

## RESEARCH

## Open Access

# On some new inequalities for differentiable co-ordinated convex functions

MA Latif<sup>1\*</sup> and SS Dragomir<sup>2</sup>

\* Correspondence:

m\_amer\_latif@hotmail.com

<sup>1</sup>College of Science, Department of Mathematics, University of Hail, Hail-2440, Saudi Arabia

Full list of author information is available at the end of the article

**Abstract**

Several new inequalities for differentiable co-ordinated convex and concave functions in two variables which are related to the left side of Hermite-Hadamard type inequality for co-ordinated convex functions in two variables are obtained.

**Mathematics Subject Classification (2000):** 26A51; 26D15

**Keywords:** convex function, co-ordinated convex function, Hermite-Hadamard's inequality, Jensen's integral inequality

**1. Introduction**

The following definition is well known in literature:

A function  $f: I \rightarrow \mathbb{R}$ ,  $\emptyset \neq I \subseteq \mathbb{R}$ , is said to be convex on  $I$  if the inequality

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y),$$

holds for all  $x, y \in I$  and  $\lambda \in [0, 1]$ .

Many important inequalities have been established for the class of convex functions, but the most famous is the Hermite-Hadamard's inequality (see for instance [1]). This double inequality is stated as:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}, \quad (1.1)$$

where  $f: I \rightarrow \mathbb{R}$ ,  $\emptyset \neq I \subseteq \mathbb{R}$  a convex function,  $a, b \in I$  with  $a < b$ . The inequalities in (1.1) are in reversed order if  $f$  is a concave function.

The inequalities (1.1) have become an important cornerstone in mathematical analysis and optimization and many uses of these inequalities have been discovered in a variety of settings. Moreover, many inequalities of special means can be obtained for a particular choice of the function  $f$ . Due to the rich geometrical significance of Hermite-Hadamard's inequality (1.1), there is growing literature providing its new proofs, extensions, refinements and generalizations, see for example [2-5] and the references therein.

Let us consider now a bidimensional interval  $\Delta =: [a, b] \times [c, d]$  in  $\mathbb{R}^2$  with  $a < b$  and  $c < d$ , a mapping  $f: \Delta \rightarrow \mathbb{R}$  is said to be convex on  $\Delta$  if the inequality

$$f(\lambda x + (1 - \lambda)z, \lambda y + (1 - \lambda)w) \leq \lambda f(x, y) + (1 - \lambda)f(z, w),$$

holds for all  $(x, y), (z, w) \in \Delta$  and  $\lambda \in [0, 1]$ .

A modification for convex functions on  $\Delta$ , which are also known as co-ordinated convex functions, was introduced by Dragomir [6,7] as follows:

A function  $f: \Delta \rightarrow \mathbb{R}$  is said to be convex on the co-ordinates on  $\Delta$  if the partial mappings  $f_y: [a, b] \rightarrow \mathbb{R}, f_y(u) = f(u, y)$  and  $f_x: [c, d] \rightarrow \mathbb{R}, f_x(v) = f(x, v)$  are convex where defined for all  $x \in [a, b], y \in [c, d]$ .

A formal definition for co-ordinated convex functions may be stated as follows:

**Definition 1.** [8] *A function  $f: \Delta \rightarrow \mathbb{R}$  is said to be convex on the co-ordinates on  $\Delta$  if the inequality*

$$f(tx + (1 - t)y, su + (1 - s)w) \leq ts f(x, u) + t(1 - s)f(x, w) + s(1 - t)f(y, u) + (1 - t)(1 - s)f(y, w),$$

holds for all  $t, s \in [0, 1]$  and  $(x, u), (y, w) \in \Delta$ .

Clearly, every convex mapping  $f: \Delta \rightarrow \mathbb{R}$  is convex on the co-ordinates. Furthermore, there exists co-ordinated convex function which is not convex, (see for example [6,7]). For recent results on co-ordinated convex functions we refer the interested reader to [6,8-13].

The following Hermite-Hadamrd type inequality for co-ordinated convex functions on the rectangle from the plane  $\mathbb{R}^2$  was also proved in [6]:

**Theorem 1.** [6] *Suppose that  $f: \Delta \rightarrow \mathbb{R}$  is co-ordinated convex on  $\Delta$ . Then one has the inequalities:*

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ & \leq \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy \right] \\ & \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \\ & \leq \frac{1}{4} \left[ \frac{1}{b-a} \int_a^b f(x, c) dx + \frac{1}{b-a} \int_a^b f(x, d) dx \right. \\ & \quad \left. + \frac{1}{d-c} \int_c^d f(a, y) dy + \frac{1}{d-c} \int_c^d f(b, y) dy \right] \\ & \leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}. \end{aligned} \tag{1.2}$$

*The above inequalities are sharp.*

In a recent article [13], Sarikaya et al. proved some new inequalities that give estimate of the difference between the middle and the rightmost terms in (1.2) for differentiable co-ordinated convex functions on rectangle from the plane  $\mathbb{R}^2$ . Motivated by notion given in [13], in the present article, we prove some new inequalities which give estimate between the middle and the leftmost terms in (1.2) for differentiable co-ordinated convex functions on rectangle from the plane  $\mathbb{R}^2$ .

