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Random fixed point theorems under mild continuity assumptions

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Abstract

In this paper, we study the existence of the random fixed points under mild continuity assumptions. The main theorems consider the almost lower semicontinuous operators defined on Banach spaces and also operators having properties weaker than lower semicontinuity. Our results either extend or improve corresponding ones present in literature.

MSC: 47H10; 47H40

Keywords: random fixed point theorem; sub-lower semicontinuity; local intersection property; upper semicontinuity

1 Introduction

Fixed point theorems are a very powerful tool of the current mathematical applications. These have been extended and generalized to study a wide class of problems arising in mechanics, physics, economics and equilibrium theory, engineering sciences, *etc.* New results concerning the deterministic or random case were obtained, for instance, in [1–14].

The main aim of this work is to establish random fixed point theorems under mild continuity assumptions. Results in this direction have also been obtained, for example, in [1–3] or [15]. Our research enables us to improve some theorems obtained recently. We prove the existence of the random fixed points for the lower semicontinuous operators and for operators having properties weaker than lower semicontinuity. By using the approximation method which is due to Ionescu Tulcea (see [16]), we provide a new proof for the random version of Ky Fan's fixed point theorem. Since the majority of our results are obtained for operators defined on Fréchet spaces, we refer the reader to the new literature concerning this topic. For instance, the authors worked also on Fréchet spaces in [17–22].

The rest of the paper is organized as follows. In the following section, some notational and terminological conventions are given. We also present, for the reader's convenience, some results on continuity and measurability of the operators. The random fixed point theorems for operators with properties weaker than lower semicontinuity are stated in Section 3. Section 4 contains random fixed point results concerning the case of lower semicontinuous operators. Section 5 is dedicated to the provision of a new proof for the random version of Ky Fan's fixed point theorem. Section 6 presents the conclusions of our research.

2 Notation and definition

Throughout this paper, we shall use the following notation:

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2^D denotes the set of all non-empty subsets of the set D. If $D \subset Y$, where Y is a topological space, $\operatorname{cl} D$ denotes the closure of D. We also denote $C(Y)$ the family of all non-empty and closed subsets of Y. A *paracompact* space is a Hausdorff topological space in which every open cover admits an open locally finite refinement. Metrizable and compact topological spaces are paracompact.

For the reader's convenience, we review a few basic definitions and results from continuity and measurability of correspondences.

Let X, Y be topological spaces and $T : X \to 2^Y$ be a correspondence. The graph of $T : X \to 2^Y$ is the set $\operatorname{Gr}(T) := \{(x,y) \in X \times Y : y \in T(x)\}$. T is said to be *upper semicontinuous* if, for each $x \in X$ and each open set V in Y with $T(x) \subset V$, there exists an open neighborhood U of x in X such that $T(y) \subset V$ for each $y \in U$. T is said to be *lower semicontinuous* if, for each $x \in X$ and each open set V in Y with $T(x) \cap V \neq \emptyset$, there exists an open neighborhood U of x in X such that $T(y) \cap V \neq \emptyset$ for each $y \in U$. $T : X \to 2^Y$ *has open lower sections* if $T^{-1}(y) := \{x \in X : y \in T(x)\}$ is open in X for each $y \in Y$. A correspondence with open lower sections is lower semicontinuous.

Let (X, d) be a metric space, C be a non-empty subset of X and $T : C \to 2^X$ be a correspondence.

We will use the following notations. We denote by $B(x, r) = \{y \in C : d(y, x) < r\}$. If B_0 is a subset of X, then we will denote $B(B_0, r) = \{y \in C : d(y, B_0) < r\}$, where $d(y, B_0) = \inf_{x \in B_0} d(y, x)$.

We say that T is *hemicompact* if each sequence $\{x_n\}$ in C has a convergent subsequence, whenever $d(x_n, T(x_n)) \to 0$ as $n \to \infty$.

Let now $(\Omega, \mathcal{F}, \mu)$ be a complete, finite measure space, and Y be a topological space. The correspondence $T : \Omega \to 2^Y$ is said to have a *measurable graph* if $\operatorname{Gr}(T) \in \mathcal{F} \otimes \alpha(Y)$, where $\alpha(Y)$ denotes the Borel σ-algebra on Y and $\otimes$ denotes the product σ-algebra. The correspondence $T : \Omega \to 2^Y$ is said to be *lower measurable* if, for every open subset V of Y, the set $T^{-1}(V) = \{\omega \in \Omega : T(\omega) \cap V \neq \emptyset\}$ is an element of $\mathcal{F}$. This notion of measurability is also called in the literature *weak measurability* or just *measurability*, in comparison with strong measurability: the correspondence $T : \Omega \to 2^Y$ is said to be *strong measurable* if, for every closed subset V of Y, the set $\{\omega \in \Omega : T(\omega) \cap V \neq \emptyset\}$ is an element of $\mathcal{F}$. In the case when Y is separable, the strong measurability coincides with the lower measurability.

Recall (see Debreu [23], p.359) that if $T : \Omega \to 2^Y$ has a measurable graph, then T is lower measurable. Furthermore, if $T(\cdot)$ is closed valued and lower measurable, then $T : \Omega \to 2^Y$ has a measurable graph.

A mapping $T : \Omega \times X \to Y$ is called a *random operator* if, for each $x \in X$, the mapping $T(\cdot, x) : \Omega \to Y$ is measurable. Similarly, a correspondence $T : \Omega \times X \to 2^Y$ is also called a random operator if, for each $x \in X$, $T(\cdot, x) : \Omega \to 2^Y$ is measurable. A measurable mapping $\xi : \Omega \to Y$ is called a *measurable selection of the operator* $T : \Omega \to 2^Y$ if $\xi(\omega) \in T(\omega)$ for each $\omega \in \Omega$. A measurable mapping $\xi : \Omega \to Y$ is called a *random fixed point* of the random operator $T : \Omega \times X \to Y$ (or $T : \Omega \times X \to 2^Y$) if for every $\omega \in \Omega$, $\xi(\omega) = T(\omega, \xi(\omega))$ (or $\xi(\omega) \in T(\omega, \xi(\omega))$).

We will need the following measurable selection theorem in order to prove our results.

Proposition 2.1 (Kuratowski-Ryll-Nardzewski selection theorem [24]) *A weakly measurable correspondence with non-empty closed values from a measurable space into a Polish space admits a measurable selector.*

3 Random fixed point theorems for operators with properties weaker than lower semicontinuity

This section is mainly dedicated to establishing the random fixed point theorems concerning the almost lower semicontinuous operators and other types of operators having properties weaker than lower semicontinuity. Our results are new in literature. They can be compared with the ones stated in [15].

