matter fields by an energy-momentum tensor for ideal fluids as in GR but with generic off-diagonal metrics[6] (encoding contributions from MGT). In N-adapted frames,

$$
\mathbf{T}_{\alpha\beta} = p\mathbf{g}_{\alpha\beta} + (\rho + p)\mathbf{v}_\alpha\mathbf{v}_\beta \tag{10}
$$

is defined for a certain (effective) energy, ρ, and for certain pressure densities, p, respectively, $\widehat{\mathbf{v}}_\alpha$ being the four-velocity of the fluid for which $\mathbf{v}_\alpha\mathbf{v}^\alpha = -1$ and $\mathbf{v}^\alpha = (0, 0, 0, 1)$ in N-adapted comoving frames/coordinates.

A tedious calculation of the N-adapted coefficients of the Ricci d-tensor for $\widehat{\mathbf{D}}$ computed for ansatz (8) and source (9) transform (6) into a system of nonlinear PDEs:

$$
\psi^{\bullet\bullet} + \psi'' = 2\,({}^{m}\tilde{\Upsilon} + {}^{f}\tilde{\Upsilon} + {}^{\mu}\tilde{\Upsilon}) = 2\,\tilde{\Upsilon}, \tag{11}
$$

$$
\phi^{\diamond}h_3^{\diamond} = 2h_3h_4\,({}^{m}\Upsilon + {}^{f}\Upsilon + {}^{\mu}\Upsilon) = 2h_3h_4\,\Upsilon, \tag{12}
$$

$$
n_i^{\diamond\diamond} + \gamma n_i^{\diamond} = 0,\ \beta w_i - \alpha_i = 0, \tag{13}
$$

[5] It is possible to construct metrics with non-Killing symmetries depending on all spacetime coordinates. This requires a more advanced and cumbersome geometric techniques; see the examples in [28–34,36] and references therein.

[6] Such metrics cannot be diagonalized by coordinate transforms because for general N-connections the anholonomy coefficients, $W^{\gamma}_{\alpha\beta}$, are not zero.

for $\alpha_i = h_3^{\diamond}\partial_i\phi,\ \beta = h_3^{\diamond}\ \phi^{\diamond},\ \gamma = (\ln|h_3|^{3/2}/|h_4|)^{\diamond}$, where

$$\phi = \ln|h_3^{\diamond}/\sqrt{|h_3 h_4|}|, \quad \text{and/ or } \Phi := e^{\phi}, \tag{14}$$

is considered as a generating function. In the above formulas, we use the following notations for the partial derivatives: $\psi^{\bullet} = \partial_1\psi = \partial\psi/\partial x^1$, $\psi' = \partial_2\psi$, $h_3^{\diamond} = \partial_4 h_3$. For simplicity, we do not study in this paper d-metrics for which $h_a^{\diamond} = 0$ and/or $\Upsilon_\mu = 0$ (such solutions in vacuum MGFT can be constructed, for instance, for f- and/or μ-modifications of black hole solutions; see the examples in [28–34]). Here we note that the relevant equations (12), (13) and the respective coefficients can be computed in a similar form if corresponding coordinates and indices are changed as $3 \to 5$ and $4 \to 6$, which allows one to extend the method for extra dimensions. Such recurrent formulas can be proven for an arbitrary finite number of extra (non-) holonomic coordinates. For simplicity, we analyze in this work only examples of off-diagonal metrics for 4-d spacetimes.

The torsionless (Levi-Civita, LC) conditions are satisfied if there are additionally imposed the conditions

$$w_i^{\diamond} = (\partial_i - w_i\partial_4)\ln\sqrt{|h_4|},\ (\partial_i - w_i\partial_4)\ln\sqrt{|h_3|} = 0,$$
$$\partial_k w_i = \partial_i w_k,\ n_i^{\diamond} = 0,\ \partial_i n_k = \partial_k n_i. \tag{15}$$

5 Exact off-diagonal solutions in MGFT

The system (11)–(13) possess an important property when (1) ψ is the solution of a two-dimensional (2-d) Poisson equation with source $2(\ldots)(x^k)$; h_3 and h_4 are related to ϕ and the sources via (14). The N-connection coefficients are determined correspondingly by integrating two times on t the equations for n_i and based on a system of first order algebraic equations for w_i. For MGTs, the procedure of finding locally anisotropic and inhomogeneous cosmological solutions is described in [36].