## 2. Main results

The following lemma is necessary and plays an important role in establishing our main results:

**Lemma 1.** Let  $f: \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  be a partial differentiable mapping on  $\Delta := [a, b] \times [c, d]$  with  $a < b, c < d$ . If  $\frac{\partial^2 f}{\partial s \partial t} \in L(\Delta)$ , then the following identity holds:

$$\begin{aligned} & \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ & - \frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx - \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy \\ & = (b-a)(d-c) \int_0^1 \int_0^1 K(t, s) \frac{\partial^2}{\partial s \partial t} f\left(ta + (1-t)b, sc + (1-s)d\right) ds dt, \end{aligned} \tag{2.1}$$

where

$$K(t, s) = \begin{cases} ts, & (t, s) \in \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right] \\ t(s-1), & (t, s) \in \left[0, \frac{1}{2}\right] \times \left[\frac{1}{2}, 1\right] \\ s(t-1), & (t, s) \in \left[\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right] \\ (t-1)(s-1), & (t, s) \in \left[\frac{1}{2}, 1\right] \times \left[\frac{1}{2}, 1\right] \end{cases}$$

*Proof.* Since

$$\begin{aligned} & (b-a)(d-c) \int_0^1 \int_0^1 K(t, s) \frac{\partial^2}{\partial s \partial t} f\left(ta + (1-t)b, sc + (1-s)d\right) ds dt \\ & = (b-a)(d-c) \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} ts \frac{\partial^2}{\partial s \partial t} f\left(ta + (1-t)b, sc + (1-s)d\right) ds dt \\ & \quad + (b-a)(d-c) \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 t(s-1) \frac{\partial^2}{\partial s \partial t} f\left(ta + (1-t)b, sc + (1-s)d\right) ds dt \\ & \quad + (b-a)(d-c) \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} s(t-1) \frac{\partial^2}{\partial s \partial t} f\left(ta + (1-t)b, sc + (1-s)d\right) ds dt \\ & \quad + (b-a)(d-c) \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 (t-1)(s-1) \frac{\partial^2}{\partial s \partial t} f\left(ta + (1-t)b, sc + (1-s)d\right) ds dt \\ & = I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{2.2}$$

Now by integration by parts, we have

$$\begin{aligned}
 I_1 &= (b-a)(d-c) \int_0^{\frac{1}{2}} t \left[ \int_0^{\frac{1}{2}} s \frac{\partial^2}{\partial s \partial t} f(ta + (1-t)b, sc + (1-s)d) ds \right] dt \\
 &= \frac{1}{4} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{1}{2} \int_0^{\frac{1}{2}} f\left(ta + (1-t)b, \frac{c+d}{2}\right) dt \\
 &\quad - \frac{1}{2} \int_0^{\frac{1}{2}} f\left(\frac{a+b}{2}, sc + (1-s)d\right) ds + \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} f(ta + (1-t)b, sc + (1-s)d) ds dt.
 \end{aligned} \tag{2.3}$$

If we make use of the substitutions  $x = ta + (1-t)b$  and  $y = sc + (1-s)d$ ,  $(t, s) \in [0, 1]^2$ , in (2.3), we observe that

$$\begin{aligned}
 I_1 &= \frac{1}{4} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{1}{2(b-a)} \int_{\frac{a+b}{2}}^b f\left(x, \frac{c+d}{2}\right) dx \\
 &\quad - \frac{1}{2(d-c)} \int_{\frac{c+d}{2}}^d f\left(\frac{a+b}{2}, y\right) dy + \frac{1}{(b-a)(d-c)} \int_{\frac{a+b}{2}}^b \int_{\frac{c+d}{2}}^d f(x, y) dy dx.
 \end{aligned}$$