Firstly we recall the following statement, which will be useful to prove the main result of this section.

Lemma 3.1 (Theorem 3.4 in [1]) *Let C be a closed separable subset of a complete metric space X, and let $T:\Omega\times C\to C(X)$ be a continuous hemicompact random operator. If, for each $\omega\in\Omega$, the set $F(\omega):=\{x\in C: x\in T(\omega,x)\}\neq\emptyset$, then T has a random fixed point.*

Now, we are presenting the almost lower semicontinuous correspondences.

Let X be a topological space and Y be a normed linear space. The correspondence $T: X\to 2^Y$ is said to be *almost lower semicontinuous* (*a.l.s.c.*) *at* $x\in X$ (see [25]), if, for any $\varepsilon>0$, there exists a neighborhood $U(x)$ of x such that $\bigcap_{z\in U(x)} B(T(z);\varepsilon)\neq\emptyset$.

T is *almost lower semicontinuous* if it is a.l.s.c. at each $x\in X$.

If Ω is a non-empty set, we say that the operator $T:\Omega\times X\to 2^Y$ is almost lower semicontinuous if, for each $\omega\in\Omega$, $T(\omega,\cdot)$ is almost lower semicontinuous.

In 1983, Deutsch and Kenderov [25] presented a remarkable characterization of a.l.s.c. correspondences as follows.

Lemma 3.2 (see [25]) *Let X be a paracompact topological space, Y be a normed vector space and $T:X\to 2^Y$ be a correspondence having convex values. Then T is a.l.s.c. if and only if, for each $\varepsilon>0$, T admits a continuous ε-approximate selection f; that is, $f:X\to Y$ is a continuous single valued function such that $f(x)\in B(T(x);\varepsilon)$ for each $x\in X$.*

The next theorem is the main result of this section. It states the existence of the random fixed points for the almost lower semicontinuous operators defined on Banach spaces.

Theorem 3.1 *Let $(\Omega,\mathcal{F})$ be a measurable space, C be a compact convex separable subset of a Banach space X and let $T:\Omega\times C\to 2^C$ be a random operator. Suppose that, for each $\omega\in\Omega$, $T(\omega,\cdot)$ is almost lower semicontinuous with non-empty convex closed values and $(T(\omega,\cdot))^{-1}: C\to 2^C$ is closed valued.*

Then T has a random fixed point.

Proof Firstly, let us define $T_n:\Omega\times C\to 2^C$ by $T_n(\omega,x)=B(T(\omega,x);1/n)$ for each $(\omega,x)\in\Omega\times C$. Since for each $\omega\in\Omega$, $T(\omega,\cdot)$ is almost lower semicontinuous, according to Lemma 3.2, for each $n\in N$, there exists a continuous function $f_n(\omega,\cdot): C\to C$ such that $f_n(\omega,x)\in T_n(\omega,x)$ for each $x\in C$. Brouwer-Schauder fixed point theorem assures that, for each $n\in N$, there exists $x_n\in C$ such that $x_n=f_n(\omega,x_n)$ and then $x_n\in T_n(\omega,x_n)$.

C is compact, then f_n is hemicompact for each $n\in\mathbb{N}$. According to Lemma 3.1, for each $n\in\mathbb{N}$, f_n has a random fixed point and then T_n has a random fixed point ξ_n, that is $\xi_n: \Omega\to C$ is measurable and $\xi_n(\omega)\in T_n(\omega,\xi_n(\omega))$ for $n\in N$.

Let $\omega\in\Omega$ be fixed. Then $d(\xi_n(\omega), T(\omega,\xi_n(\omega)))\to 0$ when $n\to\infty$ and since C is compact, $\{\xi_n(\omega)\}$ has a convergent subsequence $\{\xi_{n_k}(\omega)\}$. Let $\xi_0(\omega)=\lim_{n_k\to\infty}\xi_{n_k}(\omega)$. It follows that $\xi_0:\Omega\to C$ is measurable and for each $\omega\in\Omega$, $d(\xi_0(\omega), T(\omega,\xi_{n_k}(\omega)))\to 0$ when $n_k\to\infty$.

Let us assume that there exists $\omega \in \Omega$ such that $\xi_0(\omega) \notin T(\omega, \xi_0(\omega))$. Since $\{\xi_0(\omega)\} \cap (T(\omega, \cdot))^{-1}(\xi_0(\omega)) = \emptyset$ and X is a regular space, there exists $r_1 > 0$ such that $B(\xi_0(\omega), r_1) \cap (T(\omega, \cdot))^{-1}(\xi_0(\omega)) = \emptyset$. Consequently, for each $z \in B(\xi_0(\omega), r_1)$, we have $z \notin (T(\omega, \cdot))^{-1} \times (\xi_0(\omega))$, which is equivalent with $\xi_0(\omega) \notin T(\omega, z)$ or $\{\xi_0(\omega)\} \cap T(\omega, z) = \emptyset$. The closedness of each $T(\omega, z)$ and the regularity of X imply the existence of a real number $r_2 > 0$ such that $B(\xi_0(\omega), r_2) \cap T(\omega, z) = \emptyset$ for each $z \in B(\xi_0(\omega), r_1)$, which implies $\xi_0(\omega) \notin B(T(\omega, z); r_2)$ for each $z \in B(\xi_0(\omega), r_1)$. Let $r = \min\{r_1, r_2\}$. Hence, $\xi_0(\omega) \notin B(T(\omega, z); r)$ for each $z \in B(\xi_0(\omega), r)$, and then there exists $N^* \in \mathbb{N}$ such that for each $n_k > N^*$, $\xi_0(\omega) \notin B(T(\omega, \xi_{n_k}(\omega)); r)$ which contradicts $d(\xi_0(\omega), T(\omega, \xi_{n_k}(\omega))) \to 0$ as $n \to \infty$. It follows that our assumption is false.

Hence, for each $\omega \in \Omega$, $\xi_0(\omega) \in T(\omega, \xi_0(\omega))$, where $\xi_0 : \Omega \to C$ is measurable. We conclude that T has a random fixed point. □

Related to the almost lower semicontinuous correspondences, there are the correspondences with the local intersection property and the sub-lower semicontinuous correspondences, which differ very slightly from the first ones. We will obtain some results related to Theorem 3.1 in these cases and will introduce the reader in the topic we brought into discussion by presenting firstly the definitions.

A new theorem can be proved for the operators which satisfy the local intersection property, which is defined below.

Let X, Y be topological spaces. The correspondence $T : X \to 2^Y$ has the *local intersection property* (see [26]) if $x \in X$ with $T(x) \neq \emptyset$ implies the existence of an open neighborhood $V(x)$ of x such that $\bigcap_{z \in V(x)} T(z) \neq \emptyset$.

