Huang *Fixed Point Theory and Applications* 2014, 2014:44
http://www.fixedpointtheoryandapplications.com/content/2014/1/44

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Nonexpansive semigroups in CAT(κ) spaces

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Dedicated to Professor Wataru Takahashi on the occasion of his 70th birthday

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Abstract
The main purpose of this paper is to study Browder type convergence theorems for a nonexpansive semigroup with geometric approaches in a CAT(κ) space. Besides, we determine a necessary and sufficient condition for convergence of a Browder type iteration associated to a uniformly asymptotically regular nonexpansive semigroup on the unit sphere in an infinite-dimensional Hilbert space.
MSC: 47H20; 47H10

Keywords: nonexpansive mapping; asymptotically regular; uniformly asymptotically regular; Δ-convergence; CAT(κ) space

1 Introduction

Let (X,d) be a metric space, C a closed convex subset of X and $T : C \to C$ a mapping. Recall that T is nonexpansive on C if $d(Tx,Ty) \leq d(x,y)$, for all $x,y \in C$. We denote by $\mathfrak{F}(T)$ the fixed point set of the mapping T. A one-parameter family $\mathfrak{S} = \{T(t) : t \geq 0\}$ of self-mappings of C is called a *strongly continuous nonexpansive semigroup* on C if the following conditions are satisfied:

(i) for each $t \geq 0$, $T(t)$ is a nonexpansive mapping on C;
(ii) $T(0)x = x$, for all $x \in C$;
(iii) $T(s+t) = T(s) \circ T(t)$, for all $s,t \geq 0$;
(iv) for each $x \in C$, the mapping $t \mapsto T(t)x$ from $[0,\infty)$ into C is continuous.

Let $\mathfrak{F}(\mathfrak{S})$ denote the common fixed point set of all mappings in $\mathfrak{S}$.

There have been considerably many interesting results of iterative methods for approximating fixed points of nonexpansive mappings, nonexpansive semigroups, and their generalizations which solve some variational inequalities problems due to their various applications in several physical problems, such as in operations research, economics, and engineering design; see, *e.g.*, [1–4] and the references therein.

Suppose that X is a real Hilbert space and u is an arbitrary point of X. If T is nonexpansive on C, then for each $\alpha \in (0,1)$ there exists a unique $x_\alpha \in C$ such that $x_\alpha = \alpha u + (1-\alpha)Tx_\alpha$ because the mapping $x \mapsto \alpha u + (1-\alpha)Tx$ is a contraction. In 1967, Browder [5] was the pioneer to consider an implicit scheme and prove the following strong convergence theorem of this algorithm in a Hilbert space.

Theorem 1.1 *Let C be a bounded closed convex subset of a Hilbert space and T a nonexpansive mapping on C. Let u be an arbitrary point of C and define $x_\alpha \in C$ by*

$$x_\alpha = \alpha u + (1-\alpha)Tx_\alpha, \quad \text{for } \alpha \in (0,1).$$

Then as $\alpha \to 0$, $\{x_\alpha\}$ *converges strongly to a point of* $\mathfrak{F}(T)$ *nearest to* u.

The extension work of Browder's type convergence theorems has been tremendously studied for not only one single nonexpansive mapping, but, most significantly, a semigroup of nonexpansive mappings; see, *e.g.*, [6, 7] and the references therein.

This paper is devoted to studying Browder's type iterations for a nonexpansive semigroup in a CAT(κ) space, where $\kappa \in \mathbb{R}$, which is a specific type of metric space. Intuitively, triangles in a CAT(κ) space are 'slimmer' than corresponding 'model triangles' in a standard space of constant curvature κ (see Section 2). Complete CAT(0) spaces are often called *Hadamard spaces*. A few recent new convergence results of classical iterations on CAT(κ) spaces with $\kappa > 0$ are obtained; see, *e.g.*, [8–10] and the references therein.

In [11], Dhompongsa *et al.* extended Suzuki's result [7, Theorem 3] on common fixed points of a nonexpansive semigroup in a Hilbert space to a complete CAT(0) space.

Theorem 1.2 ([11, Theorem 4.4]) *Let* C *be a bounded closed convex subset of a* CAT(0) *space*, $\mathfrak{S} = \{T(t) : t \geq 0\}$ *a strongly continuous nonexpansive semigroup on* C, *and two sequences* $\{\alpha_n\} \subset (0,1)$, $\{t_n\} \subset (0,\infty)$ *such that* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \alpha_n/t_n = 0$. *Choose arbitrarily a point* $x_0 \in C$ *and for each* $n \in \mathbb{N}$ *let* x_n *be the fixed point of the mapping* $x \mapsto \alpha_n x_0 \oplus (1-\alpha_n)T(t_n)x$. *Then* $\mathfrak{F}(\mathfrak{S}) \neq \emptyset$ *and* $\{x_n\}$ *converges strongly to a point of* $\mathfrak{F}(\mathfrak{S})$ *nearest to* x_0.

In 2010, Acedo and Suzuki [6] proved a Browder type convergence theorem for *uniformly asymptotically regular* (*UAR* for short) nonexpansive semigroups (see Section 2) in Hilbert spaces under a weaker condition on $\{\alpha_n\}$ and $\{t_n\}$ than that in Theorem 1.2.

Theorem 1.3 ([6, Theorem 2.3]) *Let* C *be a closed convex subset of a Hilbert space*, $\mathfrak{S} = \{T(t) : t \geq 0\}$ *a UAR and strongly continuous nonexpansive semigroup on* C *such that* $\mathfrak{F}(\mathfrak{S}) \neq \emptyset$, *and two sequences* $\{\alpha_n\} \subset (0,1)$, $\{t_n\} \subset [0,\infty)$ *such that* $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \alpha_n/t_n = 0$. *Fix* $x_0 \in C$ *and define a sequence* $\{x_n\}$ *in* C *by*

$$x_n = \alpha_n x_0 + (1-\alpha_n)T(t_n)x_n.$$

Then $\{x_n\}$ *converges strongly to a point of* $\mathfrak{F}(\mathfrak{S})$ *nearest to* x_0.

The preceding two theorems lead naturally to the question of whether or not they can be extended to a CAT(κ) space with $\kappa > 0$. The purpose of this article is to investigate this question with the geometric approaches in CAT(κ) spaces.

This paper is organized as follows. In Section 2 we recall the definition of geodesic metric spaces and summarize some useful lemmas and the main properties of CAT(κ) spaces. In Section 3 we present some technical results about Δ-convergence of a sequence in a complete CAT(1) space. In Section 4 we establish our main results (Theorems 4.2, 4.3, 4.4) of Browder's iterations for nonexpansive semigroups in CAT(κ) spaces and conclude that Theorems 1.2 and 1.3 can be generalized to CAT(κ) spaces under the same respective conditions on the coefficients $\{\alpha_n\}$ and $\{t_n\}$. It is noteworthy that, however, without the *UAR* assumption, the sequence $\{x_n\}$ established in Theorem 1.2 is not necessarily convergent to the nearest point of $\mathfrak{F}(\mathfrak{S})$ to x_0 if $\lim_{n\to\infty} t_n = \hat{t} \in (0,\infty]$ even when $\lim_{n\to\infty} \alpha_n = 0$ (and therefore $\lim_{n\to\infty} \alpha_n/t_n = 0$); see Example 4.5. Furthermore, we determine a necessary and sufficient condition for a Browder type convergence theorem associated to a nonexpansive semigroup on the unit sphere in an infinite-dimensional Hilbert space. We also propose an open problem in Section 5.

2 Preliminaries

Let (X,d) be a metric space. For any subset E of X and $x \in X$, the *diameter* of E and the *distance from x to E* are defined, respectively, by

$$\operatorname{diam} E = \sup\{d(x,y) : x,y \in E\},$$
$$d(x,E) = \inf\{d(x,y) : y \in E\}.$$

We always denote the open ball and the closed ball centered at x with radius $r > 0$ by $B(x,r)$ and $\overline{B}(x,r)$, respectively.

