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Second-order duality for a nondifferentiable minimax fractional programming under generalized α-univexity

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Abstract
In this paper, we concentrate our study to derive appropriate duality theorems for two types of second-order dual models of a nondifferentiable minimax fractional programming problem involving second-order α-univex functions. Examples to show the existence of α-univex functions have also been illustrated. Several known results including many recent works are obtained as special cases.
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1 Introduction

After Schmitendorf [1], who derived necessary and sufficient optimality conditions for static minimax problems, much attention has been paid to optimality conditions and duality theorems for minimax fractional programming problems [2–17]. For the theory, algorithms, and applications of some minimax problems, the reader is referred to [18].

In this paper, we consider the following nondifferentiable minimax fractional programming problem:

$$\begin{aligned} &\text{Minimize} \quad \psi(x) = \sup_{y\in Y} \frac{f(x,y) + (x^TBx)^{1/2}}{h(x,y) - (x^TDx)^{1/2}} \\ &\text{subject to} \quad g(x) \leq 0, \end{aligned} \tag{P}$$

where Y is a compact subset of R^l, $f(\cdot,\cdot): R^n \times R^l \to R$, $h(\cdot,\cdot): R^n \times R^l \to R$ are twice continuously differentiable on $R^n \times R^l$ and $g(\cdot): R^n \to R^m$ is twice continuously differentiable on R^n, B, and D are a $n \times n$ positive semidefinite matrix, $f(x,y) + (x^TBx)^{1/2} \geq 0$, and $h(x,y) - (x^TDx)^{1/2} > 0$ for each $(x,y) \in \mathfrak{J} \times Y$, where $\mathfrak{J} = \{x \in R^n : g(x) \leq 0\}$.

Motivated by [7, 14, 15], Yang and Hou [17] formulated a dual model for fractional minimax programming problem and proved duality theorems under generalized convex functions. Ahmad and Husain [5] extended this model to nondifferentiable and obtained duality relations involving (F,α,ρ,d)-pseudoconvex functions. Jayswal [11] studied duality theorems for another two duals of (P) under α-univex functions. Recently, Ahmad *et al.* [4] derived the sufficient optimality condition for (P) and established duality relations for its dual problem under B-(p, r)-invexity assumptions. The papers [2, 4–7, 11–15, 17] involved the study of first-order duality for minimax fractional programming problems.

The concept of second-order duality in nonlinear programming problems was first introduced by Mangasarian [19]. One significant practical application of second-order dual over first-order is that it may provide tighter bounds for the value of objective function because there are more parameters involved. Hanson [20] has shown the other advantage of second-order duality by citing an example, that is, if a feasible point of the primal is given and first-order duality conditions do not apply (infeasible), then we may use second-order duality to provide a lower bound for the value of primal problem.

Recently, several researchers [3, 8–10, 16] considered second-order dual for minimax fractional programming problems. Husain *et al.* [8] first formulated second-order dual models for a minimax fractional programming problem and established duality relations involving η-bonvex functions. This work was later on generalized in [10] by introducing an additional vector r to the dual models, and in Sharma and Gulati [16] by proving the results under second-order generalized α-type I univex functions. The work cited in [3, 8, 10, 16] involves differentiable minimax fractional programming problems. Recently, Hu *et al.* [9] proved appropriate duality theorems for a second-order dual model of (P) under η-pseudobonvexity/η-quasibonvexity assumptions. In this paper, we formulate two types of second-order dual models for (P) and then derive weak, strong, and strict converse duality theorems under generalized α-univexity assumptions. Further, examples have been illustrated to show the existence of second-order α-univex functions. Our study extends some of the known results of the literature [5, 6, 11, 12, 14].

2 Notations and preliminaries

For each $(x, y) \in R^n \times R^l$ and $M = \{1, 2, \dots, m\}$, we define

$$J(x) = \{j \in M : g_j(x) = 0\},$$

$$Y(x) = \left\{ y \in Y : \frac{f(x,y) + (x^T Bx)^{1/2}}{h(x,y) - (x^T Dx)^{1/2}} = \sup_{b \in Y} \frac{f(x,b) + (x^T Bx)^{1/2}}{h(x,b) - (x^T Dx)^{1/2}} \right\},$$

$$K(x) = \Big\{ (s, t, \tilde{y}) \in N \times R^s_+ \times R^{ls} : 1 \le s \le n+1, t = (t_1, t_2, \dots, t_s) \in R^s_+,$$
$$\sum_{i=1}^{s} t_i = 1, \tilde{y} = (\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_s), \tilde{y}_i \in Y(x), i = 1, 2, \dots, s \Big\}.$$