If Ω is a non-empty set, we say that the operator $T : \Omega \times X \to 2^Y$ has local intersection property if, for each $\omega \in \Omega$, $T(\omega, \cdot)$ has the local intersection property.

We establish Theorem 3.2, which concerns operators having the local intersection property. Its proof relies on the following lemma.

Lemma 3.3 (Wu and Shen [26]) *Let X be a non-empty paracompact subset of a Hausdorff topological space E and Y be a non-empty subset of a Hausdorff topological vector space. Let $S, T : X \to 2^Y$ be correspondences which verify*:

(1) *for each $x \in X$, $S(x) \neq \emptyset$ and* $\operatorname{co} S(x) \subset T(x)$;

(2) *S has local intersection property.*

Then T has a continuous selection.

The corresponding random fixed point theorem in case of local intersection property is stated below.

Theorem 3.2 *Let $(\Omega, \mathcal{F})$ be a measurable space, C be a compact convex separable subset of a Fréchet space X and let $T : \Omega \times C \to 2^C$ be a random operator. Suppose that, for each $\omega \in \Omega$, $T(\omega, \cdot)$ has the local intersection property and is convex valued.*

Then T has a random fixed point.

Proof Since for each $\omega \in \Omega$, $T(\omega, \cdot)$ has the local intersection property, according to Lemma 3.3, there exists a continuous function $f(\omega, \cdot) : C \to C$ such that for each $x \in C$, $f(\omega, x) \in T(\omega, x)$.

According to Tihonov's fixed point theorem, there exists $x \in C$ such that $x = f(\omega, x)$ and then $x \in T(\omega, x)$.

C is compact, then f is hemicompact. According to Lemma 3.1, f has a random fixed point and then T has a random fixed point ξ, that is $\xi : \Omega \to C$ is measurable and $\xi(\omega) \in T(\omega, \xi(\omega))$. □

The sub-lower semicontinuous correspondences were defined by Zheng in [27].

Let X be a topological space and Y be a topological vector space. A correspondence $T : X \to 2^Y$ is called *sub-lower semicontinuous* [27] if, for each $x \in X$ and for each neighborhood V of 0 in Y, there exist $z \in T(x)$ and a neighborhood $U(x)$ of x in X such that, for each $y \in U(x)$, $z \in T(y) + V$.

If Ω is a non-empty set, we say that the operator $T : \Omega \times X \to 2^Y$ is sub-lower semicontinuous if, for each $\omega \in \Omega$, $T(\omega, \cdot)$ is sub-lower semicontinuous.

Zheng proved in [27] a continuous selection result for the sub-lower semicontinuous correspondences, which can be used in order to obtain Theorem 3.3. Here is his result.

Lemma 3.4 [27] *Let X be a paracompact topological space, Y be a locally convex topological vector space and let $T : X \to 2^Y$ be a correspondence with convex values. Then T is sub-lower semicontinuous if and only if, for each neighborhood V of* 0 *in Y, there exists a continuous function $f : X \to Y$ such that, for each $x \in X$, $f(x) \in T(x) + V$.*

The random fixed point existence for the sub-lower semicontinuous random operators is stated below.

Theorem 3.3 *Let (Ω, F) be a measurable space, C be a compact convex separable subset of a Fréchet space X, and let $T : \Omega \times C \to 2^C$ be a random operator. Suppose that, for each $\omega \in \Omega$, $T(\omega, \cdot)$ is sub-lower semicontinuous with non-empty convex closed values and $(T(\omega, \cdot))^{-1} : C \to 2^C$ is closed valued.*

Then T has a random fixed point.

The proof of Theorem 3.3 is similar to the one of Theorem 3.1, but it relies on Lemma 3.4.

In [28], Ansari and Yao proved a fixed point theorem for transfer open valued correspondences. Their proof is based on a continuous selection theorem which they construct. We present below their result.

Let X and Y be two topological spaces. The correspondence $T : X \to 2^Y$ is said to be *transfer open valued* (see [28]) if, for any $x \in X$ and $y \in T(x)$, there exists an $z \in X$ such that $y \in \operatorname{int}_Y T(z)$.

If Ω is a non-empty set, we say that the operator $T : \Omega \times X \to 2^Y$ is transfer open valued if, for each $\omega \in \Omega$, $T(\omega, \cdot)$ is transfer open valued.

The proof of the next lemma is included in the proof of Theorem 1 in [28], in the particular case when $I = \{1\}$, $S = T$ and K is compact.

Lemma 3.5 *Let K be a non-empty compact convex subset of a Hausdorff topological vector space E and let $T : K \to 2^K$ be a correspondence with non-empty convex values. If $K = \bigcup\{\operatorname{int}_K T^{-1}(y) : y \in K\}$ (or T^{-1} is transfer open valued), then T has a continuous selection.*

We obtain Theorem 3.4, by using Lemma 3.5 and the same argument as in the proof of Theorem 3.2.

Theorem 3.4 *Let $(\Omega, \mathcal{F})$ be a measurable space, C be a compact convex separable subset of a Fréchet space X and let $T : \Omega \times C \to 2^C$ be a random operator with non-empty convex values. Suppose that $K = \bigcup\{\mathrm{int}_K(T(\omega,\cdot))^{-1}(y) : y \in K\}$ or for each $\omega \in \Omega$, $(T(\omega,\cdot))^{-1}$ is transfer open valued.*

Then T has a random fixed point.

We also present the following result.

Theorem 3.5 *Let $(\Omega, \mathcal{F})$ be a measurable space, C be a compact convex separable subset of a Fréchet space X and let $T : \Omega \times C \to 2^C$ be a random operator with non-empty closed convex values, such that for each $\omega \in \Omega$, $(T(\omega,\cdot))^{-1} : C \to 2^C$ is closed valued. Suppose that, for each open neighborhood V of the origin and for each $\omega \in \Omega$, the correspondence $(S^{V,\omega})^{-1} : C \to 2^C$ is transfer open valued, where $S^{V,\omega}(x) = (T(\omega,x) + V) \cap C$ for each $x \in C$.*

Then T has a random fixed point.

Proof By using Lemma 3.5, we prove that for each $n \in \mathbb{N}$ and for each $\omega \in \Omega$, there exists a continuous function $f_n(\omega,\cdot) : C \to C$ such that $f_n(\omega,x) \in B(T(\omega,x); 1/n) \cap C$ for each $x \in C$. The proof is similar to the proof of Theorem 3.1. □

4 Random fixed point theorems for lower semicontinuous operators

This section is designed to extending the results established in [1] by considering lower semicontinuous operators defined on Fréchet spaces. Condition $(\mathcal{P})$, essential in the proof of the existence of the random fixed points in the quoted paper, is reformulated and new assumptions which induce it are found.

