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Fixed points for mappings satisfying some multi-valued contractions with *w*-distance

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Abstract
The existence of fixed points and iterative approximations for some nonlinear multi-valued contraction mappings in complete metric spaces with *w*-distance are proved. Two examples are included. The results presented in this paper extend, improve and unify many known results in recent literature.
MSC: 54H25; 47H10

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1 Introduction and preliminaries

In 1996, Kada *et al.* [1] introduced the concept of w-distance and got some fixed point theorems for single-valued mappings under w-distance. In 2006, Feng and Liu [2, Theorem 3.1] proved the following fixed point theorem for a multi-valued contractive mapping, which generalizes the nice fixed point theorem due to Nadler [3, Theorem 5].

Theorem 1.1 ([2]) *Let (X,d) be a complete metric space and T be a multi-valued mapping from X into $\mathrm{CL}(X)$, where $\mathrm{CL}(X)$ is the family of all nonempty closed subsets of X. Assume that*

(c_1) *the mapping $f : X \to \mathbb{R}^+$, defined by $f(x) = d(x,T(x))$, $x \in X$, is lower semi-continuous;*
(c_2) *there exist constants $b,c \in (0,1)$ with $c < b$ such that for any $x \in X$, there is $y \in T(x)$ satisfying*

$$bd(x,y) \le f(x) \quad \text{and} \quad f(y) \le cd(x,y).$$

Then T has a fixed point in X.

In 2007, Klim and Wardowski [4, Theorem 2.1] extended Theorem 1.1 and proved the following result.

Theorem 1.2 ([4]) *Let (X,d) be a complete metric space and T be a multi-valued mapping from X into $\mathrm{CL}(X)$ satisfying (c_1). Assume that*

(c_3) *there exist $b \in (0,1)$ and $\varphi : \mathbb{R}^+ \to [0,b)$ satisfying*

$$\limsup_{r\to t^+} \varphi(r) < b, \quad \forall t \in \mathbb{R}^+,$$

and for any $x \in X$, *there is* $y \in T(x)$ *satisfying*

$$bd(x,y) \le d\big(x, T(x)\big) \quad \textit{and} \quad f(y) \le \varphi\big(d(x,y)\big)d(x,y).$$

Then T *has a fixed point in* X.

In 2009 and 2010, Ćirić [5, Theorem 2.1] and Liu *et al.* [6, Theorems 2.1 and 2.3] established a few fixed point theorems for some multi-valued nonlinear contractions, which include the multi-valued contraction in Theorem 1.1 as a special case.

Theorem 1.3 ([5]) *Let* (X,d) *be a complete metric space and* T *be a multi-valued mapping from* X *into* $\mathrm{CL}(X)$ *satisfying* (c_1). *Assume that*

(c_4) *there exists a function* $\varphi : \mathbb{R}^+ \to [a,1)$, $0 < a < 1$, *satisfying*

$$\limsup_{r \to t^+} \varphi(r) < 1, \quad \forall t \in \mathbb{R}^+,$$

and for any $x \in X$, *there is* $y \in T(x)$ *satisfying*

$$\sqrt{\varphi\big(f(x)\big)}\,d(x,y) \le f(x) \quad \textit{and} \quad f(y) \le \varphi\big(f(x)\big)d(x,y).$$

Then T *has a fixed point in* X.

Theorem 1.4 ([6]) *Let* T *be a multi-valued mapping from a complete metric space* (X,d) *into* $\mathrm{CL}(X)$ *such that*

for each $x \in X$, *there exists* $y \in T(x)$ *satisfying*

$$\alpha\big(f(x)\big)d(x,y) \le f(x) \quad \textit{and} \quad f(y) \le \beta\big(f(x)\big)d(x,y),$$

where

$$B = \begin{cases} [0, \sup f(X)] & \textit{if } \sup f(X) < \infty, \\ [0, \infty) & \textit{if } \sup f(X) = \infty, \end{cases}$$

$\alpha : B \to (0,1]$ *and* $\beta : B \to [0,1)$ *satisfy that*

$$\liminf_{r \to 0^+} \alpha(r) > 0 \quad \textit{and} \quad \limsup_{r \to t^+} \frac{\beta(r)}{\alpha(r)} < 1, \quad \forall t \in \big[0, \sup f(X)\big).$$

Then

(a1) *for each* $x_0 \in X$, *there exist an orbit* $\{x_n\}_{n \in \mathbb{N}_0}$ *of* T *and* $z \in X$ *such that* $\lim_{n \to \infty} x_n = z$;

(a2) z *is a fixed point of* T *in* X *if and only if the function* $f(x) = d(x, T(x))$, $x \in X$, *is* T*-orbitally lower semi-continuous at* z.

Theorem 1.5 ([6]) *Let* T *be a multi-valued mapping from a complete metric space* (X,d) *into* $\mathrm{CL}(X)$ *such that*

for each $x \in X$, *there exists* $y \in T(x)$ *satisfying*

$$\alpha\big(d(x,y)\big)d(x,y) \le f(x) \quad \textit{and} \quad f(y) \le \beta\big(d(x,y)\big)d(x,y),$$

where

$$A = \begin{cases} [0, \operatorname{diam}(X)] & \text{if } \operatorname{diam}(X) < \infty, \\ [0, \infty) & \text{if } \operatorname{diam}(X) = \infty, \end{cases}$$

$\alpha : A \to (0,1]$ *and* $\beta : A \to [0,1)$ *satisfy that*

$$\liminf_{r\to t^+} \alpha(r) > 0 \quad \text{and} \quad \limsup_{r\to t^+} \frac{\beta(r)}{\alpha(r)} < 1, \quad \forall t \in \bigl[0, \operatorname{diam}(X)\bigr),$$

and one of α *and* β *is nondecreasing. Then*

(a1) *for each* $x_0 \in X$, *there exist an orbit* $\{x_n\}_{n\in\mathbb{N}_0}$ *of* T *and* $z \in X$ *such that* $\lim_{n\to\infty} x_n = z$;
(a2) z *is a fixed point of* T *in* X *if and only if the function* $f(x) = d(x, T(x))$, $x \in X$, *is* T*-orbitally lower semi-continuous at* z.

In 2011, Latif and Abdou [7, Theorem 2.1] generalized Theorem 1.3 and proved the following fixed point theorem for some multi-valued contractive mapping with w-distance.