Similarly, by integration by parts, we also have that

$$\begin{aligned}
 I_2 &= \frac{1}{4} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{1}{2(b-a)} \int_{\frac{a+b}{2}}^b f\left(x, \frac{c+d}{2}\right) dx \\
 &\quad - \frac{1}{2(d-c)} \int_c^{\frac{c+d}{2}} f\left(\frac{a+b}{2}, y\right) dy + \frac{1}{(b-a)(d-c)} \int_{\frac{a+b}{2}}^b \int_c^{\frac{c+d}{2}} f(x, y) dy dx, \\
 I_3 &= \frac{1}{4} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{1}{2(b-a)} \int_a^{\frac{a+b}{2}} f\left(x, \frac{c+d}{2}\right) dx \\
 &\quad - \frac{1}{2(d-c)} \int_{\frac{c+d}{2}}^d f\left(\frac{a+b}{2}, y\right) dy + \frac{1}{(b-a)(d-c)} \int_a^{\frac{a+b}{2}} \int_{\frac{c+d}{2}}^d f(x, y) dy dx
 \end{aligned}$$

and

$$\begin{aligned}
 I_4 = & \frac{1}{4} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{1}{2(b-a)} \int_a^{\frac{a+b}{2}} f\left(x, \frac{c+d}{2}\right) dx \\
 & - \frac{1}{2(d-c)} \int_c^{\frac{c+d}{2}} f\left(\frac{a+b}{2}, y\right) dy + \frac{1}{(b-a)(d-c)} \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} f(x, y) dy dx.
 \end{aligned}$$

Substitution of the  $I_1, I_2, I_3$ , and  $I_4$  in (2.2) gives the desired identity (2.1).

**Theorem 2.** Let  $f: \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  be a partial differentiable mapping on  $\Delta := [a, b] \times [c, d]$  with  $a < b, c < d$ . If  $\left| \frac{\partial^2 f}{\partial s \partial t} \right|$  is convex on the co-ordinates on  $\Delta$ , then the following inequality holds:

$$\begin{aligned}
 & \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - A \right| \\
 & \leq \frac{(b-a)(d-c)}{16} \left[ \frac{\left| \frac{\partial^2 f}{\partial s \partial t}(a, c) \right| + \left| \frac{\partial^2 f}{\partial s \partial t}(a, d) \right| + \left| \frac{\partial^2 f}{\partial s \partial t}(b, c) \right| + \left| \frac{\partial^2 f}{\partial s \partial t}(b, d) \right|}{4} \right], \tag{2.4}
 \end{aligned}$$

where

$$A = \frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy.$$

*Proof.* From Lemma 1, we have

$$\begin{aligned}
 & \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - A \right| \\
 & \leq (b-a)(d-c) \int_0^1 \int_0^1 |K(t, s)| \left| \frac{\partial^2 f}{\partial s \partial t}(ta + (1-t)b, sc + (1-s)d) \right| ds dt \tag{2.5}
 \end{aligned}$$

Since  $\left| \frac{\partial^2 f}{\partial s \partial t} \right|$  is convex on the co-ordinates on  $\Delta$ , we have

$$\begin{aligned}
 & \left| \frac{\partial^2 f}{\partial s \partial t}(ta + (1-t)b, sc + (1-s)d) \right| \leq ts \left| \frac{\partial^2 f}{\partial s \partial t}(a, c) \right| + t(1-s) \left| \frac{\partial^2 f}{\partial s \partial t}(a, d) \right| \\
 & + s(1-t) \left| \frac{\partial^2 f}{\partial s \partial t}(b, c) \right| + (1-t)(1-s) \left| \frac{\partial^2 f}{\partial s \partial t}(b, d) \right|. \tag{2.6}
 \end{aligned}$$

Substitution of (2.6) in (2.5) gives the following inequality:

$$\begin{aligned}
 & \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) dy dx + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - A \right| \\
 & \leq (b-a)(d-c) \int_0^1 \int_0^1 |K(t,s)| \left[ \left| \frac{\partial^2}{\partial s \partial t} f(a,c) \right| ts + \left| \frac{\partial^2}{\partial s \partial t} f(a,d) \right| t(1-s) \right. \\
 & \quad \left. + \left| \frac{\partial^2}{\partial s \partial t} f(b,c) \right| s(1-t) + \left| \frac{\partial^2}{\partial s \partial t} f(b,d) \right| (1-t)(1-s) \right] ds dt = (b-a)(d-c) \\
 & \quad \times \left\{ \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} ts \left[ \left| \frac{\partial^2}{\partial s \partial t} f(a,c) \right| ts + \left| \frac{\partial^2}{\partial s \partial t} f(a,d) \right| t(1-s) + \left| \frac{\partial^2}{\partial s \partial t} f(b,c) \right| s(1-t) \right. \right. \\
 & \quad \left. \left. + \left| \frac{\partial^2}{\partial s \partial t} f(b,d) \right| (1-t)(1-s) \right] ds dt + \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 t(1-s) \left[ \left| \frac{\partial^2}{\partial s \partial t} f(a,c) \right| ts \right. \right. \\
 & \quad \left. \left. + \left| \frac{\partial^2}{\partial s \partial t} f(a,d) \right| t(1-s) + \left| \frac{\partial^2}{\partial s \partial t} f(b,c) \right| s(1-t) + \left| \frac{\partial^2}{\partial s \partial t} f(b,d) \right| (1-t)(1-s) \right] ds dt \right. \\
 & \quad \left. + \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} s(1-t) \left[ \left| \frac{\partial^2}{\partial s \partial t} f(a,c) \right| ts + \left| \frac{\partial^2}{\partial s \partial t} f(a,d) \right| t(1-s) + \left| \frac{\partial^2}{\partial s \partial t} f(b,c) \right| s(1-t) \right. \right. \\
 & \quad \left. \left. + \left| \frac{\partial^2}{\partial s \partial t} f(b,d) \right| (1-t)(1-s) \right] ds dt + \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 (1-t)(1-s) \left[ \left| \frac{\partial^2}{\partial s \partial t} f(a,c) \right| ts + \right. \right. \\
 & \quad \left. \left. \left| \frac{\partial^2}{\partial s \partial t} f(a,d) \right| t(1-s) + \left| \frac{\partial^2}{\partial s \partial t} f(b,c) \right| s(1-t) + \left| \frac{\partial^2}{\partial s \partial t} f(b,d) \right| (1-t)(1-s) \right] ds dt \right\}
 \end{aligned} \tag{2.7}$$

Evaluating each integral in (2.7) and simplifying, we get (2.4). Hence the proof of the theorem is complete.

**Theorem 3.** Let  $f: \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  be a partial differentiable mapping on  $\Delta = [a, b] \times [c, d]$  with  $a < b, c < d$ . If  $\left| \frac{\partial^2 f}{\partial s \partial t} \right|^q$  is convex on the co-ordinates on  $\Delta$  and  $p, q > 1$ ,

$\frac{1}{p} + \frac{1}{q} = 1$ , then the following inequality holds:

$$\begin{aligned}
 & \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) dy dx + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - A \right| \\
 & \leq \frac{(b-a)(d-c)}{2} \left[ \frac{\left| \frac{\partial^2}{\partial s \partial t} f(a,c) \right|^q + \left| \frac{\partial^2}{\partial s \partial t} f(a,d) \right|^q + \left| \frac{\partial^2}{\partial s \partial t} f(b,c) \right|^q + \left| \frac{\partial^2}{\partial s \partial t} f(b,d) \right|^q}{4} \right]^{\frac{1}{q}},
 \end{aligned} \tag{2.8}$$

where  $A$  is as given in Theorem 2.

*Proof.* From Lemma 1, we have

$$\begin{aligned} & \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - A \right| \\ & \leq (b-a)(d-c) \int_0^1 \int_0^1 |K(t, s)| \left| \frac{\partial^2}{\partial s \partial t} f(ta + (1-t)b, sc + (1-s)d) \right| ds dt. \end{aligned} \tag{2.9}$$

Now using the well-known Hölder inequality for double integrals, we obtain

$$\begin{aligned} & \int_0^1 \int_0^1 |K(t, s)| \left| \frac{\partial^2}{\partial s \partial t} f(ta + (1-t)b, sc + (1-s)d) \right| ds dt \\ & \leq \left( \int_0^1 \int_0^1 |K(t, s)|^p ds dt \right)^{\frac{1}{p}} \left( \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(ta + (1-t)b, sc + (1-s)d) \right|^q ds dt \right)^{\frac{1}{q}}. \end{aligned} \tag{2.10}$$

Since  $\left| \frac{\partial^2 f}{\partial s \partial t} \right|^q$  is convex on the co-ordinates on  $\Delta$ , we have

$$\begin{aligned} & \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial s \partial t} f(ta + (1-t)b, sc + (1-s)d) \right|^q ds dt \\ & \leq \int_0^1 \int_0^1 \left[ ts \left| \frac{\partial^2}{\partial s \partial t} f(a, c) \right|^q + t(1-s) \left| \frac{\partial^2}{\partial s \partial t} f(a, d) \right|^q \right. \\ & \quad \left. + s(1-t) \left| \frac{\partial^2}{\partial s \partial t} f(b, c) \right|^q + (1-t)(1-s) \left| \frac{\partial^2}{\partial s \partial t} f(b, d) \right|^q \right] ds dt \\ & = \frac{\left| \frac{\partial^2}{\partial s \partial t} f(a, c) \right|^q + \left| \frac{\partial^2}{\partial s \partial t} f(a, d) \right|^q + \left| \frac{\partial^2}{\partial s \partial t} f(b, c) \right|^q + \left| \frac{\partial^2}{\partial s \partial t} f(b, d) \right|^q}{4}. \end{aligned} \tag{2.11}$$