Firstly, we recall that condition $(\mathcal{P})$, firstly introduced by Petryshyn [29] in order to prove the existence of fixed points for single valued operators, was extended to multivalued operators. We provide here the definition for the last case.

Let (X,d) be a metric space, C be a non-empty closed subset of X and $T : C \to 2^X$ be a correspondence. T is said to satisfy *condition* $(\mathcal{P})$ (see [1]) if, for every closed ball B of C with radius $r \geq 0$ and any sequence $\{x_n\}$ in C for which $d(x_n, B) \to 0$ and $d(x_n, T(x_n)) \to 0$ as $n \to \infty$, there exists $x_0 \in B$ such that $x_0 \in T(x_0)$. If Ω is any non-empty set, we say that the operator $T : \Omega \times C \to 2^X$ satisfies condition $(\mathcal{P})$ if, for each $\omega \in \Omega$, the correspondence $T(\omega,\cdot) : C \to 2^X$ satisfies condition $(\mathcal{P})$.

We also present the main result in [1], concerning operators which satisfy condition $(\mathcal{P})$. We will extend further this theorem.

Lemma 4.1 (Theorem 3.2 in [1]) *Let C be a closed separable subset of a complete metric space and let $T : \Omega \times C \to 2^X$ be a lower semicontinuous random operator, which enjoys condition $(\mathcal{P})$. Suppose that, for each $\omega \in \Omega$, the set $F(\omega) := \{x \in C : x \in T(\omega,x)\} \neq \emptyset$.*

Then T has a random fixed point.

The following lemmata are useful in order to prove Theorem 4.1.

Lemma 4.2 *Let (X,d) be a complete metric space, C be a non-empty closed separable subset of X and $T : C \to 2^X$ be a correspondence.*

(a) *Suppose that T satisfies condition $(\mathcal{P})$. If $x_0 \notin T(x_0)$, then there exists a real $r > 0$ such that $x_0 \notin B(T(x); r) \cap B(x,r)$ for each $x \in B(x_0, r)$.*

(b) *Suppose that C is locally compact and $x_0 \notin T(x_0)$ implies the existence of a real $r > 0$ such that $x_0 \notin B(T(x); r) \cap B(x, r)$ for each $x \in B(x_0, r)$. Then T satisfies condition $(\mathcal{P})$.*

Proof (a) Let us consider a correspondence T which verifies condition $(\mathcal{P})$. We will prove that T satisfies the conclusion stated at (a). For this purpose, let $x_0 \in C$ be a point such that $x_0 \notin T(x_0)$. Let us assume, by contradiction, that for each $r > 0$, there exists $x_r \in B(x_0, r)$ such that $x_0 \in B(T(x_r); r) \cap B(x_r, r)$. Therefore, for each natural number $n > 0$, there exists $x_n \in B(x_0, 1/n)$ such that $x_0 \in B(T(x_n); 1/n) \cap B(x_n, 1/n)$. Consequently, we found a sequence $\{x_n\}$ in C with the property that $d(x_n, x_0) \to 0$ and $d(x_n, T(x_n)) \to 0$ when $n \to \infty$. Since T satisfies condition $(\mathcal{P})$ for $B_0 = \{x_0\}$, it follows that $x_0 \in T(x_0)$, which contradicts $x_0 \notin T(x_0)$. This means that our assumption is false, and then there exists a real $r > 0$ such that $x_0 \notin B(T(x); r) \cap B(x, r)$ for each $x \in B(x_0, r)$.

(b) Let us consider a closed ball B_0 of C with radius $R \geq 0$ and a sequence $\{x_n\}$ in C, for which $d(x_n, B_0) \to 0$ and $d(x_n, T(x_n)) \to 0$ as $n \to \infty$. We use the sequentially compactness of the set G we will define further. Let us firstly denote, for each $n \in N$, $r_n = d(x_n, B_0)$. According to the hypotheses, the sequence $\{r_n\}$ is convergent and $\lim_{n\to\infty} r_n = 0$. Let us consider $\varepsilon > 0$, and then there exists $N(\varepsilon) \in \mathbb{N}$ such that $r_n < \varepsilon$ for each $n > N(\varepsilon)$. Since X is locally compact, the small closed balls are compact and consequently, the set G, defined as $G = \mathrm{cl}(B(B_0; \varepsilon) \cap C)$, is compact. Therefore, the sequence $\{x_n\} \subset G$ has a convergent subsequence $\{x_{n_k}\}$ and, without loss of generality, we can assume that the sequence $\{x_n\}$ is convergent. It remains to show the following assertion:

If $T : C \to 2^X$ satisfies the assumption from (b), and if there exist a closed ball B_0 and a convergent sequence $\{x_n\} \to x_0$ such that $d(x_n, B_0) \to 0$ and $d(x_n, T(x_n)) \to 0$ as $n \to \infty$ then $x_0 \in T(x_0)$ and $x_0 \in B_0$.

In order to prove this, we note that the closedness of B_0 implies $x_0 \in B_0$. Let us assume, by contradiction, that $x_0 \notin T(x_0)$. Then, according to the hypotheses, there exists $r > 0$ such that $x_0 \notin B(T(x); r) \cap B(x, r)$ for each $x \in B(x_0, r)$. The convergence of $\{x_n\}$ to x_0 implies the existence of a natural number $N(r) \in \mathbb{N}$ such that $x_n \in B(x_0, r)$ for each $n > N(r)$. Consequently, $x_0 \notin B(T(x_n); r) \cap B(x_n, r)$ for each $n > N(r)$. Since for each $n > N(r)$, $x_0 \in B(x_n, r)$, it follows that if $n > N(r)$, $x_0 \notin B(T(x_n); r)$, that is $d(x_0, T(x_n)) > r$. This fact contradicts $d(x_0, T(x_n)) \to 0$ when $n \to \infty$, which is true by the hypotheses and because $x_0 = \lim_{n\to\infty} x_n$. This means that our assumption is false, and it remains that $x_0 \in T(x_0)$. We proved that T satisfies condition $(\mathcal{P})$. □

Lemma 4.3 *Let (X, d) be a metric space, C be a non-empty closed subset of X and $T : C \to 2^X$ be a correspondence such that T and T^{-1} have closed values. Therefore, if $x_0 \notin T(x_0)$, there exists a real $r > 0$ such that $x_0 \notin B(T(x); r) \cap B(x, r)$ for each $x \in B(x_0, r)$. If, in addition, C is locally compact, then T satisfies condition $(\mathcal{P})$.*

