EURASIP Journal on Applied Signal Processing 2003:9, 914–921

The PLSI Method of Stabilizing Two-Dimensional Nonsymmetric Half-Plane Recursive Digital Filters

N. Gangatharan
Faculty of Engineering, Multimedia University, Jalan Multimedia, 63100 Cyberjaya, Selangor, Malaysia
Email: n.gangatharan@mmu.edu.my

P. S. Reddy
Faculty of Engineering, Multimedia University, Jalan Multimedia, 63100 Cyberjaya, Selangor, Malaysia
Email: subbarami.reddy@mmu.edu.my

Received 19 April 2002 and in revised form 19 February 2003

Two-dimensional (2D) recursive digital filters find applications in image processing as in medical X-ray processing. Nonsymmetric half-plane (NSHP) filters have definitely positive magnitude characteristics as opposed to quarter-plane (QP) filters. In this paper, we provide methods for stabilizing the given 2D NSHP polynomial by the planar least squares inverse (PLSI) method. We have proved in this paper that if the given 2D unstable NSHP polynomial and its PLSI are of the same degree, the PLSI polynomial is always stable, irrespective of whether the coefficients of the given polynomial have relationship among its coefficients or not. Examples are given for 2D first-order and second-order cases to prove our results. The generalization is done for the Nth order polynomial.

Keywords and phrases: PLSI, NSHP, stability, Lagrange, autocorrelation.

1. INTRODUCTION

The two-dimensional (2D) filters find numerous applications like in image processing, seismic record processing, medical X-ray processing, and so forth. The nonsymmetric half-plane (NSHP) 2D recursive filters have assured positive magnitude characteristics and so they are preferred to quarter-plane (QP) filters. But the design of stable recursive NSHP filters had been a difficult and sometimes more time-consuming problem. In this paper, we deal with the problem of stabilizing unstable NSHP 2D recursive filters by the planar least squares inverse (PLSI) approach.

The stabilization of one-dimensional (1D) recursive digital filters using least squares inverse (LSI) approach is well known [1]. However, the conjecture, made by Shanks et al. [2], known as Shanks conjecture is yet to be proved or totally disproved. Genin and Kamp [3] were the first to give a counterexample showing that the Shanks conjecture, which says that 1D technique of stabilizing can be extended to 2D case, also fails. They have taken the original unstable 2D polynomial to be of degree three in both the variables, and the corresponding PLSI polynomial of degree one was found to be unstable. Later they have produced three more counterexamples of that kind [4], where the chosen PLSI polynomial is of lower degree than the degree of the original unstable 2D polynomial. Subsequently, the modified form of the Shanks conjecture [5], which says that the PLSI polynomial should be of the same degree as the original unstable polynomial, has also been proven to be not true [6].

Two of the methods which were extensions from the 1D system theory to 2D case for stabilizing 2D recursive digital filters—namely, the discrete Hilbert transform (DHT) method and the PLSI method—have been left unsolved and not much work was reported on these in 1980s.

Recently, in [7, 8] the problem facing the DHT method was resolved. Now it is clear that the DHT method of stabilizing unstable filters works only when the original unstable polynomial is devoid of zeros on the unit circle in the 1D case and on the unit bicircle in the 2D case.

In fact, in [8], a new method of stabilizing multidimensional ($N > 2$) recursive digital filters has been presented. This new method boils down to the same DHT method when applied to 1D and 2D filters.

More recently, in [9] a complete solution to the PLSI method of stabilization was reported. It was proved that a restriction of the kind imposed on the DHT method on the original 2D polynomial is not needed for the PLSI method to work, but a different type of restriction is necessary for the PLSI method, especially for QP filters.

In this paper, we present a method of stabilizing the given

2D NSHP unstable polynomial through the PLSI polynomial approach. It is interesting to note that the PLSI polynomial is always stable provided that the degree of the PLSI polynomial is the same as the given polynomial, whatever may be the relationship among the coefficients in the given polynomial.

Definition 1. A 2D NSHP polynomial of degree N is given by $A(Z_1, Z_2) = \sum^N \sum^N_{R+\oplus} a_{mn} Z_1^m Z_2^n$, where $R + \oplus = \{m \geq 0, n \geq 0\} \cup \{m > 0, n < 0\}$.

The main difference between QP and NSHP filters comes by the way in which the output masks are defined. The output mask of NSHP is more general than that of QP filters. Hence NSHP filters will be superior to QP filters. Based on the region of support R, there are eight classes of NSHP filters. However, all our discussions are based on $R + \oplus$ filter [10].

In Section 2, we discuss the basic definition of PLSI polynomial for the QP polynomial and NSHP polynomial. We then briefly mention the method of obtaining the PLSI polynomials. In Section 3, we discuss the existence of maximum "b_{00}" for the first order and the second order which in turn results in stable polynomial. In Section 4, we present numerical examples for the first order and second order and prove that the PLSI is stable.

2. OBTAINING PLSI POLYNOMIAL FOR NSHP POLYNOMIALS

In this section, we discuss the basic definition of the PLSI polynomials and the method for obtaining the PLSI in general.

Definition 2. In the case of QP filters, if $A(Z_1, Z_2) = \sum_{i=0}^N \sum_{j=0}^N a_{ij} Z_1^i Z_2^j$ is a given 2D polynomial of degree N, then the polynomial $B(Z_1, Z_2) = \sum_{i=0}^M \sum_{j=0}^M b_{ij} Z_1^i Z_2^j$ of the degree M forms the PLSI of $A(Z_1, Z_2)$ if

$$B(Z_1, Z_2) \approx \frac{1}{A(Z_1, Z_2)} \quad \text{for } |Z_1| = 1,\ |Z_2| = 1. \tag{1}$$