We fix the sum of nontrivial constants $\check{\Lambda} = {}^m\check{\Lambda} + {}^f\check{\Lambda} + {}^\mu\check{\Lambda}$ and re-define the generating function, $\Phi \longleftrightarrow \check{\Phi}$, using formulas

$$\check{\Lambda}\check{\Phi}^2 = \left[\Phi^2|\Upsilon| + \int dt\ \Phi^2|\Upsilon|^{\diamond}\right],$$
$$\Phi^2 = \frac{|\check{\Lambda}|}{|\Upsilon|^2}\int dt\ \check{\Phi}^2|\Upsilon|, \tag{16}$$

where $(\Phi^2)^{\diamond}/|\Upsilon| = (\check{\Phi}^2)^{\diamond}/\check{\Lambda}$. In order to solve the second equation in (15), $(\partial_i - w_i\partial_4)\ln\sqrt{|h_3|} = 0$, the generating function Φ must be chosen to satisfy the conditions $(\partial_i\Phi)^{\diamond} = \partial_i\Phi^{\diamond}$. We can parameterize the solutions for the system (12) and (14) in the form $h_3[\check{\Phi}] = \frac{\check{\Phi}^2}{4|\check{\Lambda}|}$ and $h_4[\check{\Phi}] = \frac{(\check{\Phi}^{\diamond})^2}{\check{\Lambda}\Phi^2} = \frac{|\check{\Phi}^{\diamond}\ \Upsilon|^2}{\check{\Lambda}|\check{\Lambda}|\int dt\ \check{\Phi}^2\ |\Upsilon|}$.

We find in explicit form solutions of algebraic equations in (13) and the conditions $\partial_k w_i = \partial_i w_k$ from the second line in (15) if

$$w_i = \partial_i\Phi/\Phi^{\diamond} = \partial_i\widetilde{A}, \tag{17}$$

with a nontrivial function $\widetilde{A}(x^k, t)$ depending on generating function Φ via a first order Pfaff system. Integrating two times on t in (13), we express

$$n_k = {}_1n_k + {}_2n_k\int dy^4\ h_4/(\sqrt{|h_3|})^3, \tag{18}$$

where ${}_1n_k(x^i)$ and ${}_2n_k(x^i)$ are integration functions. To generate LC-configurations we take ${}_2n_k = 0$ and ${}_1n_k = \partial_k n(x^i)$.

Putting together the above formulas, we conclude that generic off-diagonal quadratic elements

$$ds^2 = e^{\psi(x^k,[{}^m\tilde{\Upsilon}+{}^f\tilde{\Upsilon}+{}^\mu\tilde{\Upsilon}])}[(dx^1)^2 + (dx^2)^2]$$
$$+\frac{\check{\Phi}^2[dy^3 + \partial_k n\ dx^k]^2}{4|{}^m\check{\Lambda} + {}^f\check{\Lambda} + {}^\mu\check{\Lambda}|}$$
$$\pm\frac{|[\check{\Phi}^{\diamond}]\,[{}^m\Upsilon + {}^f\Upsilon + {}^\mu\Upsilon]|^2}{|{}^m\check{\Lambda} + {}^f\check{\Lambda} + {}^\mu\check{\Lambda}|^{\frac{3}{2}}\int dt\ \check{\Phi}^2\,|{}^m\Upsilon + {}^f\Upsilon + {}^\mu\Upsilon|}$$
$$\times(dt + \partial_i\widetilde{A}[\check{\Phi}]\,dx^i)^2 \tag{19}$$

determine generic off-diagonal solutions of the field equations in MGFT. For well-defined assumptions on the Killing symmetry on ∂_3 and imposed at the end zero torsion conditions such metrics belong to the integral variety of the system (11)–(15). We can generate exact solutions in "pure" f-modified gravity if put ${}^\mu\Upsilon = {}^\mu\Lambda = 0$. If $\Lambda \neq 0$, we can nonholonomically induce a nontrivial ${}^\mu\Upsilon$. Inverse nonlinear transforms are possible if we change mutually the left labels μ with f.