Proof Let us consider $x_0 \in C$ such that $x_0 \notin T(x_0)$. Since $\{x_0\} \cap T^{-1}(x_0) = \emptyset$ and X is a regular space, there exists $r_1 > 0$ such that $B(x_0, r_1) \cap B(T^{-1}(x_0); r_1) = \emptyset$, and then $B(x_0, r_1) \cap T^{-1}(x_0) = \emptyset$. Consequently, for each $x \in B(x_0, r_1)$, we have $x \notin T^{-1}(x_0)$, which is equivalent with $x_0 \notin T(x)$ or $\{x_0\} \cap T(x) = \emptyset$. The closedness of each $T(x)$ and the regularity of X imply the existence of a real number $r_2 > 0$ such that $B(x_0, r_2) \cap T(x) = \emptyset$ for each $x \in B(x_0, r_1)$, which implies $x_0 \notin B(T(x); r_2)$ for each $x \in B(x_0, r_1)$. Let $r = \min\{r_1, r_2\}$. Hence, $x_0 \notin B(T(x); r)$ for each $x \in B(x_0, r)$, and thus the conclusion is fulfilled. In view of Lemma 4.2, the last assertion is true. □

The following theorem states the existence of the random fixed points for lower semicontinuous correspondences defined on locally compact complete metric spaces.

Theorem 4.1 *Let* $(\Omega, \mathcal{F})$ *be a measurable space, C be a closed separable subset of a locally compact complete metric space and let* $T : \Omega \times C \to 2^X$ *be a lower semicontinuous random operator with closed values. Suppose that, for each* $\omega \in \Omega$, $(T(\omega, \cdot))^{-1} : X \to 2^C$ *is closed valued and the set* $F(\omega) := \{x \in C : x \in T(\omega, x)\} \neq \emptyset$.

Then T has a random fixed point.

Proof Since C is locally compact and for each $\omega \in \Omega$, $T(\omega, \cdot) : C \to 2^X$ and $(T(\omega, \cdot))^{-1} : X \to 2^C$ have closed values, by applying Lemma 4.3, we find that T satisfies condition $(\mathcal{P})$. All the assumptions of Lemma 4.1 are fulfilled, then T has a random fixed point. □

The existence of the random fixed points remains valid if for each $\omega \in \Omega$, $(T(\omega, \cdot))^{-1} : X \to 2^C$ is lower semicontinuous. In this case, we establish Theorem 4.2.

Theorem 4.2 *Let* $(\Omega, \mathcal{F})$ *be a measurable space, C be a closed separable subset of a locally compact complete metric space and let* $T : \Omega \times C \to 2^X$ *be an operator with closed values. Suppose that, for each* $\omega \in \Omega$, $(T(\omega, \cdot))^{-1} = U(\omega, \cdot) : X \to 2^C$ *is lower semicontinuous and closed valued, for each* $x \in X$, $U(\cdot, x) : \Omega \to 2^C$ *is measurable and the set* $F(\omega) := \{x \in C : x \in T(\omega, x)\} \neq \emptyset$.

Then T has a random fixed point.

Proof According to Theorem 4.1, there exists a measurable mapping $\xi : \Omega \to C$ such that for each $\omega \in \Omega$, $\xi(\omega) \in (T(\omega, \cdot))^{-1}(\xi(\omega))$, that is, for each $\omega \in \Omega$, $\xi(\omega) \in T(\omega, \xi(\omega))$. Therefore, we obtained a random fixed point for T. □

The next result, due to Michael, is very important in the theory of continuous selections.

Lemma 4.4 (Michael [30]) *Let X be a* T_1, *paracompact space. If Y is a Banach space, then each lower semicontinuous convex closed valued correspondence* $T : X \to 2^Y$ *admits a continuous selection.*

By using the above lemma, we obtain Theorem 4.3.

Theorem 4.3 *Let* $(\Omega, \mathcal{F})$ *be a measurable space, C be a compact convex separable subset of a Banach space and let* $T : \Omega \times C \to 2^C$ *be a lower semicontinuous random operator with closed convex values, such that for each* $\omega \in \Omega$, $(T(\omega, \cdot))^{-1} : C \to 2^C$ *is closed valued.*

Then T has a random fixed point.

Proof Since C is compact and for each $\omega \in \Omega$, $T(\omega, \cdot) : C \to 2^C$ and $(T(\omega, \cdot))^{-1} : C \to 2^C$ have closed values, by applying Lemma 4.3, we find that T satisfies condition $(\mathcal{P})$.

According to Michael's selection theorem, there exists a continuous function $f : C \to C$ such that f is continuous and $f(x) \in T(\omega, x)$. According to Brouwer-Schauder fixed point theorem, there exist deterministic fixed points for f, then the set $F(\omega) := \{x \in C : x \in T(\omega, x)\} \neq \emptyset$. All the assumptions of Theorem 4.1 are fulfilled, then T has a random fixed point. □

Remark 4.1 Theorem 4.3 can be compared with Theorem 3.9 in [3]. This last result also refers to the existence of random fixed points for lower semicontinuous operators and its proof is based on Michael's selection theorem. A comparison of the assumptions of the two theorems makes the object of the research in [31].

Fierro *et al.* showed in [1] (the proof of Theorem 3.1) that if T is lower semicontinuous, then it satisfies the following condition, which we call $(*)$, condition necessary to prove the existence of random fixed points. We denote $\mathrm{Fix}(T) = \{x \in C : x \in T(x)\}$.

Definition 4.1 Let (X, d) be a complete metric space, C be a non-empty closed separable subset of X and $Z = \{z_n\}$ be a countable dense subset of C. We say that the correspondence $T : C \to 2^X$ satisfies condition $(*)$ if, for each closed ball B_0 in X with the property that $B_0 \cap \mathrm{Fix}(T) \neq \emptyset$, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $d(z_{n_k}, B_0) < 1/k$ and $d(z_{n_k}, T(z_{n_k})) < 1/k$ for each $k \in \mathbb{N}$.

Our work will consider simpler assumptions which imply condition $(*)$. Now, we are introducing condition α.

Definition 4.2 Let X be a topological vector space and C be a non-empty subset of X.

(1) We say that the correspondence $T : C \to 2^X$ satisfies condition α, if for each $x_0 \in \mathrm{Fix}(T)$, T is sub-lower semicontinuous in x_0.
(2) If Ω is a non-empty set, we say that the operator $T : \Omega \times C \to 2^X$ satisfies condition α if, for each $\omega \in \Omega$, the correspondence $T(\omega, \cdot) : C \to 2^X$ satisfies condition α.

