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Results in Mathematics



# Free Cyclic Submodules in the Context of the Projective Line

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**Abstract.** We discuss the free cyclic submodules over an associative ring  $R$  with unity. Special attention is paid to those which are generated by outliers. This paper describes all orbits of such submodules in the ring of lower triangular  $3 \times 3$  matrices over a field  $F$  under the action of the general linear group. Besides rings with outliers generating free cyclic submodules, there are also rings with outliers generating only torsion cyclic submodules and without any outliers. We give examples of all cases.

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**Keywords.** Admissible pairs, free cyclic submodules, projective line, non-unimodular pairs, outliers.

## 1. Introduction

In Benz [3] describes classical geometries of Möbius, Laguerre and Minkowski, using the notion of the projective line over a ring. Veldkamp in [19] points out that the assumption of a ring to be of stable rank 2, allows to generalize many properties from classical projective geometry over a field. They both define the projective line by unimodular pairs.

To admit a wider class of rings, Herzer [11] defines a point of the projective line as a cyclic submodule generated by an admissible pair. Hence points of  $\mathbb{P}(R)$  are elements of an orbit under the action of the  $GL_2(R)$ . In the present paper we adopt this convention as well. This approach without any assumptions leads to the existence of points of  $\mathbb{P}(R)$  properly contained in another point. Blunck and Havlicek remark that avoiding this bizarre situation is equivalent to the assumption that a ring is Dedekind-finite, see [6, Proposition 2.2].

Havlicek and Saniga [15] propose to consider another type of free cyclic submodules, i.e. represented by pairs not contained in any cyclic submodule generated by a unimodular pair (so-called outliers). In this note, we show that the class of non-unimodular free cyclic submodules can be wide. We find four orbits of such submodules in the ring  $T_3$  of lower triangular  $3 \times 3$  matrices over a field  $F$  under the action of  $GL_2(T_3)$ . On the other hand there are classes of rings without outliers (e.g. semisimple rings, Proposition 5) and rings such that outliers generate only torsion submodules (e.g. finite commutative rings, Theorem 5). This answers the question posed in [12] about outliers in finite rings. We remark also that there are infinite rings with non-unimodular free cyclic submodules properly contained in unimodular ones (Proposition 4). Furthermore, we show that if  $R$  is a finite ring and non-unimodular  $R(a, b) \subset R^2$  is free, then  $(a, b)$  is an outlier. In that case, there is no need to check the condition from the definition.

Using the classification of finite rings from [9, 10], we find all rings up to order  $p^4$ ,  $p$  prime, with outliers generating free cyclic submodules.

The problem to completely characterize rings with outliers, especially generating free cyclic submodules, is still open.

## 2. Preliminaries

Throughout this paper we shall only consider associative rings with 1 ( $1 \neq 0$ ). The group of invertible elements of the ring  $R$  will be denoted by  $R^*$ . If  $R$  is a ring, the expression  $R^2$  will mean a left free module over  $R$ . If  $(a, b) \in R^2$ , then the set:  $R(a, b) = \{(\alpha a, \alpha b); \alpha \in R\}$  is a *left cyclic submodule* of  $R^2$ . It is called *free* if the equation  $(ra, rb) = (0, 0)$  implies that  $r = 0$ , i.e.  $R(a, b)$  is non-torsion. We assume that  $R$  satisfies the invariant basis property (IBP) [8]. For such rings the basis of a cyclic submodule  $R(a, b) \subset R^2$  is always of cardinality 1 and any invertible matrix with entries in  $R$  has square size, i.e. it belongs to the general linear group  $GL_n(R)$  for some natural number  $n$ .

The general linear group  $GL_2(R)$  acts in natural way (from the right) on the free left  $R$ -module  $R^2$ .

**Definition 1** [6]. *The projective line over  $R$*  is the orbit

$$\mathbb{P}(R) := R(1, 0)^{GL_2(R)}$$

of the free cyclic submodule  $R(1, 0)$  under the action of  $GL_2(R)$ .