Definition 2.1 Let $\zeta : X \to R$ $(X \subseteq R^n)$ be a twice differentiable function. Then ζ is said to be second-order α-univex at $u \in X$, if there exist $\eta : X \times X \to R^n$, $b_0 : X \times X \to R_+$, $\phi_0 : R \to R$, and $\alpha : X \times X \to R_+ \backslash \{0\}$ such that for all $x \in X$ and $p \in R^n$, we have

$$b_0 \phi_0 \left[\zeta(x) - \zeta(u) + \frac{1}{2} p^T \nabla^2 \zeta(u) p \right]$$
$$\ge \alpha(x,u) \eta^T(x,u) \big[\nabla \zeta(u) + \nabla^2 \zeta(u) p \big].$$

Example 2.1 Let $\zeta : X \to R$ be defined as $\zeta(x) = e^x + \sin^2 x + x^2$, where $X = (-1, \infty)$. Also, let $\phi_0(t) = t + 18$, $b_0(x,u) = u + 1$, $\alpha(x,u) = \frac{u^2+2}{x+1}$ and $\eta(x,u) = x + u$. The function ζ is second-order α-univex at $u = 1$, since

$$\begin{aligned}
&b_0\phi_0\left[\zeta(x) - \zeta(u) + \frac{1}{2}p^T\nabla^2\zeta(u)p\right] - \alpha(x,u)\eta^T(x,u)\left[\nabla\zeta(u) + \nabla^2\zeta(u)p\right] \\
&\quad = 2\left(e^x + \sin^2 x + x^2\right) + 1.521 + 3.886(p - 1.5)^2 \\
&\quad \geq 0 \quad \text{for all } x \in X \text{ and } p \in R.
\end{aligned}$$

But every α-univex function need not be invex. To show this, consider the following example.

Example 2.2 Let $\Omega : X = (0,\infty) \to R$ be defined as $\Omega(x) = -x^2$. Let $\phi_0(t) = -t$, $b_0(x,u) = \frac{1}{u}$, $\alpha(x,u) = 2u$, and $\eta(x,u) = \frac{1}{2u}$. Then we have

$$\begin{aligned}
&b_0\phi_0\left[\Omega(x) - \Omega(u) + \frac{1}{2}p^T\nabla^2\Omega(u)p\right] - \alpha(x,u)\eta^T(x,u)\left[\nabla\Omega(u) + \nabla^2\Omega(u)p\right] \\
&\quad = \frac{1}{u}\left[x^2 + (p+u)^2\right] \geq 0 \quad \text{for all } x, u \in X \text{ and } p \in R.
\end{aligned}$$

Hence, the function Ω is second-order α-univex but not invex, since for $x = 3$, $u = 2$, and $p = 1$, we obtain

$$\Omega(x) - \Omega(u) + \frac{1}{2}p^T\nabla^2\Omega(u)p - \eta^T(x,u)\left[\nabla\Omega(u) + \nabla^2\Omega(u)p\right] = -4.5 < 0.$$

Lemma 2.1 (Generalized Schwartz inequality) *Let B be a positive semidefinite matrix of order n. Then, for all $x, w \in R^n$,*

$$x^TBw \leq \left(x^TBx\right)^{1/2}\left(w^TBw\right)^{1/2}.$$

The equality holds if $Bx = \lambda Bw$ for some $\lambda \geq 0$.

Following Theorem 2.1 ([13], Theorem 3.1) will be required to prove the strong duality theorem.

Theorem 2.1 (Necessary condition) *If x^* is an optimal solution of problem (P) satisfying $x^{*T}Bx^* > 0$, $x^{*T}Dx^* > 0$, and $\nabla g_j(x^*)$, $j \in J(x^*)$ are linearly independent, then there exist $(s^*, t^*, \tilde{y}) \in K(x^*)$, $k_0 \in R_+$, $w, v \in R^n$ and $\mu^* \in R^m_+$ such that*

$$\sum_{i=1}^{s^*} t_i^*\left\{\nabla f\left(x^*, \tilde{y}_i\right) + Bw - k_0\left(\nabla h\left(x^*, \tilde{y}_i\right) - Dv\right)\right\} + \sum_{j=1}^{m} \mu_j^*\nabla g_j\left(x^*\right) = 0, \tag{2.1}$$