To obtain $B(Z_1, Z_2)$, just like in 1D case, we first form

$$C(Z_1, Z_2) = A(Z_1, Z_2) \times B(Z_1, Z_2) \approx 1 \tag{2}$$

and then we form an error function (see [11]) E from $C(Z_1, Z_2)$, where

$$C(Z_1, Z_2) = \sum_{i=0}^{M+N} \sum_{j=0}^{M+N} c_{ij} Z_1^i Z_2^j = 1 \tag{3}$$

as

$$E = (1 - c_{00})^2 + \sum_i \sum_j c_{ij}^2. \tag{4}$$

We then differentiate E with respect to each unknown coefficient b_{ij} and equate each $\partial E/\partial b_{ij}$ to zero to get a set of linear algebraic equations of the form

$$\overline{T}\,\overline{b} = \overline{a}. \tag{5}$$

In (5), $\overline{T}$ is a square matrix of order $(M + 1) \times (M + 1)$ made up of the 2D autocorrelation functions of $A(Z_1, Z_2)$ as its elements, $\overline{b}$ is an $(M+1) \times 1$ column matrix of coefficients of $B(Z_1, Z_2)$ like

$$\overline{b} = \{b_{00}, b_{01}, \ldots, b_{mm}\}^t, \tag{6}$$

and $\overline{a}$ is also an $(M + 1) \times 1$ column matrix like

$$\overline{a} = \{a_{00}, 0, 0, \ldots, 0\}^t. \tag{7}$$

Definition 3. A 2D NSHP polynomial

$$B(Z_1, Z_2) = \sum^N \sum^N_{R+\oplus} b_{mn} Z_1^m Z_2^n \tag{8}$$

of degree N is the PLSI of $A(Z_1, Z_2) = \sum^N \sum^N_{R+\oplus} a_{mn} Z_1^m Z_2^n$, also of degree N, where $R + \oplus = \{m \geq 0, n \geq 0\} \cup \{m > 0, n < 0\}$ if (1) holds.

It may be noted that the coefficients b_{mn}'s of $B(Z_1, Z_2)$ can be obtained by solving (5) for the vector $\overline{b}$, but the formulation of (5) as explained in Section 1 is rather very tedious, especially for larger values of M and N.

Here, we indicate the method (see [12]) that can be used to form (5) by using form-preserving 1D polynomials.

Definition 4. A 1D polynomial $A_1(z) = \sum_{k=0}^N a_k Z^k$ is the form-preserving polynomial of a 2D QP polynomial $A(Z_1, Z_2)$:

$$A(Z_1, Z_2) = \sum_{m=0}^{N_1} \sum_{n=0}^{N_2} a_{mn} Z_1^m Z_2^n \tag{9}$$

if for every integer set (m, n) in $A(Z_1, Z_2)$ there exists a unique k such that $a_k = a_{mn}$.

It has been proved in [13] that $A_1(Z) = A(Z_1^L, Z)$ forms the form-preserving polynomial of $A(Z_1, Z_2)$ if $L \geq (N_2 + 1)$. It has also been proved in [13] that if $B(Z_1, Z_2)$ and $A(Z_1, Z_2)$ are two 2D polynomials as defined in Definition 2, then if

$$C(Z_1, Z_2) = A(Z_1, Z_2) \times B(Z_1, Z_2), \tag{10}$$

$C(Z) = C(Z^L, Z)$ will be a 1D form-preserving polynomial of $C(Z_1, Z_2)$ if

$$L \geq N + M + 1. \tag{11}$$

We quoted this concept of form-preserving polynomials because later we are going to use these form preserving 1D polynomials for formulating the matrix $\overline{T}$ of (5) as well as for testing the stability or instability of the PLSI polynomials.

Theorem 1. *If $B(Z_1, Z_2)$ is the PLSI polynomial of $A(Z_1, Z_2)$, then $B_1(Z)$ is the LSI of $A_1(Z)$ if $L = M+N+1$, where $B_1(Z) = B(Z^L, Z)$ and $A_1(Z) = A(Z^L, Z)$.*

The above theorem has been proved in [13]. Obtaining the coefficients of 1D LSI polynomial $B_1(Z)$ corresponding to the 1D polynomial $A_1(Z)$ is very easy. Since $\overline{T}$ matrix can be mechanically written down in terms of the coefficients $A_1(Z)$ [1], this method of deriving the 2D PLSI polynomial $B(Z_1, Z_2)$, corresponding to the given 2D polynomial $A(Z_1, Z_2)$, is highly recommended. Once we form the $\overline{T}$ matrix of (5), of course after deleting certain rows and columns of a corresponding coefficient matrix pertaining to $A_1(Z)$, we can easily solve (5) for $\overline{b}$, the column vector of coefficients of $B(Z_1, Z_2)$.

We now elaborate on the deletion of certain rows and corresponding columns from the coefficient matrix, mechanically written from the coefficients of the polynomial $A_1(Z)$. We obviously know that $B_1(Z)$ which is a form-preserving polynomial of $B(Z_1, Z_2)$ will be lacunary with some terms corresponding to certain powers of Z being absent. This is because $B_1(Z)$ is what we get as $B(Z^L, Z)$, with L value being $M + N + 1$ which is much more than $(M + 1)$. So when we frame the matrix equation mechanically for $A_1(Z)$ and the corresponding LSI $B_1(Z)$ of the following type:

$$\overline{T_1}\,\overline{b_1} = \overline{a_1}, \tag{12}$$

the column vector $\overline{b_1}$ does contain some zeros; and while arriving at (5), we have to delete some rows and corresponding columns of $\overline{T_1}$, some rows of $\overline{b_1}$ corresponding to zero coefficients in $B_1(Z)$, and some rows of $\overline{a_1}$.

Also if $N > M$, the last $N - M$ rows and corresponding columns of $\overline{T_1}$ are to be deleted. The minimum error (see [1]) is

$$E_{\min} = 1 - b_{00}a_{00}. \tag{13}$$

3. OBTAINING PLSI POLYNOMIAL FOR FIRST-ORDER NSHP POLYNOMIALS

Theorem 2. *If $B(Z_1, Z_2) = \sum^N \sum_{R+\oplus}^N b_{mn} Z_1^m Z_2^n$ is a 2D NSHP polynomial of degree N, then its form-preserving 1D polynomial $B_1(Z) = B(Z^L, Z)$, when $L = 4N + 1$, will have the same autocorrelation coefficients as the $B(Z_1, Z_2)$.*

This theorem can be used to our advantage whenever we want to form the autocorrelation coefficients of a 2D polynomial since obtaining these coefficients from 1D polynomial $B_1(Z)$ is very simple. What we get from (5) will be the same as what we get from the 1D polynomial $B_1(Z)$ by deleting proper rows and columns from (12) as mentioned earlier.