Also, we notice that

$$\begin{aligned} \int_0^1 \int_0^1 |K(t, s)|^p ds dt &= \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} t^p s^p ds dt + \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 t^p (1-s)^p ds dt \\ & \quad + \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} s^p (1-t)^p ds dt + \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 (1-t)^p (1-s)^p ds dt \\ & = \frac{4}{(p+1)^2} \left(\frac{1}{2}\right)^{2(p+1)}. \end{aligned} \tag{2.12}$$

Using (2.11) and (2.12) in (2.10), we obtain

$$\int_0^1 \int_0^1 |K(t,s)| \left| \frac{\partial^2}{\partial s \partial t} f(ta + (1-t)b, sc + (1-s)d) \right| ds dt$$

$$\leq \frac{1}{4(p+1)^p} \left[ \frac{\left| \frac{\partial^2}{\partial s \partial t} f(a,c) \right|^q + \left| \frac{\partial^2}{\partial s \partial t} f(a,d) \right|^q + \left| \frac{\partial^2}{\partial s \partial t} f(b,c) \right|^q + \left| \frac{\partial^2}{\partial s \partial t} f(b,d) \right|^q}{4} \right]^{\frac{1}{q}}.$$

Utilizing the last inequality in (2.9) gives us (2.8). This completes the proof of the theorem.

Now we state our next result in:

**Theorem 4.** Let  $f: \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  be a partial differentiable mapping on  $\Delta = [a, b] \times [c, d]$  with  $a < b, c < d$ . If  $\left| \frac{\partial^2 f}{\partial s \partial t} \right|^q$  is convex on the co-ordinates on  $\Delta$  and  $q \geq 1$ , then the following inequality holds:

$$\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) dy dx + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - A \right|$$

$$\leq \frac{(b-a)(d-c)}{16} \left[ \frac{\left| \frac{\partial^2}{\partial s \partial t} f(a,c) \right|^q + \left| \frac{\partial^2}{\partial s \partial t} f(a,d) \right|^q + \left| \frac{\partial^2}{\partial s \partial t} f(b,c) \right|^q + \left| \frac{\partial^2}{\partial s \partial t} f(b,d) \right|^q}{4} \right]^{\frac{1}{q}}, \tag{2.13}$$

where  $A$  is as given in Theorem 2.

*Proof.* By using Lemma 1, we have that the following inequality:

$$\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_a^d f(x,y) dy dx + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - A \right|$$

$$\leq (b-a)(d-c) \int_0^1 \int_0^1 |K(t,s)| \left| \frac{\partial^2}{\partial s \partial t} f(ta + (1-t)b, sc + (1-s)d) \right| ds dt. \tag{2.14}$$

By the power mean inequality, we have

$$\int_0^1 \int_0^1 |K(t,s)| \left| \frac{\partial^2}{\partial s \partial t} f(ta + (1-t)b, sc + (1-s)d) \right| ds dt$$

$$\leq \left( \int_0^1 \int_0^1 |K(t,s)| ds dt \right)^{1-\frac{1}{q}}$$

$$\times \left( \int_0^1 \int_0^1 |K(t,s)| \left| \frac{\partial^2}{\partial s \partial t} f(ta + (1-t)b, sc + (1-s)d) \right|^q ds dt \right)^{\frac{1}{q}} \tag{2.15}$$

$$= \left( \frac{1}{16} \right)^{1-\frac{1}{q}} \left( \int_0^1 \int_0^1 |K(t,s)| \left| \frac{\partial^2}{\partial s \partial t} f(ta + (1-t)b, sc + (1-s)d) \right|^q ds dt \right)^{\frac{1}{q}}.$$



Using the fact that  $\left| \frac{\partial^2 f}{\partial s \partial t} \right|^q$  is convex on the co-ordinates on  $\Delta$ , we get

$$\begin{aligned} & \left| \frac{\partial^2}{\partial s \partial t} f(ta + (1-t)b, sc + (1-s)d) \right|^q \\ &= ts \left| \frac{\partial^2}{\partial s \partial t} f(a, c) \right|^q + t(1-s) \left| \frac{\partial^2}{\partial s \partial t} f(a, d) \right|^q + s(1-t) \left| \frac{\partial^2}{\partial s \partial t} f(b, c) \right|^q \\ &+ (1-t)(1-s) \left| \frac{\partial^2}{\partial s \partial t} f(b, d) \right|^q \end{aligned}$$

