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Research Article

Existence and Data Dependence of Fixed Points and Strict Fixed Points for Contractive-Type Multivalued Operators

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The purpose of this paper is to present several existence and data dependence results of the fixed points of some multivalued generalized contractions in complete metric spaces. As for application, a continuation result is given.

1. Introduction

Throughout this paper, the standard notations and terminologies in nonlinear analysis (see [14, 15]) are used. For the convenience of the reader we recall some of them.

Let (X,d) be a metric space. By $\widetilde{B}(x_0,r)$ we denote the closed ball centered in $x_0 \in X$ with radius $r > 0$.

Also, we will use the following symbols:

$$\begin{gathered} P(X) := \{Y \subset X \mid Y \text{ is nonempty}\}, \qquad P_{\mathrm{cl}}(X) := \{Y \in P(X) \mid Y \text{ is closed}\}, \\ P_b(X) := \{Y \in P(X) \mid Y \text{ is bounded}\}, \qquad P_{b,\mathrm{cl}}(X) := P_{\mathrm{cl}}(X) \cap P_b(X). \end{gathered} \tag{1.1}$$

Let A and B be nonempty subsets of the metric space (X,d). The gap between these sets is

$$D(A,B) = \inf\{d(a,b) \mid a \in A,\ b \in B\}. \tag{1.2}$$

In particular, $D(x_0,B) = D(\{x_0\},B)$ (where $x_0 \in X$) is called the distance from the point x_0 to the set B.

The Pompeiu-Hausdorff generalized distance between the nonempty closed subsets A and B of the metric space (X,d) is defined by the following formula:

$$H(A,B) := \max\left\{\sup_{a\in A}\inf_{b\in B} d(a,b), \sup_{b\in B}\inf_{a\in A} d(a,b)\right\}. \tag{1.3}$$

If $A,B \in P_{b,\mathrm{cl}}(X)$, then one denotes

$$\delta(A,B) := \sup\{d(a,b) \mid a\in A,\ b\in B\}. \tag{1.4}$$

The symbol $T: X \to P(Y)$ denotes a set-valued operator from X to Y. We will denote by $\operatorname{Graph}(T) := \{(x,y)\in X\times Y \mid y\in T(x)\}$ the graph of T. Recall that the set-valued operator is called closed if $\operatorname{Graph}(T)$ is a closed subset of $X\times Y$.

For $T: X\to P(X)$ the symbol $\operatorname{Fix}(T) := \{x\in X \mid x\in T(x)\}$ denotes the fixed point set of the set-valued operator T, while $S\operatorname{Fix}(T) := \{x\in X \mid \{x\} = T(x)\}$ is the strict fixed point set of T.

If (X,d) is a metric space, $T: X\to P_{\mathrm{cl}}(X)$ is called a multivalued a-contraction if $a\in\,]0,1[$ and $H(T(x_1),T(x_2)) \le a\cdot d(x_1,x_2)$, for each $x_1,x_2\in X$.

In the same setting, an operator $T: X\to P_{\mathrm{cl}}(X)$ is a multivalued weakly Picard operator (briefly MWP operator) (see [15]) if for each $x\in X$ and each $y\in T(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that

(i) $x_0 = x$, $x_1 = y$,

(ii) $x_{n+1}\in T(x_n)$, for all $n\in\mathbb{N}$,

(iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of T.

Any multivalued a-contraction or any multivalued Reich-type operator (see Reich [10]) are examples of MWP operators. For other examples and results, see Petruşel [9].

Also, let us mention that a sequence $(x_n)_{n\in\mathbb{N}}$ in X satisfying the condition (ii) from the previous definition is called the sequence of successive approximations of T starting from $x_0\in X$.

The following result was proved in the work of Feng and Liu (see [5]).

Theorem 1.1 (Feng, Liu). *Let (X,d) be a complete metric space, $T: X\to P_{\mathrm{cl}}(X)$ and $q>1$. Consider $S_q(x) := \{y\in T(x) \mid d(x,y)\le q\cdot D(x,T(x))\}$. Suppose that T satisfies the following condition:*

(1.1) *there is $a<1/q$ such that for each $x\in X$ there is $y\in S_q(x)$ satisfying*

$$D(y,T(y)) \le a\cdot d(x,y). \tag{1.5}$$

Also, suppose that the function $p: X\to\mathbb{R}$, $p(x) := D(x,T(x))$ is lower semicontinuous. Then $\operatorname{Fix} T \neq \emptyset$.