Example 1. Let $B(Z_1, Z_2) = \sum^1 \sum_{R+\oplus}^1 b_{mn} Z_1^m Z_2^n$ be a 2D first-order NSHP polynomial. This can be written as follows:

$$B(Z_1, Z_2) = b_{00} + b_{01}Z_2 + b_{10}Z_1 + b_{11}Z_1Z_2 + bZ_1Z_2^{-1}. \tag{14}$$

Then, $B_1(Z) = B(Z^{4N+1}, Z) = B(Z^5, Z)$ will be equal to

$$B_1(Z) = b_{00} + b_{01}Z + 0 + 0 + bZ^4 + b_{10}Z^5 + b_{11}Z^6. \tag{15}$$

The autocorrelation coefficients r_j's of $B_1(Z)$ can be written down as follows:

$$\begin{aligned} (b_{00})^2 + (b_{01})^2 + (b)^2 + (b_{10})^2 + (b_{11})^2 &= r_0, \\ b_{00}b_{01} + bb_{10} + b_{10}b_{11} &= r_1, \\ *bb_{11} &= r_2, \\ *b_{10}b &= r_3, \\ b_{00}b + b_{01}b_{10} &= r_4, \\ b_{00}b_{10} + b_{01}b_{11} &= r_5, \\ b_{00}b_{11} &= r_6, \end{aligned} \tag{16}$$

where $*$ indicates the equations that do not contain b_{00}.

The autocorrelation equations given in (16) are seven in number. It may be noted that two of these equations do not contain the constant coefficients b_{00} of $B(Z_1, Z_2)$. It is easy to verify that $B(Z_1, Z_2)$ has the same autocorrelation coefficients as in (16).

In general, for a polynomial of Nth degree in both variables, $2N^2$ number of autocorrelation equations does not contain the constant coefficient b_{00} out of the total $4N^2 + 2N + 2$ equations.

The following theorem is proved in [14].

Theorem 3. *A 2D first-quadrant polynomial $B(Z_1, Z_2)$ of degree N is stable if and only if $B(Z^{2N+1}, Z)$ is stable.*

It may be noted that if, in $B(Z_1, Z_2)$, the degree of Z_1 is M and of Z_2 is N, then $B(Z_1, Z_2)$ is stable if and only if $B(Z^L, Z)$, where $L = M + N + 1$ is stable.

We consider the NSHP polynomial (14).

In order to determine the stability of this NSHP polynomial, we first map this into first-quadrant filter by finding out the minimum critical angle sector. Once the NSHP is mapped into the first-quadrant filter, then the stability can be determined for the first-quadrant filter as given in Theorem 3.

The corresponding QP polynomial corresponding to $B(Z_1, Z_2)$ [10] is

$$G(Z_1, Z_2) = b_{00} + b_{01}Z_2 + b_{10}Z_1Z_2 + b_{11}Z_1Z_2^2 + bZ_1. \tag{17}$$

According to Theorem 3, the form-preserving 1D polynomial to be tested for stability is

$$G(Z) = b_{00} + b_{01}Z + b_{10}Z^5 + b_{11}Z^6 + bZ^4. \tag{18}$$

The same polynomial $G(Z)$ can be obtained from $B(Z_1, Z_2)$ as $B(Z^{4N+1}, Z)$, where $N = 1$. If the degree of Z_1 is different from Z_2, then the transformation is $B(Z^{2M+2N+1}, Z)$. Thus we have the following theorem.

Theorem 4. *An NSHP polynomial $B(Z_1, Z_2)$ of degree N is stable if and only if its form-preserving polynomial $B(Z^{4N+1}, Z)$ is stable.*

In (16), since $B(Z_1, Z_2)$ is a PLSI of the constant coefficient 2D NSHP polynomial, b_{00} is supposed to have its highest value with the corresponding autocorrelation coefficients being r_0, r_1, r_2, r_3, r_4, r_5, and r_6. If we want to make sure that it is indeed the highest possible value, we can use the Lagrange multiplier method of optimization that is to be discussed later. This is because, according to (13), the PLSI will be stable only if the error is the minimum, which requires b_{00} to be maximum.

4. EXISTENCE OF MAXIMUM FOR 2D FIRST- AND SECOND-ORDER PLSI POLYNOMIAL OF THE NSHP POLYNOMIAL

We have seen in earlier sections that if $B(Z_1, Z_2) = \sum^1 \sum_{R+\oplus}^1 b_{ij} Z_1^i Z_2^j$ is a 2D first-order NSHP polynomial, then the form-preserving 1D polynomial $B_1(Z) = B(Z^L, Z)$, when $L = 4N+1$, will have the same autocorrelation coefficients as the $B(Z_1, Z_2)$.

In order to prove that the PLSI polynomial $B(Z_1, Z_2)$ is stable, we have to show or prove the existence of a maximum (optimum) value for its constant b_{00}. So, we discuss in this section Lagrange multiplier method of optimization and the existence of solution for the equations. First, we arrive at a figure for the number of unknowns for each case and finally we generalize for Nth order case.

Example 2 (first-order case). Let

$$B(Z_1, Z_2) = \sum^{1} \sum_{R+\oplus}^{1} b_{ij} Z_1^i Z_2^j \tag{19}$$

be the given first-order polynomial. This can be written as (14).

The form-preserving 1D polynomial $B(Z_1, Z_2)$ becomes

$$B_1(Z) = B(Z^{4N+1}, Z) = B(Z^5, Z) \quad (\text{since } N = 1), \tag{20}$$

$$B_1(Z) = b_{00} + b_{01}Z + 0 + 0 + bZ^4 + b_{10}Z^5 + b_{11}Z^6. \tag{21}$$

It has seven autocorrelation functions r_s's as given in (16), where $r_s = \sum_{r=0}^{N} b_r b_{r+s}$, $s = 0, 1, 2, \ldots, N$.