Theorem 1.6 ([7]) *Let* (X, d) *be a complete metric space with a* w*-distance* w, *and let* T *be a multi-valued mapping from* X *into* $\mathrm{CL}(X)$. *Assume that*

(c_5) *the mapping* $f : X \to \mathbb{R}^+$, *defined by* $f_w(x) = w(x, T(x))$, $x \in X$, *is lower semi-continuous*;
(c_6) *there exists a function* $\varphi : \mathbb{R}^+ \to [b, 1)$, $0 < b < 1$, *satisfying*

$$\limsup_{r\to t^+} \varphi(r) < 1, \quad \forall t \in \mathbb{R}^+$$

and for any $x \in X$, *there is* $y \in T(x)$ *satisfying*

$$\sqrt{\varphi\bigl(f_w(x)\bigr)}\, w(x, y) \le f_w(x) \quad \text{and} \quad f_w(y) \le \varphi\bigl(f_w(x)\bigr) w(x, y).$$

Then there exists $v_0 \in X$ *such that* $f_w(v_0) = 0$. *Further, if* $w(v_0, v_0) = 0$, *then* $v_0 \in T(v_0)$.

The purpose of this paper is to prove the existence of fixed points and iterative approximations for some multi-valued contractive mappings with w-distance. Two examples with uncountably many points are included. The results presented in this paper extend, improve and unify Theorem 3.1 in [2], Theorem 2.1 in [4], Theorems 2.1 and 2.2 in [5], Theorems 2.1 and 2.3 in [6], Theorems 2.1-2.3 and 2.5 in [7], Theorem 6 in [8], Theorems 2.2 and 2.4 in [9] and Theorems 3.1-3.4 in [10].

Throughout this paper, we assume that $\mathbb{R}^+ = [0, \infty)$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ denotes the set of all positive integers.

Definition 1.7 ([1]) A function $w : X \times X \to \mathbb{R}^+$ is called a w-distance in X if it satisfies the following:

(w_1) $w(x, z) \le w(x, y) + w(y, z)$, $\forall x, y, z \in X$;

(w_2) for each $x \in X$, a mapping $w(x, \cdot) : X \to \mathbb{R}^+$ is lower semi-continuous, that is, if $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in X with $\lim_{n\to\infty} y_n = y \in X$, then $w(x, y) \le \liminf_{n\to\infty} w(x, y_n)$;

(w_3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $w(z, x) \le \delta$ and $w(z, y) \le \delta$ imply $d(x, y) \le \varepsilon$.

For any $u \in X$, $D \subseteq X$, w-distance w and $T : X \to \mathrm{CL}(X)$, put

$$d(u, D) = \inf_{y\in D} d(u, y), \qquad w(u, D) = \inf_{y\in D} w(u, y),$$

$$f(u) = d\bigl(u, T(u)\bigr), \qquad f_w(u) = w\bigl(u, T(u)\bigr),$$

$$\mathrm{diam}(X) = \sup\bigl\{d(x, y) : x, y \in X\bigr\}, \qquad \mathrm{diam}(X_w) = \sup\bigl\{w(x, y) : x, y \in X\bigr\},$$

$$A_w = \begin{cases} [0, \mathrm{diam}(X_w)] & \text{if } \mathrm{diam}(X_w) < \infty, \\ [0, \infty) & \text{if } \mathrm{diam}(X_w) = \infty \end{cases}$$

and

$$B_w = \begin{cases} [0, \sup f_w(X)] & \text{if } \sup f_w(X) < \infty, \\ [0, \infty) & \text{if } \sup f_w(X) = \infty. \end{cases}$$

A sequence $\{x_n\}_{n\in\mathbb{N}_0}$ in X is called *an orbit* of T at $x_0 \in X$ if $x_n \in T(x_{n-1})$ for all $n \in \mathbb{N}$. A function $g : X \to \mathbb{R}^+$ is said to be *T-orbitally lower semi-continuous* at $z \in X$ if $g(z) \le \liminf_{n\to\infty} g(x_n)$ for each orbit $\{x_n\}_{n\in\mathbb{N}_0} \subset X$ of T with $\lim_{n\to\infty} x_n = z$. A function $\varphi : A_w \to \mathbb{R}^+$ is called *subadditive* in A_w if $\varphi(s + t) \le \varphi(s) + \varphi(t)$ for all $s, t \in A_w$. A function $\varphi : A_w \to \mathbb{R}^+$ is called *strictly inverse* in A_w if $\varphi(t) < \varphi(s)$ implies that $t < s$.

Lemma 1.8 ([11]) *Let (X, d) be a metric space with a w-distance w and $D \in \mathrm{CL}(X)$. Suppose that there exists $u \in X$ such that $w(u, u) = 0$. Then $w(u, D) = 0$ if and only if $u \in D$.*

2 Fixed point theorems

In this section we prove the existence of fixed points and iterative approximations for some nonlinear multi-valued contraction mappings in complete metric spaces with w-distance.

Theorem 2.1 *Let (X, d) be a complete metric space, w be a w-distance in X and T be a multi-valued mapping from X into $\mathrm{CL}(X)$ such that*

$$\begin{aligned} &\text{for each } x \in X, \text{there exists } y \in T(x) \text{ satisfying} \\ &\alpha\bigl(f_w(x)\bigr)\varphi\bigl(w(x, y)\bigr) \le f_w(x) \quad \text{and} \quad f_w(y) \le \beta\bigl(f_w(x)\bigr)\psi\bigl(w(x, y)\bigr), \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} &\alpha \text{ and } \beta \text{ are functions from } B_w \text{ into } (0, 1] \text{ and } [0, 1), \text{respectively, with} \\ &\beta(0) < \alpha(0), \qquad \liminf_{r\to 0^+} \alpha(r) > 0 \quad \text{and} \quad \limsup_{r\to t^+} \frac{\beta(r)}{\alpha(r)} < 1, \quad \forall t \in B_w, \end{aligned} \tag{2.2}$$

$$\begin{aligned} &\varphi \text{ and } \psi \text{ are functions from } A_w \text{ into } \mathbb{R}^+ \text{ with } \psi(t) \le \varphi(t), \quad \forall t \in A_w \quad \text{and} \\ &\varphi \text{ is subadditive in } A_w \text{ and satisfies that either} \end{aligned} \tag{2.3}$$

$$\varphi \text{ is strictly inverse in } A_w, \varphi(0) = 0, \varphi(t) > 0, \quad \forall t \in A_w \setminus \{0\} \tag{2.4}$$

or

$$\varphi \text{ is strictly increasing in } A_w \text{ and } \lim_{t\to 0^+} \varphi^{-1}(t) = 0, \text{ where } \varphi^{-1} \text{ stands for the inverse function of } \varphi. \tag{2.5}$$

Then

(a1) *for each* $x_0 \in X$, *there exists an orbit* $\{x_n\}_{n\in\mathbb{N}_0}$ *of* T *such that* $\lim_{n\to\infty} x_n = u_0$ *for some* $u_0 \in X$;

(a2) $f_w(u_0) = 0$ *if and only if the function* f_w *is* T*-orbitally lower semi-continuous at* u_0;