Let C be a closed convex subset of X and let $\mathfrak{S} = \{T(t) : t \geq 0\}$ be a family of self-mappings of C. Then $\mathfrak{S}$ is called

(i) *asymptotically regular* on C if for any $h \geq 0$ and any $x \in C$,

$$\lim_{t\to\infty} d\big(T(h)T(t)x, T(t)x\big) = 0;$$

(ii) *uniformly asymptotically regular* (in short *UAR*) on C if for any $h \geq 0$ and any bounded subset D of C,

$$\lim_{t\to\infty} \sup_{x\in D} d\big(T(h)T(t)x, T(t)x\big) = 0.$$

For $x,y \in X$, a *geodesic path* joining x to y (or a *geodesic* from x to y) is an isometric mapping $\gamma : [a,b] \subset \mathbb{R} \to X$ such that $\gamma(a) = x$, $\gamma(b) = y$, *i.e.*, $d(\gamma(t),\gamma(t')) = |t - t'|$, for all $t,t' \in [a,b]$. Therefore $d(x,y) = b - a$. The image of γ is called a *geodesic* (*segment*) from x to y and we shall denote a definite choice of this geodesic segment by $[x,y]$. We remark that composing γ with a translation (this is still an isometry), one can always choose the interval $[a,b]$ to be $[0,\ell]$, where $\ell = b - a$. A point $z = \gamma(t)$ in the geodesic $[x,y]$ will be written as $z = (1-\lambda)x \oplus \lambda y$, where $\lambda = (t-a)/(b-a)$, and so $d(z,x) = \lambda d(x,y)$ and $d(z,y) = (1-\lambda)d(x,y)$.

Let $r > 0$. The metric space (X,d) is said to be

(i) a *geodesic* (*metric*) *space* if any two points in X are joined by a geodesic;
(ii) *uniquely geodesic* if there is exactly one geodesic joining x to y for all $x,y \in X$;
(iii) *r-geodesic space* if any two points $x,y \in X$ with $d(x,y) < r$ are joined by a geodesic;
(iv) *r-uniquely geodesic* if any two points $x,y \in X$ with $d(x,y) < r$ are joined by a unique geodesic in X.

A subset C of X is *convex* if every pair of points $x,y \in C$ can be joined by a geodesic in X and the image of every such geodesic is contained in C. If this condition holds for all points $x,y \in C$ with $d(x,y) < r$, then C is said to be *r-convex*.

The n-dimensional sphere $\mathbb{S}^n$ is the set $\{x = (x_1,\ldots,x_{n+1}) \in \mathbb{R}^{n+1} : (x \mid x) = 1\}$, where $(\cdot \mid \cdot)$ denote the Euclidean scalar product. It is endowed with the following metric: Let $d_{\mathbb{S}^n}$:

$\mathbb{S}^n \times \mathbb{S}^n \to \mathbb{R}$ be the function that assigns to each pair $(A,B) \in \mathbb{S}^n \times \mathbb{S}^n$ the unique number $d(A,B) \in [0,\pi]$ such that

$$\cos d_{\mathbb{S}^n}(A,B) = (A \mid B).$$

Then $d_{\mathbb{S}^n}$ is a metric [12, I.2.1].

Definition 2.1 Given a real number κ, we denote by M_κ^n the following metric spaces:

(i) if $\kappa = 0$, then M_0^n is the Euclidean space $\mathbb{R}^n$;
(ii) if $\kappa > 0$, then M_κ^n is obtained from the sphere $\mathbb{S}^n$ by multiplying the distance function by $1/\sqrt{\kappa}$;
(iii) if $\kappa < 0$, then M_κ^n is obtained from the sphere $\mathbb{H}^n$ by multiplying the distance function by $1/\sqrt{-\kappa}$, where $\mathbb{H}^n$ is the hyperbolic n-space.

It is well known that M_κ^n is a geodesic metric space. If $\kappa \leq 0$, then M_κ^n is uniquely geodesic. If $\kappa > 0$, then M_κ^n is $\pi/\sqrt{\kappa}$-uniquely geodesic, and any open ball (respectively, closed) ball of radius $\leq \pi/(2\sqrt{\kappa})$ (respectively, $<\pi/(2\sqrt{\kappa})$) in X is convex [12, I.2.11]. The diameter of M_κ^n will be denoted D_κ and thus D_κ is $\pi/\sqrt{\kappa}$ if $\kappa > 0$, and ∞ otherwise.

Given two distinct points $A, B \in \mathbb{S}^n$ with $d(A,B) = \ell < \pi$ there is a natural way to parameterize a unique geodesic joining A to B: consider the path $c(t) = (\cos t)A + (\sin t)u$, $t \in [0,\ell]$, where the initial vector $u \in \mathbb{R}^{n+1}$ is the unit vector in the direction of $B - (A \mid B)A$. We shall refer to the image of c as a minimal great arc joining A to B.

The *spherical angle* between two minimal great arcs issuing from a point of $\mathbb{S}^n$, with the initial vectors u and v, say, is the unique number $\alpha \in [0,\pi]$ such that $\cos\alpha = (u \mid v)$. A *spherical triangle* $\triangle$ in $\mathbb{S}^n$ consists of a choice of three distinct points (its vertices) $A, B, C \in \mathbb{S}^n$, and three minimal great arcs (its sides), one joining each other of vertices. The *vertex angle* at C is defined to be the spherical angle between the sides of $\triangle$ joining C to A and C to B.

Proposition 2.2 (The Spherical Law of Cosines [12, I.2.13]) *Let a geodesic triangle in M_κ^n ($\kappa > 0$) have sides a, b, c and angles α, β, γ at the vertices opposite to the sides of length a, b, c, respectively. Then*

$$\cos(\sqrt{\kappa}c) = \cos(\sqrt{\kappa}a)\cos(\sqrt{\kappa}b) + \sin(\sqrt{\kappa}a)\sin(\sqrt{\kappa}b)\cos\gamma.$$

In particular, fixing a, b and κ, c is a strictly increasing function of γ, varying from $|a-b|$ to $a+b$ as γ varies from 0 to π.

A *geodesic triangle* $\triangle$ in a metric space X consists of three points $p, q, r \in X$, its *vertices*, and a choice of three geodesic segments $[p,q]$, $[q,r]$, $[r,p]$ joining them, its *sides*. Such a geodesic triangle will be denoted $\triangle([p,q],[q,r],[r,p])$ or (less accurately if X is not uniquely geodesic) $\triangle(p,q,r)$. If a point $x \in X$ lies in the union of $[p,q]$, $[q,r]$ and $[r,p]$, then we write $x \in \triangle$.

A triangle $\overline{\triangle} = \triangle(\bar{p},\bar{q},\bar{r})$ in M_κ^2 is called a *comparison triangle* for $\triangle = \triangle([p,q],[q,r],[r,p])$ in X if $d_{M_\kappa^2}(\bar{p},\bar{q}) = d(p,q)$, $d_{M_\kappa^2}(\bar{q},\bar{r}) = d(q,r)$ and $d_{M_\kappa^2}(\bar{r},\bar{p}) = d(r,p)$. Such a triangle $\overline{\triangle} \subset M_\kappa^2$ always exists if the perimeter $d(p,q) + d(q,r) + d(r,p)$ of $\triangle$ is less than $2D_\kappa$; it is unique up to an isometry of M_κ^2 [12, I.2.14]. A point $\bar{x} \in [\bar{q},\bar{r}]$ is called a *comparison point* for $x \in [q,r]$ if $d_{M_\kappa^2}(\bar{q},\bar{x}) = d(q,x)$.

A geodesic triangle $\triangle$ in X is said to satisfy the $\mathrm{CAT}(\kappa)$ *inequality* if, given a comparison triangle $\overline{\triangle} \subset M_\kappa^2$ for $\triangle$, for all $x, y \in \triangle$,

$$d(x,y) \le d_{M_\kappa^2}(\bar{x},\bar{y}),$$

where $\bar{x}, \bar{y} \in \overline{\triangle}$ are respective comparison points of x, y.