In other words, the points of  $\mathbb{P}(R)$  are those free cyclic submodules of  $R^2$  which possess a free cyclic complement. This leads to a definition of admissibility.

**Definition 2.** A pair  $(a, b) \in R^2$  is *admissible*, if there exist elements  $c, d \in R$  such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(R),$$

i.e.  $R(a, b)$  is a free cyclic submodule which has a free cyclic complement. If  $R$  is commutative, then the condition mentioned above is equivalent to

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in R^*.$$

Therefore  $\mathbb{P}(R) = \{R(a, b) \subset R^2; (a, b) \text{ admissible}\}$ . As we mentioned before, in an earlier definition of the projective line over commutative ring used by Benz [3], the points of the projective line are cyclic submodules generated by unimodular pairs.

**Definition 3.** A pair  $(a, b) \in R^2$  is *right unimodular*, if there exist elements  $x, y \in R$  such that

$$ax + by = 1.$$

Analogously,  $(a, b) \in R^2$  is *left unimodular*, if there exist elements  $x, y \in R$  such that

$$xa + yb = 1.$$

From now on, whenever we will write ‘unimodularity’, we always mean ‘right unimodularity’. We also call the cyclic submodule  $R(a, b)$  unimodular, if  $(a, b)$  is.

*Remark 1.* Obviously, the admissibility implies the unimodularity and if  $(a, b) \in R^2$  is unimodular, then  $R(a, b)$  is a free cyclic submodule of  $R^2$ .

In contrast to the cyclic submodules generated by admissible pairs, other free cyclic submodules do not have a free cyclic complement in  $R^2$ .

The following simple remark describes unimodularity in terms of (right) ideals.

*Remark 2.* Let  $R$  be a ring and  $a, b \in R$ . The following statements are equivalent:

1.  $aR + bR = R$ .
2. There exist elements  $x, y \in R$  such that  $ax + by = 1$ .
3. There is no proper right ideal  $I$  such that  $a, b \in I$ .

*Proof.* 1.  $\Leftrightarrow$  2. See [19].

2.  $\Rightarrow$  3. Suppose that there exist  $x, y \in R$  such that  $ax + by = 1$  and let  $I$  be a right ideal such that  $a, b \in I$ . Of course,  $ax \in I$  for all  $x \in R$  and  $by \in I$  for all  $y \in R$ . Consequently  $(ax + by) \in I$  for all  $x, y \in R$ . Thus  $1 \in I$ , and therefore  $I = R$ .

3.  $\Rightarrow$  2. Assume that  $ax + by \neq 1$  for all  $x, y \in R$ , then  $\{ax + by; x, y \in R\} = aR + bR \neq R$ . So,  $aR + bR$  is a proper right ideal which contradicts 3.  $\square$

In general, cyclic submodules generated by unimodular (resp. admissible) pairs can be also generated by non-unimodular (resp. non-admissible) ones. In special cases cyclic submodule generated by unimodular (resp. admissible) pair cannot have non-unimodular (resp. non-admissible) generators.

We substitute ‘admissible’ by ‘unimodular’ in [6, Proposition 2.1] and we get:

**Proposition 1.** *Let  $(x, y) \in R^2$  be unimodular and let  $r \in R$ . Put  $(a, b) := r(x, y)$ . Then*

1.  *$r$  is left invertible if, and only if,  $R(x, y) = R(a, b)$ .*
2.  *$r$  is right invertible if, and only if,  $(a, b)$  is unimodular.*

*Proof.* 1. “ $\Rightarrow$ ” If there exists an  $s \in R$  such that  $sr = 1$ , then  $s(a, b) = (x, y)$ . Hence  $R(x, y) = R(a, b)$ .