(a3) $u_0 \in T(u_0)$ *provided that* $w(u_0, u_0) = 0 = f_w(u_0)$;

(a4) T *has a fixed point in* X *if for each orbit* $\{z_n\}_{n\in\mathbb{N}_0}$ *of* T *in* X *and* $v \in X$ *with* $v \notin T(v)$, *one of the following conditions is satisfied*:

$$\inf\bigl\{w(z_n, v) + \varphi\bigl(w(z_n, z_{n+1})\bigr) : n \in \mathbb{N}_0\bigr\} > 0; \tag{2.6}$$

$$\inf\bigl\{w(z_n, v) + w\bigl(z_n, T(z_n)\bigr) : n \in \mathbb{N}_0\bigr\} > 0. \tag{2.7}$$

Proof Firstly, we prove (a1). Let

$$\gamma(t) = \frac{\beta(t)}{\alpha(t)}, \quad \forall t \in B_w. \tag{2.8}$$

It follows from (2.1) that for each $x_0 \in X$, there exists $x_1 \in T(x_0)$ satisfying

$$\alpha\bigl(f_w(x_0)\bigr)\varphi\bigl(w(x_0, x_1)\bigr) \le f_w(x_0) \quad \text{and} \quad f_w(x_1) \le \beta\bigl(f_w(x_0)\bigr)\psi\bigl(w(x_0, x_1)\bigr),$$

which together with (2.3) and (2.8) yields that

$$\begin{aligned} f_w(x_1) &\le \beta\bigl(f_w(x_0)\bigr)\psi\bigl(w(x_0, x_1)\bigr) \le \beta\bigl(f_w(x_0)\bigr)\varphi\bigl(w(x_0, x_1)\bigr) \\ &\le \beta\bigl(f_w(x_0)\bigr)\frac{f_w(x_0)}{\alpha(f_w(x_0))} = \gamma\bigl(f_w(x_0)\bigr)f_w(x_0). \end{aligned}$$

Continuing this process, we choose easily an orbit $\{x_n\}_{n\in\mathbb{N}_0}$ of T satisfying

$$\begin{aligned} &x_{n+1} \in T(x_n), \quad \alpha\bigl(f_w(x_n)\bigr)\varphi\bigl(w(x_n, x_{n+1})\bigr) \le f_w(x_n) \quad \text{and} \\ &f_w(x_{n+1}) \le \beta\bigl(f_w(x_n)\bigr)\psi\bigl(w(x_n, x_{n+1})\bigr), \quad \forall n \in \mathbb{N}_0. \end{aligned} \tag{2.9}$$

It follows from (2.3), (2.8) and (2.9) that

$$\begin{aligned} f_w(x_{n+1}) &\le \beta\bigl(f_w(x_n)\bigr)\psi\bigl(w(x_n, x_{n+1})\bigr) \le \beta\bigl(f_w(x_n)\bigr)\varphi\bigl(w(x_n, x_{n+1})\bigr) \\ &\le \beta\bigl(f_w(x_n)\bigr)\frac{f_w(x_n)}{\alpha(f_w(x_n))} = \gamma\bigl(f_w(x_n)\bigr)f_w(x_n), \quad \forall n \in \mathbb{N}_0. \end{aligned} \tag{2.10}$$

Now we claim that

$$\lim_{n\to\infty} f_w(x_n) = 0. \tag{2.11}$$

Notice that the ranges of α and β, (2.2) and (2.8) ensure that

$$0 \le \gamma(t) < 1, \quad \forall t \in B_w. \tag{2.12}$$

Using (2.10) and (2.12), we conclude that $\{f_w(x_n)\}_{n\in\mathbb{N}_0}$ is a nonnegative and nonincreasing sequence, which means that there is a constant $a \ge 0$ satisfying

$$\lim_{n\to\infty} f_w(x_n) = a. \tag{2.13}$$

Suppose that $a > 0$. Using (2.2), (2.8), (2.10), (2.12) and (2.13), we obtain that

$$\begin{aligned} a &= \limsup_{n\to\infty} f_w(x_{n+1}) \le \limsup_{n\to\infty}\bigl[\gamma\bigl(f_w(x_n)\bigr)f_w(x_n)\bigr] \\ &\le \limsup_{n\to\infty}\gamma\bigl(f_w(x_n)\bigr)\limsup_{n\to\infty} f_w(x_n) \\ &\le a\limsup_{r\to a^+}\gamma(r) < a, \end{aligned}$$

which is a contradiction. Thus $a = 0$, that is, (2.11) holds.

Next we claim that $\{x_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence. Put

$$b = \limsup_{n\to\infty}\gamma\bigl(f_w(x_n)\bigr) \quad \text{and} \quad c = \liminf_{n\to\infty}\alpha\bigl(f_w(x_n)\bigr). \tag{2.14}$$

It follows from (2.2), (2.8), (2.12) and (2.14) that

$$0 \le b < 1 \quad \text{and} \quad c > 0. \tag{2.15}$$

Let $p \in (0, c)$ and $q \in (b, 1)$. Because of (2.14) and (2.15), we deduce that there exists some $n_0 \in \mathbb{N}$ such that

$$\gamma\bigl(f_w(x_n)\bigr) < q \quad \text{and} \quad \alpha\bigl(f_w(x_n)\bigr) > p, \quad \forall n \ge n_0,$$

which together with (2.9) and (2.10) yields that

$$f_w(x_{n+1}) \le q f_w(x_n) \quad \text{and} \quad \varphi\bigl(w(x_n, x_{n+1})\bigr) \le \frac{f_w(x_n)}{p}, \quad \forall n \ge n_0,$$

which implies that

$$f_w(x_{n+1}) \le q^{n+1-n_0} f_w(x_{n_0}) \quad \text{and} \quad \varphi\bigl(w(x_n, x_{n+1})\bigr) \le \frac{f_w(x_{n_0})}{p} q^{n-n_0}, \quad \forall n \ge n_0. \tag{2.16}$$

By means of (w_1), (2.3) and (2.16), we deduce that

$$\begin{aligned} \varphi\bigl(w(x_n, x_m)\bigr) &\le \sum_{k=n}^{m-1}\varphi\bigl(w(x_k, x_{k+1})\bigr) \le \sum_{k=n}^{m-1}\frac{f_w(x_{n_0})}{p} q^{k-n_0} \\ &\le \frac{f_w(x_{n_0})}{p(1-q)} q^{n-n_0}, \quad \forall m > n \ge n_0. \end{aligned} \tag{2.17}$$

Given $\varepsilon > 0$, denote by δ the constant in (w_3) corresponding to ε. Assume that (2.4) holds. It follows from $\varphi(\delta) > 0$ and $q \in (b,1)$ that there exists a positive integer $N \geq n_0$ satisfying

$$\frac{f_w(x_{n_0})}{p(1-q)} q^{n-n_0} < \varphi(\delta), \quad \forall n \geq N. \tag{2.18}$$

Combining (2.17) and (2.18), we infer that

$$\max\{\varphi(w(x_N, x_m)), \varphi(w(x_N, x_n))\} \leq \frac{f_w(x_{n_0})}{p(1-q)} q^{n-n_0} < \varphi(\delta), \quad \forall m > n \geq N,$$

which together with (2.4) guarantees that

$$\max\{w(x_N, x_m), w(x_N, x_n)\} < \delta, \quad \forall m > n > N. \tag{2.19}$$

It follows from (w_3) and (2.19) that

$$d(x_m, x_n) \leq \varepsilon, \quad \forall m > n > N. \tag{2.20}$$

It is clear that (2.20) yields that $\{x_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence.