and hence, we obtain

$$\begin{aligned} & \int_0^1 \int_0^1 |K(t, s)| \left| \frac{\partial^2}{\partial s \partial t} f(ta + (1-t)b, sc + (1-s)d) \right|^q ds dt \\ & \leq \int_0^1 \int_0^1 |K(t, s)| \left[ ts \left| \frac{\partial^2}{\partial s \partial t} f(a, c) \right|^q + t(1-s) \left| \frac{\partial^2}{\partial s \partial t} f(a, d) \right|^q \right. \\ & \quad \left. + s(1-t) \left| \frac{\partial^2}{\partial s \partial t} f(b, c) \right|^q + (1-t)(1-s) \left| \frac{\partial^2}{\partial s \partial t} f(b, d) \right|^q \right] ds dt \\ & = \frac{1}{64} \left[ \left| \frac{\partial^2}{\partial s \partial t} f(a, c) \right|^q + \left| \frac{\partial^2}{\partial s \partial t} f(a, d) \right|^q + \left| \frac{\partial^2}{\partial s \partial t} f(b, c) \right|^q + \left| \frac{\partial^2}{\partial s \partial t} f(b, d) \right|^q \right]. \end{aligned}$$

Therefore (2.15) becomes

$$\begin{aligned} & \int_0^1 \int_0^1 |K(t, s)| \left| \frac{\partial^2}{\partial s \partial t} f(ta + (1-t)b, sc + (1-s)d) \right| ds dt \\ & \leq \frac{1}{16} \left[ \frac{\left| \frac{\partial^2}{\partial s \partial t} f(a, c) \right|^q + \left| \frac{\partial^2}{\partial s \partial t} f(a, d) \right|^q + \left| \frac{\partial^2}{\partial s \partial t} f(b, c) \right|^q + \left| \frac{\partial^2}{\partial s \partial t} f(b, d) \right|^q}{4} \right]^{\frac{1}{q}} \quad (2.16) \end{aligned}$$

Substitution of (2.16) in (2.14), we obtain (2.13). Hence the proof is complete.

**Remark 1.** Since  $2^p > p + 1$  if  $p > 1$  and accordingly

$$\frac{1}{4} < \frac{1}{2(p+1)\frac{1}{p}}$$

and hence we have that the following inequality:

$$\frac{1}{16} < \frac{1}{4} \cdot \frac{1}{4} < \frac{1}{2(p+1)\frac{1}{p}} \cdot \frac{1}{2(p+1)\frac{1}{p}} = \frac{1}{4(p+1)\frac{2}{p}},$$

and as a consequence we get an improvement of the constant in Theorem 3.

Following theorem is about concave functions on the co-ordinates on  $\Delta$ :

**Theorem 5.** Let  $f: \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  be a partial differentiable mapping on  $\Delta: = [a, b] \times [c, d]$  with  $a < b, c < d$ . If  $\left| \frac{\partial^2 f}{\partial s \partial t} \right|^q$  is concave on the co-ordinates on  $\Delta$  and  $q \geq 1$ , then we have the inequality:

$$\begin{aligned} & \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - A \right| \\ & \leq \frac{(b-a)(d-c)}{64} \left[ \left| \frac{\partial^2 f}{\partial s \partial t} \left(\frac{a+2b}{3}, \frac{c+2d}{3}\right) \right| + \left| \frac{\partial^2 f}{\partial s \partial t} \left(\frac{a+2b}{3}, \frac{2c+d}{3}\right) \right| \right. \\ & \quad \left. \left| \frac{\partial^2 f}{\partial s \partial t} \left(\frac{2a+b}{3}, \frac{c+2d}{3}\right) \right| + \left| \frac{\partial^2 f}{\partial s \partial t} \left(\frac{2a+b}{3}, \frac{2c+d}{3}\right) \right| \right], \end{aligned} \quad (2.17)$$

where  $A$  is as defined in Theorem 2.

*Proof.* By the concavity of  $\left| \frac{\partial^2 f}{\partial s \partial t} \right|^q$  on the co-ordinates on  $\Delta$  and power mean inequality, we note that the following inequality holds:

$$\begin{aligned} \left| \frac{\partial^2 f}{\partial s \partial t} (\lambda x + (1-\lambda)y, v) \right|^q & \geq \lambda \left| \frac{\partial^2 f}{\partial s \partial t} (x, v) \right|^q + (1-\lambda) \left| \frac{\partial^2 f}{\partial s \partial t} (y, v) \right|^q \\ & \geq \left( \lambda \left| \frac{\partial^2 f}{\partial s \partial t} (x, v) \right| + (1-\lambda) \left| \frac{\partial^2 f}{\partial s \partial t} (y, v) \right| \right)^q, \end{aligned}$$

for all  $x, y \in [a, b], \lambda \in [0, 1]$  and for fixed  $v \in [c, d]$ .