“ $\Leftarrow$ ”

If  $R(x, y) = R(a, b)$ , then there exists an  $s \in R$  such that  $s(a, b) = (x, y)$ , and so  $sr(x, y) = (x, y)$ . There are  $x', y' \in R$  with  $1 = xx' + yy' = srxx' + sryy' = sr$ . So  $r$  is left invertible.

2. Suppose that  $(x, y) \in R^2$  is unimodular, then there exist  $x', y' \in R$  with  $xx' + yy' = 1$ . Let  $r(x, y) = (a, b)$  for some  $r \in R$ .

“ $\Rightarrow$ ”

If  $r$  is right invertible, then  $rs = 1$  for some  $s \in R$ . Hence  $(a, b)$  is unimodular:

$$a(x's) + b(y's) = rx(x's) + ry(y's) = r((xx') + (yy'))s = rs = 1.$$

“ $\Leftarrow$ ”

If  $(a, b)$  is unimodular, then there exist  $a', b' \in R$  with  $aa' + bb' = 1$ , which implies that  $r$  has a right inverse:

$$aa' + bb' = (rx)a' + (ry)b' = r(xa' + yb') = 1.$$

□

Rings with the property  $ab = 1 \Rightarrow ba = 1$  are called *Dedekind-finite*. On account of the above proposition and of the Proposition 2.1 (2) [6] we obtain:

**Corollary 1.** *If  $R$  is Dedekind-finite, then any cyclic submodule  $R(a, b)$  that is generated by a unimodular (resp. admissible) pair does not have non-unimodular (resp. non-admissible) generators.*

As we know, each admissible pair  $(a, b) \in R^2$  is unimodular. What about the converse implication? There are examples of rings where unimodularity does not imply admissibility [7, Remark 5.1]. However, it is also known that if  $R$  is a ring of stable rank 2 (for example, local rings and matrix rings over fields), then admissibility and unimodularity are equivalent and  $R$  is Dedekind-finite [6, Remark 2.4]. So finite or commutative rings satisfy this property as well. In case of such rings, the projective line can be described by using unimodularity or admissibility interchangeably.

Let us introduce the following temporary notation:

(F) Any nonzero element of the ring  $R$  is either invertible or a zero divisor.

It is known that any finite ring satisfies (F). Additionally, if  $R$  satisfies (F), then the ring  $M_n(R)$  ( $n \geq 1$ ) fulfills this condition as well.

**Corollary 2.** *Let  $R$  satisfy (F),  $(x, y) \in R^2$  be unimodular and  $(a, b) = r(x, y)$ . Then the following are equivalent:*

1.  $r \in R$  is invertible.
2.  $R(x, y) = R(a, b)$ .
3.  $(a, b)$  is unimodular.

*Remark 3.* Let  $R$  be a ring and let  $(a, b) \in R^2$ . If there exists  $(x, y) \in R^2$  and a left zero divisor  $r \in R$  such that  $(a, b) = r(x, y)$ , then  $R(a, b)$  is not a free cyclic submodule.

*Proof.* Suppose that the above assumptions are satisfied. Hence there exists nonzero  $\alpha \in R$  such that  $\alpha r = 0$ , which yields:

$$\alpha(a, b) = \alpha(rx, ry) = \alpha r(x, y) = (0, 0).$$

□

Condition (F) implies that the same free cyclic submodule can be generated by two pairs precisely when they are left-proportional by an invertible element of  $R$ .

**Proposition 2.** *Let  $R$  satisfy (F). If a pair  $(a, b) \in R^2$  generates a free cyclic submodule, then all cyclic submodules that contain  $(a, b)$  are free.*

*Proof.* Assume that there exist  $(x, y) \in R^2$  and  $r \in R$  such that  $(a, b) = r(x, y)$ . According to Remark 3, this  $r$  is an invertible element of  $R$ , hence  $R(a, b) = R(x, y)$ . □

The next class of cyclic submodules, which can be considered in the context of the projective line, is the one proposed by Havlicek and Saniga in [15].