It should be noted that above classes of metrics can be extended to describe exact solutions with nonholonomically induced torsion $\widehat{\mathcal{T}} = \{\widehat{\mathbf{T}}^{\alpha}_{\ \beta\gamma}[\check{\Phi}, \tilde{\Upsilon}, \Upsilon, \check{\Lambda}]\}$ of $\widehat{\mathbf{D}}$. We substitute in (19) $\partial_k n \to n_k(x^i, t)$ (18) and take instead of (17) the value $w_i = \partial_i\Phi/\Phi^{\diamond}$. It is possible to re-write all coefficients in terms of the generating function Φ, or in terms of $\check{\Phi}$. The LC conditions (15) are not satisfied for such configurations.[7]

6 On properties of off-diagonal solutions in MGFT and GR

The metrics (19) describe locally anisotropic and inhomogeneous spacetimes determined by certain classes of generating functions $\check{\Phi}(x^i, t)$ and $\psi(x^k, [{}^m\tilde{\Upsilon} + {}^f\tilde{\Upsilon} + {}^\mu\tilde{\Upsilon}])$; sources

[7] Such torsion fields are different from those in Einstein–Cartan, gauge and/or string gravity where additional field equations and sources are considered to define the torsion dynamics.

${}^{m}\Upsilon(x^i,t), {}^{f}\Upsilon(x^i,t), {}^{\mu}\Upsilon(x^i,t)$ and ${}^{m}\tilde{\Upsilon}(x^i)$, ${}^{f}\tilde{\Upsilon}(x^i), {}^{\mu}\tilde{\Upsilon}(x^i)$, and integration functions like $\partial_k n(x^k)$; and effective cosmological constants ${}^{m}\check{\Lambda}, {}^{f}\check{\Lambda}, {}^{\mu}\check{\Lambda}$, which can be considered as integration constants. These values and one of the $\pm$ should be fixed such that they are compatible with the observational data. We can generate inhomogeneous cosmological metrics taking certain limits $\check{\Phi}(x^i,t) \to \check{\Phi}(t)$ and for respective sources $\Upsilon(x^i,t) \to \Upsilon(t)$. Such solutions generalize the class of known anisotropic solutions of Bianchi cosmology to configurations; the coefficients of metrics are not subject to typical symmetric conditions for those spacetimes and, in our approach, may encode geometric and physical data for MGFT interactions.

Fixing, for instance, ${}^{\mu}\check{\Lambda} = {}^{\mu}\tilde{\Upsilon} = {}^{\mu}\Upsilon = 0$, i.e. for the zero mass of the graviton, the metrics (19) reproduce certain results of $f(\widehat{\mathbf{R}})$ gravity and cosmology theories; see [36] and references therein. So, at least for $\mu = 0$, by introducing a conformal factor ω before h_3, h_4 in the above formulas, redefining the generating functions, and for small off-diagonal coefficients, we reproduce nonholonomic deformations of ΛCDM universes.