The purpose of this paper is to study the existence and data dependence of the fixed points and strict fixed points for some self and nonself multivalued operators satisfying to some generalized Feng-Liu-type conditions.

Our results are in connection with the theory of MWP operators (see [9, 15]) and they generalize some fixed point and strict fixed point principles for multivalued operators given in [3–5, 7, 8, 10–13].

2. Fixed points

Let (X,d) be a metric space, $T : X \to P_{cl}(X)$ a multivalued operator, and $q > 1$. Define $S_q(x) := \{y \in T(x) \mid d(x,y) \le q \cdot D(x,T(x))\}$. Obviously $S_q(x) \ne \emptyset$, for each $x \in X$ and S_q is a multivalued selection of T.

Our first main result is the following theorem.

Theorem 2.1. *Let (X,d) be a complete metric space, $x_0 \in X$, $r > 0$, $q > 1$, and $T : X \to P_{cl}(X)$ a multivalued operator. Suppose that*

(i) *there exists $a \in \mathbb{R}_+$ with $aq < 1$ such that for each $x \in \widetilde{B}(x_0,r)$ there exists $y \in S_q(x)$ having the property*

$$D(y,T(y)) \le a \cdot d(x,y), \tag{2.1}$$

(ii) *T is closed or the function $p : X \to \mathbb{R}_+$, $p(x) := D(x,T(x))$ is lower semicontinuous,*

(iii) *$D(x_0,T(x_0)) \le ((1-aq)/q) \cdot r$.*

Then $\operatorname{Fix}(T) \cap \widetilde{B}(x_0,r) \ne \emptyset$.

Proof. From (i) and (iii) there is $x_1 \in T(x_0)$ such that $d(x_0,x_1) \le qD(x_0,T(x_0)) < (1-aq)r$ and $D(x_1,T(x_1)) \le ad(x_0,x_1) \le aqD(x_0,T(x_0))$. Hence $x_1 \in \widetilde{B}(x_0,r)$. Next, we can find $x_2 \in T(x_1)$ such that $d(x_1,x_2) \le qD(x_1,T(x_1)) \le aqd(x_0,x_1) < aq(1-aq) \cdot r$ and $D(x_2,T(x_2)) \le ad(x_1,x_2) \le aqD(x_1,T(x_1)) \le (aq)^2D(x_0,T(x_0))$. As a consequence, $d(x_0, x_2) \le d(x_0,x_1) + d(x_1,x_2) \le (1-aq)r + aq(1-aq)r = (1-(aq)^2)r$ and so $x_2 \in \widetilde{B}(x_0,r)$.

Inductively we get a sequence $(x_n)_{n\in\mathbb{N}}$ having the following properties:

(a) $x_{n+1} \in T(x_n)$, $n \in \mathbb{N}$;

(b) $d(x_n,x_{n+1}) \le (aq)^n d(x_0,x_1)$, $d(x_0,x_n) \le (1-(aq)^n)r$, $n \in \mathbb{N}$;

(c) $D(x_n,T(x_n)) \le (aq)^n \cdot D(x_0,T(x_0))$, $n \in \mathbb{N}$.

From (b) we have that $(x_n)_{n\in\mathbb{N}}$ converges to $x^* \in \widetilde{B}(x_0,r)$.

From (a) and the fact that $\operatorname{Graph}T$ is closed we obtain $x^* \in \operatorname{Fix}T$.

From (c) and the fact that p is lower semicontinuous we have $p(x_n) \le (aq)^n \cdot p(x_0)$, for each $n \in \mathbb{N}$. Since $aq < 1$, we immediately deduce that the sequence $(p(x_n))$ is convergent to 0, as $n \to +\infty$. Then $0 \le p(x^*) \le \liminf_{n\to+\infty} p(x_n) = 0$. So, $p(x^*) = 0$ and then $x^* \in T(x^*)$. □

Remark 2.2. The above result is a local version of the main result in [5, Theorem 3.1] see Theorem 1.1. In particular, Theorem 1.1 follows from Theorem 2.1 by taking $r := +\infty$. Theorem 2.1 also extends some results from [3, 4, 7–9], and so forth.