$$f\left(x^*, \tilde{y}_i\right) + \left(x^{*T}Bx^*\right)^{1/2} - k_0\left(h\left(x^*, \tilde{y}_i\right) - \left(x^{*T}Dx^*\right)^{1/2}\right) = 0, \quad i = 1, 2, \ldots, s^*, \tag{2.2}$$

$$\sum_{j=1}^{m} \mu_j^* g_j\left(x^*\right) = 0, \tag{2.3}$$

$$t_i^* \geq 0 \ \left(i = 1, 2, \ldots, s^*\right), \qquad \sum_{i=1}^{s^*} t_i^* = 1, \tag{2.4}$$

$$w^TBw \leq 1, \qquad v^TDv \leq 1, \qquad \left(x^{*T}Bx^*\right)^{1/2} = x^{*T}Bw, \qquad \left(x^{*T}Dx^*\right)^{1/2} = x^{*T}Dv. \tag{2.5}$$

In the above theorem, both matrices B and D are positive semidefinite at x^*. If either $x^{*T}Bx^*$ or $x^{*T}Dx^*$ is zero, then the functions involved in the objective of problem (P) are not differentiable. To derive necessary conditions under this situation, for $(s^*, t^*, \tilde{y}) \in K(x^*)$, we define

$$Z_{\tilde{y}}(x^*) = \{z \in R^n : z^T \nabla g_j(x^*) \leq 0, j \in J(x^*), \text{ with any one of the next conditions (i)-(iii) holds}\}.$$

(i) $x^{*T}Bx^* > 0, \quad x^{*T}Dx^* = 0$

$$\Rightarrow \quad z^T\left(\sum_{i=1}^{s^*} t_i^*\left\{\nabla f(x^*, \tilde{y}_i) + \frac{Bx^*}{(x^{*T}Bx^*)^{1/2}} - k_0 \nabla h(x^*, \tilde{y}_i)\right\}\right) + \left(z^T(k_0^2 D)z\right)^{1/2} < 0,$$

(ii) $x^{*T}Bx^* = 0, \quad x^{*T}Dx^* > 0$

$$\Rightarrow \quad z^T\left(\sum_{i=1}^{s^*} t_i^*\left\{\nabla f(x^*, \tilde{y}_i) - k_0\left(\nabla h(x^*, \tilde{y}_i) - \frac{Dx^*}{(x^{*T}Dx^*)^{1/2}}\right)\right\}\right) + \left(z^T Bz\right)^{1/2} < 0,$$

(iii) $x^{*T}Bx^* = 0, \quad x^{*T}Dx^* = 0$

$$\Rightarrow \quad z^T\left(\sum_{i=1}^{s^*} t_i^*\left\{\nabla f(x^*, \tilde{y}_i) - k_0 \nabla h(x^*, \tilde{y}_i)\right\}\right) + \left(z^T(k_0^2 D)z\right)^{1/2} + \left(z^T Bz\right)^{1/2} < 0.$$

If in addition, we insert the condition $Z_{\tilde{y}}(x^*) = \phi$, then the result of Theorem 2.1 still holds.

For the sake of convenience, let

$$\psi_1(\cdot) = \xi_1(\cdot) + \sum_{j=1}^{m} \mu_j\left(g_j(\cdot) - g_j(z)\right) \tag{2.6}$$

and

$$\psi_2(\cdot) = \left[\sum_{i=1}^{s} t_i\left(h(z, \tilde{y}_i) - z^T Dv\right)\right]\left[\sum_{i=1}^{s} t_i\left(f(\cdot, \tilde{y}_i) + (\cdot)^T Bw\right) + \sum_{j=1}^{m} \mu_j g_j(\cdot)\right] - \left[\sum_{i=1}^{s} t_i\left(f(z, \tilde{y}_i) + z^T Bw\right) + \sum_{j=1}^{m} \mu_j g_j(z)\right]\left[\sum_{i=1}^{s} t_i\left(h(\cdot, \tilde{y}_i) - (\cdot)^T Dv\right)\right],$$

where

$$\xi_1(\cdot) = \sum_{i=1}^{s} t_i\left[\left(h(z, \tilde{y}_i) - z^T Dv\right)\left(f(\cdot, \tilde{y}_i) + (\cdot)^T Bw\right) - \left(f(z, \tilde{y}_i) + z^T Bw\right)\left(h(\cdot, \tilde{y}_i) - (\cdot)^T Dv\right)\right].$$