**Definition 4** [12, Definition 9]. A pair  $(a, b) \in R^2$  that is not contained in any cyclic submodule generated by a unimodular pair is called *an outlier*.

In the last section, will be needed one more concept. Recall that a monomorphism of modules  $f : M' \rightarrow M$  is called *split* if there exists  $g : M \rightarrow M'$  such that  $g \circ f = 1_{M'}$ .

### 3. Free Cyclic Submodules Generated by Non-unimodular Pairs

For some rings there are free cyclic submodules which are generated only by non-unimodular pairs. We establish some connections between outliers and non-principal ideals [12].

**Proposition 3.** *Let  $(a, b) \in R^2$  be non-unimodular. If the right ideal  $aR + bR$  is non-principal, then  $(a, b)$  is an outlier.*

*Proof.* Assume that  $(a, b)$  is not an outlier. By Definition 4. there exist  $\alpha \in R$  and an unimodular pair  $(x, y) \in R^2$  such that  $(a, b) = \alpha(x, y)$ . Hence  $aR + bR = \alpha xR + \alpha yR = \alpha(xR + yR) = \alpha R$ , which completes the proof.  $\square$

**Corollary 3** [12, Theorem 13]. *Let  $(a, b) \in R^2$  be non-unimodular. Then  $(a, b)$  is an outlier, if one of the following conditions is satisfied:*

1. *There does not exist a principal proper right ideal  $\alpha R$  such that  $a, b \in \alpha R$ .*
2.  *$aR + bR \subsetneq \alpha R$  for all principal proper right ideals  $\alpha R$  such that  $a, b \in \alpha R$ .*

**Theorem 1.** *Let  $R$  satisfy (F).*

1. *If there exists a principal proper right ideal  $\alpha R$  containing  $a$  and  $b$ , then  $R(a, b)$  is a torsion cyclic submodule.*
2. *If  $R(a, b) \subset R^2$  is non-unimodular and free, then  $(a, b)$  is an outlier.*

*Proof.* 1. Suppose that the above assumptions are satisfied. Then there exist  $c, d \in R$  such that  $a = \alpha c, b = \alpha d$  and nonzero  $r \in R$  with  $r\alpha = 0$ . We thus get  $r(a, b) = r(\alpha c, \alpha d) = (0, 0)$ , which is our claim.

2. If  $R(a, b)$  is a free cyclic submodule, then  $r \in R$  is invertible if  $(a, b) = r(x, y)$  for some  $(x, y) \in R^2$ , which follows from Remark 3. Hence  $(a, b)$  is an outlier (see Corollary 2).  $\square$

*Example 1.* Consider the ring  $R = \left\{ \begin{bmatrix} a & 0 & 0 \\ b & a & 0 \\ c & 0 & d \end{bmatrix}; a, b, c, d \in GF(2) \right\}$ . Let

$$\begin{aligned} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} &= A, \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = B, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = C, \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} &= D, \\ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} &= I, \quad \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = J, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = K, \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} &= 0. \end{aligned}$$

There are exactly two right ideals which are not principal:  $I_1 = \{0, I, J, K\}$ ,  $I_2 = \{0, A, B, C, D, I, J, K\}$ . By Proposition 3. pairs of matrices which are generators of the right ideals  $I_1, I_2$ , are outliers. For example,  $\text{gen}(I, A) = \text{gen}(K, A) = \text{gen}(A, D) = \text{gen}(A, B) = I_2$ . An easy calculation shows that each of them generates a free cyclic submodule, for instance,

$$\begin{aligned}
 & \begin{bmatrix} a & 0 & 0 \\ b & a & 0 \\ c & 0 & d \end{bmatrix} \left( \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right) \\
 &= \left( \begin{bmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ d & 0 & 0 \end{bmatrix}, \begin{bmatrix} a & 0 & 0 \\ b+a & a & 0 \\ c+d & 0 & 0 \end{bmatrix} \right) \\
 &= \left( \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \Leftrightarrow \begin{bmatrix} a & 0 & 0 \\ b & a & 0 \\ c & 0 & d \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
 \end{aligned}$$