Assume that (2.5) holds. Since φ is strictly increasing, so does φ^{-1}. It follows from (2.5) and $q \in (b,1)$ that there exists a positive integer $N \geq n_0$ satisfying

$$\varphi^{-1}\left(\frac{f_w(x_{n_0})}{p(1-q)} q^{n-n_0}\right) < \delta, \quad \forall n \geq N,$$

which together with (2.5) and (2.17) means that

$$w(x_n, x_m) = \varphi^{-1}(\varphi(w(x_n, x_m))) \leq \varphi^{-1}\left(\frac{f_w(x_{n_0})}{p(1-q)} q^{n-n_0}\right) < \delta, \quad \forall m > n \geq N,$$

which ensures that (2.19) and (2.20) hold. Consequently, $\{x_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence.

It follows from completeness of (X, d) that there is some $u_0 \in X$ such that $\lim_{n\to\infty} x_n = u_0$.

Secondly, we prove (a2). Suppose that f_w is T-orbitally lower semi-continuous at u_0. Let $\{x_n\}_{n\in\mathbb{N}_0}$ be the orbit of T defined by (2.9) and satisfy (2.11). It follows from (2.11) that

$$0 \leq w(u_0, T(u_0)) = f_w(u_0) \leq \liminf_{n\to\infty} f_w(x_n) = 0,$$

which means that $f_w(u_0) = 0$. Conversely, suppose that $f_w(u_0) = 0$ for some $u_0 \in X$. Let $\{y_n\}_{n\in\mathbb{N}_0}$ be an arbitrary orbit of T in X with $\lim_{n\to\infty} y_n = u_0$. It follows that

$$f_w(u_0) = 0 \leq \liminf_{n\to\infty} f_w(y_n),$$

that is, f_w is T-orbitally lower semi-continuous at u_0.

Thirdly, we prove (a3). Note that $T(u_0)$ is closed and

$$w(u_0, u_0) = 0 = f_w(u_0) = w(u_0, T(u_0)).$$

It follows from Lemma 1.8 that $u_0 \in T(u_0)$.

Finally, we prove (a4). Assume that $\{x_n\}_{n\in\mathbb{N}_0}$ is the orbit of T defined by (2.9) and that it satisfies (2.11), (2.16), (2.17) and $\lim_{n\to\infty} x_n = u_0 \in X$. Clearly, (2.16) and $q \in (b,1)$ mean that

$$\lim_{n\to\infty} \varphi\big(w(x_n, x_{n+1})\big) = 0. \tag{2.21}$$

Now we claim that

$$\lim_{n\to\infty} w(x_n, u_0) = 0. \tag{2.22}$$

In order to prove (2.22), we consider two possible cases as follows.

Case 1. Assume that (2.4) holds. Let $\varepsilon > 0$ be given. Notice that $\varphi(\varepsilon) > 0$ and $q \in (b,1)$. It follows that there exists a positive integer $N > n_0$ satisfying

$$\frac{f_w(x_{n_0})}{p(1-q)} q^{n-n_0} < \varphi(\varepsilon), \quad \forall n \geq N,$$

which together with (2.17) yields that

$$\varphi\big(w(x_n, x_m)\big) \leq \frac{f_w(x_{n_0})}{p(1-q)} q^{n-n_0} < \varphi(\varepsilon), \quad \forall m > n \geq N.$$

Since φ is strictly inverse, it follows that

$$w(x_n, x_m) < \varepsilon, \quad \forall m > n \geq N.$$

Letting $m \to \infty$ in the above inequality and using (w_2), we get that

$$w(x_n, u_0) \leq \liminf_{m\to\infty} w(x_n, x_m) \leq \varepsilon, \quad \forall n \geq N,$$

that is, (2.22) holds.

Case 2. Assume that (2.5) holds. It follows from (2.5) and (2.17) that

$$w(x_n, x_m) = \varphi^{-1}\big(\varphi\big(w(x_n, x_m)\big)\big) \leq \varphi^{-1}\left(\frac{f_w(x_{n_0})}{p(1-q)} q^{n-n_0}\right), \quad \forall m > n \geq n_0,$$

which together with (w_2) and (2.5) ensures that

$$w(x_n, u_0) \leq \liminf_{m\to\infty} w(x_n, x_m) \leq \varphi^{-1}\left(\frac{f_w(x_{n_0})}{p(1-q)} q^{n-n_0}\right) \to 0 \quad \text{as } n \to \infty,$$

that is, (2.22) holds.

Suppose that $u_0 \notin T(u_0)$. Let $v = u_0$ and $z_n = x_n$ for each $n \in \mathbb{N}_0$. Assume that (2.6) holds. Making use of (2.6), (2.21) and (2.22), we conclude that

$$0 < \inf\big\{w(x_n, u_0) + \varphi\big(w(x_n, x_{n+1})\big) : n \in \mathbb{N}_0\big\} = 0,$$

which is a contradiction. Assume that (2.7) holds. By virtue of (2.7), (2.11) and (2.22), we infer that

$$0 < \inf\big\{w(x_n, u_0) + w(x_n, x_{n+1}) : n \in \mathbb{N}_0\big\} = 0,$$

which is also a contradiction. Consequently, $u_0 \in T(u_0)$. This completes the proof. □

Theorem 2.2 *Let* (X,d) *be a complete metric space, w be a w-distance in X and T be a multi-valued mapping from X into* CL(X) *such that* (2.3) *and one of* (2.4) *and* (2.5) *hold and*

$$\begin{aligned}&\text{for each } x \in X, \text{there exists } y \in T(x) \text{ satisfying}\\&\alpha\big(w(x,y)\big)\varphi\big(w(x,y)\big) \le f_w(x) \quad \text{and} \quad f_w(y) \le \beta\big(w(x,y)\big)\psi\big(w(x,y)\big),\end{aligned} \tag{2.23}$$

where

$$\begin{aligned}&\alpha \text{ and } \beta \text{ are functions from } A_w \text{ into } (0,1] \text{ and } [0,1), \text{respectively, with}\\&\beta(0) < \alpha(0), \qquad \liminf_{r\to 0^+} \alpha(r) > 0 \quad \text{and} \quad \limsup_{r\to t^+} \frac{\beta(r)}{\alpha(r)} < 1, \quad \forall t \in A_w\end{aligned} \tag{2.24}$$

and

$$\text{one of } \alpha \text{ and } \beta \text{ is nondecreasing in } A_w. \tag{2.25}$$