3 Model I

In this section, we consider the following second-order dual problem for (P):

$$\max_{(s,t,\widetilde{y})\in K(z)} \sup_{(z,\mu,w,v,p)\in H_1(s,t,\widetilde{y})} F(z), \tag{DM1}$$

where $F(z) = \sup_{y\in Y} \frac{f(z,y)+(z^TBz)^{1/2}}{h(z,y)-(z^TDz)^{1/2}}$ and $H_1(s,t,\widetilde{y})$ denotes the set of all $(z,\mu,w,v,p) \in R^n \times R^m_+ \times R^n \times R^n \times R^n$ satisfying

$$\nabla\psi_1(z) + \nabla^2\psi_1(z)p = 0, \tag{3.1}$$

$$\sum_{j=1}^{m} \mu_j g_j(z) - \frac{1}{2}p^T\nabla^2\psi_1(z)p \geq 0, \tag{3.2}$$

$$\begin{aligned} &w^TBw \leq 1, \qquad v^TDv \leq 1, \\ &\left(z^TBz\right)^{1/2} = z^TBw, \qquad \left(z^TDz\right)^{1/2} = z^TDv. \end{aligned} \tag{3.3}$$

If the set $H_1(s,t,\widetilde{y}) = \phi$, we define the supremum of $F(z)$ over $H_1(s,t,\widetilde{y})$ equal to $-\infty$.

Remark 3.1 If $p = 0$, then using (3.3), the above dual model reduces to the problems studied in [6, 11, 12]. Further, if B and D are zero matrices of order n, then (DM1) becomes the dual model considered in [14].

Next, we establish duality relations between primal (P) and dual (DM1).

Theorem 3.1 (Weak duality) *Let x and $(z,\mu,w,v,s,t,\widetilde{y},p)$ are feasible solutions of (P) and (DM1), respectively. Assume that*

(i) $\psi_1(\cdot)$ is second-order α-univex at z,

(ii) $\phi_0(a) \geq 0 \Rightarrow a \geq 0$ and $b_0(x,z) > 0$.

Then

$$\sup_{\widetilde{y}\in Y} \frac{f(x,\widetilde{y}) + (x^TBx)^{1/2}}{h(x,\widetilde{y}) - (x^TDx)^{1/2}} \geq F(z).$$

Proof Assume on contrary to the result that

$$\sup_{\widetilde{y}\in Y} \frac{f(x,\widetilde{y}) + (x^TBx)^{1/2}}{h(x,\widetilde{y}) - (x^TDx)^{1/2}} < F(z). \tag{3.4}$$

Since $\widetilde{y}_i \in Y(z)$, $i = 1,2,\dots,s$, we have

$$F(z) = \frac{f(z,\widetilde{y}_i) + (z^TBz)^{1/2}}{h(z,\widetilde{y}_i) - (z^TDz)^{1/2}}. \tag{3.5}$$

From (3.4) and (3.5), for $i = 1,2,\dots,s$, we get

$$\frac{f(x,\widetilde{y}_i) + (x^TBx)^{1/2}}{h(x,\widetilde{y}_i) - (x^TDx)^{1/2}} \leq \sup_{\widetilde{y}\in Y} \frac{f(x,\widetilde{y}) + (x^TBx)^{1/2}}{h(x,\widetilde{y}) - (x^TDx)^{1/2}} < \frac{f(z,\widetilde{y}_i) + (z^TBz)^{1/2}}{h(z,\widetilde{y}_i) - (z^TDz)^{1/2}}.$$

This further from $t_i \geq 0$, $i = 1, 2, \ldots, s$, $t \neq 0$ and $\widetilde{y}_i \in Y(z)$, we obtain

$$\sum_{i=1}^{s} t_i \Big[\big(h(z, \widetilde{y}_i) - (z^T Dz)^{1/2} \big) \big(f(x, \widetilde{y}_i) + (x^T Bx)^{1/2} \big) - \big(f(z, \widetilde{y}_i) + (z^T Bz)^{1/2} \big) \times \big(h(x, \widetilde{y}_i) - (x^T Dx)^{1/2} \big) \Big] < 0. \tag{3.6}$$

Now,

$$\begin{aligned}
\xi_1(x) &= \sum_{i=1}^{s} t_i \Big[\big(h(z, \widetilde{y}_i) - z^T Dv \big) \big(f(x, \widetilde{y}_i) + x^T Bw \big) \\
&\quad - \big(f(z, \widetilde{y}_i) + z^T Bw \big) \big(h(x, \widetilde{y}_i) - x^T Dv \big) \Big] \\
&\leq \sum_{i=1}^{s} t_i \Big[\big(h(z, \widetilde{y}_i) - (z^T Dz)^{1/2} \big) \big(f(x, \widetilde{y}_i) + (x^T Bx)^{1/2} \big) \\
&\quad - \big(f(z, \widetilde{y}_i) + (z^T Bz)^{1/2} \big) \big(h(x, \widetilde{y}_i) - (x^T Dx)^{1/2} \big) \Big] \quad \big(\text{using Lemma 2.1 and (3.3)}\big) \\
&< 0 \quad \big(\text{from (3.6)}\big).
\end{aligned}$$