As for application, a homotopy result can be proved.

Theorem 2.3. *Let (X,d) be a complete metric space, U an open subset of X, and $q > 1$. Suppose that $G : \overline{U} \times [0,1] \to P_{cl}(X)$ is a closed multivalued operator such that the following conditions are satisfied:*

(a) *$x \notin G(x,t)$, for each $x \in \partial U$ and each $t \in [0,1]$;*

(b) *there exists $a \in \mathbb{R}_+$ with $aq < 1$, such that for each $t \in [0,1]$ and each $x \in \overline{U}$ there exists $y \in \overline{U} \cap S_q(x,t)$ (where $S_q(x,t) := \{y \in G(x,t) \mid d(x,y) \le q \cdot D(x,G(x,t))\}$),* *with the property*

$$D(y,G(y,t)) \leq a \cdot d(x,y); \tag{2.2}$$

(c) *there exists a continuous increasing function* $\phi : [0,1] \to \mathbb{R}$ *such that*

$$H(G(x,t),G(x,s)) \leq |\phi(t)-\phi(s)| \; \forall t,s \in [0,1] \text{ and each } x \in \overline{U}. \tag{2.3}$$

Then $G(\cdot,0)$ *has a fixed point if and only if* $G(\cdot,1)$ *has a fixed point.*

Proof. Suppose $G(\cdot,0)$ has a fixed point. Define

$$Q := \{(t,x) \in [0,1] \times U \mid x \in G(x,t)\}. \tag{2.4}$$

Obviously $Q \neq \emptyset$. Consider on Q a partial order defined as follows:

$$(t,x) \leq (s,y) \quad \text{iff } t \leq s, \quad d(x,y) \leq \frac{2q}{1-aq} \cdot [\phi(s)-\phi(t)]. \tag{2.5}$$

Let M be a totally ordered subset of Q and consider $t^* := \sup\{t \mid (t,x) \in M\}$. Consider a sequence $(t_n,x_n)_{n\in\mathbb{N}^*} \subset M$ such that $(t_n,x_n) \leq (t_{n+1},x_{n+1})$ and $t_n \to t^*$, as $n \to +\infty$. Then

$$d(x_m,x_n) \leq \frac{2q}{1-aq} \cdot [\phi(t_m)-\phi(t_n)], \quad \text{for each } m,n \in \mathbb{N}^*, \; m > n. \tag{2.6}$$

When $m,n \to +\infty$ we obtain $d(x_m,x_n) \to 0$ and so $(x_n)_{n\in\mathbb{N}^*}$ is Cauchy. Denote by $x^* \in X$ its limit. Then $x_n \in G(x_n,t^*)$, $n \in \mathbb{N}^*$ and G closed imply that $x^* \in G(x^*,t^*)$. Also, from (a) $x^* \in U$. Hence $(t^*,x^*) \in Q$. Since M is totally ordered, we get $(t,x) \leq (t^*,x^*)$, for each $(t,x) \in M$. Thus (t^*,x^*) is an upper bound of M. Hence Zorn's lemma applies and Q admits a maximal element $(t_0,x_0) \in Q$. We claim that $t_0 = 1$. This will finish the proof.

Suppose the contrary, that is, $t_0 < 1$. Choose $r > 0$ and $t \in]t_0,1]$ such that $\widetilde{B}(x_0,r) \subset U$ and $r := (2q/(1-aq)) \cdot [\phi(t)-\phi(t_0)]$.

Then the $D(x_0,G(x_0,t)) \leq D(x_0,G(x_0,t_0)) + H(G(x_0,t_0),G(x_0,t)) \leq 0 + [\phi(t)-\phi(t_0)] = (1-aq)r/2q < (1-aq)r/q$.