Then (a1)-(a4) *hold.*

Proof Firstly, we prove (a1). Let

$$\gamma(t) = \frac{\beta(t)}{\alpha(t)}, \quad \forall t \in A_w. \tag{2.26}$$

Notice that the ranges of α and β, (2.24) and (2.26) ensure that

$$0 \le \gamma(t) < 1, \quad \forall t \in A_w. \tag{2.27}$$

It follows from (2.23) that for each $x_0 \in X$, there exists $x_1 \in T(x_0)$ satisfying

$$\alpha\big(w(x_0,x_1)\big)\varphi\big(w(x_0,x_1)\big) \le f_w(x_0) \quad \text{and} \quad f_w(x_1) \le \beta\big(w(x_0,x_1)\big)\psi\big(w(x_0,x_1)\big),$$

which together with (2.3) and (2.26) means that

$$\begin{aligned}f_w(x_1) &\le \beta\big(w(x_0,x_1)\big)\psi\big(w(x_0,x_1)\big) \le \beta\big(w(x_0,x_1)\big)\varphi\big(w(x_0,x_1)\big)\\&\le \beta\big(w(x_0,x_1)\big)\frac{f_w(x_0)}{\alpha(w(x_0,x_1))} = \gamma\big(w(x_0,x_1)\big)f_w(x_0).\end{aligned}$$

Continuing this process, we choose easily an orbit $\{x_n\}_{n\in\mathbb{N}_0}$ of T satisfying

$$\begin{aligned}&x_{n+1} \in T(x_n), \quad \alpha\big(w(x_n,x_{n+1})\big)\varphi\big(w(x_n,x_{n+1})\big) \le f_w(x_n) \quad \text{and}\\&f_w(x_{n+1}) \le \beta\big(w(x_n,x_{n+1})\big)\psi\big(w(x_n,x_{n+1})\big), \quad \forall n \in \mathbb{N}_0,\end{aligned} \tag{2.28}$$

which together with (2.3) and (2.26) gives that

$$\begin{aligned}f_w(x_{n+1}) &\le \beta\big(w(x_n,x_{n+1})\big)\psi\big(w(x_n,x_{n+1})\big) \le \beta\big(w(x_n,x_{n+1})\big)\varphi\big(w(x_n,x_{n+1})\big)\\&\le \beta\big(w(x_n,x_{n+1})\big)\frac{f_w(x_n)}{\alpha(w(x_n,x_{n+1}))} = \gamma\big(w(x_n,x_{n+1})\big)f_w(x_n), \quad \forall n \in \mathbb{N}_0\end{aligned} \tag{2.29}$$

and

$$\begin{aligned}\varphi\big(w(x_{n+1},x_{n+2})\big) &\le \frac{f_w(x_{n+1})}{\alpha(w(x_{n+1},x_{n+2}))}\\ &\le \frac{\beta(w(x_n,x_{n+1}))}{\alpha(w(x_{n+1},x_{n+2}))}\psi\big(w(x_n,x_{n+1})\big), \quad \forall n\in\mathbb{N}_0.\end{aligned} \tag{2.30}$$

Now we claim that

$$w(x_{n+1},x_{n+2}) \le w(x_n,x_{n+1}), \quad \forall n\in\mathbb{N}_0. \tag{2.31}$$

Suppose that there exists $n_0\in\mathbb{N}_0$ satisfying

$$w(x_{n_0+1},x_{n_0+2}) > w(x_{n_0},x_{n_0+1}). \tag{2.32}$$

Let (2.4) hold. It follows from (2.3), (2.25), (2.26), (2.30) and (2.32) that

$$\begin{aligned}\varphi\big(w(x_{n_0+1},x_{n_0+2})\big) &\le \frac{\beta(w(x_{n_0},x_{n_0+1}))}{\alpha(w(x_{n_0+1},x_{n_0+2}))}\psi\big(w(x_{n_0},x_{n_0+1})\big)\\ &\le \max\big\{\gamma\big(w(x_{n_0},x_{n_0+1})\big),\gamma\big(w(x_{n_0+1},x_{n_0+2})\big)\big\}\varphi\big(w(x_{n_0},x_{n_0+1})\big).\end{aligned} \tag{2.33}$$

If $\varphi(w(x_{n_0},x_{n_0+1})) = 0$, it follows from (2.33) that $\varphi(w(x_{n_0+1},x_{n_0+2})) = 0$. Thus (2.4) and (2.32) guarantee that

$$0 \le w(x_{n_0},x_{n_0+1}) < w(x_{n_0+1},x_{n_0+2}) = 0,$$

which is a contradiction; if $\varphi(w(x_{n_0},x_{n_0+1})) > 0$, (2.4), (2.26), (2.27) and (2.33) yield that

$$\begin{aligned}\varphi\big(w(x_{n_0+1},x_{n_0+2})\big) &\le \max\big\{\gamma\big(w(x_{n_0},x_{n_0+1})\big),\gamma\big(w(x_{n_0+1},x_{n_0+2})\big)\big\}\varphi\big(w(x_{n_0},x_{n_0+1})\big)\\ &< \varphi\big(w(x_{n_0},x_{n_0+1})\big).\end{aligned} \tag{2.34}$$

Since φ is strictly inverse, it follows from (2.32) and (2.34) that

$$w(x_{n_0+1},x_{n_0+2}) < w(x_{n_0},x_{n_0+1}) < w(x_{n_0+1},x_{n_0+2}),$$

which is impossible.

Let (2.5) hold. Notice that φ is strictly increasing. It follows from (2.3), (2.26), (2.27), (2.30) and (2.32) that

$$\begin{aligned}\varphi\big(w(x_{n_0+1},x_{n_0+2})\big) &\le \frac{\beta(w(x_{n_0},x_{n_0+1}))}{\alpha(w(x_{n_0+1},x_{n_0+2}))}\psi\big(w(x_{n_0},x_{n_0+1})\big)\\ &\le \max\big\{\gamma\big(w(x_{n_0},x_{n_0+1})\big),\gamma\big(w(x_{n_0+1},x_{n_0+2})\big)\big\}\varphi\big(w(x_{n_0},x_{n_0+1})\big)\\ &\le \varphi\big(w(x_{n_0},x_{n_0+1})\big)\\ &< \varphi\big(w(x_{n_0+1},x_{n_0+2})\big),\end{aligned}$$

which is absurd. Hence (2.31) holds. That is, $\{w(x_n, x_{n+1})\}_{n\in\mathbb{N}_0}$ is a nonincreasing and nonnegative sequence. It follows that $\lim_{n\to\infty} w(x_n, x_{n+1}) = d$ for some $d \geq 0$.