Therefore,

$$\xi_1(x) < 0 = \xi_1(z). \tag{3.7}$$

By hypothesis (i), we have

$$b_0 \phi_0 \left[\psi_1(x) - \psi_1(z) + \frac{1}{2} p^T \nabla^2 \psi_1(z) p \right] \geq \alpha(x, z) \eta^T(x, z) \big\{ \nabla \psi_1(z) + \nabla^2 \psi_1(z) p \big\}.$$

This follows from (3.1) that

$$b_0 \phi_0 \left[\psi_1(x) - \psi_1(z) + \frac{1}{2} p^T \nabla^2 \psi_1(z) p \right] \geq 0$$

which using hypothesis (ii) yields

$$\psi_1(x) - \psi_1(z) + \frac{1}{2} p^T \nabla^2 \psi_1(z) p \geq 0.$$

This further from (2.6), (3.2), and the feasibility of x implies

$$\xi_1(x) \geq -\sum_{j=1}^{m} \mu_j g_j(x) \geq 0 = \xi_1(z).$$

This contradicts (3.7), hence the result. □

Theorem 3.2 (Strong duality) *Let x^* be an optimal solution for (P) and let $\nabla g_j(x^*)$, $j \in J(x^*)$ be linearly independent. Then there exist $(s^*, t^*, \widetilde{y}^*) \in K(x^*)$ and $(x^*, \mu^*, w^*, v^*, p^* = 0) \in H_1(s^*, t^*, \widetilde{y}^*)$, such that $(x^*, \mu^*, w^*, v^*, s^*, t^*, \widetilde{y}^*, p^* = 0)$ is feasible solution of (DM1) and the two* *objectives have same values. If, in addition, the assumptions of Theorem 3.1 hold for all feasible solutions* $(x, \mu, w, v, s, t, \widetilde{y}, p)$ *of (DM1), then* $(x^*, \mu^*, w^*, v^*, s^*, t^*, \widetilde{y}^*, p^* = 0)$ *is an optimal solution of (DM1).*

Proof Since x^* is an optimal solution of (P) and $\nabla g_j(x^*)$, $j \in J(x^*)$ are linearly independent, then by Theorem 2.1, there exist $(s^*, t^*, \widetilde{y}^*) \in K(x^*)$ and $(x^*, \mu^*, w^*, v^*, p^* = 0) \in H_1(s^*, t^*, \widetilde{y}^*)$ such that $(x^*, \mu^*, w^*, v^*, s^*, t^*, \widetilde{y}^*, p^* = 0)$ is feasible solution of (DM1) and the two objectives have same values. Optimality of $(x^*, \mu^*, w^*, v^*, s^*, t^*, \widetilde{y}^*, p^* = 0)$ for (DM1), thus follows from Theorem 3.1. □

Theorem 3.3 (Strict converse duality) *Let* x^* *be an optimal solution to (P) and* $(z^*, \mu^*, w^*, v^*, s^*, t^*, \widetilde{y}^*, p^*)$ *be an optimal solution to (DM1). Assume that*

(i) $\psi_1(\cdot)$ *is strictly second-order* α*-univex at* z^**,*
(ii) $\{\nabla g_j(x^*), j \in J(x^*)\}$*, are linearly independent,*
(iii) $\phi_0(a) > 0 \Rightarrow a > 0$ *and* $b_0(x^*, z^*) > 0$*.*

Then $z^* = x^*$*.*

Proof By the strict α-univexity of $\psi_1(\cdot)$ at z^*, we get

$$\begin{aligned} & b_0(x^*, z^*)\phi_0\left[\psi_1(x^*) - \psi_1(z^*) + \frac{1}{2}p^{*T}\nabla^2\psi_1(z^*)p^*\right] \\ & \quad > \alpha(x^*, z^*)\eta^T(x^*, z^*)\{\nabla\psi_1(z^*) + \nabla^2\psi_1(z^*)p^*\} \end{aligned}$$