There is one principal right ideal  $I_3$  which contains eight elements, including:  $I, J, K$ . Notice that the right ideal  $IR + JR = \{0, I\} + \{0, J\} = \{0, I, J, K\} \subsetneq I_3$ . Although the pairs  $(I, J), (J, I)$  are outliers, they do not generate free cyclic submodules by Theorem 1.1. By the same methods it follows that the pairs  $(I, K), (K, I), (K, J), (J, K)$  are outliers too and do not generate free cyclic submodules. In consequence, there are two different kinds of outliers. 24 of them generate 6 free cyclic submodules and 6 others do not.

*Example 2.* Choose the pair  $v = \left( \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right)$  over the ring  $R$

of matrices  $\left\{ \begin{bmatrix} a & 0 & 0 \\ b & a & 0 \\ c & 0 & d \end{bmatrix}; a, b, c, d \in GF(p), p \text{ prime} \right\}$ .  $v$  is right and left non-unimodular over  $R$ . The left cyclic submodule  $Rv$  is free and  $v$  is an outlier. But the right cyclic submodule  $vR$  is torsion and  $v$  is not an outlier. So the definition of outlier is not symmetric.

*Example 3 [14, 15].* Consider the ring  $T$  of ternions over the commutative field  $F$ , i.e. the ring of upper triangular  $2 \times 2$  matrices with entries from  $F$ . The free cyclic submodules of  $T^2$  fall into two distinct orbits under the action of the  $GL_2(T)$ :

$$O_1 = T \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right)^{GL_2(T)} \quad O_2 = T \left( \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right)^{GL_2(T)}.$$

The first orbit makes up the projective line  $\mathbb{P}(T)$ , the second one is the orbit of free cyclic submodules generated by outliers.

Let  $R = T_3$  be the ring of lower triangular  $3 \times 3$  matrices with entries from an arbitrary commutative field  $F$ .

**Theorem 2.** *Under the action of the general linear group  $GL_2(T_3)$  the free cyclic submodules of  $T_3^2$  fall into 5 distinct orbits. The pairs generating free cyclic submodules of  $T_3^2$  fall into  $4 + |F|$  distinct orbits with the following representatives and the right ideals generated by them:*

1.  $\left\{ \left[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right], I_1 = \left\{ \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}; a, b, c, d, e, f \in F \right\};$
2.  $\left\{ \left[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right], I_2 = \left\{ \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & 0 \end{bmatrix}; a, b, c, d, e \in F \right\};$
3.  $\left\{ \left[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & e & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right], e \in F \right\},$   
 $I_3 = \left\{ \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & ec & 0 \end{bmatrix}; a, b, c, d \in F \right\};$
4.  $\left\{ \left[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right], I_4 = \left\{ \begin{bmatrix} a & 0 & 0 \\ b & 0 & 0 \\ d & e & 0 \end{bmatrix}; a, b, d, e \in F \right\};$
5.  $\left\{ \left[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right], I_5 = \left\{ \begin{bmatrix} a & 0 & 0 \\ b & 0 & 0 \\ d & e & f \end{bmatrix}; a, b, d, e, f \in F \right\}.$

*Proof.* Clearly, all unimodular pairs are in the orbit

$$\left[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right]^{GL_2(T_3)}.$$

Let now  $\left[ \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}, \begin{bmatrix} a' & 0 & 0 \\ b' & c' & 0 \\ d' & e' & f' \end{bmatrix} \right], a, b, c, d, e, f, a', b', c', d', e', f' \in F:$

1. be non-unimodular;
2. generate a free cyclic submodule.