Next lemma shows that condition α is stronger than condition $(*)$.

Lemma 4.5 *Let (X, d) be a complete metric space and C a non-empty closed separable subset of X. If the correspondence $T : C \to 2^X$ satisfies condition α, then T satisfies $(*)$.*

Proof Let $Z = \{z_n\}$ be a countable dense subset of C. Let B_0 be a closed ball in X such that $B_0 \cap \mathrm{Fix}(T) \neq \emptyset$. Then there exists $x_0 \in X$ with the property that $x_0 \in B_0 \cap C$ and $x_0 \in T(x_0)$. According to condition α, for each $k \in \mathbb{N}$, there exists an open neighborhood $U_k(x_0)$ of x_0 such that $x_0 \in B(T(x), 1/k)$ for each $x \in U_k(x_0)$. Then $x_0 \in B(T(x), 1/k)$ for each $x \in B(x_0, 1/k) \cap U_k(x_0)$ and thus the intersection $B(x_0, 1/k) \cap U_k(x_0) \cap B(T(x), 1/k)$ is non-empty for each $x \in B(x_0, 1/k) \cap U_k(x_0)$. Since $B(x_0, 1/k) \cap U_k(x_0)$ and $B(T(x), 1/k)$ are open sets, $B(x_0, 1/k) \cap U_k(x_0) \cap B(T(x), 1/k) \neq \{x_0\}$. Therefore, for each $k \in \mathbb{N}$, we can choose $z_{n_k} \in B(x_0, 1/k) \cap C \cap Z$, $z_{n_k} \neq x_0$. Consequently, for each $k \in \mathbb{N}$, $z_{n_k} \in B(B_0, 1/k) \cap C \cap Z$ and $d(z_{n_k}, T(z_{n_k})) < 1/k$, that is T satisfies $(*)$. □

We establish the following random fixed point theorem.

Theorem 4.4 *Let $(\Omega, \mathcal{F})$ be a measurable space, C be a closed separable subset of a complete metric space and let $T : \Omega \times C \to 2^X$ be a random operator which enjoys conditions $(\mathcal{P})$ and $(*)$. Suppose that, for each $\omega \in \Omega$, the set $F(\omega) := \{x \in C : x \in T(\omega, x)\} \neq \emptyset$.*

Then T has a random fixed point.

Proof Let $Z = \{z_n\}$ be a countable dense subset of C. Let us define $F : \Omega \to 2^C$ by $F(\omega) = \{x \in C : x \in T(\omega, x)\}$. We will prove the measurability of F. In order to do this, we consider

B_0 an arbitrary closed ball of C, and let us denote

$$L(B_0) := \bigcap_{k=1}^{\infty} \bigcup_{z \in B(B_0;1/k) \cap Z} \{\omega \in \Omega : d(z, T(\omega, z)) < 1/k\}.$$

Now, we are proving that $F^{-1}(B_0) = L(B_0)$.

Firstly, let us consider $\omega \in F^{-1}(B_0)$ and hence there exists $x_0 \in B_0$ such that $x_0 \in (T(\omega, \cdot))^{-1}(x_0)$.

Since T satisfies condition $(*)$, for each $k \in \mathbb{N}$, there exists $z_{n_k} \in B(B_0, 1/k) \cap Z$ such that $d(z_{n_k}, T(\omega, z_{n_k})) < 1/k$. Therefore, $\omega \in L(B_0)$ and then $F^{-1}(B_0) \subseteq L(B_0)$.

The rest of the proof is similar to the corresponding one in Theorem 3.1 in [1]. Therefore, F is measurable with non-empty closed values, and according to the Kuratowski and Ryll-Nardzewski Proposition 2.1, F has a measurable selection $\xi : \Omega \to C$ such that $\xi(\omega) \in T(\omega, (\xi, \omega))$ for each $\omega \in \Omega$. □

We state the following consequence of Theorem 4.4 and Lemma 4.5.

Corollary 4.1 *Let $(\Omega, \mathcal{F})$ be a measurable space, C be a closed separable subset of a complete metric space and let $T : \Omega \times C \to 2^X$ be a random operator which enjoys conditions $(\mathcal{P})$ and α. Suppose that, for each $\omega \in \Omega$, the set $F(\omega) := \{x \in C : x \in T(\omega, x)\} \neq \emptyset$.*

Then T has a random fixed point.

Based on Theorem 4.4 and Lemma 4.3, we obtain the next corollary.

Corollary 4.2 *Let $(\Omega, \mathcal{F})$ be a measurable space, C be a closed separable subset of a locally compact complete metric space and let $T : \Omega \times C \to 2^X$ be a random operator which enjoys condition α and has closed values, such that for each $\omega \in \Omega$, $(T(\omega, \cdot))^{-1} : C \to 2^C$ is closed valued. Suppose that, for each $\omega \in \Omega$, the set $F(\omega) := \{x \in C : x \in T(\omega, x)\} \neq \emptyset$.*

Then T has a random fixed point.

Proof Since for each $\omega \in \Omega$, $T(\omega, \cdot) : C \to 2^X$ and $(T(\omega, \cdot))^{-1} : X \to 2^C$ are closed valued and C is locally compact, then T fulfills condition $(\mathcal{P})$. In order to complete the proof, we apply Theorem 4.4. □

We will use further the following notation.

Notation 4.1 Let X be a vector space, C be a non-empty subset of X and $T : C \to 2^X$ be a correspondence. We denote $S_T : C \to 2^X$ the correspondence defined by $S_T(x) = \mathrm{co}\{T(x), x\}$ for each $x \in C$.

We notice that for each $x \in X$, $T(x) \subseteq S_T(x)$ and $T(x) = S_T(x)$ if only if $x \in T(x)$ and $T(x)$ is convex. We also notice that $x \in S_T(x)$ for each $x \in X$.

Now, we define the notion of a local approximating pair of correspondences.

Definition 4.3 Let X be a topological space and Y be a topological vector space. Let $S, T : X \to 2^Y$ be correspondences. We say that (T, S) is a local approximating pair if $T(x) \subseteq S(x)$ for each $x \in X$ and if $S(x_0) = T(x_0)$, then, for each open neighborhood of the origin V, there exists $U(x_0)$, an open neighborhood of x_0 such that $S(x) \subseteq T(x) + V$ for each $x \in U(x_0)$.

Lemma 4.6 gives a new condition which implies $(*)$.