The metrics (19) do not have, in general, a simple physical interpretation. Choosing the integration constants, we can extract (for instance) Kasner type solutions with dynamical chaos etc.; see examples in [28–34] and references therein. A rigorous study of nonperturbative and nonlinear effects of such generic off-diagonal dynamical systems even for small μ is necessary (this is a matter of further research). Here we note that the nonholonomic nonlinear coupling with re-definition of generating functions by formulas (16), and by off-diagonal coefficients of (19), encodes geometric and physical data for MGFT into effective Einstein spaces. This follows from the fact that such solutions are equivalent (up to frame/coordinate transforms) to the equations $\check{\mathbf{R}}_{\mu\nu} = \check{\Lambda}\check{\mathbf{g}}_{\alpha\beta}$. This motivates equivalent re-definitions of the sources $\Upsilon_{\mu\nu} \to \hat{\Upsilon}_{\mu\nu} \to ({}^{m}\check{\Lambda} + {}^{f}\check{\Lambda} + {}^{\mu}\check{\Lambda})\check{\mathbf{g}}_{\alpha\beta}$ as we supposed in (10). Considering solitonic configurations, we can polarize or "open" for a period of time some modes of massive gravity and then "switch off" such interactions and "pump" certain induced f-modified effects into off-diagonal coefficients of Einstein metrics with redefined cosmological constants and generating functions.

7 Scale factors and off-diagonal deformations of FLRW metrics

Let us introduce a new time coordinate $\widehat{t}$, where $t = t(x^i, \widehat{t})$ and $\sqrt{|h_4|}\partial t/\partial \widehat{t}$, and a scale factor $\widehat{a}(x^i, \widehat{t})$ when the d-metric (19) can be represented in the form

$$ds^2 = \widehat{a}^2(x^i,\widehat{t})[\eta_i(x^k,\widehat{t})(dx^i)^2 + \widehat{h}_3(x^k,\widehat{t})(\mathbf{e}^3)^2 - (\widehat{\mathbf{e}}^4)^2], \quad (20)$$

where $\eta_i = \widehat{a}^{-2}e^{\psi}, \widehat{a}^2\widehat{h}_3 = h_3, \mathbf{e}^3 = dy^3 + \partial_k n\, dx^k, \widehat{\mathbf{e}}^4 = d\widehat{t} + \sqrt{|h_4|}(\partial_i t + w_i)$. Small off-diagonal deformations can be modeled with a small parameter ε, with $0 \leq \varepsilon < 1$, where

$$\eta_i \simeq 1 + \varepsilon\chi_i(x^k,\widehat{t}), \partial_k n \simeq \varepsilon\widehat{n}_i(x^k), \\ \sqrt{|h_4|}(\partial_i t + w_i) \simeq \varepsilon\widehat{w}_i(x^k,\widehat{t}). \quad (21)$$

We can choose a subclass of generating functions and sources when $\widehat{a}(x^i,\widehat{t}) \to \widehat{a}(t), \widehat{h}_3(x^i,\widehat{t}) \to \widehat{h}_3(\widehat{t})$ etc. Such conditions, or conditions of type (21), have to be imposed after a locally anisotropic solution was constructed in explicit form. This results in new classes of solutions even in diagonal limits because of the generic nonlinear and nonholonomic character of off-diagonal systems in MGFT. For $\varepsilon \to 0$ and $\widehat{a}(x^i,\widehat{t}) \to \widehat{a}(t)$, we obtain scaling factors which are very different from those in Friedmann–Lemaître–Robertson–Walker, FLRW, cosmology with GR solutions. Nevertheless, they mimic such cosmological models with re-defined interaction parameters and possible small off-diagonal deformations of cosmological evolution for modified gravity theories as we analyzed in detail in [36]. In this work, we consider effective sources encoding contributions from massive gravity, with $\widehat{a}^2\widehat{h}_3 = \frac{\check{\Phi}^2}{4|\check{\Lambda}|}$, where $\frac{\check{\Phi}^2}{\Phi^2} = \frac{|{}^{m}\Upsilon + {}^{f}\Upsilon + {}^{\mu}\Upsilon| + \int dt\, \Phi^2 |{}^{m}\Upsilon + {}^{f}\Upsilon + {}^{\mu}\Upsilon|^{\diamond}}{{}^{m}\check{\Lambda} + {}^{f}\check{\Lambda} + {}^{\mu}\check{\Lambda}}$.