We obtain from 2. that  $a \neq 0$  or  $a' \neq 0$ , and then we assume  $a \neq 0$ . Multiplying by the invertible matrix

$$\left[ \begin{bmatrix} a^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} a^{-1}a' & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \right]$$

$0$   $I$

gives a pair  $\left[ \begin{bmatrix} 1 & 0 & 0 \\ b_1 & c & 0 \\ d_1 & e & f \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ b'_1 & c' & 0 \\ d'_1 & e' & f' \end{bmatrix} \right]$ . Here  $I$  is the identity matrix.



We consider now all possibilities:

Case 1.  $c \neq 0$ . The result of multiplication

$$\left[ \left[ \begin{bmatrix} 1 & 0 & 0 \\ b_1 & c & 0 \\ d_1 & e & f \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ b'_1 & c' & 0 \\ d'_1 & e' & f' \end{bmatrix} \right] \left[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & c^{-1} & 0 \\ 0 & 0 & 1 \\ & & & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -c^{-1}c' & 0 \\ 0 & 0 & 0 \\ & & & I \end{bmatrix} \right]$$

is a pair  $\left[ \left[ \begin{bmatrix} 1 & 0 & 0 \\ b_1 & 1 & 0 \\ d_1 & e_1 & f \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ b'_1 & 0 & 0 \\ d'_1 & e'_1 & f' \end{bmatrix} \right]$ . From 1. we get  $f = f' = 0$ .

Case 1.1.  $b'_1 \neq 0$ . We multiply  $\left[ \left[ \begin{bmatrix} 1 & 0 & 0 \\ b_1 & 1 & 0 \\ d_1 & e_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ b'_1 & 0 & 0 \\ d'_1 & e'_1 & 0 \end{bmatrix} \right]$  by the invertible matrix

$$\left[ \begin{array}{cc} I & 0 \\ \left[ \begin{array}{ccc} -b_1^{-1}b_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] & \left[ \begin{array}{ccc} b_1^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \end{array} \right],$$

which gives a pair  $\left[ \left[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ d_2 & e_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ d'_2 & e'_1 & 0 \end{bmatrix} \right]$ . We know from 2. that  $e_1 \neq d'_2$ .

Case 1.1.1.  $e'_1 \neq 0$ . We multiply again last pair by the invertible matrix

$$\left[ \begin{array}{cc} I & \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \left[ \begin{array}{ccc} 0 & 0 & 0 \\ -e_1^{-1}d_2 & -e_1^{-1}e_1 & 0 \\ 0 & 0 & 0 \end{array} \right] & \begin{bmatrix} 1 & 0 & 0 \\ e_1^{-1}(e_1 - d'_2) & e_1^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \right]$$

and finally we obtain representative of the orbit:  $\left[ \left[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right]$ .

In the same manner (considering other cases and multiplying by invertible matrices) we get all orbits of pairs generating free cyclic submodules of  $T_3^2$ .

It is easy to check that they are distinct, i.e. there is no invertible matrix that converts one orbit to another. Multiplying the representatives of the orbits with invertible elements of  $T_3$  (from the left) immediately yields that all free cyclic submodules generated by pairs from point 3. are in the same orbit:

$$\left[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -e & 1 \end{bmatrix} \left[ \left[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & e & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right] \right] = \left[ \left[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right] \right].$$

□

**Corollary 4.** *Two pairs  $(x, y), (w, z) \in T_3^2$  generating free cyclic submodules are in the same  $GL_2(T_3)$ -orbit if, and only if, the right ideals generated by  $x, y \in T_3$  and by  $w, z \in T_3$  coincide.*

*Proof.* Let  $I_{(x,y)}$  denote the right ideal of  $T_3$  which is generated by  $x$  and  $y$ .  
 “ $\Rightarrow$ ”

If pairs  $(x, y), (w, z) \in T_3^2$  are in the same  $GL_2(T_3)$ -orbit, then there exists a matrix  $A \in GL_2(T_3)$  such that  $(x, y)A = (w, z)$ . This gives  $I_{(w,z)} \subseteq I_{(x,y)}$ .