Then the multivalued operator $G(\cdot,t) : \widetilde{B}(x_0,r) \to P_{cl}(X)$ satisfies all the assumptions of Theorem 2.1. Hence there exists a fixed point $x \in \widetilde{B}(x_0,r)$ for $G(\cdot,t)$. Thus $(t,x) \in Q$. Since

$$d(x_0,x) \leq r = \frac{2q}{1-aq} \cdot [\phi(t)-\phi(t_0)], \tag{2.7}$$

we immediately get $(t_0,x_0) < (t,x)$. This is a contradiction with the maximality of (t_0,x_0). □

Remark 2.4. Theorem 2.3 extends the main theorem in the work of Frigon and Granas [6]. See also Agarwal et al. [1] and Chiş and Precup [2] for some similar results or possibilities for extension.

Another fixed point result is the following.

Theorem 2.5. *Let (X,d) be a complete metric space, $x_0 \in X$, $r > 0$, $q > 1$, and $T : X \to P_{cl}(X)$ a multivalued operator. Suppose that*

(i) *there exists $a,b \in \mathbb{R}_+$ with $aq + b < 1$ such that for each $x \in \widetilde{B}(x_0,r)$ there exists $y \in S_q(x)$ having the property*

$$D(y,T(y)) \leq a \cdot d(x,y) + b \cdot D(x,T(x)), \tag{2.8}$$

(ii) *T is closed or the function $p : X \to \mathbb{R}_+$, $p(x) := D(x,T(x))$ is lower semicontinuous,*

(iii) *$D(x_0,T(x_0)) < ((1-(aq+b))/q) \cdot r$.*

Then $\text{Fix}(T) \cap \widetilde{B}(x_0,r) \neq \emptyset$.

Proof. By (i) and (iii) we deduce the existence of an element $x_1 \in T(x_0)$ such that $d(x_0, x_1) \leq qD(x_0,T(x_0)) < (1-(aq+b))r$ and $D(x_1,T(x_1)) \leq ad(x_0,x_1) + bD(x_0,T(x_0)) \leq (aq+b)D(x_0,T(x_0))$.

Inductively we obtain $(x_n)_{n\in\mathbb{N}}$ a sequence of successive approximations of T satisfying, for each $n \in \mathbb{N}$, the following relations:

(1) $d(x_n,x_{n+1}) \leq q(aq+b)^n \cdot D(x_0,T(x_0))$, $d(x_0,x_n) \leq (1-(aq+b)^n) \cdot r$,

(2) $D(x_n,T(x_n)) \leq (aq+b)^n \cdot D(x_0,T(x_0))$.

The rest of the proof runs as before and so the conclusion follows. □

Remark 2.6. The above result generalizes the fixed point result in the work of Rus [12], where the following graphic contraction condition is involved: there is $a,b \in \mathbb{R}_+$ with $a+b<1$ such that $H(T(x),T(y)) \leq a \cdot d(x,y) + bD(x,T(x))$, for each $x \in X$ and each $y \in T(x)$.

A data dependence result is the following.

Theorem 2.7. *Let (X,d) be a complete metric space, $T_1,T_2 : X \to P_{cl}(X)$ multivalued operators, and $q_1,q_2 > 1$. Suppose that*

(i) *there exist $a_i,b_i \in \mathbb{R}_+$ with $a_iq_i + b_i < 1$ such that for each $x \in X$ there exists $y \in S_{q_i}(x)$ having the property*

$$D(y,T_i(y)) \leq a_i \cdot d(x,y) + b_i \cdot D(x,T_i(x)), \quad \textit{for } i \in \{1,2\}; \tag{2.9}$$

(ii) *there exists $\eta > 0$ such that $H(T_1(x),T_2(x)) \leq \eta$, for each $x \in X$;*

(iii) *T_i is closed or the function $p_i : X \to \mathbb{R}_+$, $p_i(x) := D(x,T_i(x))$ is lower semicontinuous, for $i \in \{1,2\}$.*

Then

(a) $\text{Fix}(T_i) \in P_{cl}(X)$, *for* $i \in \{1,2\}$,

(b) $H(\text{Fix}(T_1),\text{Fix}(T_2)) \leq \max_{i\in\{1,2\}}\{q_i/(1-(a_iq_i+b_i))\} \cdot \eta$.