Definition 2.3 If $\kappa \le 0$, then X is called a $\mathrm{CAT}(\kappa)$ space if X is a geodesic space all of whose geodesic triangles satisfy the $\mathrm{CAT}(\kappa)$ inequality.

If $\kappa > 0$, then X is called a $\mathrm{CAT}(\kappa)$ space if X is D_κ-geodesic and all geodesic triangles in X of perimeter less than $2D_\kappa$ satisfy the $\mathrm{CAT}(\kappa)$ inequality.

In particular, Hilbert spaces are CAT(0). A $\mathrm{CAT}(\kappa)$ space is a $\mathrm{CAT}(\kappa')$ space for every $\kappa' \ge \kappa$. A $\mathrm{CAT}(\kappa)$ space X is (D_κ-)uniquely geodesic (if $\kappa > 0$) and any open (respectively, closed) ball of radius $\le D_\kappa/2$ (respectively, $< D_\kappa/2$) in X is convex [12, II.1.4].

Lemma 2.4 *Let* (X,d) *be a* $\mathrm{CAT}(\kappa)$ *space and let* $\alpha, \beta \in [0,1]$. *Then*

(i) [12, II.2. Exercise 2.3(1)] *for* $p \in X$ *and* $x, y \in B(p, D_\kappa/2)$, *we have*

$$d\bigl(\alpha x \oplus (1-\alpha)y, p\bigr) \le \alpha d(x,p) + (1-\alpha)d(y,p);$$

(ii) *for* $x, y \in X$, *we have*

$$d\bigl(\alpha x \oplus (1-\alpha)y, \beta x \oplus (1-\beta)y\bigr) = |\alpha - \beta| d(x,y);$$

(iii) [10, Lemma 3.3] *if* $\kappa > 0$, *for* $x, y, z \in X$ *with* $\max\{d(x,y), d(y,z), d(x,z)\} < M \le D_\kappa/2$, *we have*

$$d\bigl(\alpha x \oplus (1-\alpha)y, \alpha x \oplus (1-\alpha)z\bigr) \le \frac{\sin[(1-\alpha)M]}{\sin M} d(y,z).$$

Let p, q, r be three distinct points of X with $d(p,q) + d(q,r) + d(r,p) < 2D_\kappa$. The κ-*comparison angle* between q and r at p, denoted $\angle_p^{(\kappa)}(q,r)$, is the angle at $\bar{p}$ in a comparison triangle $\triangle(\bar{p},\bar{q},\bar{r}) \subset M_\kappa^2$ for $\triangle(p,q,r)$.

Let $\gamma : [0,a] \to X$ and let $\gamma' : [0,a'] \to X$ be two geodesic paths with $\gamma(0) = \gamma'(0)$. Given $t \in (0,a]$ and $t' \in (0,a']$, we consider the comparison triangle $\overline{\triangle}(\gamma(0), \gamma(t), \gamma'(t'))$ and the κ-comparison angle $\angle_{\gamma(0)}^{(\kappa)}(\gamma(t), \gamma'(t'))$. The (Alexandrov) *angle* or the *upper angle* between the geodesic paths γ and γ' is the number $\angle(\gamma,\gamma') \in [0,\pi]$ defined by

$$\angle\bigl(\gamma,\gamma'\bigr) = \limsup_{t,t' \to 0} \angle_{\gamma(0)}^{(\kappa)}\bigl(\gamma(t), \gamma'\bigl(t'\bigr)\bigr) = \lim_{\epsilon \to 0} \sup_{0<t,t'<\epsilon} \angle_{\gamma(0)}^{(\kappa)}\bigl(\gamma(t), \gamma'\bigl(t'\bigr)\bigr).$$

If X is uniquely geodesic, $p \ne q$ and $p \ne r$, the *angle* of $\triangle(p,q,r)$ in X at p is the (Alexandrov) angle between the geodesic segments $[p,q]$ and $[p,r]$ issuing from p and is denoted $\angle_p(q,r)$.

Proposition 2.5 ([12, I.1.13, 1.14 and II.1.8]) *Let* X *be a metric space and let* γ, γ', γ'' *be three geodesics issuing from a common point. Then*

(i) $\angle(\gamma,\gamma') = \angle(\gamma',\gamma)$;

(ii) $\angle(\gamma, \gamma') \leq \angle(\gamma, \gamma'') + \angle(\gamma', \gamma'')$;
(iii) *if* $\gamma : [-a, a] \to X$ *is a geodesic with* $a > 0$, *and if* $\gamma', \gamma'' : [0, a] \to X$ *are defined by* $\gamma'(t) = \gamma(-t)$ *and* $\gamma''(t) = \gamma(t)$, *then* $\angle(\gamma', \gamma'') = \pi$.

3 Basic properties and Δ-convergence

This section contains a number of primary results in [8] which are crucial to the study of our problem. The following proposition states very useful properties of the metric projection in a complete CAT(1) space.

Proposition 3.1 ([8, Proposition 3.5]) *Let* X *be a complete* CAT(1) *space and let* $C \subset X$ *be nonempty closed and* π*-convex. Suppose that* $x \in X$ *such that* $d(x, C) < \pi/2$. *Then the following are satisfied*:
(i) *There exists a unique point* $P_C x \in C$ *such that* $d(x, P_C x) = d(x, C)$.
(ii) *If* $x \notin C$ *and* $y \in C$ *with* $y \neq P_C x$, *then* $\angle_{P_C x}(x, y) \geq \pi/2$.
(iii) *If* $\operatorname{diam}(X) \leq \pi$, *then for any* $y \in C$,

$$d(P_C x, P_C y) = d(P_C x, y) \leq d(x, y).$$

The mapping P_C of X onto C in Proposition 3.1 is called the metric projection.

The next result shows the existence property of fixed points for a nonexpansive mapping.

Proposition 3.2 ([8, Theorem 3.9]) *Let* X *be a complete* CAT(1) *space such that* $\operatorname{diam} X < \pi/2$. *Then every nonexpansive mapping* $T : X \to X$ *has at least one fixed point*.

The rest of this section is devoted to presenting several closely related characterizations of Δ-convergence. For this purpose, we start with some basic definitions of an asymptotic radius and an asymptotic center. Let $\{x_n\}$ be a bounded sequence in a complete CAT(1) space X. For $x \in X$ and $C \subset X$, let $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$. The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\};$$

the *asymptotic radius* $r_C(\{x_n\})$ *with respective to* C of $\{x_n\}$ is given by

$$r_C(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\};$$

the *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is given by the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\};$$

the *asymptotic center* $A_C(\{x_n\})$ *with respective to* C of $\{x_n\}$ is given by the set

$$A_C(\{x_n\}) = \{x \in C : r(x, \{x_n\}) = r_C(\{x_n\})\}.$$

Proposition 3.3 ([8, Proposition 4.1]) *Let* X *be a complete* CAT(1) *space and* $C \subset X$ *nonempty closed and* π*-convex. If* $\{x_n\}$ *be a sequence in* X *such that* $r_C(\{x_n\}) < \pi/2$, *then* $A_C(\{x_n\})$ *consists of exactly one point*.

Definition 3.4 A sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of every subsequence of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n\to\infty} x_n = x$ and x is called the Δ-limit of $\{x_n\}$.

The next result is an immediate consequence of the preceding proposition.

Proposition 3.5 ([8, Corollary 4.4]) *Let X be a complete* CAT(1) *space and $\{x_n\}$ a sequence in X. If $r(\{x_n\}) < \pi/2$, then $\{x_n\}$ has a Δ-convergent subsequence.*

4 Main results

Since the validity of all our results, including the proofs as well, on CAT(1) spaces can be restored on any CAT(κ) space with $\kappa > 0$ by rescaling without major changes, we will pay our attention to CAT(1) spaces. In addition, when we deal with a CAT(κ) space, the hypothesis $d(x_0, \mathfrak{F}(T)) < \pi/4$ for each theorem in this section is replaced by $d(x_0, \mathfrak{F}(T)) < D_\kappa/4$ and so can be dropped if $\kappa \le 0$ (in this case, $D_\kappa = \infty$; refer to Section 2 for the definition).