Including $B_1(Z)$, there are totally 2^N number of 1D polynomials (in general) which has the same autocorrelation coefficients r_s's as that of $B(Z)$. Out of these 2^N number of 1D polynomials which are said to form a family, only one polynomial is stable satisfying the condition

$$B(Z) \neq 0, \quad |Z| \leq 1. \tag{22}$$

The stable polynomial is the one which has the maximum value (magnitude) for its constant term. To test the stability, we discuss below the Lagrange multiplier method.

In this method, one has to maximize a function f as

$$f = b'_{00} \tag{23}$$

satisfying the constraints g_i given as

$$g_i = \sum_{r=0} b'_r b'_{r+s} - r_s = 0, \quad s = 0, 1, 2, \ldots, N, \tag{24}$$

where

$$r_s = \sum_{r=0}^{N} b_r b_{r+s}, \quad s = 0, 1, 2, \ldots, N, \tag{25}$$

that is,

$$g_i = 0, \quad i = 0, 1, 2, \ldots, N. \tag{26}$$

For the sake of clarity, we briefly discuss the method as follows. Form the Lagrange function

$$L(b'_{00}, \lambda_j) = f + \sum_{j=0}^{N} \lambda_j g_j, \tag{27}$$

where λ_j are the Lagrange multipliers. Then form

$$\frac{\partial L(b'_{00}, \lambda_j)}{\partial b'_{00}} = 0, \tag{28}$$

$$\frac{\partial L(b'_{00}, \lambda_j)}{\partial \lambda_j} = 0, \quad j = 0, 1, 2, \ldots, N. \tag{29}$$

Equation (28) is called Lagrange equation.

Now,

$$L = f + \lambda_0 g_0 + \lambda_1 g_1 + \lambda_2 g_2 + \cdots + \lambda_6 g_6; \tag{30}$$

$$\begin{aligned} L = b'_{00} &+ \lambda_0 \Big[(b'_{00})^2 + (b'_{01})^2 + (b')^2 + (b'_{10})^2 \\ &+ (b'_{11})^2 - r_0 \Big] \\ &+ \lambda_1 [b'_{00}b'_{01} + b'b'_{10} + b'_{10}b'_{11} - r_1] + \lambda_2 [b'b'_{11} - r_2] \\ &+ \lambda_3 [b'_{10}b' - r_3] + \lambda_4 [b'_{00}b' + b'_{01}b'_{10} - r_4] \\ &+ \lambda_5 [b'_{00}b'_{10} + b'_{01}b'_{11} - r_5] + \lambda_6 [b'_{00}b'_{11} - r_6]; \end{aligned} \tag{31}$$

$$\frac{\partial L}{\partial b'_{00}} = 1 + 2b'_{00}\lambda_0 + b'_{01}\lambda_1 + b'\lambda_4 + b'_{10}\lambda_5 + b'_{11}\lambda_6; \tag{32}$$

$$\begin{aligned}
\frac{\partial L}{\partial \lambda_0} &= L\lambda_0 \\
&= (b'_{00})^2 + (b'_{01})^2 + (b')^2 + (b'_{10})^2 + (b'_{11})^2 - r_0 \\
&= 0, \\
\frac{\partial L}{\partial \lambda_1} &= L\lambda_1 = b'_{00}b'_{01} + b'b'_{10} + b'_{10}b'_{11} - r_1 = 0, \\
\frac{\partial L}{\partial \lambda_2} &= L\lambda_2 = b'b'_{11} - r_2 = 0, \\
\frac{\partial L}{\partial \lambda_3} &= L\lambda_3 = b'_{10}b' - r_3 = 0, \\
\frac{\partial L}{\partial \lambda_4} &= L\lambda_4 = b'_{00}b' + b'_{01}b'_{10} - r_4 = 0, \\
\frac{\partial L}{\partial \lambda_5} &= L\lambda_5 = b'_{00}b'_{10} + b'_{01}b'_{11} - r_5 = 0, \\
\frac{\partial L}{\partial \lambda_6} &= L\lambda_6 = b'_{00}b'_{11} - r_6 = 0.
\end{aligned} \tag{33}$$

There are eight constraint equations including (33) and the Lagrange equation (32). We have 5 b'_{ij}'s and $5\lambda_j$'s as unknowns with the total of 10. In the above formulas, we have considered the number of λ_j's as only 5 because we do not have to assign λ_j for the constraint equation which does not contain b'_{00}. Thus we have 10 unknowns and 8 equations which can be easily solved, and hence the optimum b'_{00} exists. So the PLSI is stable.

Example 3 (second-degree case). Let

$$\begin{aligned}
B(Z_1, Z_2) = {} & b_{00} + b_{01}Z_2 + b_{02}Z_2^2 + b_{10}Z_1 \\
& + b_{11}Z_1Z_2 + b_{12}Z_1Z_2^2 + b_{20}Z_1^2 \\
& + b_{21}Z_1^2Z_2 + b_{22}Z_1^2Z_2^2 + b_{1-1}Z_1Z_2^{-1} \\
& + b_{2-1}Z_1^2Z_2^{-1} + b_{1-2}Z_1Z_2^{-2} + b_{2-2}Z_1^2Z_2^{-2}.
\end{aligned} \tag{34}$$

The form-preserving 1D polynomial of the NSHP PLSI polynomial $B(Z_1, Z_2)$ is $B(Z)$ and is obtained using the transform

$$\begin{aligned}
B(Z) &= B(Z^{4N+1}, Z), \quad N = 2, \\
B(Z) &= b_{00} + b_{01}Z + b_{02}Z^2 + 0 + 0 + 0 + 0 + b_{1-2}Z^7 \\
&\quad + b_{1-1}Z^8 + b_{10}Z^9 + b_{11}Z^{10} + b_{12}Z^{11} + 0 + 0 + 0 \\
&\quad + 0 + b_{2-2}Z^{16} + b_{2-1}Z^{17} + b_{20}Z^{18} + b_{21}Z^{19} + b_{22}Z^{20}.
\end{aligned} \tag{35}$$

Form the Lagrange function

$$L(b'_{00}, \lambda_j) = f + \sum_{j=0}^{N} \lambda_j g_j + \cdots, \tag{36}$$

where λ_j's are Lagrange multipliers. To find the optimum and hence stable PLSI, obtain the constraint equations and unknowns.