Now we claim that (2.11) holds. Using (2.27) and (2.29), we conclude that $\{f_w(x_n)\}_{n\in\mathbb{N}_0}$ is a nonnegative and nonincreasing sequence. Consequently, (2.13) is satisfied for some $a \geq 0$. Suppose that $a > 0$. Using (2.13), (2.24), (2.27) and (2.29), we obtain that

$$\begin{aligned} a &= \limsup_{n\to\infty} f_w(x_{n+1}) \leq \limsup_{n\to\infty}\bigl[\gamma\bigl(w(x_n, x_{n+1})\bigr) f_w(x_n)\bigr] \\ &\leq \limsup_{n\to\infty}\gamma\bigl(w(x_n, x_{n+1})\bigr) \limsup_{n\to\infty} f_w(x_n) \leq a \limsup_{t\to d^+}\gamma(t) \\ &< a, \end{aligned}$$

which is a contradiction. Thus $a = 0$, that is, (2.11) holds.

Next we claim that $\{x_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence. Put

$$b = \limsup_{n\to\infty}\gamma\bigl(w(x_n, x_{n+1})\bigr) \quad \text{and} \quad c = \liminf_{n\to\infty}\alpha\bigl(w(x_n, x_{n+1})\bigr). \tag{2.35}$$

It follows from (2.24), (2.27), (2.29) and (2.35) that (2.15) holds. Let $p \in (0, c)$ and $q \in (b, 1)$. Because of (2.15) and (2.35), we deduce that there exists some $n_0 \in \mathbb{N}$ such that

$$\gamma\bigl(w(x_n, x_{n+1})\bigr) < q \quad \text{and} \quad \alpha\bigl(w(x_n, x_{n+1})\bigr) > p, \quad \forall n \geq n_0,$$

which together with (2.28) and (2.29) yields that

$$f_w(x_{n+1}) \leq q f_w(x_n) \quad \text{and} \quad \varphi\bigl(w(x_n, x_{n+1})\bigr) \leq \frac{f_w(x_n)}{p}, \quad \forall n \geq n_0.$$

The rest of the proof is similar to that of Theorem 2.1 and is omitted. This completes the proof. □

3 Remarks and illustrative examples

In this section we construct two nontrivial examples to illustrate the results in Section 2.

Remark 3.1 Theorem 2.1 extends Theorem 3.1 in [2], Theorem 2.1 in [5], Theorem 2.1 in [6], Theorems 2.1 and 2.2 in [7], Theorems 2.2 and 2.4 in [9], and Theorems 3.1 and 3.2 in [10]. Example 3.2 below shows that Theorem 2.1 extends substantially Theorem 3.1 in [2] and Theorem 2.1 in [5] and differs from Theorems 5 and 6 in [8] and Theorem 2.1 in [4].

Example 3.2 Let $X = [0,1] \cup \{\frac{6}{5}\}$ be endowed with the Euclidean metric $d = |\cdot|$ and $u_0 = 0$. Define $w : X \times X \to \mathbb{R}^+$, $T : X \to \mathrm{CL}(X)$, $\alpha : [0, \frac{1}{4}] \to (0, 1]$, $\beta : [0, \frac{1}{4}] \to [0, 1)$ and $\varphi, \psi : [0, \frac{6}{5}] \to \mathbb{R}^+$ by

$$\begin{aligned} & w(x, y) = y, \quad \forall x, y \in X, \\ & T(x) = \begin{cases} \{\frac{x}{4}\}, & \forall x \in [0, \frac{2}{5}) \cup (\frac{2}{5}, 1], \\ \{\frac{1}{10}, \frac{1}{3}\}, & \forall x \in \{\frac{2}{5}, \frac{6}{5}\}, \end{cases} \\ & \alpha(t) = \frac{8+t}{9}, \qquad \beta(t) = \frac{2+t}{3}, \quad \forall t \in \left[0, \frac{1}{4}\right] \end{aligned}$$

and

$$\varphi(t) = t, \qquad \psi(t) = \min\{t, |1-t|\}, \quad \forall t \in \left[0, \frac{6}{5}\right].$$

It is easy to see that $A_w = [0, \frac{6}{5}]$, $B_w = [0, \frac{1}{4}]$, (2.3), (2.4) and (2.5) hold and

$$f_w(x) = w(x, T(x)) = \begin{cases} \frac{x}{4}, & \forall x \in [0, \frac{2}{5}) \cup (\frac{2}{5}, 1], \\ \frac{1}{10}, & \forall x \in \{\frac{2}{5}, \frac{6}{5}\}, \end{cases}$$

is T-orbitally lower semi-continuous at u_0,

$$\beta(0) = \frac{2}{3} < \frac{8}{9} = \alpha(0), \qquad \liminf_{r\to 0^+} \alpha(r) = \frac{8}{9} > 0,$$
$$\limsup_{r\to t^+} \frac{\beta(r)}{\alpha(r)} = \limsup_{r\to t^+}\left(\frac{2+r}{3}\cdot\frac{9}{8+r}\right) = \frac{6+3t}{8+t} < 1, \quad \forall t \in \left[0, \frac{1}{4}\right].$$

For $x \in [0, \frac{2}{5}) \cup (\frac{2}{5}, 1]$, there exists $y = \frac{x}{4} \in T(x) = \{\frac{x}{4}\}$ satisfying

$$\alpha(f_w(x))\varphi(w(x,y)) = \frac{8+\frac{x}{4}}{9}\cdot\frac{x}{4} \le \frac{x}{4} = f_w(x)$$

and

$$f_w(y) = \frac{x}{16} \le \frac{2+\frac{x}{4}}{3}\cdot\min\left\{\frac{x}{4}, 1-\frac{x}{4}\right\} = \beta(f_w(x))\psi(w(x,y)).$$

For $x \in \{\frac{2}{5}, \frac{6}{5}\}$, there exists $y = \frac{1}{10} \in T(x) = \{\frac{1}{10}, \frac{1}{3}\}$ satisfying

$$\alpha(f_w(x))\varphi(w(x,y)) = \frac{8+\frac{1}{10}}{9}\cdot\frac{1}{10} \le \frac{1}{10} = f_w(x)$$

and

$$f_w(y) = \frac{1}{40} \le \frac{2+\frac{1}{10}}{3}\cdot\min\left\{\frac{1}{10}, 1-\frac{1}{10}\right\} = \beta(f_w(x))\psi(w(x,y)).$$

Put $v \in X \setminus \{0\}$ and $\{z_n\}_{n\in\mathbb{N}_0}$ is an orbit of T in X. It is easy to verify that $\lim_{n\to\infty} z_n = u_0 = 0$ and

$$\begin{aligned} &\inf\{w(z_n, v) + \varphi(w(z_n, z_{n+1})) : n \in \mathbb{N}_0\} \\ &\quad = \inf\{v + z_{n+1} : n \in \mathbb{N}_0\} \\ &\quad = v + u_0 = v > 0. \end{aligned}$$

Hence (2.1), (2.2) and (2.6) hold, that is, the conditions of Theorem 2.1 are fulfilled. Thus Theorem 2.1 guarantees that (a1)-(a4) hold. Moreover, T has a fixed point $u_0 = 0 \in X$.