To verify our result, the following basic property of asymptotical regularity is required; also *cf.* [6, Proposition 2.1] in a topological vector space.

Lemma 4.1 *Let C be a subset of a metric space (X,d) and let $\mathfrak{S} = \{T(t) : t \ge 0\}$ be a family of self-mappings of C such that $T(s+t) = T(s) \circ T(t)$, for all $s, t \ge 0$. If $\mathfrak{S}$ is asymptotically regular, then*

$$\mathfrak{F}\big(T(t)\big) = \mathfrak{F}(\mathfrak{S}), \quad \textit{for all } t > 0.$$

Proof Fix $t > 0$. Then $\mathfrak{F}(\mathfrak{S}) \subset \mathfrak{F}(T(t))$. To prove $\mathfrak{F}(T(t)) \subset \mathfrak{F}(\mathfrak{S})$, let x be a fixed point of $T(t)$. For $h \ge 0$, we obtain

$$\begin{aligned} d\big(T(h)x, x\big) &= \lim_{k\to\infty} d\big(T(h) \circ T(t)^k x, T(t)^k x\big) \\ &= \lim_{k\to\infty} d\big(T(h+tk)x, T(tk)x\big) \\ &= \lim_{s\to\infty} d\big(T(h+s)x, T(s)x\big) \\ &= 0 \end{aligned}$$

and therefore $x \in \mathfrak{F}(\mathfrak{S})$. □

Let (X,d) be a complete CAT(1) space, C a closed π-convex subset of X and $T : C \to C$ a nonexpansive mapping. Then $\mathfrak{F}(T)$ is closed. Also it is seen that $\mathfrak{F}(T)$ is π-convex. Indeed, for $x, y \in \mathfrak{F}(T)$ with $d(x,y) < \pi$, we have

$$d\big(x, T\big(\lambda x \oplus (1-\lambda)y\big)\big) \le d\big(x, \lambda x \oplus (1-\lambda)y\big) = (1-\lambda)d(x,y)$$

and similarly $d(y, T(\lambda x \oplus (1-\lambda)y)) \le \lambda d(x,y)$. Thus both equalities must hold and hence $\lambda x \oplus (1-\lambda)y$ and $T(\lambda x \oplus (1-\lambda)y)$ belong to the unique geodesic $[x,y]$. This implies that $T(\lambda x \oplus (1-\lambda)y) = \lambda x \oplus (1-\lambda)y$.

Now consider a family $\mathfrak{S} = \{T(t) : t \geq 0\}$ of nonexpansive self-mappings of C such that $\mathfrak{F}(\mathfrak{S}) \neq \emptyset$. From the previous remark, each fixed point set $\mathfrak{F}(T(t))$ is closed and convex, hence so is $\mathfrak{F}(\mathfrak{S})$. Fix $x_0 \in C$ with $0 < d(x_0, \mathfrak{F}(\mathfrak{S})) < \pi/4$. Let $p = P_{\mathfrak{F}(\mathfrak{S})}x_0$ (the uniqueness follows from Proposition 3.1(i)) and $\epsilon = d(x_0, p)$. Then for each $t \geq 0$, $T(t)$ maps the closed π-convex set $C \cap \overline{B}(p, \epsilon)$ into itself. For any $\alpha \in (0,1]$ and $t \geq 0$, define the mapping $S_{\alpha,t} : C \cap \overline{B}(p,\epsilon) \to C \cap \overline{B}(p,\epsilon)$ by $S_{\alpha,t}x = \alpha x_0 \oplus (1-\alpha)T(t)x$. Observe that $S_{\alpha,t}$ is a contraction. In fact, for $x, y \in C \cap \overline{B}(p,\epsilon)$, we apply Lemma 2.4(iii) to get

$$d(S_{\alpha,t}x, S_{\alpha,t}y) \leq \sin\left[(1-\alpha)\frac{\pi}{2}\right] d\big(T(t)x, T(t)y\big) \leq \sin\left[(1-\alpha)\frac{\pi}{2}\right] d(x,y).$$

Then $S_{\alpha,t}$ has a unique fixed point in $C \cap \overline{B}(p,\epsilon)$.

Let $[t]$ denote the maximum integer no greater than t. We now extend Theorem 1.3 in Hilbert spaces to CAT(κ) spaces as the following result.

Theorem 4.2 *Let X be a complete* CAT(1) *space, C a closed π-convex subset of X, $\mathfrak{S} = \{T(t) : t \geq 0\}$ a UAR and strongly continuous nonexpansive semigroup on C with $\mathfrak{F}(\mathfrak{S}) \neq \emptyset$, and two sequences $\{\alpha_n\} \subset (0,1]$, $\{t_n\} \subset (0,\infty)$ such that $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \alpha_n/t_n = 0$. Choose arbitrarily a point $x_0 \in C$ such that $d(x_0, \mathfrak{F}(\mathfrak{S})) < \pi/4$. Let $p = P_{\mathfrak{F}(\mathfrak{S})}x_0$ and $\epsilon = d(x_0, p)$. Define a sequence $\{x_n\}$ in $C \cap \overline{B}(p,\epsilon)$ by the implicit iteration*

$$x_n = \alpha_n x_0 \oplus (1-\alpha_n)T(t_n)x_n. \tag{4.1}$$

Then $\{x_n\}$ converges strongly to the point p of $\mathfrak{F}(\mathfrak{S})$ nearest to x_0.

Proof If $x_0 \in \mathfrak{F}(\mathfrak{S})$, then $\{x_n\}$ reduces to the constant sequence $\{x_0, x_0, \ldots\}$. Suppose that $x_0 \notin \mathfrak{F}(\mathfrak{S})$. We shall prove that any subsequence of $\{x_n\}$ contains a subsequence converging strongly to p from which it follows that $\{x_n\}$ also converges strongly to the point p. Let $\{y_n\}$ be any subsequence of $\{x_n\}$. Proposition 3.5 asserts that $\{y_n\}$ has a Δ-convergent subsequence. By passing to a subsequence we may assume that $\{y_n\}$ Δ-converges to a point y. Let $\{\beta_n\}$ and $\{s_n\}$ be the respective corresponding subsequences of $\{\alpha_n\}$ and $\{t_n\}$.

Observe that $y \in \mathfrak{F}(\mathfrak{S})$. To see this, take $\hat{s} = \limsup_{n\to\infty} s_n$. There are three cases:

(i) $\hat{s} = 0$,
(ii) $0 < \hat{s} < \infty$,
(iii) $\hat{s} = \infty$.

By passing to a subsequence if necessary it may be assumed that $\hat{s} = \lim_{n\to\infty} s_n$. We discuss each case as follows.

Suppose that $\hat{s} = 0$. Fix $t \geq 0$ and we obtain

$$\begin{aligned} d\big(T(t)y, y_n\big) &\leq d\left(T(t)y, T\left(\left[\frac{t}{s_n}\right]s_n\right)y\right) + d\left(T\left(\left[\frac{t}{s_n}\right]s_n\right)y, T\left(\left[\frac{t}{s_n}\right]s_n\right)y_n\right) \\ &\quad + \sum_{j=1}^{[t/s_n]} d\big(T\big((j-1)s_n\big)y_n, T(js_n)y_n\big) \\ &\leq d\left(T\left(t - \left[\frac{t}{s_n}\right]s_n\right)y, y\right) + d(y, y_n) + \left[\frac{t}{s_n}\right] d\big(y_n, T(s_n)y_n\big) \\ &\leq \max_{0\leq s\leq s_n} d\big(T(s)y, y\big) + d(y, y_n) + \frac{t\beta_n}{s_n} d\big(x_0, T(s_n)y_n\big). \end{aligned}$$

This implies that

$$\limsup_{n\to\infty} d\bigl(T(t)y, y_n\bigr) \le \limsup_{n\to\infty} d(y, y_n);$$

hence $T(t)y = y$ for all $t \ge 0$, that is, $y \in \mathfrak{F}(\mathfrak{S})$.