Next we multiply last equation by  $A^{-1}$ , which yields  $(x, y) = (w, z)A^{-1}$ , and, in consequence  $I_{(x,y)} \subseteq I_{(w,z)}$ . The result is  $I_{(x,y)} = I_{(w,z)}$ .  
 “ $\Leftarrow$ ”

This is straightforward from Theorem 2. □

In case of rings without  $(F)$  there are also non-unimodular free cyclic submodules that are not generated by outliers.

**Proposition 4.** *If  $R$  is a (commutative) PID, then non-unimodular free cyclic submodules are generated by non-outliers.*

*Proof.* We use the fact that  $\gcd(a, b) = 1$  implies  $\text{gen}(a, b) = R$  for any elements  $a, b$  of a proper commutative PID  $R$  (see [16]).

Suppose that  $(a, b) \in R^2$  is non-unimodular. If  $\gcd(a, b) = d$ , then  $a = dr_1, b = dr_2$  with  $\gcd(r_1, r_2) = 1$ . Hence  $(r_1, r_2)$  is unimodular and  $(a, b) \in R(r_1, r_2)$ , so  $(a, b)$  is a non-outlier. □

### 4. Rings Without Non-unimodular Free Cyclic Submodules

There are some rings  $R$  such that free cyclic submodules  $R(a, b)$  are generated only by admissible pairs  $(a, b) \in R^2$ . They all make up the projective line  $\mathbb{P}(R)$ , for instance, fields or finite local rings [12, Theorem 20. 1]. In case of these rings all free cyclic submodules can be written as the points of the projective line:  $\mathbb{P}(R) = \{R(1, x); x \in R\} \cup \{R(d, 1); d \in I\}$ , where  $I$  denotes the unique maximal ideal of  $R$ .

**Proposition 5.** *Let  $R$  be a semisimple ring. A cyclic submodule  $R(a, b)$  is free if, and only if  $(a, b) \in R^2$  is admissible.*

*Proof.* This follows from the fact that the monomorphism

$$R \longrightarrow R(a, b) : r \mapsto r(a, b)$$

is split [1, Corollary 13.10.]. □

Another class of rings without non-unimodular free cyclic submodules comprises the finite principal ideal rings [12, Theorem 23].

Now we consider commutative finite rings.

**Theorem 3** [17, 4, VI.2]. *Let  $R$  be a commutative finite ring. There exist local rings  $R_1, R_2, \dots, R_n$  such that*

$$R = R_1 \times R_2 \times \dots \times R_n.$$

**Theorem 4.** *Let  $R$  be a direct product of rings  $R_1, R_2, \dots, R_n$ .*

1. *A pair  $((a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_n)) \in R^2$  is unimodular if, and only if, pairs*

$$(a_1, b_1) \in R_1^2, (a_2, b_2) \in R_2^2, \dots (a_n, b_n) \in R_n^2$$

*are unimodular.*

2. *A pair  $((a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_n)) \in R^2$  is admissible if, and only if, pairs*

$$(a_1, b_1) \in R_1^2, (a_2, b_2) \in R_2^2, \dots (a_n, b_n) \in R_n^2$$

*are admissible.*

3. *A pair  $((a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_n)) \in R^2$  is an outlier if, and only if, there exists  $i \in \{1, 2, \dots, n\}$  such that  $(a_i, b_i) \in R_i^2$  is an outlier.*
4.  *$R((a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_n))$  is a free cyclic submodule of  $R^2$  if, and only if,*

$$R_1(a_1, b_1), R_2(a_2, b_2), \dots R_n(a_n, b_n)$$

*are free cyclic submodules of  $R_1^2, R_2^2, \dots, R_n^2$ .*

*Proof.* We give the proof only for the point 3., the others are simple consequences of the definitions.