Lemma 4.6 *Let C be a non-empty closed separable subset of a Fréchet space X. Let $T : C \to 2^X$ be a correspondence such that (T, S_T) is a local approximating pair. Then T satisfies $(*)$.*

Proof Let $Z = \{z_n\}$ be a countable dense subset of C. Let us consider $x_0 \in X$ such that $x_0 \in T(x_0)$, $\{x_0\}$ being included in a closed ball B_0, and the neighborhoods of the origin of the following type: $V_k = B(0, 1/k)$, $k \in \mathbb{N}$. For each $k \in N$, there exists a neighborhood U_k of x_0 such that $U_k \subset B_0 \cap B(x_0, 1/k)$ and $S_T(x) = \operatorname{co}\{T(x) \cup x\} \subset T(x) + V_k$ for each $x \in U_k$. For each $k \in \mathbb{N}$, we can pick $z_{n_k} \in U_k \cap Z$. Then, for each $k \in \mathbb{N}$, $d(z_{n_k}, x_0) < 1/k$ and $d(z_{n_k}, T(z_{n_k})) < 1/k$. □

The next result is a consequence of Theorem 4.4 and Lemma 4.6.

Corollary 4.3 *Let (Ω, F) be a measurable space, C be a closed separable subset of Fréchet space and let $T : \Omega \times C \to 2^X$ be a random operator such that (T, S_T) is a local approximating pair. Assume that T enjoys condition $(\mathcal{P})$ and for each $\omega \in \Omega$, the set $F(\omega) := \{x \in C : x \in T(\omega, x)\} \neq \emptyset$.*

Then T has a random fixed point.

5 Random fixed point theorems for upper semicontinuous correspondences

The main aim of this section is to provide a new proof for the random version of Ky Fan's fixed point theorem. Our result is distinguished by the fact that $(\Omega, \mathcal{F})$ is only a measurable space, without having other additional properties. The proof is based on the upper approximation technique, which is due to Ionescu Tulcea [16]. The notions related to this topic are presented below.

Let X, Y be topological spaces and $T : X \to 2^Y$ be a correspondence. The correspondence $\overline{T} : X \to 2^Y$ is defined by $\overline{T}(x) := \{y \in Y : (x, y) \in \operatorname{cl}_{X\times Y} \operatorname{Gr}(T)\}$ for each $x \in X$ (the set $\operatorname{cl}_{X\times Y} \operatorname{Gr}(T)$ is called the adherence of the graph of T). It is easy to see that $\operatorname{cl} T(x) \subseteq \overline{T}(x)$ for each $x \in X$.

Let X be a topological space and Y be a topological vector space. A correspondence $T : X \to 2^Y$ is *quasi-regular* (see [16]) if: (1) it has open lower sections, that is, $T^{-1}(y)$ is open in X for each $y \in Y$; (2) $T(x)$ is non-empty and convex for each $x \in X$ and (3) $\overline{T}(x) = \operatorname{cl} T(x)$ for each $x \in X$. T is called *regular* (see [16]) if it is quasi-regular and it has an open graph.

In [16] Ionescu Tulcea defined the notion of upper approximating family for a correspondence.

Let X be a non-empty set, Y be a non-empty subset of topological vector space E and $T : X \to 2^Y$. A family $(T_j)_{j\in J}$ of correspondences between X and Y, indexed by a non-empty filtering set J (denote by $\leq$ the order relation in J) is an *upper approximating family for T* [16] if:

(1) $T(x) \subset T_j(x)$ for each $x \in X$ and for each $j \in J$;
(2) for each $j \in J$ there exists $j^* \in J$ such that, for each $h \geq j^*$ and $h \in J$, $T_h(x) \subset T_j(x)$ for each $x \in X$;
(3) for each $x \in X$ and $V \in ß$, where ß is a base of neighborhood of 0 in E, there exists $j_{x,V} \in J$ such that $T_h(x) \subset T(x) + V$ if $h \in J$ and $j_{x,V} \leq h$.

By (1)-(3) we deduce that

(4) for each $x \in X$ and $k \in J$, $T(x) \subset \bigcap_{j\in J} T_j(x) = \bigcap_{k\leq j, k\in J} T_j(x) \subset \operatorname{cl} T(x) \subset \overline{T}(x)$.

Conditions for the existence of an approximating family for an upper semicontinuous correspondence are given in the following lemma. We recall that a subset X of a locally convex topological vector space E has *the property* (K) (see [16]) if, for every compact subset D of X, the set $\operatorname{co} D$ is relatively compact in E.

Lemma 5.1 (see [16]) *Let X be a paracompact space and let Y be a non-empty closed convex subset in a Hausdorff locally convex topological vector space and with the property (K). Let $T: X \to 2^Y$ be a compact and upper semicontinuous correspondence with non-empty convex compact values. Then there exists a filtering set J such that there exists a family $(T_j)_{j\in J}$ of correspondences between X and Y with the following properties*:

(1) *for each $j \in J$, T_j is regular*;
(2) *$(T_j)_{j\in J}$ and $(\overline{T}_j)_{j\in J}$ are upper approximating family for T*;
(3) *for each $j \in J$, $\overline{T}_j$ is continuous if Y is compact.*

Theorem 5.1 is a random version of the Ky Fan fixed point theorem. A new proof is provided. Our result is distinguished by the fact that $(\Omega, \mathcal{F})$ must be only a measurable space, without any additional properties.

Theorem 5.1 *Let $(\Omega, \mathcal{F})$ be a measurable space, C be a compact convex separable subset of a Fréchet space X and let $T: \Omega \times C \to 2^C$ be a random operator. Suppose that, for each $\omega \in \Omega$, $T(\omega, \cdot)$ is upper semicontinuous with non-empty convex compact values.*

Then T has a random fixed point.

Proof Let $Z = \{z_n\}$ be a countable dense subset of C. Let us define $F: \Omega \to 2^C$ by $F(\omega) = \{x \in C : x \in T(\omega, x)\}$. We will prove the measurability of F. In order to do this, we consider B_0 an arbitrary closed ball of C, and let us denote

$$L(B_0) := \bigcap_{k=1}^{\infty} \bigcup_{z \in B(B_0, 1/k) \cap Z} \bigl\{\omega \in \Omega : d\bigl(z, T(\omega, z)\bigr) < 1/k\bigr\}.$$

Now, we are proving that $F^{-1}(B_0) = L(B_0)$.

Firstly, let us consider $\omega \in F^{-1}(B_0)$ and hence there exists $x_0 \in B_0$ such that $x_0 \in T(\omega, x_0)$. Since $T(\omega, \cdot)$ satisfies the assumptions of Lemma 5.1, there exists a filtering set J such that there exists an upper approximating family $(T_j(\omega, \cdot))_{j\in J}$ of correspondences between C and X, every $T_j(\omega, \cdot)$ being regular. For $j \in J$, each $T_j(\omega, \cdot)$ is lower semicontinuous, then it fulfills condition $(*)$. We will prove that condition $(*)$ is also fulfilled by $T(\omega, \cdot)$.