If $0 < \hat{s} < \infty$, then

$$\begin{aligned} d\bigl(T(\hat{s})y, y_n\bigr) &\le d\bigl(T(\hat{s})y, T(s_n)y\bigr) + d\bigl(T(s_n)y, T(s_n)y_n\bigr) + d\bigl(T(s_n)y_n, y_n\bigr) \\ &\le d\bigl(y, T\bigl(|s_n - \hat{s}|\bigr)y\bigr) + d(y, y_n) + d\bigl(T(s_n)y_n, y_n\bigr) \\ &\le d\bigl(y, T\bigl(|s_n - \hat{s}|\bigr)y\bigr) + d(y, y_n) + \beta_n d\bigl(x_0, T(s_n)y_n\bigr) \end{aligned}$$

and thus

$$\limsup_{n\to\infty} d\bigl(T(\hat{s})y, y_n\bigr) \le \limsup_{n\to\infty} d(y, y_n).$$

Hence $T(\hat{s})y = y$ from which it follows that $y \in \mathfrak{F}(\mathfrak{S})$ by Lemma 4.1.

If $\hat{s} = \infty$, fix $t \ge 0$. For all sufficiently large n we get

$$\begin{aligned} d\bigl(T(t)y, y_n\bigr) &\le d\bigl(T(t)y, T(t)y_n\bigr) + d\bigl(T(t)y_n, y_n\bigr) \\ &\le d(y, y_n) + \beta_n d\bigl(T(t)y_n, x_0\bigr) + (1-\beta_n) d\bigl(T(t)y_n, T(s_n)y_n\bigr) \\ &\le d(y, y_n) + \beta_n d\bigl(T(t)y_n, x_0\bigr) + (1-\beta_n) d\bigl(y_n, T(s_n - t)y_n\bigr) \\ &\le d(y, y_n) + \beta_n d\bigl(T(t)y_n, x_0\bigr) + \beta_n(1-\beta_n) d\bigl(x_0, T(s_n - t)y_n\bigr) \\ &\quad + (1-\beta_n)^2 d\bigl(T(s_n)y_n, T(s_n - t)y_n\bigr) \\ &\le d(y, y_n) + \beta_n d\bigl(T(t)y_n, x_0\bigr) + \beta_n(1-\beta_n) d\bigl(x_0, T(s_n - t)y_n\bigr) \\ &\quad + (1-\beta_n)^2 d\bigl(T(s_n - t + t)y_n, T(s_n - t)y_n\bigr) \end{aligned}$$

which yields

$$\limsup_{n\to\infty} d\bigl(T(t)y, y_n\bigr) \le \limsup_{n\to\infty} d(y, y_n)$$

since

$$\limsup_{n\to\infty} d\bigl(T(s_n - t + t)y_n, T(s_n - t)y_n\bigr) \le \lim_{n\to\infty} \sup_{x \in C\cap\overline{B}(p,\epsilon)} d\bigl(T(h_n + t)x, T(h_n)x\bigr) = 0,$$

where $h_n = s_n - t$. It follows that $T(t)y = y$ and therefore $y \in \mathfrak{F}(\mathfrak{S})$. This finishes the proof that y is a common fixed point of all mappings in $\mathfrak{S}$.

Next, we claim that $\{y_n\}$ converges strongly to y. We suppose on the contrary that

$$r\bigl(\{y_n\}\bigr) = \sigma > 0.$$

From $y = T(s_n)y$, we obtain

$$\begin{aligned} d(y, y_n) &\le \beta_n d(y, x_0) + (1-\beta_n) d\bigl(y, T(s_n)y_n\bigr) \\ &\le \beta_n d(y, x_0) + (1-\beta_n) d(y, y_n), \end{aligned}$$

and hence taking the limit superior as $n \to \infty$ yields

$$\limsup_{n\to\infty} d\big(y, T(s_n)y_n\big) = \limsup_{n\to\infty} d(y, y_n) = \sigma, \tag{4.2}$$

since $\lim_{n\to\infty} \beta_n = 0$. Recall that

$$d(x_0, y_n) = (1 - \beta_n) d\big(x_0, T(s_n)y_n\big) < d\big(x_0, T(s_n)y_n\big), \quad \text{for all } n.$$

Then

$$d\big(y_n, T(s_n)y_n\big) = d\big(x_0, T(s_n)y_n\big) - d(x_0, y_n) > 0, \quad \text{for all } n.$$

Recall that $x_0 \notin \mathfrak{F}(\mathfrak{S})$. We have $y \neq x_0$ and hence

$$\limsup_{n\to\infty} d(x_0, y_n) > \sigma. \tag{4.3}$$

According to (4.2) and (4.3), by passing to a subsequence again we may assume that

$$d(x_0, y_n) > 0, \qquad d(y, y_n) > 0, \qquad d\big(y, T(s_n)y_n\big) > 0, \quad \text{for all } n.$$

Let $\triangle(\bar{x}_0, \bar{y}, \overline{T(s_n)y_n})$ be a comparison triangle for $\triangle(x_0, y, T(s_n)y_n)$ in $\mathbb{S}^2$. Observe that

$$\angle_{\bar{y}_n}(\bar{x}_0, \bar{y}) \geq \pi/2. \tag{4.4}$$

For, contrarily, if $\angle_{\bar{y}_n}(\bar{x}_0, \bar{y}) < \pi/2$, since $\angle_{\bar{y}_n}(\bar{x}_0, \overline{T(s_n)y_n}) = \pi$, then $\angle_{\bar{y}_n}(\bar{y}, \overline{T(s_n)y_n}) > \pi/2$; see Proposition 2.5. By the law of cosines in $\mathbb{S}^2$ (Proposition 2.2), since $d(y_n, T(s_n)y_n) > 0$, it follows that

$$\cos d_{\mathbb{S}^2}\big(\bar{y}, \overline{T(s_n)y_n}\big) < \cos d_{\mathbb{S}^2}(\bar{y}, \bar{y}_n),$$

which is a contradiction because $\overline{T(s_n)}$ is nonexpansive. Notice that

$$0 < d(y, y_n) \leq d_{\mathbb{S}^2}(\bar{y}, \bar{y}_n).$$

Hence (4.4) implies that

$$\cos d_{\mathbb{S}^2}(\bar{x}_0, \bar{y}) \leq \cos d_{\mathbb{S}^2}(\bar{x}_0, \bar{y}_n) \cos d_{\mathbb{S}^2}(\bar{y}, \bar{y}_n) < \cos d_{\mathbb{S}^2}(\bar{x}_0, \bar{y}_n),$$

or equivalently,

$$d_{\mathbb{S}^2}(\bar{x}_0, \bar{y}_n) < d_{\mathbb{S}^2}(\bar{x}_0, \bar{y}).$$

Similarly,

$$d_{\mathbb{S}^2}(\bar{y}, \bar{y}_n) < d_{\mathbb{S}^2}(\bar{x}_0, \bar{y}).$$

The previous two inequalities, together with the CAT(κ) inequality, yield

$$d(x_0, y_n) < d(x_0, y), \qquad d(y, y_n) < d(x_0, y), \quad \text{for all } n. \tag{4.5}$$

Let $\triangle(\bar{x}_0, \bar{y}, \bar{y}_n)$ be a comparison triangle for $\triangle(x_0, y, y_n)$ in $\mathbb{S}^2$. Since $d_{\mathbb{S}^2}(\bar{y}_n, [\bar{x}_0, \bar{y}]) < \pi/2$, Proposition 3.1(i) assures that there is a unique point $\bar{u}_n \in [\bar{x}_0, \bar{y}]$ nearest to $\bar{y}_n$. Let $u_n \in [x_0, y]$ such that $d(x_0, u_n) = d_{\mathbb{S}^2}(\bar{x}_0, \bar{u}_n)$ and $d(u_n, y) = d_{\mathbb{S}^2}(\bar{u}_n, \bar{y})$. By passing to a subsequence again it may be assumed that $\{\bar{u}_n\}$ and $\{u_n\}$ converge, respectively, to $\bar{u} \in [\bar{x}_0, \bar{y}]$ and $u \in [x_0, y]$. Since

$$\begin{aligned}
\sigma &= \limsup_{n\to\infty} d(y, y_n) \le \limsup_{n\to\infty} d(u, y_n) \\
&\le \limsup_{n\to\infty} d_{\mathbb{S}^2}(\bar{u}, \bar{u}_n) + \limsup_{n\to\infty} d_{\mathbb{S}^2}(\bar{u}_n, \bar{y}_n) \\
&= \limsup_{n\to\infty} d_{\mathbb{S}^2}(\bar{u}_n, \bar{y}_n) \\
&\le \limsup_{n\to\infty} d_{\mathbb{S}^2}(\bar{y}, \bar{y}_n) \\
&= \limsup_{n\to\infty} d(y, y_n),
\end{aligned} \tag{4.6}$$

this guarantees that $u = y$.