Proof. (a) By Theorem 2.1 we have that $\text{Fix}\,T_i \neq \emptyset$, for $i \in \{1,2\}$. Also, $\text{Fix}\,T_i$ is closed, for $i \in \{1,2\}$. Indeed, for example, let $(u_n)_{n\in\mathbb{N}} \in \text{Fix}\,T_1$, such that $u_n \to u$, as $n \to +\infty$. Then, when T_1 is closed, the conclusion follows. When $p_1(x) := D(x,T_1(x))$ is lower semicontinuous we have $0 \leq p_1(u) \leq \liminf_{n\to+\infty} p_1(u_n) = 0$. Hence $p_1(u) = 0$ and so $u \in \text{Fix}\,T_1$.

(b) For the second conclusion, let $x_0^* \in \operatorname{Fix} T_1$. Then there exists $x_1 \in S_{q_2}(x_0^*)$ with $D(x_1, T_2(x_1)) \le a_2 \cdot d(x_0^*, x_1) + b_2 \cdot D(x_0^*, T_2(x_0^*))$. Hence $d(x_0^*, x_1) \le q_2 \cdot D(x_0^*, T_2(x_0^*))$ and $D(x_1, T_2(x_1)) \le (a_2 q_2 + b_2) \cdot D(x_0^*, T_2(x_0^*))$. Inductively we get a sequence $(x_n)_{n\in\mathbb{N}}$ with the following properties:

(1) $x_0 = x_0^* \in \operatorname{Fix} T_1$,

(2) $d(x_n, x_{n+1}) \le q_2(a_2q_2 + b_2)^n \cdot D(x_0^*, T_2(x_0^*))$, $n \in \mathbb{N}$,

(3) $D(x_n, T_2(x_n)) \le (a_2q_2 + b_2)^n \cdot D(x_0^*, T_2(x_0^*))$, $n \in \mathbb{N}$.

From (2) we have

$$d(x_n, x_{n+m}) \le q_2(a_2q_2 + b_2)^n \cdot \frac{1 - (a_2q_2 + b_2)^m}{1 - (a_2q_2 + b_2)} D(x_0^*, T_2(x_0^*)). \tag{2.10}$$

Hence $(x_n)_{n\in\mathbb{N}}$ is Cauchy and so it converges to an element $u_2^* \in X$. As in the proof of Theorem 2.1, from (3) we immediately get that $u_2^* \in \operatorname{Fix} T_2$. When $m \to +\infty$ in the above relation, we obtain $d(x_n, u_2^*) \le (q_2(a_2q_2 + b_2)^n / (1 - (a_2q_2 + b_2))) D(x_0^*, T_2(x_0^*))$, for each $n \in \mathbb{N}$.

For $n = 0$ we get $d(x_0, u_2^*) \le q_2/(1 - (a_2q_2 + b_2)) D(x_0^*, T_2(x_0^*))$.

As a consequence

$$d(x_0, u_2^*) \le \frac{q_2}{1 - (a_2q_2 + b_2)} \cdot H(T_1(x_0^*), T_2(x_0^*)) \le \frac{q_2}{1 - (a_2q_2 + b_2)} \cdot \eta. \tag{2.11}$$

In a similar way we can prove that for each $y_0^* \in \operatorname{Fix} T_2$ there exists $u_1^* \in \operatorname{Fix} T_1$ such that $d(y_0, u_1^*) \le q_1/(1 - (a_1q_1 + b_1)) \cdot \eta$. The proof is complete. □

Remark 2.8. Theorem 2.7 gives (for $b_i = 0$, $i \in \{1, 2\}$) a data dependence result for the fixed point set of a generalized contraction in Feng and Liu sense, see [5].

Remark 2.9. The condition $D(T(x), T(y)) \le a \cdot d(x, y)$, for each $x, y \in X$, does not imply the existence of a fixed point for a multivalued operator $T : X \to P_{cl}(X)$. Take for example $X := [1, +\infty]$ and $T(x) := [2x, +\infty[$ see also [10]. On the other hand, if $X := \{0, 1\} \cup \{k^n \mid n \in \mathbb{N}^*\}$ (with $k \in]0, 1[$) and $T : X \to P_{cl}(X)$ given by

$$T(x) = \begin{cases} \{0, k\}, & \text{if } x = 0, \\ \{k^{n+1}, 1\}, & \text{if } x = k^n \ (n \in \mathbb{N}), \end{cases} \tag{2.12}$$

then T does not satisfies the hypothesis of Nadler's theorem, but satisfies the condition $D(y, T(y)) \le a \cdot d(x, y) + b \cdot D(x, T(x))$, for each $(x, y) \in \operatorname{Graph} T$ and $\operatorname{Fix} T = \{0\}$.