which in view of (3.1) and hypothesis (iii) give

$$\psi_1(x^*) - \psi_1(z^*) + \frac{1}{2}p^{*T}\nabla^2\psi_1(z^*)p^* > 0.$$

Using (2.6), (3.2), and feasibility of x^* in above, we obtain

$$\xi_1(x^*) > 0 = \xi_1(z^*). \tag{3.8}$$

Now, we shall assume that $z^* \neq x^*$ and reach a contradiction. Since x^* and $(z^*, \mu^*, w^*, v^*, s^*, t^*, \widetilde{y}^*, p^*)$ are optimal solutions to (P) and (DM1), respectively, and $\{\nabla g_j(x^*), j \in J(x^*)\}$, are linearly independent, by Theorem 3.2, we get

$$\sup_{\widetilde{y}^* \in Y} \frac{f(x^*, \widetilde{y}^*) + (x^{*T}Bx^*)^{1/2}}{h(x^*, \widetilde{y}^*) - (x^{*T}Dx^*)^{1/2}} = F(z^*). \tag{3.9}$$

Since $\widetilde{y}_i^* \in Y(z^*)$, $i = 1, 2, \ldots, s^*$, we have

$$F(z^*) = \frac{f(z^*, \widetilde{y}_i^*) + (z^{*T}Bz^*)^{1/2}}{h(z^*, \widetilde{y}_i^*) - (z^{*T}Dz^*)^{1/2}}. \tag{3.10}$$

By (3.9) and (3.10), we get

$$\begin{aligned} & \left[\left(h(z^*, \widetilde{y}_i^*) - \left(z^{*T}Dz^*\right)^{1/2}\right)\left(f(x^*, \widetilde{y}_i^*) + \left(x^{*T}Bx^*\right)^{1/2}\right)\right. \\ & \quad \left. - \left(f(z^*, \widetilde{y}_i^*) + \left(z^{*T}Bz^*\right)^{1/2}\right)\left(h(x^*, \widetilde{y}_i^*) - \left(x^{*T}Dx^*\right)^{1/2}\right)\right] \leq 0, \end{aligned}$$

for all $i = 1, 2, \ldots, s^*$ and $\widetilde{y}_i^* \in Y$. From $\widetilde{y}_i^* \in Y(z^*) \subset Y$ and $t^* \in R_+^{s^*}$, with $\sum_{i=1}^{s^*} t_i^* = 1$, we obtain

$$\sum_{i=1}^{s^*} t_i^* \big[\big(h(z^*, \widetilde{y}_i^*) - (z^{*T} D z^*)^{1/2} \big) \big(f(x^*, \widetilde{y}_i^*) + (x^{*T} B x^*)^{1/2} \big) - \big(f(z^*, \widetilde{y}_i^*) + (z^{*T} B z^*)^{1/2} \big) \big(h(x^*, \widetilde{y}_i^*) - (x^{*T} D x^*)^{1/2} \big) \big] \le 0. \tag{3.11}$$

From Lemma 2.1, (3.3), and (3.11), we have

$$\begin{aligned}
\xi_1(x^*) &= \sum_{i=1}^{s^*} t_i^* \big[\big(h(z^*, \widetilde{y}_i^*) - z^{*T} D v^* \big) \big(f(x^*, \widetilde{y}_i^*) + x^{*T} B w^* \big) \\
&\quad - \big(f(z^*, \widetilde{y}_i^*) + z^{*T} B w^* \big) \big(h(x^*, \widetilde{y}_i^*) - x^{*T} D v^* \big) \big] \\
&\le \sum_{i=1}^{s^*} t_i^* \big[\big(h(z^*, \widetilde{y}_i^*) - (z^{*T} D z^*)^{1/2} \big) \big(f(x^*, \widetilde{y}_i^*) + (x^{*T} B x^*)^{1/2} \big) \\
&\quad - \big(f(z^*, \widetilde{y}_i^*) + (z^{*T} B z^*)^{1/2} \big) \big(h(x^*, \widetilde{y}_i^*) - (x^{*T} D x^*)^{1/2} \big) \big] \\
&\le 0 = \xi_1(z^*),
\end{aligned}$$

which contradicts (3.8), hence the result. □

4 Model II

In this section, we consider another dual problem to (P):

$$\max_{(s,t,\widetilde{y}) \in K(z)} \sup_{(z,\mu,w,v,p) \in H_2(s,t,\widetilde{y})} \frac{\sum_{i=1}^{s} t_i (f(z, \widetilde{y}_i) + (z^T B z)^{1/2}) + \sum_{j=1}^{m} \mu_j g_j(z)}{\sum_{i=1}^{s} t_i (h(z, \widetilde{y}_i) - (z^T D z)^{1/2})}, \tag{DM2}$$

where $H_2(s, t, \widetilde{y})$ denotes the set of all $(z, \mu, w, v, p) \in R^n \times R_+^m \times R^n \times R^n \times R^n$ satisfying

$$\nabla \psi_2(z) + \nabla^2 \psi_2(z) p = 0, \tag{4.1}$$

$$p^T \nabla^2 \psi_2(z) p \le 0, \tag{4.2}$$

$$w^T B w \le 1, \qquad v^T D v \le 1, \qquad (z^T B z)^{1/2} = z^T B w, \qquad (z^T D z)^{1/2} = z^T D v. \tag{4.3}$$