Assume that  $(a_i, b_i) \in R_i^2$  are not outliers for all  $i \in \{1, 2, \dots, n\}$ . Equivalently, we can say that there exist unimodular pairs  $(x_i, y_i) \in R_i^2$  and  $r_i \in R$  such that  $(a_i, b_i) = r_i(x_i, y_i)$  for all  $i \in \{1, 2, \dots, n\}$ . So we can write:

$$\begin{aligned} ((a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_n)) &= ((r_1x_1, r_2x_2, \dots, r_nx_n), \\ (r_1y_1, r_2y_2, \dots, r_ny_n)) &= (r_1, r_2, \dots, r_n)((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)). \end{aligned}$$

In the light of the point 1.  $((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n))$  is unimodular. According to Definition 4.  $((a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_n))$  is not an outlier. □

**Theorem 5.** *If  $R$  is a commutative finite ring, then outliers do not generate free cyclic submodules.*

*Proof.* Let  $R$  be a commutative finite ring. In the light of Theorem 3.  $R$  is a direct product of local rings  $R_1, R_2, \dots, R_n$ . Write  $a_i, b_i \in R_i$  for all  $i \in \{1, 2, \dots, n\}$ .

An equivalent formulation of the above theorem is now:

$R((a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_n))$  is a free cyclic submodule of  $R^2$  if, and only if,  $((a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_n))$  is unimodular.

“ $\Rightarrow$ ” Assume that  $R((a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_n))$  is a free cyclic submodule of  $R^2$ . According to Theorem 4.  $R_i(a_i, b_i)$  is a free cyclic submodule of  $R_i^2$  for all  $i \in \{1, 2, \dots, n\}$ . As we know,  $(a_i, b_i)$  is unimodular for all  $i \in \{1, 2, \dots, n\}$ . Theorem 4. now yields  $((a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_n))$  is unimodular.

“ $\Leftarrow$ ” Follows from Remark 1. □

**Corollary 5.** *If  $R$  is a commutative finite ring, then all free cyclic submodules make up the projective line  $\mathbb{P}(R)$ .*

*Example 4.* Let us consider the finite noncommutative ring  $R$  of order  $p^4$  with characteristic  $p^2, p$  prime. The additive group of  $R$  is equal to  $R^+ = Z_{p^2} \oplus Z_p \oplus Z_p$  with a basis  $\{1, t, y\}$ . The multiplication in the ring  $R$  is uniquely determined by the relations  $t^2 = 0, y^2 = y, ty = 0, yt = t$ , [9].

We have  $(1 - t - y)(r + st + hy) + t(r' + s't + h'y) = r + (-r + r')t - ry$  for some  $0 \leq r, r' \leq p^2 - 1, 0 \leq s, h, s', h' \leq p - 1$ , hence the pair  $(1 - t - y, t)$  is non-unimodular. It is easily seen that  $R(1 - t - y, t)$  is free. On account of Theorem 1.  $(1 - t - y, t)$  is an outlier.  $R$  is an example of a ring non-embeddable into any ring of matrices over  $GF(p^2)$ , [18].

Now we are able to describe all finite rings up to order  $p^4, p$  prime, with outliers generating free cyclic submodules. Since any finite ring with identity is isomorphic to direct sum of rings with identity of prime power order (see [17]) and according to Theorems 4 and 5, we may restrict ourselves to the study of noncommutative indecomposable rings (see [1, p. 99]) up to order  $p^4, p$  prime. By direct calculation, taking into account classification theorems [9, 10] we find that there are exactly four such rings for any  $p$ :

- of order  $p^3$ : the ring of ternions over  $GF(p)$ ;
- of order  $p^4$ :
  - with characteristic  $p$ :

$$\left\{ \begin{bmatrix} a & 0 & 0 \\ b & a & 0 \\ c & 0 & d \end{bmatrix}; a, b, c, d \in GF(p) \right\}; \left\{ \begin{bmatrix} a & c & d \\ 0 & b & 0 \\ 0 & 0 & b \end{bmatrix}; a, b, c, d \in GF(p) \right\};$$

- with characteristic  $p^2$ —the ring from Example 4.

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