Hence,

$$\left| \frac{\partial^2 f}{\partial s \partial t} (\lambda x + (1-\lambda)y, v) \right| \geq \lambda \left| \frac{\partial^2 f}{\partial s \partial t} (x, v) \right| + (1-\lambda) \left| \frac{\partial^2 f}{\partial s \partial t} (y, v) \right|,$$

for all  $x, y \in [a, b], \lambda \in [0, 1]$  and for fixed  $v \in [c, d]$ .

Similarly, we can show that

$$\left| \frac{\partial^2 f}{\partial s \partial t} (u, \lambda z + (1-\lambda)w) \right| \geq \lambda \left| \frac{\partial^2 f}{\partial s \partial t} (u, z) \right| + (1-\lambda) \left| \frac{\partial^2 f}{\partial s \partial t} (u, w) \right|,$$

for all  $z, w \in [c, d], \lambda \in [0, 1]$  and for fixed  $u \in [a, d]$ , thus  $\left| \frac{\partial^2 f}{\partial s \partial t} \right|^q$  is concave on the co-ordinates on  $\Delta$ .

It is clear from Lemma 1 that

$$\begin{aligned}
 & \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, \gamma) d\gamma dx + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - A \right| \\
 & \leq (b-a)(d-c) \int_0^1 \int_0^1 |K(t, s)| \left| \frac{\partial^2}{\partial s \partial t} f(ta + (1-t)b, sc + (1-s)d) \right| ds dt \\
 & = (b-a)(d-c) \left[ \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} st \left| \frac{\partial^2}{\partial s \partial t} f(ta + (1-t)b, sc + (1-s)d) \right| ds dt \right. \\
 & \quad + \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 t(1-s) \left| \frac{\partial^2}{\partial s \partial t} f(ta + (1-t)b, sc + (1-s)d) \right| ds dt \\
 & \quad + \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} s(1-t) \left| \frac{\partial^2}{\partial s \partial t} f(ta + (1-t)b, sc + (1-s)d) \right| ds dt \\
 & \quad \left. + \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 (1-t)(1-s) \left| \frac{\partial^2}{\partial s \partial t} f(ta + (1-t)b, sc + (1-s)d) \right| ds dt \right]. \tag{2.18}
 \end{aligned}$$

Since  $\left| \frac{\partial^2 f}{\partial s \partial t} \right|$  is concave on the co-ordinates, we have, by Jensen's inequality for integrals, that:

$$\begin{aligned}
 & \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} st \left| \frac{\partial^2}{\partial s \partial t} f(ta + (1-t)b, sc + (1-s)d) \right| ds dt \\
 & = \int_0^{\frac{1}{2}} t \left[ \int_0^{\frac{1}{2}} s \left| \frac{\partial^2}{\partial s \partial t} f(ta + (1-t)b, sc + (1-s)d) \right| ds \right] dt \\
 & \leq \int_0^{\frac{1}{2}} t \left( \int_0^{\frac{1}{2}} s ds \right) \left| \frac{\partial^2}{\partial s \partial t} f \left( ta + (1-t)b, \frac{\int_0^{\frac{1}{2}} s(sc + (1-s)d) ds}{\int_0^{\frac{1}{2}} s ds} \right) \right| dt \\
 & = \frac{1}{8} \int_0^{\frac{1}{2}} t \left| \frac{\partial^2}{\partial s \partial t} f \left( ta + (1-t)b, \frac{c+2d}{3} \right) \right| dt \\
 & \leq \frac{1}{8} \left( \int_0^{\frac{1}{2}} t dt \right) \left| \frac{\partial^2}{\partial s \partial t} f \left( \frac{\int_0^{\frac{1}{2}} t(ta + (1-t)b) dt}{\int_0^{\frac{1}{2}} t dt}, \frac{c+2d}{3} \right) \right| \\
 & = \frac{1}{64} \left| \frac{\partial^2}{\partial s \partial t} f \left( \frac{a+2b}{3}, \frac{c+2d}{3} \right) \right|. \tag{2.19}
 \end{aligned}$$