Now we show that Theorem 2.1 in [5] is unapplicable in proving the existence of fixed points for the multi-valued mapping T. Otherwise there exists a function $\varphi : \mathbb{R}^+ \to [a, 1)$,

$0 < a < 1$, such that

$$\limsup_{r \to t^+} \varphi(r) < 1, \quad \forall t \in \mathbb{R}^+, \tag{3.1}$$

and for any $x \in X$ there is $y \in T(x)$ satisfying

$$\sqrt{\varphi\big(f(x)\big)}d(x,y) \le f(x) \tag{3.2}$$

and

$$f(y) \le \varphi\big(f(x)\big)d(x,y). \tag{3.3}$$

Note that

$$f(x) = d\big(x, T(x)\big) = \begin{cases} \frac{3}{4}x, & \forall x \in [0, \frac{2}{5}) \cup (\frac{2}{5}, 1], \\ \frac{1}{15}, & x = \frac{2}{5}, \\ \frac{13}{15}, & x = \frac{6}{5}. \end{cases}$$

Put $x = \frac{2}{5}$. For $y \in T(x) = \{\frac{1}{10}, \frac{1}{3}\}$, we discuss two cases as follows.

Case 1. $y = \frac{1}{10}$. It follows from (3.2) and (3.3) that

$$\frac{3}{10}\sqrt{\varphi\left(\frac{1}{15}\right)} = \sqrt{\varphi\left(f\left(\frac{2}{5}\right)\right)}d\left(\frac{2}{5}, \frac{1}{10}\right) = \sqrt{\varphi\big(f(x)\big)}d(x,y) \le f(x) = f\left(\frac{2}{5}\right) = \frac{1}{15}$$

and

$$\frac{3}{40} = f\left(\frac{1}{10}\right) = f(y) \le \varphi\big(f(x)\big)d(x,y) = \varphi\left(f\left(\frac{2}{5}\right)\right)d\left(\frac{2}{5}, \frac{1}{10}\right) = \frac{3}{10}\varphi\left(\frac{1}{15}\right),$$

which imply that

$$0.25 = \frac{1}{4} \le \varphi\left(\frac{1}{15}\right) \le \frac{4}{81} = 0.049,$$

which is impossible.

Case 2. $y = \frac{1}{3}$. It follows from (3.3) that

$$\frac{1}{4} = f\left(\frac{1}{3}\right) = f(y) \le \varphi\big(f(x)\big)d(x,y) = \varphi\left(f\left(\frac{2}{5}\right)\right)d\left(\frac{2}{5}, \frac{1}{3}\right) = \frac{1}{15}\varphi\left(\frac{1}{15}\right),$$

which together with $\varphi(\mathbb{R}^+) \subseteq [a, 1)$ yields that

$$\frac{15}{4} \le \varphi\left(\frac{1}{15}\right) < 1,$$

which is absurd.

Next we show that Theorem 5 in [8] is useless in proving the existence of fixed points for the multi-valued mapping T. Otherwise there exists a function $\varphi : \mathbb{R}^+ \to [0,1)$ such that (3.1) holds, and for any $x \in X$ there is $y \in T(x)$ satisfying

$$d(x,y) \le \big(2 - \varphi\big(d(x,y)\big)\big) f(x) \tag{3.4}$$

and

$$f(y) \le \varphi\big(d(x,y)\big) d(x,y). \tag{3.5}$$

Put $x = \frac{2}{5}$. For $y \in T(x) = \{\frac{1}{10}, \frac{1}{3}\}$, we discuss two cases as follows.

Case 1. $y = \frac{1}{10}$. It follows from (3.4) that

$$\frac{3}{10} = d\left(\frac{2}{5}, \frac{1}{10}\right) = d(x,y) \le \big(2 - \varphi\big(d(x,y)\big)\big) f(x) = \left(2 - \varphi\left(\frac{3}{10}\right)\right) \frac{1}{15},$$

which together with $\varphi(\mathbb{R}^+) \subseteq [0,1)$ yields that

$$0 \le \varphi\left(\frac{3}{10}\right) \le -\frac{5}{2} < 0,$$

which is a contradiction.

Case 2. $y = \frac{1}{3}$. It follows from (3.4) that

$$\frac{1}{4} = f\left(\frac{1}{3}\right) = f(y) \le \varphi\big(d(x,y)\big) d(x,y) = \varphi\left(d\left(\frac{2}{5}, \frac{1}{3}\right)\right) d\left(\frac{2}{5}, \frac{1}{3}\right) = \frac{1}{15} \varphi\left(\frac{1}{15}\right),$$

which together with $\varphi(\mathbb{R}^+) \subseteq [0,1)$ gives that

$$\frac{15}{4} \le \varphi\left(\frac{1}{15}\right) < 1,$$

which is impossible.

Finally we show that Theorem 6 in [8] is futile in proving the existence of fixed points for the multi-valued mapping T. Otherwise there exist functions $\varphi : \mathbb{R}^+ \to (0,1)$, $b : \mathbb{R}^+ \to [b,1)$, $b > 0$ such that

$$\varphi(t) < b(t), \qquad \limsup_{r \to t^+} \varphi(r) < \limsup_{r \to t^+} b(r), \quad \forall t \in \mathbb{R}^+, \tag{3.6}$$

and for any $x \in X$, there is $y \in T(x)$ satisfying (3.5) and

$$b\big(d(x,y)\big) d(x,y) \le f(x). \tag{3.7}$$

Put $x = \frac{2}{5}$. For $y \in T(x) = \{\frac{1}{10}, \frac{1}{3}\}$, we discuss two cases as follows.