The generating functions, sources, and parameters in these formulas determine integral varieties (i.e. general solutions) of certain systems of nonlinear PDE. Such values have to be fixed, which results in certain physical values compatible with experimental data. Following the procedure from section 5 of [36], we can derive a corresponding effective field theory; see also references therein.

8 Reconstructing off-diagonal cosmological models in MGFT

Let us consider a model when the gravitational Lagrange density (4) is chosen $\mathbf{f}(\tilde{\mathbf{R}}) = \widehat{\mathbf{R}} + \mathbf{M}({}^{\mu}\mathbf{T})$, where ${}^{\mu}\mathbf{T} := \mathbf{T} + 2\,\mu^2(3 - \mathrm{tr}\sqrt{\mathbf{g}^{-1}\mathbf{q}} - \det\sqrt{\mathbf{g}^{-1}\mathbf{q}})$. We denote ${}^{1}\mathbf{M} := \mathbf{dM}/\mathbf{d}\,{}^{\mu}\mathbf{T}$ and $\widehat{H} := \widehat{a}^{\diamond}/\widehat{a}$ for a limit $\widehat{a}(x^i,\widehat{t}) \to \widehat{a}(t)$ taken for a solution (20) and consider that an observer is in a nonholonomic basis (1) with $N_i^a = \{n_i, w_i(t)\}$ for a nontrivial off-diagonal vacuum with effective polarizations $\eta_\alpha(t)$. It should be emphasized that $\widehat{a}(t)$ is different from $\mathring{a}(t)$ for a standard FLRW cosmology.

The cosmological scenarios are tested in terms of the redshift $1 + z = \widehat{a}^{-1}(t)$ for and ${}^{\mu}T = {}^{\mu}T(z)$, with a new "shift" derivative where (for instance, for a function $s(t)$) $s^{\diamond} = -(1+z)H\partial_z$. We can derive MGFT off-diagonal deformed FLRW equations following the procedure considered for the formulas (63) and (64) in [36]. It is described by a set of three equations

$$
\begin{aligned}
&3\widehat{H}^2 + \frac{1}{2}[\mathbf{f}(z) + \mathbf{M}(z)] - \kappa^2 \rho(z) = 0, \\
&-3\widehat{H}^2 + (1+z)\widehat{H}(\partial_z \widehat{H}) - \frac{1}{2}\{\mathbf{f}(z) + \mathbf{M}(z) \\
&\quad +3(1+z)\widehat{H}^2\} = 0, \\
&\rho(z)\, \partial_z\, \mathbf{f} = 0.
\end{aligned} \tag{22}
$$

Re-defining the generating function, we fix the condition $\partial_z\ {}^1\mathbf{M}(z) = 0$ and satisfy the condition $\partial_z\ \mathbf{f} = 0$, which allows nonzero densities in certain adapted frames of references. The functional $\mathbf{M}({}^{\mu}\mathbf{T})$ encodes degrees of freedom of mass gravity for the evolution of the energy density where $\rho = \rho_0 a^{-3(1+\varpi)} = \rho_0(1+z)a^{3(1+\varpi)}$. This is taken for the dust matter approximation ϖ and $\rho \sim (1+z)^3$.

Using (22), it is possible to elaborate reconstruction procedures for nontrivial μ in a form similar to that in [36–39]. For instance, it is well known that any FLRW cosmology can be realized in a specific $f(R)$ gravity. Here we analyze how specific MGFTs and the FLRW cosmology can be encoded into off-diagonal deformations. Let us introduce the "e-folding" variable $\zeta := \ln a/a_0 = -\ln(1+z)$ considered instead of the cosmological time t. We take $\mathbf{f}(\tilde{\mathbf{R}})$ as in (4), use $\widehat{\Upsilon}(x^i,\zeta) = {}^m\Upsilon(x^i,\zeta) + {}^f\Upsilon(x^i,\zeta) + {}^{\mu}\Upsilon(x^i,\zeta)$ instead of (10) and parameterize the geometric objects with dependencies on (x^i,ζ) (in particular, only on ζ), for corresponding generating functions (16), where $\partial_\zeta = \partial/\partial\zeta$ with $s^{\diamond} = \widehat{H}\partial_\zeta s$ for any function s. The matter energy density ρ is (22).