3. Strict fixed points

Let (X, d) be a metric space, $T : X \to P_{b,cl}(X)$ a multivalued operator, and $q > 1$. Define $M_q(x) := \{y \in T(x) \mid \delta(x, T(x)) \le q \cdot d(x, y)\}$. Obviously, M_q is a multivalued selection of T and $M_q(x) \ne \emptyset$, for each $x \in X$.

We have the following theorem.

Theorem 3.1. *Let (X,d) be a complete metric space, $T : X \to P_b(X)$ a multivalued operator and $q > 1$. Suppose*

(3.1) *there exists $a \in \mathbb{R}_+$ with $aq < 1$ such that for each $x \in X$ there exists $y \in M_q(x)$ having the property*

$$\delta(y,T(y)) \leq a \cdot \max\left\{\delta(x,T(x)), \frac{1}{2}[D(x,T(y))\right\}. \tag{3.1}$$

If the function $r : X \to \mathbb{R}_+$, $r(x) := \delta(x,T(x))$ is lower semicontinuous, then $S\mathrm{Fix}(T) \neq \emptyset$.

Proof. Let $x_0 \in X$. If $\delta(x_0,T(x_0)) = 0$ we are done. Suppose that $\delta(x_0,T(x_0)) > 0$. Then there exists $x_1 \in M_q(x_0)$ such that $\delta(x_1,T(x_1)) \leq a \cdot \max\{\delta(x_0,T(x_0)),(1/2) \cdot D(x_0,T(x_1))\} \leq \max\{a/(2-a),aq\}d(x_0,x_1)$.

Inductively we construct a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximation of T with $\delta(x_n,T(x_n)) \leq q \cdot d(x_n,x_{n+1})$, for each $n \in \mathbb{N}$. Then $d(x_n,x_{n+1}) \leq \delta(x_n,T(x_n)) \leq a \cdot \max\{\delta(x_{n-1},T(x_{n-1})),(1/2) \cdot D(x_{n-1},T(x_n))\} \leq a \cdot \max\{q \cdot d(x_{n-1},x_n),(1/2) \cdot D(x_{n-1},T(x_n))\} \leq \max\{a/(2-a),aq\} \cdot d(x_{n-1},x_n)$. Since $\alpha := \max\{a/(2-a),aq\} < 1$, we immediately get that the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent in the complete metric space (X,d). Denote by x^* its limit.

We also have that $r(x_{n+1}) \leq q \cdot \alpha^n \cdot d(x_0,x_1)$. When $n \to +\infty$ we obtain $\lim_{n\to+\infty} r(x_n) = 0$. From the lower semicontinuity of r we conclude $0 \leq r(x^*) \leq \liminf_{n\to+\infty} r(x_n) = 0$. Hence $\delta(x^*,T(x^*)) = 0$ and so $x^* \in S\mathrm{Fix}\,T$. □

Remark 3.2. The above result generalizes some strict fixed point results, given by Reich in [10, 11], Rus in [12, 13] and Ćirić in [3]. In particular, (3.1) implies the Ćirić-type condition on the graph of T.

Remark 3.3. If X is a metric space, the condition

$$\delta(T(x),T(y)) \leq a \cdot d(x,y), \quad \text{for each } x,y \in X, \tag{3.2}$$

necessarily implies that T is singlevalued. This is not the case, if T satisfies the condition

$$\delta(y,T(y)) \leq a \cdot \max\left\{d(x,y),\delta(x,T(x)), \frac{1}{2}[D(x,T(y))+D(y,T(x))]\right\}, \tag{3.3}$$

for each $(x,y) \in X$. Take for example $X := [0,1]$ and $T(x) := [0,x/4]$. Then $S\mathrm{Fix}\,T = \{0\}$ see also [10].

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