Since $(T_j(\omega, \cdot))_{j\in J}$ is an approximating family for $T(\omega, \cdot)$, for each $x \in C$ and $r > 0$, there exists $j_{x,r} \in J$ such that $T_i(\omega, x) \subset B(T(\omega, x); r)$ if $i \in J$ and $j_{x,r} \leq i$. For $U = \bigcup_{n\geq 1} B(B_0; 1/n) \cap C$ and every $r > 0$, there exists $j_{U,r} \in J$ such that $T_i(\omega, x) \subset B(T(\omega, x); r)$ if $i \in J$ and $j_{U,r} \leq i$.

If $x_0 \in T(\omega, x_0)$, then $x_0 \in T_i(\omega, x_0)$ for each $i \in J$. For each $k \in \mathbb{N}$, there exist $j_{U,k} \in J$ and $i_k \in J$ such that $j_{U,k} \leq i_k$ and $T_{i_k}(\omega, x) \subset B(T(\omega, x); 1/2k)$ for each $x \in U$. Since $T_{i_k}(\omega, \cdot)$ is lower semicontinuous, there exists $\{x_{n_h}^{i_k}\} \in C \cap Z$ such that for each $h \in \mathbb{N}$, $d(x_{n_h}^{i_k}, B_0) < 1/2h$ and $d(x_{n_h}^{i_k}, T_{i_h}(\omega, x_{n_h}^{i_k})) < 1/2h$. Consequently, $d(x_{n_k}^{i_k}, T(\omega, x_{n_k}^{i_k})) < 1/k$ for each $k \in \mathbb{N}$. Let us construct the sequence $\{z_{n_k}\} \in C \cap Z$ such that for each $k \in \mathbb{N}$, $z_{n_k} = x_{n_k}^{i_k}$. Therefore, we found a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $z_{n_k} \in B(B_0; 1/k)$ and $d(z_{n_k}, T(\omega, z_{n_k})) < 1/k$. Hence, $\omega \in L(B_0)$ and the inclusion $F^{-1}(B_0) \subseteq L(B_0)$ is proven.

Let $\omega \in \Omega$, let B_0 be a closed ball of C and $\{x_n\}$ be a sequence in C such that $d(x_n, B) \to 0$ and $d(x_n, T(\omega, x_n)) \to 0$ when $n \to \infty$. The set C is compact, then we can assume that $\{x_n\}$ is a convergent sequence. Let $\lim_{n\to\infty} x_n = x_0 \in B_0$. Since $d(x_n, T(\omega, x_n)) \to 0$ when $n \to \infty$, for each $n \in \mathbb{N}$, there exists $y_n \in T(\omega, x_n)$ such that $d(x_n, y_n) \to 0$ when $n \to \infty$. This fact assures the convergence of the sequence $\{y_n\}$ and $\lim_{n\to\infty} y_n = x_0$. Now, we are using the upper semicontinuity of T and we conclude that for each $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that $T(\omega, x_n) \subseteq B(T(\omega, x_0); \varepsilon)$ for each $n > N(\varepsilon)$. It follows that for each $\varepsilon > 0$, $x_0 \in B(T(\omega, x_0); \varepsilon)$, which implies $x_0 \in T(\omega, x_0)$.

We showed that the operator T satisfies condition $(\mathcal{P})$.

We prove that $F : \Omega \to 2^C$ is measurable and has closed values, by following the same line as in the proof of Theorem 3.1 in [1].

In addition, we mention that since C is compact and convex, and for each $\omega \in \Omega$, $T(\omega, \cdot)$ is upper semicontinuous with non-empty convex compact values, then, according to the Ky Fan fixed point theorem, the set $F(\omega) := \{x \in C : x \in T(\omega, x)\} \neq \emptyset$.

Consequently, F is measurable with non-empty closed values. According to the Kuratowski and Ryll-Nardzewski Proposition 2.1, F has a measurable selection $\xi : \Omega \to C$ such that $\xi(\omega) \in T(\omega, (\xi, \omega))$ for each $\omega \in \Omega$. We proved the existence of a random fixed point for T. □

We also establish a new random fixed point result concerning the upper semicontinuous random operators.

Corollary 5.1 *Let $(\Omega, \mathcal{F})$ be a measurable space, C be a compact convex separable subset of a Fréchet space X, and let $T : \Omega \times C \to 2^C$ be an operator. Suppose that, for each $\omega \in \Omega$, $(T(\omega, \cdot))^{-1} = U(\omega, \cdot) : C \to 2^C$ is upper semicontinuous with non-empty convex compact values and for each $x \in X$, $U(\cdot, x) : \Omega \to 2^C$ is measurable .*

Then T has a random fixed point.

Proof According to Theorem 5.1, there exists a measurable mapping $\xi : \Omega \to C$ such that for every $\omega \in \Omega$, $\xi(\omega) \in (T(\omega, \cdot))^{-1}(\xi(\omega))$, that is, for every $\omega \in \Omega$, $\xi(\omega) \in T(\omega, \xi(\omega))$. Therefore, we obtained a random fixed point for T. □

In the end of the paper we state the following theorem, which is a consequence of Theorem 5.1.

Theorem 5.2 *Let $(\Omega, \mathcal{F})$ be a measurable space, C be a compact convex separable subset of a Fréchet space X, and let $T : \Omega \times C \to 2^C$. Suppose that $\overline{T} : \Omega \times C \to 2^C$ is a random operator and for each $\omega \in \Omega$, $\overline{T}(\omega, \cdot)$ has non-empty convex values, where $\overline{T}(\omega, \cdot)$ is defined by $\overline{T}(\omega, x) := \{y \in C : (x, y) \in \mathrm{cl}_{C\times C} \operatorname{Gr}(T(\omega, \cdot))\}$ for each $x \in C$.*

Then there exists a measurable mapping $\xi : \Omega \to C$ such that $\xi(\omega) \in \overline{T}(\omega, (\xi, \omega))$ for each $\omega \in \Omega$.

Proof For each $\omega \in \Omega$, $\overline{T}(\omega, \cdot)$ is upper semicontinuous with non-empty convex compact values, and we can apply Theorem 5.1. □

6 Concluding remarks

We have proven the existence of random fixed points for almost lower semicontinuous and lower semicontinuous operators defined on Fréchet spaces. Our research extends some results which exist in the literature. It is an interesting problem to find new types of operators which satisfy weak continuity properties and have random fixed points.

Competing interests

The author declares that they have no competing interests.

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