On the other hand, according to (4.6), by passing to a subsequence we may suppose that

$$d_{\mathbb{S}^2}(\bar{u}_n, \bar{y}_n) > \frac{\sigma}{2}, \quad \text{for all } n.$$

Let $a_n = d_{\mathbb{S}^2}(\bar{x}_0, \bar{u}_n)$, $b_n = d_{\mathbb{S}^2}(\bar{u}_n, \bar{y}_n)$, $c_n = d_{\mathbb{S}^2}(\bar{x}_0, \bar{y}_n)$ and $\gamma_n = \angle_{\bar{u}_n}(\bar{x}_0, \bar{y}_n)$. Then by (4.5), $b_n > \sigma/2$ and $c_n < d_{\mathbb{S}^2}(\bar{x}_0, \bar{y})$. Observe that $u_n \neq x_0$. Otherwise, since $y \neq x_0$, we have $\angle_{\bar{x}_0}(\bar{y}, \bar{y}_n) = \angle_{\bar{u}_n}(\bar{y}, \bar{y}_n) \ge \pi/2$ which implies that

$$d(y, y_n) = d_{\mathbb{S}^2}(\bar{y}, \bar{y}_n) \ge d_{\mathbb{S}^2}(\bar{x}_0, \bar{y}) = d(x_0, y).$$

But this contradicts to (4.5). Therefore $u_n \neq x_0$ and so $\gamma_n \ge \pi/2$ by Proposition 3.1(ii). The law of cosines in $\mathbb{S}^2$ yields

$$\begin{aligned}
\cos c_n &= \cos a_n \cos b_n + \sin a_n \sin b_n \cos \gamma_n \\
&\le \cos a_n \cos b_n.
\end{aligned}$$

Since $\sigma/2 < b_n \le c_n < d_{\mathbb{S}^2}(\bar{x}_0, \bar{y}) < \pi/2$, this implies that

$$\cos a_n \ge \frac{\cos c_n}{\cos b_n} > \frac{\cos d_{\mathbb{S}^2}(\bar{x}_0, \bar{y})}{\cos(\sigma/2)} > \cos d_{\mathbb{S}^2}(\bar{x}_0, \bar{y}).$$

Hence

$$d_{\mathbb{S}^2}(\bar{x}_0, \bar{u}_n) = a_n < \cos^{-1}\left(\frac{\cos d_{\mathbb{S}^2}(\bar{x}_0, \bar{y})}{\cos(\sigma/2)}\right) < d_{\mathbb{S}^2}(\bar{x}_0, \bar{y}).$$

It follows that

$$d(y,u_n) = d_{\mathbb{S}^2}(\bar{y},\bar{u}_n) = d_{\mathbb{S}^2}(\bar{x}_0,\bar{y}) - d_{\mathbb{S}^2}(\bar{x}_0,\bar{u}_n) > d_{\mathbb{S}^2}(\bar{x}_0,\bar{y}) - \delta > 0,$$

where $\delta = \cos^{-1}[\cos d_{\mathbb{S}^2}(\bar{x}_0,\bar{y})/\cos(\sigma/2)]$. Taking the limit as $n \to \infty$ yields $u \neq y$, which is a contradiction. Thus $\sigma = 0$ and so $\{y_n\}$ converges strongly to y, as claimed.

It remains to prove that $y = P_{\mathfrak{F}(\mathfrak{S})}x_0$. Let $q \in \mathfrak{F}(\mathfrak{S})$ and consider a comparison triangle $\triangle(\bar{x}_0, \bar{q}, \overline{T(s_n)y_n})$ for $\triangle(x_0, q, T(s_n)y_n)$. From

$$d_{\mathbb{S}^2}\bigl(\bar{q}, \overline{T(s_n)y_n}\bigr) \leq d_{\mathbb{S}^2}(\bar{q},\bar{y}_n),$$

it is seen that $\angle_{\bar{y}_n}(\bar{q}, \overline{T(s_n)y_n}) \leq \pi/2$. Hence $\angle_{\bar{y}_n}(\bar{x}_0,\bar{q}) \geq \pi/2$ and so

$$d(x_0,y_n) = d_{\mathbb{S}^2}(\bar{x}_0,\bar{y}_n) \leq d_{\mathbb{S}^2}(\bar{x}_0,\bar{q}) = d(x_0,q).$$

We then take the limit as $n \to \infty$ and obtain

$$d(x_0,y) \leq d(x_0,q),$$

which shows that y is the nearest point in $\mathfrak{F}(\mathfrak{S})$ to x_0. Consequently, we conclude that $\{x_n\}$ converges strongly to $P_{\mathfrak{F}(\mathfrak{S})}x_0$, which completes the proof. □

It is worthy emphasizing that a uniformly asymptotical regularity hypothesis of Theorem 4.2 is superfluous when $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \alpha_n/t_n = 0$; see case (i) in the proof of Theorem 4.2. Thus, we state the result as follows.

Theorem 4.3 *Let X be a complete* CAT(1) *space, C a closed π-convex subset of X, $\mathfrak{S} = \{T(t) : t \geq 0\}$ a strongly continuous nonexpansive semigroup on C with $\mathfrak{F}(\mathfrak{S}) \neq \emptyset$, and two sequences $\{\alpha_n\} \subset (0,1]$, $\{t_n\} \subset (0,\infty)$ such that* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \alpha_n/t_n = 0$. *Choose arbitrarily a point $x_0 \in C$ such that $d(x_0,\mathfrak{F}(\mathfrak{S})) < \pi/4$. Let $p = P_{\mathfrak{F}(\mathfrak{S})}x_0$ and $\epsilon = d(x_0,p)$. Then the sequence $\{x_n\}$ in $C \cap \overline{B}(p,\epsilon)$ defined by* (4.1) *converges strongly to the point p.*

Furthermore, if X is a CAT(0) space (recall that $D_0 = \infty$) in Theorem 4.3, then the corresponding assumption $d(x_0,\mathfrak{F}(\mathfrak{S})) < \pi/4$ is $d(x_0,\mathfrak{F}(\mathfrak{S})) < \infty$, which can be dropped. Moreover, the sequence $\{x_n\}$ defined by (4.1) is bounded. In fact, since

$$\begin{aligned} d(p,x_n) &\leq \alpha_n d(p,x_0) + (1-\alpha_n) d\bigl(p, T(t_n)x_n\bigr) \\ &\leq \alpha_n d(p,x_0) + (1-\alpha_n) d(p,x_n), \end{aligned}$$

this shows that $d(p,x_n) \leq d(p,x_0)$. Then $\{x_n\}$ is bounded and so is $\{T(t)x_n : t \geq 0, n \in \mathbb{N}\}$. We restate Theorem 1.2 without assuming boundedness of C as follows.