If the set $H_2(s, t, \widetilde{y})$ is empty, we define the supremum in (DM2) over $H_2(s, t, \widetilde{y})$ equal to $-\infty$.

Remark 4.1 If $p = 0$, then using (4.3), the above dual model becomes the dual model considered in [5, 11, 12]. In addition, if B and D are zero matrices of order n, then (DM2) reduces to the problem studied in [14].

Now, we obtain the following appropriate duality theorems between (P) and (DM2).

Theorem 4.1 (Weak duality) *Let x and $(z, \mu, w, v, s, t, \widetilde{y}, p)$ are feasible solutions of (P) and (DM2), respectively. Suppose that the following conditions are satisfied:*

(i) $\psi_2(\cdot)$ is second-order α-univex at z,

(ii) $\phi_0(a) \ge 0 \Rightarrow a \ge 0$ and $b_0(x, z) > 0$.

Then

$$\sup_{\widetilde{y}\in Y}\frac{f(x,\widetilde{y})+(x^TBx)^{1/2}}{h(x,\widetilde{y})-(x^TDx)^{1/2}}\geq\frac{\sum_{i=1}^{s}t_i(f(z,\widetilde{y}_i)+(z^TBz)^{1/2})+\sum_{j=1}^{m}\mu_jg_j(z)}{\sum_{i=1}^{s}t_i(h(z,\widetilde{y}_i)-(z^TDz)^{1/2})}.$$

Proof Assume on contrary to the result that

$$\sup_{\widetilde{y}\in Y}\frac{f(x,\widetilde{y})+(x^TBx)^{1/2}}{h(x,\widetilde{y})-(x^TDx)^{1/2}}<\frac{\sum_{i=1}^{s}t_i(f(z,\widetilde{y}_i)+(z^TBz)^{1/2})+\sum_{j=1}^{m}\mu_jg_j(z)}{\sum_{i=1}^{s}t_i(h(z,\widetilde{y}_i)-(z^TDz)^{1/2})}$$

or

$$\begin{aligned}&\big(f(x,\widetilde{y}_i)+\big(x^TBx\big)^{1/2}\big)\Bigg[\sum_{i=1}^{s}t_i\big(h(z,\widetilde{y}_i)-\big(z^TDz\big)^{1/2}\big)\Bigg]\\&\quad<\big(h(x,\widetilde{y}_i)-\big(x^TDx\big)^{1/2}\big)\Bigg[\sum_{i=1}^{s}t_i\big(f(z,\widetilde{y}_i)+\big(z^TBz\big)^{1/2}\big)+\sum_{j=1}^{m}\mu_jg_j(z)\Bigg],\\&\quad\forall\widetilde{y}_i\in Y(z),i=1,2,\ldots,s.\end{aligned}$$

Using $t_i\geq 0$, $i=1,2,\ldots,s$ and (4.3) in above, we have

$$\begin{aligned}&\sum_{i=1}^{s}t_i\big(f(x,\widetilde{y}_i)+\big(x^TBx\big)^{1/2}\big)\Bigg[\sum_{i=1}^{s}t_i\big(h(z,\widetilde{y}_i)-z^TDv\big)\Bigg]\\&\quad<\sum_{i=1}^{s}t_i\big(h(x,\widetilde{y}_i)-\big(x^TDx\big)^{1/2}\big)\Bigg[\sum_{i=1}^{s}t_i\big(f(z,\widetilde{y}_i)+z^TBw\big)+\sum_{j=1}^{m}\mu_jg_j(z)\Bigg].\end{aligned}\tag{4.4}$$