With respect to N-adapted frames(1), we can repeat all computations leading to Eqs. (2)–(7) in [37] and prove that MGFTs with $\mathbf{f}(\tilde{\mathbf{R}})$ realize a FLRW like cosmological model. The nonholonomic field equation corresponding to the first FLRW equation is

$$
\begin{aligned}
\mathbf{f}(\tilde{\mathbf{R}}) = &(\widehat{H}^2 + \widehat{H}\ \partial_\zeta \widehat{H})\partial_\zeta[\mathbf{f}(\tilde{\mathbf{R}})] \\
&-36\widehat{H}^2[4\widehat{H} + (\partial_\zeta \widehat{H})^2 + \widehat{H}\partial^2_{\zeta\zeta}\widehat{H}]\partial^2_{\zeta\zeta}\mathbf{f}(\tilde{\mathbf{R}})] + \kappa^2\rho.
\end{aligned}
$$

Introducing an effective quadratic Hubble rate, $\tilde{\kappa}(\zeta) := \widehat{H}^2(\zeta)$, where $\zeta = \zeta(\tilde{\mathbf{R}})$ for certain parameterizations, this equation transforms into

$$
\begin{aligned}
\mathbf{f} = &-18\tilde{\kappa}(\zeta)[\partial^2_{\zeta\zeta}\tilde{\kappa}(\zeta) + 4\partial_\zeta\tilde{\kappa}(\zeta)]\frac{\mathrm{d}^2\mathbf{f}}{\mathrm{d}\tilde{\mathbf{R}}^2} \\
&+6\left[\tilde{\kappa}(\zeta) + \frac{1}{2}\partial_\zeta\tilde{\kappa}(\zeta)\right]\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}\tilde{\mathbf{R}}} \\
&+2\rho_0 a_0^{-3(1+\varpi)} a^{-3(1+\varpi)\zeta(\widehat{\mathbf{R}})}.
\end{aligned} \tag{23}
$$

Off-diagonal cosmological models are determined by metrics of type (20), $t \to \zeta$, and a functional $\mathbf{f}(\tilde{\mathbf{R}})$ used for computing $\widehat{\Upsilon}$ and $\check{\Phi}$. Such nonlinear systems can be described effectively by the field equations for an (nonholonomic) Einstein space $\check{\mathbf{R}}^{\alpha}{}_{\beta} = \check{\Lambda}\delta^{\alpha}{}_{\beta}$. The value $d\mathbf{f}/d\tilde{\mathbf{R}}$ and higher derivatives vanish for any functional dependence $\mathbf{f}(\check{\Lambda})$ with $\partial_\zeta\check{\Lambda} = 0$. As we work with off-diagonal configurations, the recovering procedure simplifies substantially in such cases.

9 An example of reconstruction of MGFT and nonholonomically deformed Einstein spaces reproducing the Λ CDM era

We consider any $\widehat{a}(\zeta)$ and $\widehat{H}(\zeta)$ determined by an off-diagonal solution (20), with respect to correspondingly N-adapted frames. The analog of the FLRW equation for ΛCDM cosmology is

$$
\begin{aligned}
3\kappa^{-2}\widehat{H}^2 &= 3\kappa^{-2}H_0^2 + \rho_0\widehat{a}^{-3} \\
&= 3\kappa^{-2}H_0^2 + \rho_0 a_0^{-3}e^{-3\zeta},
\end{aligned} \tag{24}
$$

where H_0 and ρ_0 are fixed to have certain constant values. Such assumptions are considered after the coefficients of off-diagonal solutions are found and where the dependencies on (x^i,ζ) are changed into dependencies on ζ. The values with "hat" are generated via a corresponding re-definition of the generating functions and the effective sources. The first term on the r.h.s. is related to an effective cosmological constant $\check{\Lambda}$ (9) which appears in re-definition (16). For this model, the second term in (24) describes, in general, an inhomogeneous distribution of cold dark mater (CDM). The similarity with the diagonalizable cosmological models in GR is kept if $\check{\Lambda} = 12H_0^2$ to survive in the limit $w_i, n_i \to 0$, for certain approximations of type (21).