In a similar way, we also have that

$$\int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 t(1-s) \left| \frac{\partial^2}{\partial s \partial t} f(ta + (1-t)b, sc + (1-s)d) \right| ds dt \leq \frac{1}{64} \left| \frac{\partial^2}{\partial s \partial t} f\left(\frac{a+2b}{3}, \frac{2c+d}{3}\right) \right|, \quad (2.20)$$

$$\int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} s(1-t) \left| \frac{\partial^2}{\partial s \partial t} f(ta + (1-t)b, sc + (1-s)d) \right| ds dt \leq \frac{1}{64} \left| \frac{\partial^2}{\partial s \partial t} f\left(\frac{2a+b}{3}, \frac{c+2d}{3}\right) \right| \quad (2.21)$$

and

$$\int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 (1-t)(1-s) \left| \frac{\partial^2}{\partial s \partial t} f(ta + (1-t)b, sc + (1-s)d) \right| ds dt \leq \frac{1}{64} \left| \frac{\partial^2}{\partial s \partial t} f\left(\frac{2a+b}{3}, \frac{c+2d}{3}\right) \right|. \quad (2.22)$$

By making use of (2.19)-(2.22) in (2.18), we get the desired result. This completes the proof.

#### Acknowledgements

This article is in final form and no version of it will be submitted for publication elsewhere.

#### Author details

<sup>1</sup>College of Science, Department of Mathematics, University of Hail, Hail-2440, Saudi Arabia <sup>2</sup>School of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia

#### Authors' contributions

MAL and SSD carried out the design of the study and performed the analysis. Both of the authors read and approved the final version of the manuscript.

#### Competing interests

The authors declare that they have no competing interests.

Received: 13 October 2011 Accepted: 15 February 2012 Published: 15 February 2012

#### References

1. Pečarić, JE, Proschan, F, Tong, YL: *Convex Functions, Partial Ordering and Statistical Applications*. Academic Press, New York (1991)
2. Alomari, M, Darus, M, Dragomir, SS: Inequalities of Hermite-Hadamard's type for functions whose derivatives absolute values are quasi-convex. *RGMI Res Rep Collect.* **12**(suppl. 14) (2009)
3. Dragomir, SS, Agarwal, RP: Two inequalities for differentiable mappings and applications to special means of real numbers and to Trapezoidal formula. *Appl Math Lett.* **11**(5):91-95 (1998). doi:10.1016/S0893-9659(98)00086-X
4. Kirmaci, US: Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula. *Appl Math Comput.* **147**, 137-146 (2004). doi:10.1016/S0096-3003(02)00657-4
5. Pearce, CME, Pečarić, JE: Inequalities for differentiable mappings with applications to special means and quadrature formula. *Appl Math Lett.* **13**, 51-55 (2000)

6. Dragomir, SS: On Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane. *Taiwanese J Math.* **4**, 775–788 (2001)
7. Dragomir, SS, Pearce, CEM: Selected Topics on Hermite-Hadamard Inequalities and Applications. RGMIA Monographs, Victoria University (2000) Online: [http://www.staff.vu.edu.au/RGMIA/monographs/hermite\\_hadamard.html](http://www.staff.vu.edu.au/RGMIA/monographs/hermite_hadamard.html)
8. Latif, MA, Alomari, M: Hadamard-type inequalities for product two convex functions on the co-ordinates. *Int Math Forum.* **4**(47):2327–2338 (2009)
9. Alomari, M, Darus, M: Co-ordinated  $s$ -convex function in the first sense with some Hadamard-type inequalities. *Int J Contemp Math Sci.* **3**(32):1557–1567 (2008)
10. Alomari, M, Darus, M: On the Hadamard's inequality for log-convex functions on the coordinates. *J Inequal Appl* **13** (2009). (Article ID 283147)
11. Latif, MA, Alomari, M: On the Hadamard-type inequalities for  $h$ -convex functions on the co-ordinates. *Int J Math Anal.* **3**(33):1645–1656 (2009)
12. Özdemir, ME, Set, E, Sarikaya, MZ: New some Hadamard's type inequalities for coordinated  $m$ -convex and  $(\alpha, m)$ -convex functions. *RGMIA Res Rep Collect* **13** (2010). Supplement, (Article 4)
13. Sarikaya, MZ, Set, E, Özdemir, ME, Dragomir, SS: Some new Hadamard's type inequalities for co-ordinated convex functions. arXiv:1005.0700v1 [math.CA]

doi:10.1186/1029-242X-2012-28

**Cite this article as:** Latif and Dragomir: On some new inequalities for differentiable co-ordinated convex functions. *Journal of Inequalities and Applications* 2012 **2012**:28.

**Submit your manuscript to a SpringerOpen<sup>®</sup> journal and benefit from:**

- ▶ Convenient online submission
- ▶ Rigorous peer review
- ▶ Immediate publication on acceptance
- ▶ Open access: articles freely available online
- ▶ High visibility within the field
- ▶ Retaining the copyright to your article

---

Submit your next manuscript at ▶ [springeropen.com](http://springeropen.com)

---