The constraint equations are

$$\frac{\partial L}{\partial b'_{00}}, \frac{\partial L}{\partial \lambda_0}, \frac{\partial L}{\partial \lambda_2}, \ldots, \frac{\partial L}{\partial \lambda_{20}}. \tag{37}$$

As seen above, there are 21+1=22 constraint equations including one Lagrange equation. We have 13 b_{ij}'s as unknowns and, in addition, 13 λ_j's making 26 unknowns as a total. In the above, we considered the number of λ_j's as only 13 because we do not have to assign a λ_j for the constraint equation which does not contain b'_{00}. Thus we have 26 unknowns and 22 equations which can be easily solved, and hence the optimum b'_{00} exists. So, the PLSI is always stable.

Example 4 (Nth-order case). For the 2D NSHP polynomial of Nth order, the total number of constraint equations is $4N^2 + 2N + 2$, and out of this $2N^2$ number of equations do not contain b_{00}.

But the number of the unknowns λ_j's is $2N^2 + 2N + 1$ and b_{ij} is $2N^2 + 2N + 1$ and hence the total is $4N^2 + 4N + 2$. (The highest order of the form-preserving 1D polynomial is $4N^2 + 2N$ for the Nth-order NSHP polynomial.)

Since $4N^2 + 4N + 2 > 4N^2 + 2N + 2$, the number of unknowns are more than the number of equations and it can be easily solved, and hence the optimum b'_{00} exists. So the PLSI polynomial is stable.

In Examples 2, 3, and 4, we have simply stated that the equations can be solved and hence the optimum b'_{00} exists. Take, for instance, Example 2, the only unique solution for the set of (33), since it contains less number of the unknowns b_{ij}'s than the number of equations, is the one obtainable after solving the set of (5)

The vector solution $\overline{b_1}$ gives us all the coefficients b_{ij}'s of the PLSI polynomial. But when we couple (32) with (33), if we have more numbers of unknowns than the numbers of equations, then the sets of (32) and (33) together can be solved by a computer-aided nonlinear optimization method by forming and minimizing an artificial objective function

$$F = \sum_{i=0} \lambda_i. \tag{38}$$

In the computer-aided optimization method of solving nonlinear equation, one will be assured of a real solution if the total number of the unknowns, namely, b_{ij}'s and λ_j's, is greater than the number of equations by at least one. This is because the programmer has the freedom to choose the value of at least one coefficient as he likes. And if the value of this one coefficient (unknown), say that of b'_{00}, is chosen to be the same as b_{00}, which we already got when we solved (5), we will arrive at the same unique solution as mentioned before. This solution will also satisfy (32). It may be noted that we are not trying to solve (32) and (33) together manually or by using computer. Our interest is in establishing theoretically the fact that an optimum solution for these equations, which will be the same as we had already got by solving (5), does exist. This ensures the stability of the PLSI polynomial. On the other hand, if the number of the unknowns in (32) and (33) is not greater than the number of equations, the nonlinear computer-aided optimization, since the programmer has no degree of freedom, sometimes may not give us any real solution at all or it may give some other real solution other than what we had got by solving (5). If this is the case, the PLSI polynomial we have already got will not be stable.

The value of b'_{00} which we get after using the computer-aided nonlinear optimization method is the maximum and it is equal to b_{00} provided that the programmer has the freedom, namely, the number of unknowns greater than the number of equations by at least one. We know that b_{00} has to be maximum for $E_{\min}$ in (13) to be really the least and positive with a_{00} being taken as positive.

In the case of 1D, the LSI polynomial is always stable when its constant term is the maximum. Similarly, if we can ensure that, also in the case of 2D polynomial, the PLSI polynomial has its maximum constant term, the PLSI will be stable. This is what we have ensured.

Since [14] contains Theorem 3 which enables us to test the stability of a 2D QP polynomial in a simple way, its availability or unavailability now does not make much difference. One can always use the already established methods [10] to test the stability of a 2D QP polynomial by first transforming the NSHP polynomial into a QP polynomial. But in two numerical examples presented in the paper, we used the simple stability test given in [14] successfully.

5. STABILITY OF 2D NSHP PLSI POLYNOMIALS

In this section, we present examples for 2D NSHP PLSI polynomial of first-and second-order and check their stabilities.

Example 5. Consider the following 2D NSHP polynomial of order 1:

$$A(Z_1, Z_2) = 0.9Z_1Z_2^{-1} + 0.3 + 0.6Z_2 + 0.6Z_1 + 0.8Z_1Z_2. \quad (39)$$

Let the PLSI polynomial be $B(Z_1, Z_2) = b_{00} + b_{01}Z_2 + b_{10}Z_1 + b_{11}Z_1Z_2 + bZ_1Z_2^{-1}$.

The form-preserving 1D polynomial $A(Z)$ of $A(Z_1, Z_2)$ is obtained by using the transform

$$Z_2 = Z, \qquad Z_1 = Z^{4N+1} = Z^5 \quad (\text{since } N = 1). \quad (40)$$

Thus,

$$A(Z) = 0.8Z^6 + 0.6Z^5 + 0.9Z^4 + 0 + 0 + 0.6Z + 0.3. \quad (41)$$

The polynomial $A(Z)$ is unstable as some of the roots are inside the unit circle.