Now,

$$\begin{aligned}\psi_2(x)&=\Bigg[\sum_{i=1}^{s}t_i\big(f(x,\widetilde{y}_i)+x^TBw\big)+\sum_{j=1}^{m}\mu_jg_j(x)\Bigg]\Bigg[\sum_{i=1}^{s}t_i\big(h(z,\widetilde{y}_i)-z^TDv\big)\Bigg]\\&\quad-\Bigg[\sum_{i=1}^{s}t_i\big(h(x,\widetilde{y}_i)-x^TDv\big)\Bigg]\Bigg[\sum_{i=1}^{s}t_i\big(f(z,\widetilde{y}_i)+z^TBw\big)+\sum_{j=1}^{m}\mu_jg_j(z)\Bigg]\\&\leq\Bigg[\sum_{i=1}^{s}t_i\big(f(x,\widetilde{y}_i)+\big(x^TBx\big)^{1/2}\big)+\sum_{j=1}^{m}\mu_jg_j(x)\Bigg]\Bigg[\sum_{i=1}^{s}t_i\big(h(z,\widetilde{y}_i)-z^TDv\big)\Bigg]\\&\quad-\Bigg[\sum_{i=1}^{s}t_i\big(h(x,\widetilde{y}_i)-\big(x^TDx\big)^{1/2}\big)\Bigg]\Bigg[\sum_{i=1}^{s}t_i\big(f(z,\widetilde{y}_i)+z^TBw\big)+\sum_{j=1}^{m}\mu_jg_j(z)\Bigg]\\&\quad\big(\text{from Lemma 2.1 and (4.3)}\big)\\&<\sum_{i=1}^{s}t_i\big(h(z,\widetilde{y}_i)-z^TDv\big)\sum_{j=1}^{m}\mu_jg_j(x)\quad\big(\text{using (4.4)}\big)\\&\leq 0\quad\Bigg(\text{since }\sum_{i=1}^{s}t_i\big(h(z,\widetilde{y}_i)-z^TDv\big)>0\text{ and }\sum_{j=1}^{m}\mu_jg_j(x)\leq 0\Bigg).\end{aligned}$$

Hence,

$$\psi_2(x) < 0 = \psi_2(z). \tag{4.5}$$

Now, by the second-order α-univexity of $\psi_2(\cdot)$ at z, we get

$$b_0\phi_0\left[\psi_2(x) - \psi_2(z) + \frac{1}{2}p^T\nabla^2\psi_2(z)p\right] \geq \eta^T(x,z)\alpha(x,z)\left\{\nabla\psi_2(z) + \nabla^2\psi_2(z)p\right\}$$

which using (4.1) and hypothesis (ii) give

$$\psi_2(x) - \psi_2(z) + \frac{1}{2}p^T\nabla^2\psi_2(z)p \geq 0.$$

This from (4.2) follows that

$$\psi_2(x) \geq \psi_2(z)$$

which contradicts (4.5). This proves the theorem. □

By a similar way, we can prove the following theorems between (P) and (DM2).

Theorem 4.2 (Strong duality) *Let x^* be an optimal solution for (P) and let $\nabla g_j(x^*)$, $j \in J(x^*)$ be linearly independent. Then there exist $(s^*, t^*, \widetilde{y}^*) \in K(x^*)$ and $(x^*, \mu^*, w^*, v^*, p^* = 0) \in H_2(s^*, t^*, \widetilde{y}^*)$, such that $(x^*, \mu^*, w^*, v^*, s^*, t^*, \widetilde{y}^*, p^* = 0)$ is feasible solution of (DM2) and the two objectives have same values. If, in addition, the assumptions of weak duality hold for all feasible solutions $(x, \mu, w, v, s, t, \widetilde{y}, p)$ of (DM2), then $(x^*, \mu^*, w^*, v^*, s^*, t^*, \widetilde{y}^*, p^* = 0)$ is an optimal solution of (DM2).*

Theorem 4.3 (Strict converse duality) *Let x^* and $(z^*, \mu^*, w^*, v^*, s^*, t^*, \widetilde{y}^*, p^*)$ are optimal solutions of (P) and (DM2), respectively. Assume that*

(i) $\psi_2(\cdot)$ is strictly second-order α-univex at z,
(ii) $\{\nabla g_j(x^*), j \in J(x^*)\}$ are linearly independent,
(iii) $\phi_0(a) > 0 \Rightarrow a > 0$ and $b_0(x^*, z^*) > 0$.

Then $z^* = x^*$.

5 Concluding remarks

In the present work, we have formulated two types of second-order dual models for a nondifferentiable minimax fractional programming problems and proved appropriate duality relations involving second-order α-univex functions. Further, examples have been illustrated to show the existence of such type of functions. Now, the question arises whether or not the results can be further extended to a higher-order nondifferentiable minimax fractional programming problem.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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