Theorem 4.4 *Let X be a complete* CAT(0) *space, C a closed convex subset of X, $\mathfrak{S} = \{T(t) : t \geq 0\}$ a strongly continuous nonexpansive semigroup on C with $\mathfrak{F}(\mathfrak{S}) \neq \emptyset$, and two sequences $\{\alpha_n\} \subset (0,1]$, $\{t_n\} \subset (0,\infty)$ such that* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \alpha_n/t_n = 0$. *Choose arbitrarily a point $x_0 \in C$. Then the sequence $\{x_n\}$ in C defined by* (4.1) *converges strongly to a point of $\mathfrak{F}(\mathfrak{S})$ nearest to x_0.*

Nonetheless, Theorem 4.2 has a uniformly asymptotical regularity hypothesis that cannot in general be removed when $\lim_{n\to\infty} t_n = \hat{t} \in (0,\infty]$ as illustrated in the following example.

Example 4.5 Consider the complex plane $(\mathbb{C}, |\cdot|)$, where $|\cdot|$ is the absolute value, so that it is a complete CAT(0) space. For $t \geq 0$, define a self-mapping $T(t)$ of $\mathbb{C}$ by $T(t)(z) = ze^{it}$, $z \in \mathbb{C}$. Then $\mathfrak{S} = \{T(t) : t \geq 0\}$ is a strongly continuous nonexpansive semigroup with $\mathfrak{F}(\mathfrak{S}) = \{0\}$. The family $\mathfrak{S}$ is not *UAR* on $\mathbb{C}$ because

$$\lim_{t\to\infty} \sup_{|z|=1} \bigl|T(\pi)T(t)(z) - T(t)(z)\bigr| = \lim_{t\to\infty} \sup_{|z|=1} \bigl|ze^{i(t+\pi)} - ze^{it}\bigr| = 2.$$

Let $\alpha_n = 1/n$, $t_n = n\pi$, $n \in \mathbb{N}$, so that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} \alpha_n/t_n = 0$. Choose $z_0 = 1$. Define a sequence $\{z_n\}$ in $\mathbb{C}$ by (4.1), that is,

$$z_n = \begin{cases} \frac{1}{2n-1}, & n \text{ is odd}, \\ 1, & n \text{ is even}. \end{cases}$$

Hence the sequence $\{z_n\}$ is divergent.

Next if we take $\alpha_n = 1/n$, $t_n = (1/n) + 2\pi$, $n \in \mathbb{N}$, then $\lim_{n\to\infty} t_n = 2\pi$ and $\lim_{n\to\infty} \alpha_n/t_n = 0$. Choose $z_0 = 1$ and define a sequence $\{z_n\}$ in $\mathbb{C}$ by (4.1) to get

$$z_n = \frac{1/n}{1 - [1 - (1/n)]e^{i/n}}.$$

However, the sequence $\{z_n\}$ converges to $(1+i)/2 \notin \mathfrak{F}(\mathfrak{S})$.

These two examples explain that strong convergence of the sequence defined by (4.1) in Theorem 4.2 ceases to be true without the *UAR* assumption if $\lim_{n\to\infty} t_n = \hat{t} \in (0,\infty]$.

The next example shows that the condition $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \alpha_n/t_n = 0$ in Theorem 4.2 is also necessary when X is the unit sphere in ℓ_2, the infinite-dimensional Hilbert space of square-summable sequences.

Example 4.6 Let $\mathbb{S}_{\ell_2}$ be the unit sphere in ℓ_2 endowed with the following intrinsic metric $d : \mathbb{S}_{\ell_2} \times \mathbb{S}_{\ell_2} \to \mathbb{R}$: for $x, y \in \mathbb{S}_{\ell_2}$, $d(x,y) \in [0,\pi]$ such that

$$\cos d(x,y) = (x \mid y) = \sum_{k=1}^{\infty} x(k)y(k).$$

This space $(\mathbb{S}_{\ell_2}, d)$ is a CAT(1) space.

Take one element $v = (1,0,\ldots,0,\ldots)$ of the canonical basis of $\mathbb{S}_{\ell_2}$. Let $C = \{x \in \mathbb{S}_{\ell_2} : d(x,v) \leq r\}$ be a closed ball centered at v in $\mathbb{S}_{\ell_2}$, where $0 < r < \pi/4$. For $t \geq 0$, define a mapping $T(t) : C \to C$ by

$$T(t)(x) = \bigl(1 - e^{-t}\bigr)v \oplus e^{-t}x,$$

that is,

$$\bigl(T(t)(x)\bigr)(1) = \cos\bigl[e^{-t} \cos^{-1} x(1)\bigr], \tag{4.7}$$

and for $k = 2, 3, \ldots,$

$$\big(T(t)(x)\big)(k) = \begin{cases} 0, & \text{if } x(1) = 1, \\ \frac{x(k)\sin[e^{-t}\cos^{-1}x(1)]}{\sqrt{1-x(1)^2}}, & \text{if } x(1) \neq 1. \end{cases} \tag{4.8}$$

For $x, y \in C$, from Lemma 2.4(iii), we obtain

$$\begin{aligned} d\big(T(t)x, T(t)y\big) &= d\big(\big(1-e^{-t}\big)v \oplus e^{-t}x, \big(1-e^{-t}\big)v \oplus e^{-t}y\big) \\ &\le \frac{\sin(e^{-t}\pi/2)}{\sin\pi/2} d(x,y) \\ &\le d(x,y). \end{aligned}$$

Then $T(t)$ is nonexpansive. In fact, v is the unique fixed point of $T(t)$ for $t > 0$. Furthermore, it is seen that $\mathfrak{S} = \{T(t) : t \ge 0\}$ is a *UAR* and strongly continuous nonexpansive semigroup on C.

Choose $u = (u_k) \in C$ with $0 < u_1 < 1$ so that $0 < \cos^{-1} u_1 = d(u,v) \le r$. For $\alpha \in (0,1)$ and $t \in (0,\infty)$, define $x(\alpha,t)$ by

$$\begin{aligned} x(\alpha,t) &= \alpha u \oplus (1-\alpha)T(t)x(\alpha,t) \\ &= \cos\big[(1-\alpha)d\big(u, T(t)x(\alpha,t)\big)\big]u + \sin\big[(1-\alpha)d\big(u, T(t)x(\alpha,t)\big)\big]\frac{A(\alpha,t)}{\|A(\alpha,t)\|}, \end{aligned} \tag{4.9}$$

where

$$A(\alpha,t) = T(t)x(\alpha,t) - \big(u \mid T(t)x(\alpha,t)\big)u.$$

We remark that $\|A(\alpha,t)\|^2 = 1 - (u \mid T(t)x(\alpha,t))^2$. Let $\{\alpha_n\} \subset (0,1]$, $\{t_n\} \subset (0,\infty)$ be two sequences and define a sequence $\{x_n\}$ in $\overline{B}(v, \cos^{-1} u_1) \subset C$ by

$$x_n = x(\alpha_n, t_n) = \alpha_n u \oplus (1-\alpha_n)T(t_n)x_n.$$

We shall prove that if the sequence $\{x_n\}$ converges to the common fixed point v of $\mathfrak{S}$, then $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \alpha_n/t_n = 0$.