The LSI of $A(Z)$ is $B(Z)$ and to find out $B(Z)$, we compute autocorrelation coefficients of $A(Z)$ as follows:

$$\begin{aligned} \gamma_0 &= 2.26,\\ \gamma_1 &= 1.2,\\ \gamma_2 &= 0.72,\\ \gamma_3 &= 0.54,\\ \gamma_4 &= 0.63,\\ \gamma_5 &= 0.66,\\ \gamma_6 &= 0.24. \end{aligned} \quad (42)$$

Now we form (12) as follows:

$$\begin{bmatrix} 2.26 & 1.2 & 0.72 & 0.54 & 0.63 & 0.66 & 0.24\\ 1.2 & 2.26 & 1.2 & 0.72 & 0.54 & 0.63 & 0.66\\ 0.72 & 1.2 & 2.26 & 1.2 & 0.72 & 0.54 & 0.63\\ 0.54 & 0.72 & 1.2 & 2.26 & 1.2 & 0.72 & 0.54\\ 0.63 & 0.54 & 0.72 & 1.2 & 2.26 & 1.2 & 0.72\\ 0.16 & 0.63 & 0.54 & 0.72 & 1.2 & 2.26 & 1.2\\ 0.24 & 0.66 & 0.63 & 0.54 & 0.72 & 1.2 & 2.26 \end{bmatrix} \begin{bmatrix} b_{00}\\ b_{01}\\ 0\\ 0\\ b\\ b_{10}\\ b_{11} \end{bmatrix} = \begin{bmatrix} 0.3\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0 \end{bmatrix}. \quad (43)$$

The column vector $\overline{b_1}$ does contain zeros and while arriving at (5), we have to delete some rows and corresponding columns of $\overline{T_1}$, some rows of $\overline{b_1}$ corresponding to zero coefficients in $B(Z)$, and some zeros of $\overline{a_1}$.

After deleting the second and third columns, the corresponding rows of $\overline{T_1}$, and the corresponding rows of $\overline{b_1}$ and $\overline{a_1}$, we get

$$\begin{bmatrix} 2.26 & 1.2 & 0.63 & 0.66 & 0.24\\ 1.2 & 2.26 & 0.54 & 0.63 & 0.66\\ 0.63 & 0.54 & 2.26 & 1.2 & 0.72\\ 0.16 & 0.63 & 1.2 & 2.26 & 1.2\\ 0.24 & 0.66 & 0.72 & 1.2 & 2.26 \end{bmatrix} \begin{bmatrix} b_{00}\\ b_{01}\\ b\\ b_{10}\\ b_{11} \end{bmatrix} = \begin{bmatrix} 0.3\\ 0\\ 0\\ 0\\ 0 \end{bmatrix},$$

$$\begin{bmatrix} b_{00}\\ b_{01}\\ b\\ b_{10}\\ b_{11} \end{bmatrix} = \begin{bmatrix} 2.26 & 1.2 & 0.63 & 0.66 & 0.24\\ 1.2 & 2.26 & 0.54 & 0.63 & 0.66\\ 0.63 & 0.54 & 2.26 & 1.2 & 0.72\\ 0.16 & 0.63 & 1.2 & 2.26 & 1.2\\ 0.24 & 0.66 & 0.72 & 1.2 & 2.26 \end{bmatrix}^{-1} \begin{bmatrix} 0.3\\ 0\\ 0\\ 0\\ 0 \end{bmatrix}, \quad (44)$$

$$\begin{bmatrix} b_{00}\\ b_{01}\\ b\\ b_{10}\\ b_{11} \end{bmatrix} = \begin{bmatrix} 0.1995\\ -0.10038\\ -0.02343\\ -0.03636\\ 0.03492 \end{bmatrix},$$

$$\begin{aligned} B(Z) = {} & 0.03492Z^6 - 0.03636Z^5 - 0.02343Z^4\\ & + 0 + 0 - 0.10038Z + 0.1995. \end{aligned}$$

The 2D PLSI is

$$\begin{aligned} B(Z_1, Z_2) &= -0.02343 Z_1 Z_2^{-1} + 0.1995 - 0.10038 Z_2 \\ &\quad - 0.03636 Z_1 + 0.03492 Z_1 Z_2. \end{aligned} \tag{45}$$

The 2D PLSI polynomial is found to be stable. This is because we have 8 equations and 10 unknown coefficients and hence the optimum exists.

Example 6. Consider the following 2D NSHP polynomial of order 2:

$$\begin{aligned} A(Z_1, Z_2) &= 0.6 + 0.9Z_2 + 0.3Z_2^2 + 0.9Z_1 \\ &\quad + 1.5Z_1Z_2 + 0.9Z_1Z_2^2 + 0.3Z_1^2 \\ &\quad + 0.9Z_1^2Z_2 + 0.6Z_1^2Z_2^2 + 0.6Z_1^2Z_2^2 \\ &\quad + 0.5Z_1Z_2^{-1} + 0.8Z_1^2Z_2^{-1} + 0.7Z_1Z_2^{-2} + Z_1^2Z_2^{-2}. \end{aligned} \tag{46}$$

As can be seen, we have assumed centrosymmetry among the coefficients in the QP.

Let the PLSI polynomial be $B(Z_1, Z_2)$, where (34) holds. The form-preserving 1D polynomial $A(Z)$ of $A(Z_1, Z_2)$ is obtained by using the transform $Z_2 = Z$ and $Z_1 = Z^{4N+1} = Z^9$ (since $N = 2$):

$$\begin{aligned} A(Z) &= 0.6 + 0.9Z + 0.3Z^2 + 0 + 0 + 0 + 0 + 0.7Z^7 \\ &\quad + 0.5Z^8 + 0.9Z^9 + 1.5Z^{10} + 0.9Z^{11} + 0 + 0 + 0 \\ &\quad + 0 + Z^{16} + 0.8Z^{17} + 0.3Z^{18} + 0.9Z^{19} + 0.6Z^{20}. \end{aligned} \tag{47}$$

The polynomial $A(Z)$ is unstable.

Now, the LSI of $A(Z)$ is $B(Z)$ and to find out $B(Z)$, we compute the autocorrelation function of $A(Z)$ as given in Example 5 for first order.