Case 1. $y = \frac{1}{10}$. It follows from (3.7) and (3.5) that

$$\frac{3}{10} b\left(\frac{3}{10}\right) = b\left(d\left(\frac{2}{5}, \frac{1}{10}\right)\right) d\left(\frac{2}{5}, \frac{1}{10}\right) = b\big(d(x,y)\big) d(x,y) \le f(x) = f\left(\frac{2}{5}\right) = \frac{1}{15}$$

and

$$\frac{3}{40} = f\left(\frac{1}{10}\right) = f(y) \le \varphi\big(d(x,y)\big)d(x,y) = \frac{3}{10}\varphi\left(\frac{3}{10}\right),$$

which together with (3.6) means that

$$b\left(\frac{3}{10}\right) \le \frac{2}{9} < \frac{1}{4} \le \varphi\left(\frac{3}{10}\right) < b\left(\frac{3}{10}\right),$$

which is absurd.

Case 2. $y = \frac{1}{3}$. It follows from (3.5) that

$$\frac{1}{4} = f\left(\frac{1}{3}\right) = f(y) \le \varphi\big(d(x,y)\big)d(x,y) = \varphi\left(d\left(\frac{2}{5},\frac{1}{3}\right)\right)d\left(\frac{2}{5},\frac{1}{3}\right) = \frac{1}{15}\varphi\left(\frac{1}{15}\right),$$

which together with $\varphi(\mathbb{R}^+) \subseteq [0,1)$ gives that

$$\frac{15}{4} \le \varphi\left(\frac{1}{15}\right) < 1,$$

which is impossible.

Observe that Theorem 6 in [8] extends Theorem 3.1 in [2], Theorem 2.1 in [4] and Theorem 2.2 in [5]. It follows that Theorem 3.1 in [2], Theorem 2.1 in [4] and Theorem 2.2 in [5] are not applicable in proving the existence of fixed points for the multi-valued mapping T.

Remark 3.3 Theorem 2.2 extends, improves and unifies Theorem 3.1 in [2], Theorem 2.1 in [4], Theorem 2.2 in [5], Theorem 2.3 in [6], Theorems 2.3 and 2.5 in [7], Theorem 6 in [8], and Theorems 3.3 and 3.4 in [10]. The following example reveals that Theorem 2.2 generalizes indeed the corresponding results in [2, 4, 5, 8].

Example 3.4 Let $X = [0,\infty)$ be endowed with the Euclidean metric $d = |\cdot|$ and $p \ge 1$ be a constant. Put $u_0 = 0$. Define $w : X \times X \to \mathbb{R}^+$, $T : X \to \mathrm{CL}(X)$, $\alpha : [0,\infty) \to (0,1]$ and $\varphi, \psi : [0,\infty) \to \mathbb{R}^+$ by $\beta : [0,\infty) \to [0,1)$ by

$$w(x,y) = y^p, \quad \forall x,y \in X,$$

$$T(x) = \begin{cases} [\frac{x^2}{2}, \frac{x}{2}], & \forall x \in [0,1), \\ [\frac{1}{9}, \frac{1}{4}], & \forall x \in [1,\infty), \end{cases}$$

$$\alpha(t) = \frac{5 + t^{\frac{1}{p}}}{10}, \qquad \beta(t) = \frac{3 + t^{\frac{1}{p}}}{10}, \quad \forall t \in [0,\infty)$$

and

$$\varphi(t) = t, \quad \forall t \in [0,\infty), \qquad \psi(t) = \begin{cases} t, & \forall t \in [0,1), \\ \frac{1}{2}, & \forall t \in [1,\infty). \end{cases}$$

It is easy to see that $A_w = [0,\infty)$, (2.3), (2.4) and (2.5) hold, w is a w-distance in X and

$$f_w(x) = w\big(x, T(x)\big) = \begin{cases} (\frac{x^2}{2})^p, & \forall x \in [0,1), \\ \frac{1}{9^p}, & \forall x \in [1,\infty) \end{cases}$$

is T-orbitally lower semi-continuous in X, α and β are nondecreasing,

$$\beta(0) = \frac{3}{10} < \frac{1}{2} = \alpha(0), \qquad \liminf_{r\to 0^+} \alpha(r) = \frac{1}{2} > 0$$

and

$$\limsup_{r\to t^+} \frac{\beta(r)}{\alpha(r)} = \frac{3 + t^{\frac{1}{p}}}{5 + t^{\frac{1}{p}}} < 1, \quad \forall t \in A_w.$$

Put $x \in [0,1)$ and $y = \frac{x^2}{2} \in T(x)$. Note that

$$5 + y \le 10 \quad \text{and} \quad \left(\frac{y}{2}\right)^p \le \frac{1}{4^p} \le \frac{3+y}{10}$$

imply that

$$\alpha\big(w(x,y)\big)\varphi\big(w(x,y)\big) = \frac{5+y}{10} \cdot y^p \le y^p = f_w(x)$$

and

$$f_w(y) = \left(\frac{y^2}{2}\right)^p \le \frac{3+y}{10} \cdot y^p = \beta\big(w(x,y)\big)\psi\big(w(x,y)\big).$$

Put $x \in [1,\infty)$ and $y = \frac{1}{9} \in T(x) = [\frac{1}{9}, \frac{1}{4}]$. It follows that

$$\alpha\big(w(x,y)\big)\varphi\big(w(x,y)\big) = \frac{5 + \frac{1}{9}}{10} \cdot \frac{1}{9^p} \le \frac{1}{9^p} = f_w(x)$$

and

$$f_w(y) = \frac{1}{182^p} \le \frac{3 + \frac{1}{9}}{10} \cdot \frac{1}{9^p} = \beta\big(w(x,y)\big)\psi\big(w(x,y)\big).$$

Let $v \in X \setminus \{0\}$ and $\{z_n\}_{n\in\mathbb{N}_0}$ be an orbit of T. It is easy to verify that $\lim_{n\to\infty} z_n = 0$ and

$$\begin{aligned} &\inf\big\{w(z_n, v) + \varphi\big(w(z_n, z_{n+1})\big) : n \in \mathbb{N}_0\big\} \\ &\quad = \inf\big\{v^p + z_{n+1}^p : n \in \mathbb{N}_0\big\} = v^p > 0. \end{aligned}$$

That is, (2.6) and (2.23)-(2.25) hold. Thus the conditions of Theorem 2.2 are satisfied. Consequently, Theorem 2.2 ensures that (a1)-(a4) hold and $u_0 = 0$ is a fixed point of the multi-valued mapping T in X.

Notice that

$$f(x) = d\big(x, T(x)\big) = \begin{cases} \frac{x}{2}, & \forall x \in [0,1), \\ x - \frac{1}{4}, & \forall x \in [1,\infty) \end{cases}$$

and

$$\liminf_{x\to 1} f(x) = \frac{1}{2} < \frac{3}{4} = f(1),$$

which implies that f is not lower semi-continuous at 1. Thus Theorem 3.1 in [2], Theorem 2.1 in [4], Theorem 2.2 in [5] and Theorem 6 in [8] could not be used to judge the existence of fixed points of the multi-valued mapping T in X.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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