First, assume that $\beta = \limsup_{n\to\infty} \alpha_n$. Since $\lim_{n\to\infty} x_n(1) = 1$ and $|x_n(k)| \le \sqrt{1 - x_n(1)^2}$ for $k \ge 2$, we obtain from (4.7) and (4.8) that

$$\lim_{n\to\infty} \big(T(t_n)(x_n)\big)(1) = 1, \qquad \lim_{n\to\infty} \big(T(t_n)(x_n)\big)(k) = 0.$$

Therefore $\lim_{n\to\infty} T(t_n)(x_n) = v$, which implies that

$$\lim_{n\to\infty} \big(u \mid T(t_n)(x_n)\big) = u_1, \qquad \lim_{n\to\infty} d\big(u, T(t_n)x_n\big) = \cos^{-1} u_1; \tag{4.10}$$

hence

$$\lim_{n\to\infty} A(\alpha_n, t_n)(1) = 1 - u_1^2, \qquad \lim_{n\to\infty} \big\|A(\alpha_n, t_n)\big\| = \sqrt{1 - u_1^2}. \tag{4.11}$$

Let $\theta = d(u,v) = \cos^{-1} u_1$ so that $\sin\theta = \sqrt{1-u_1^2}$. Using (4.9), (4.10), and (4.11), we have

$$\begin{aligned}
1 &= \lim_{n\to\infty} x_n(1) \\
&= \liminf_{n\to\infty} \biggl\{ u_1 \cos\bigl[(1-\alpha_n) d\bigl(u, T(t_n)x_n\bigr)\bigr] + \frac{A(\alpha_n,t_n)(1)}{\|A(\alpha_n,t_n)\|} \sin\bigl[(1-\alpha_n) d\bigl(u, T(t_n)x_n\bigr)\bigr] \biggr\} \\
&\le \limsup_{n\to\infty} \bigl\{ u_1 \cos\bigl[(1-\alpha_n) d\bigl(u, T(t_n)x_n\bigr)\bigr] \bigr\} \\
&\quad + \liminf_{n\to\infty} \biggl\{ \frac{A(\alpha_n,t_n)(1)}{\|A(\alpha_n,t_n)\|} \sin\bigl[(1-\alpha_n) d\bigl(u, T(t_n)x_n\bigr)\bigr] \biggr\} \\
&= \cos\theta \cos\bigl[(1-\beta)\theta\bigr] + \sin\theta \sin\bigl[(1-\beta)\theta\bigr] \\
&= \cos(\beta\theta).
\end{aligned}$$

This shows that $\beta = 0$ because $\theta > 0$ and so $\lim_{n\to\infty} \alpha_n = 0$.

To prove that $\lim_{n\to\infty} \alpha_n/t_n = 0$, we recall the inequality $t \ge 1 - e^{-t}$ for all $t \ge 0$. Since

$$t_n d(v,x_n) \ge \bigl(1 - e^{-t_n}\bigr) d(v,x_n) = d\bigl(x_n, T(t_n)(x_n)\bigr) = \alpha_n d\bigl(u, T(t_n)(x_n)\bigr),$$

it follows that

$$\begin{aligned}
0 &= \lim_{n\to\infty} d(v,x_n) \\
&\ge \limsup_{n\to\infty} \biggl[\frac{\alpha_n}{t_n} d\bigl(u, T(t_n)(x_n)\bigr) \biggr] \\
&= \theta \limsup_{n\to\infty} \frac{\alpha_n}{t_n}.
\end{aligned}$$

Consequently, $\limsup_{n\to\infty} \alpha_n/t_n = 0$ and therefore $\lim_{n\to\infty} \alpha_n/t_n = 0$ as desired.

The following result is an immediate consequence of Theorem 4.2 and Example 4.6.

Theorem 4.7 *Let X be the unit sphere in an infinite-dimensional Hilbert space, and let $\{\alpha_n\} \subset (0,1]$, $\{t_n\} \subset (0,\infty)$ be two sequences. Then the following conditions are equivalent*:

(i) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \alpha_n/t_n = 0$.

(ii) *Let C be a closed convex subset of X with* $\operatorname{diam} C < \pi/2$, $\mathfrak{S} = \{T(t) : t \ge 0\}$ *a UAR and strongly continuous nonexpansive semigroup on C with $\mathfrak{F}(\mathfrak{S}) \neq \emptyset$, $x_0 \in C$ such that $d(x_0, \mathfrak{F}(\mathfrak{S})) < \pi/4$, $p = P_{\mathfrak{F}(\mathfrak{S})} x_0$, $\epsilon = d(x_0,p)$ and $\{x_n\}$ is a sequence in $C \cap \overline{B}(p,\epsilon)$ defined by* (4.1).

Then $\{x_n\}$ converges strongly to p.

5 Remark

A *semitopological semigroup S* is a semigroup equipped with a Hausdorff topology such that for each $t \in S$ the mappings $s \mapsto ts$ and $s \mapsto st$ from S into S are continuous. A semitopological semigroup S is *left* (*respectively, right*) *reversible* if any two closed right (respectively, left) ideals of S have nonvoid intersection, *i.e.*, $\overline{aS} \cap \overline{bS} \neq \emptyset$ (respectively, $\overline{Sa} \cap \overline{Sb} \neq \emptyset$), for $a, b \in S$, where $\overline{E}$ denotes the closure of a set E in a topological space. The class S of all left reversible semitopological semigroups includes all groups, all commuting semigroups, and all normal left amenable semitopological semigroups, *i.e.*, the space $CB(S)$ of bounded continuous functions on S has a left invariant mean; see [2, 3, 13–15]. If S is left (respectively, right) reversible, $(S,\preceq)$ is a directed system when the binary relation $\preceq$ on S is defined by $a \preceq b$ if and only if $\{a\} \cup \overline{aS} \supset \{b\} \cup \overline{bS}$ (respectively, $\{a\} \cup \overline{Sa} \supset \{b\} \cup \overline{Sb}$), $a, b \in S$.

Let S be a semitopological semigroup and C a closed convex subset of a metric space (X,d). A family $\mathfrak{S} = \{T(s) : s \in S\}$ is called a *representation* of S on C if for each $s \in S$, $T(s)$ is a mapping from C into C and $T(st) = T(s) \circ T(t)$, for all $s, t \in S$. The representation is called a *strongly continuous nonexpansive semigroup* on C (or a *continuous representation* of S *as nonexpansive mappings* on C) if the following conditions are satisfied (see [3]):

(i) for each $t \in S$, $T(t)$ is a nonexpansive mapping on C;

(ii) for each $x \in C$, the mapping $t \mapsto T(t)x$ from S into C is continuous.

The representation $\mathfrak{S}$ of S is called

(i) *asymptotically regular* on C if for any $h \in S$ and any $x \in C$,

$$\lim_{t\in S} d\bigl(T(h)T(t)x, T(t)x\bigr) = 0;$$

(ii) *uniformly asymptotically regular* (in short *UAR*) on C if for any $h \in S$ and any bounded subset D of C,

$$\lim_{t\in S} \sup_{x\in D} d\bigl(T(h)T(t)x, T(t)x\bigr) = 0.$$

Problem 5.1 Let X be a complete CAT(1) space, C a closed π-convex subset of X, S a commutative (or left reversible) semitopological semigroup, and $\mathfrak{S} = \{T(t) : t \in S\}$ a continuous representation of S as nonexpansive mappings on C. Can we obtain the result analogous to Theorem 4.2 corresponding to a continuous representation $\mathfrak{S} = \{T(t) : t \in S\}$ of S as nonexpansive mappings on C?

To answer this problem, the following conjecture is required and hence needs to be verified.

Conjecture 5.2 *Let S be a commutative* (*or left reversible*) *semitopological semigroup, C a subset of a metric space (X,d) and $\mathfrak{S} = \{T(t) : t \in S\}$ a representation of S on C. If $\mathfrak{S}$ is asymptotically regular, then*

$$\mathfrak{F}\bigl(T(t)\bigr) = \mathfrak{F}(\mathfrak{S}), \quad \textit{for all } t \in S.$$

Competing interests
The author declares that she has no competing interests.

Acknowledgements
This research was supported by a grant NSC 102-2115-M-259-004 from the National Science Council of Taiwan. The author, therefore, thanks the NSC financial support. The author would like to express the most sincere thanks to the referees for their careful reading of the manuscript, important comments and the citation of Ref. [13]. The author is also very grateful to Professor Anthony Lau for giving valuable remarks and Refs. [14] and [3], and for suggesting presenting Problem 5.1 for future studies.

Received: 18 October 2013 Accepted: 27 January 2014 Published: 20 Feb 2014

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10.1186/1687-1812-2014-44
Cite this article as: Huang: **Nonexpansive semigroups in** CAT(κ) **spaces.** *Fixed Point Theory and Applications* 2014, 2014:44