The autocorrelation functions are

$$\begin{aligned} \gamma_0 &= 8.77, \\ \gamma_1 &= 6.16, \\ \gamma_2 &= 3.57, \\ \gamma_3 &= 2.88, \\ \gamma_4 &= 1.23, \\ \gamma_5 &= 1.11, \\ \gamma_6 &= 3.00, \\ \gamma_7 &= 3.51, \\ \gamma_8 &= 4.04, \\ \gamma_9 &= 5.42, \\ \gamma_{10} &= 4.13, \\ \gamma_{11} &= 1.71, \\ \gamma_{12} &= 0.93, \\ \gamma_{13} &= 0.42, \\ \gamma_{14} &= 0.30, \\ \gamma_{15} &= 1.14, \\ \gamma_{16} &= 1.14, \\ \gamma_{17} &= 1.02, \\ \gamma_{18} &= 1.17, \\ \gamma_{19} &= 1.08, \\ \gamma_{20} &= 0.36. \end{aligned} \tag{48}$$

There are 21 autocorrelation coefficients. Now, we form (12).

Here $\overline{T_1}$ matrix has an order of 21×21.

The column vector $\overline{b_1}$ does contain some zeros and, while arriving at (5), we have to delete some rows and corresponding columns of $\overline{T_1}$, some rows of $\overline{b_1}$ corresponding to zero coefficients in $B(Z)$, and some zeros of $\overline{a_1}$.

After deleting 8 columns containing 0's, the corresponding rows of $\overline{T_1}$, and the corresponding rows of $\overline{b_1}$ and $\overline{a_1}$, we get

$$\begin{bmatrix} 8.77 & 6.16 & 3.57 & 3.51 & 4.04 & 5.42 & 4.13 & 1.71 & 1.14 & 1.02 & 1.17 & 1.08 & 0.36 \\ 6.16 & 8.77 & 6.16 & 3.00 & 3.51 & 4.04 & 5.42 & 4.13 & 1.14 & 1.14 & 1.02 & 1.17 & 1.08 \\ 3.57 & 6.16 & 8.77 & 1.11 & 3.00 & 3.51 & 4.04 & 5.42 & 0.30 & 1.14 & 1.14 & 1.02 & 1.17 \\ 3.51 & 3 & 1.11 & 8.77 & 6.16 & 3.57 & 2.88 & 1.23 & 5.42 & 4.13 & 1.71 & 0.93 & 0.42 \\ 4.04 & 3.51 & 3 & 6.16 & 8.77 & 6.16 & 3.57 & 2.88 & 4.04 & 5.42 & 4.13 & 1.71 & 0.93 \\ 5.42 & 4.04 & 3.51 & 3.57 & 6.16 & 8.77 & 6.16 & 3.57 & 3.51 & 4.04 & 5.42 & 4.13 & 1.71 \\ 4.13 & 5.42 & 4.04 & 2.88 & 3.57 & 6.16 & 8.77 & 6.16 & 3 & 3.51 & 4.04 & 5.42 & 4.13 \\ 1.71 & 4.13 & 5.42 & 1.23 & 2.88 & 3.57 & 6.16 & 8.77 & 1.11 & 3.00 & 3.51 & 4.04 & 5.42 \\ 1.14 & 1.14 & 0.30 & 5.42 & 4.04 & 3.51 & 3 & 1.11 & 8.77 & 6.16 & 3.57 & 2.88 & 1.23 \\ 1.02 & 1.14 & 1.14 & 4.13 & 5.42 & 4.04 & 3.51 & 3 & 6.16 & 8.77 & 6.16 & 3.57 & 2.88 \\ 1.17 & 1.02 & 1.14 & 1.71 & 4.13 & 5.42 & 4.04 & 3.51 & 3.57 & 6.16 & 8.77 & 6.16 & 3.57 \\ 1.08 & 1.17 & 1.02 & 0.93 & 1.71 & 4.13 & 5.42 & 4.04 & 2.88 & 3.57 & 6.16 & 8.77 & 6.16 \\ 0.36 & 1.08 & 1.17 & 0.42 & 0.93 & 1.71 & 4.13 & 5.42 & 1.23 & 2.88 & 3.57 & 6.16 & 8.77 \end{bmatrix} \begin{bmatrix} b_{00} \\ b_{01} \\ b_{02} \\ b_{1-2} \\ b_{1-1} \\ b_{10} \\ b_{11} \\ b_{12} \\ b_{2-2} \\ b_{2-1} \\ b_{20} \\ b_{21} \\ b_{22} \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} b_{00} \\ b_{01} \\ b_{02} \\ b_{1-2} \\ b_{1-1} \\ b_{10} \\ b_{11} \\ b_{12} \\ b_{2-2} \\ b_{2-1} \\ b_{20} \\ b_{21} \\ b_{22} \end{bmatrix} = \begin{bmatrix} 0.53944 \\ -0.33972 \\ 0.0516 \\ 0.02502 \\ 0.0078 \\ -0.04992 \\ -0.0039 \\ -0.0588 \\ 0.0561 \\ -0.03534 \\ 0.0681 \\ -0.01218 \\ 0.05784 \end{bmatrix}. \tag{49}$$

Now, the PLSI $B(Z)$ is

$$\begin{aligned} B(Z) = {} & 0.53944 - 0.339727 + 0.0516Z^2 + 0 + 0 + 0 + 0 \\ & + 0.2502Z^7 + 0.0078Z^8 - 0.04992Z^9 - 0.0039Z^{10} \\ & - 0.0588Z^{11} + 0 + 0 + 0 + 0 + 0.0561Z^{16} - 0.03534Z^{17} \\ & + 0.0681Z^{18} - 0.01218Z^{19} + 0.05784Z^{20}. \end{aligned} \tag{50}$$

The PLSI $B(Z_1, Z_2)$ is found to be stable even though the original NSHP polynomial $A(Z_1, Z_2)$ has centrosymmetry among the coefficients in the QP. This is because we have 22 constraint equations and 26 unknown coefficients, and hence the optimum exists.