The effective quadratic Hubble rate and the modified scalar curvature, $\tilde{\mathbf{R}}$, are computed using (24), respectively,

$$
\begin{aligned}
&\tilde{\kappa}(\zeta) := H_0^2 + \kappa^2\rho_0 a_0^{-3}e^{-3\zeta} \quad \text{and} \\
&\tilde{\mathbf{R}} = 3\partial_\zeta\tilde{\kappa}(\zeta) + 12\tilde{\kappa}(\zeta) = 12H_0^2 + \kappa^2\rho_0 a_0^{-3}e^{-3\zeta}.
\end{aligned}
$$

Equation (23) transforms into

$$
\begin{aligned}
&X(1-X)\frac{\mathrm{d}^2\mathbf{f}}{\mathrm{d}X^2} + [\chi_3 - (\chi_1 + \chi_2 + 1)X]\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}X} \\
&-\chi_1\chi_2\mathbf{f} = 0,
\end{aligned} \tag{25}
$$

for certain constants, for which $\chi_1 + \chi_2 = \chi_1\chi_2 = -1/6$ and $\chi_3 = -1/2$ where $3\zeta = -\ln[\kappa^{-2}\rho_0^{-1}a_0^3(\tilde{\mathbf{R}} - 12H_0^2)]$ and $X := -3 + \tilde{\mathbf{R}}/3H_0^2$. The solutions of such equations with constant coefficients and for different types of scalar curvatures were found in [37] and [36] as Gauss hypergeometric functions. Similarly, we denote $\mathbf{f} = F(X) := F(\chi_1,\chi_2,\chi_3;X)$, where for some constants A and B,

$$
\begin{aligned}
F(X) = &AF(\chi_1,\chi_2,\chi_3;X) + BX^{1-\chi_3}F(\chi_1 - \chi_3 + 1, \\
&\chi_2 - \chi_3 + 1, 2 - \chi_3; X).
\end{aligned}
$$

This provides a proof of the statement that MGFT can indeed describe ΛCDM scenarios without the need of an effective cosmological constant.

10 Final remarks

One of the most interesting results of applications of the AFDM [28–34] to nonlinear MGFTs systems is that via a redefinition of generating functions and effective sources we can mimic f modifications and massive gravity effects. This is possible by modeling modified theories via off-diagonal interactions in effective Einstein spaces. Such models are generically nonlinear, parametric, and are considered with respect to nonholonomic frames which allows one to decouple and integrate the associated PDEs in general forms.

There is a proof of the absence of FLRW cosmology in massive gravity (see section 2.1 in [40]). The proof follows for homogeneous and isotropic ansatz for metrics in certain models of massive theory. In this paper, we studied more general constructions both for modified gravity functionals and off-diagonal locally anisotropic and inhomogeneous metrics. Our solutions describe massive gravity effects encoded both in effective matter sources and in off-diagonal deformations. Even for very special cases when $f(\tilde{\mathbf{R}})$ is linear on $\tilde{\mathbf{R}}$ such contributions are not trivial because such a scalar curvature is computed not for the Levi-Civita connection but for a nonholonomically deformed ansatz. Considering holonomic configurations, we can reproduce the general results [1] or model cosmological scenarios from [2–4]. For nonlinear systems, it is very important when certain assumptions and additional constraints are considered. If some "simplifications" or approximations are made at the very beginning, we formulate certain conclusions about properties of a theory and even follow a procedure of finding of solutions. But we can also eliminate a number of other types of solutions and various nonlinear characteristics. In our approach, we elaborated a more general and more realistic model with generic off-diagonal effects with certain stability configurations and off-diagonal modifications of FLRW cosmology generated by effective sources in nonlinear massive gravity.

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