6. CONCLUSIONS

In this paper, we dealt with the stabilization of 2D NSHP polynomials by the PLSI approach. The PLSI $B(Z_1, Z_2)$ will be stable provided that the degree of the given polynomial $A(Z_1, Z_2)$ and that of $B(Z_1, Z_2)$ are the same. In the case of QP filters, if there is symmetry among the coefficients, either in the original polynomial or the corresponding PLSI, then the PLSI need not be stable if the order is greater than two. This is because the number of constraint equations will be more than the number of unknowns in the optimization. Therefore, a restriction is there in the stabilization of QP PLSI polynomial. However, in NSHP, the PLSI will definitely be stable, irrespective of the degree provided that it has the same order as the original polynomial.

REFERENCES

[1] E. A. Robinson, *Statistical Communication and Detection*, Griffin, London, UK, 1967.

[2] J. L. Shanks, S. Treitel, and J. Justice, "Stability and synthesis of two-dimensional recursive filters," *IEEE Transactions on Audio and Electro acoustics*, vol. 20, no. 2, pp. 115–128, 1972.

[3] Y. Genin and Y. Kamp, "Counter example in the least-square inverse stabilization of 2-D recursive filters," *Electronics Letters*, vol. 11, pp. 130–131, July 1975.

[4] Y. Genin and Y. Kamp, "Comments on 'On the stability of the least mean-square inverse process in two-dimensional digital filters'," *IEEE Trans. Acoustics, Speech, and Signal Processing*, vol. 25, no. 1, pp. 92–93, 1977.

[5] E. I. Jury, "An overview of Shanks' conjecture and comments on its validity," in *Proc. 10th Asilomor Conf. Circuits Systems and Computers*, Pacific Grove, Calif, USA, November 1976.

[6] A. H. Kayran and R. King, "Comments on least-squares inverse polynomials and a counter example for a Jury's conjecture," *Electronics Letters*, vol. 16, no. 21, pp. 795–796, 1980.

[7] N. Damera-Venkata, M. Venkataraman, M. S. Hrishikesh, and P. S. Reddy, "Stabilization of 2-D recursive digital filters by the DHT method," *IEEE Trans. on Circuits and Systems II: Analog and Digital Signal Processing*, vol. 46, no. 1, pp. 85–88, 1999.

[8] N. Damera-Venkata, M. Venkataraman, M. S. Hrishikesh, and P. S. Reddy, "A new transform for the stabilization and stability testing of multidimensional recursive digital filters," *IEEE Trans. on Circuits and Systems II: Analog and Digital Signal Processing*, vol. 47, no. 9, pp. 965–968, 2000.

[9] E. M. A. Gnanamuthu, N. Gangatharan, and P. S. Reddy, "The PLSI method of stabilizing 2-D recursive digital filters—A complete solution," Submitted to IEEE journal of circuits and systems.

[10] T. S. Huang, Ed., *Two-Dimensional Digital Signal Processing I. Linear Filters*, vol. 42 of *Topics in Applied Physics*, Springer-Verlag, Berlin, Germany, 1981.

[11] E. I. Jury, R. V. Kolavenu, and B. D. O. Anderson, "Stabilization of certain two dimensional recursive digital filters," *Proc. IEEE*, vol. 65, pp. 887–892, June 1977.

[12] S. S. Rao, *Optimization Theory and Applications*, Wiley Eastern Limited, New Delhi, India, 1987.

[13] P. S. Reddy, D. Reddy, and M. Swamy, "Proof of a modified form of Shank's conjecture on the stability of 2-D planar least square inverse polynomials and its implications," *IEEE Trans. Circuits and Systems*, vol. 31, no. 12, pp. 1009–1015, 1984.

[14] P. Rangarajan, P. Muthukumaraswamy, N Gangatharan, and P. S. Reddy, "A simple stability test for 2-D recursive digital filters," submitted to International Journal of Circuit Theory and Applications.

N. Gangatharan was born in Kanyakumari, India, 1966. He received the B.E. degree in electronics and communication engineering in 1988 and the M.E. degree in microwave and optical engineering in 1990, both from Madurai Kamaraj University, India. He is a second ranker in the M.E. degree examinations of Madurai Kamaraj University in 1990. He received his second M.E. degree in computer science and engineering from Manonmaniam Sundaranar University, Tirunelveli, India in 1997 and his MBA degree in finance and marketing from Madurai Kamaraj University, India in 1999. He was on the faculty of Electronics and Communication Engineering at the Indian Engineering College, Vadakkangulam, India, as a Professor during 1997–2001. In 2001, he joined as a Lecturer the Faculty of Engineering, Multimedia University, Cyberjaya Campus, Malaysia. Since 2001, he has been working toward the Ph.D. degree at the same university. His current research interests concern digital signal processing, particularly in the stabilization of multidimensional recursive digital filters. His other research areas are in multimedia signal processing and QoS for networked multimedia systems.

P. S. Reddy was born in India. He received his B.E. degree (Honors) in electrical engineering with a gold medal from Andhra University, India, 1964. He is a first ranker in the M.S. degree examinations in engineering at University of Madras, India, in 1966. He received his Ph.D. degree in electrical engineering from Indian Institute of Technology (IIT), Madras, India in 1972. He was a Senior Fellow of the Indian government during 1964–1967, during which he was trained for a teaching profession at Guindy Engineering College, India. During 1968–1990, he was on the faculty of IIT, Madras, India, where he was a Professor. He was a Postdoctoral Fellow at the University of Aachen, Germany, during 1972–1973, and at Concordia University, Montreal, Canada, during 1980–1981. He was a Visiting Scientist at Concordia University during 1988–1991. During 1991–2001, he has been a Professor and Chairman of the Department of Electrical and Computer Engineering (ECE) at various engineering colleges in India. He worked as a Professor at Multimedia University, Malaysia, during 2001–2002. He is currently a Postgraduate Professor in the Department of ECE at SRM Engineering College, Chennai, India. He has published about 45 research papers in international journals. His research interests are in the areas of digital signal processing and communication engineering.