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# Spectral curve for open strings attached to the $Y = 0$ brane

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Abstract: The concept of spectral curve is generalized to open strings in AdS/CFT with integrability preserving boundary conditions. Our definition is based on the logarithms of the eigenvalues of the open monodromy matrix and makes possible to determine all the analytic, symmetry and asymptotic properties of the quasimomenta. We work out the details of the whole construction for the $Y = 0$ brane boundary condition. The quasimomenta of open circular strings are explicitly calculated. We use the asymptotic solutions of the $Y$-system and the boundary Bethe Ansatz equations to recover the spectral curve in the strong coupling scaling limit. Using the curve the quasiclassical fluctuations of some open string solutions are also studied.

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## 1 Introduction

One of the greatest progresses in contemporary theoretical physics is the AdS/CFT correspondence [1–3]. In the most analyzed version it relates the spectrum of IIB strings in the $AdS_5 \times S^5$ background to the scaling dimensions of single trace operators of the maximally supersymmetric four dimensional gauge theory. The integrability, which shows up in the 't Hooft limit, allows a complete characterization and exact determination of the full spectrum [4]. This characterization is different in the weak and strong coupling regimes.

In the strong coupling or (semi)classical domain integrability manifests itself by the existence of a spectral parameter dependent flat connection [5]. The corresponding parallel transporter can be evaluated on a non-trivial closed loop, which defines the monodromy matrix, whose trace is time independent and generates an infinite family of conserved charges. Even more, the logarithm of the eigenvalues of the monodromy matrix (quasi momenta) form an eight-sheeted Riemann surface: the spectral curve [6]. The spectral curve provides a very elegant description of the finite energy classical configurations. By requiring the right analytical properties for the quasi momenta allows to find the classical curve for each solution and to determine its energy without explicitly constructing the solution itself [7]. Moreover, the curve can be used to characterize the small fluctuations around the classical solutions and by this way to describe their semiclassical corrections [8].

The quantum spectrum of particles in a large volume can be described by specifying their dispersion relations and momentum quantization conditions, called the asymptotic Bethe Ansatz equations [9]. These equations are valid for any coupling provided the volume is large. The obtained spectrum can be compared at weak coupling to the spectrum of the dilation operator of gauge theory, while at strong coupling to the energies of the classical string solutions. The spectral curve can be also recovered in the strong coupling limit as Bethe roots condense and form the expected cuts [10].

The complete quantum description of the spectrum valid for any coupling and volume is given in terms of the Y-system [11–13]. The large volume solution of this Y-system is related in a simple way to the asymptotic Bethe Ansatz equations and the spectral curve can be recovered in the strong coupling limit [14, 15].

The AdS/CFT correspondence which relates the scaling dimensions of single trace operators to the energies of closed string states relates also the scaling dimensions of determinant type operators and the energies of open string states. Open strings end on D-branes and a careful choice of the brane can ensure integrability in the 't Hooft limit [16, 17].

The classical integrability of open strings can be shown by constructing the analogue of the monodromy matrix. In the open case the parallel transporter can be used to move from one boundary to the other. At the boundary gluing automorphism has to be introduced, such that when combined with the transporter a so called double row monodromy matrix is obtained [18, 19]. Its trace is time independent and generates an infinite family of conserved charges [19].

The goal of the present paper is to generalize the spectral curve construction from the closed case to the open one and use it to characterize the open string spectrum. We define the spectral curve via the logarithm of the eigenvalues of the double row monodromy matrix. This definition determines the analytical properties of the spectral curve (including the cut structure, the poles with prescribed residues and its infinite asymptotics).

We then analyze the curve form different point of views. For simplicity we restrict the investigations for the $Y = 0$ brane boundary conditions as in this case both the asymptotical Bethe Ansatz and the Y-system solutions are available [23, 27]. We obtain and characterize the curve as the semi-classical limit of these descriptions. We also construct explicitly the spectral curve for the BMN state and for circular open strings. Finally, we show how the spectrum of small fluctuations can be determined.

The paper is organized as follows: we start in the next section by the Lagrangian definition of the model. We follow the notation of [19], where the monodromy matrix was constructed. We then introduce the quasimomenta and list its properties such as symmetries, asymptotics and singularity structure. In section 3 we provide the explicit BMN and circular strings solutions and calculate the quasi momenta from first principles. Section 4 shows how one can derive the spectral curve from the boundary Y-system, while section 5 contains the analogous derivation starting from the asymptotic BA equations. We analyze the small fluctuations in the language of the spectral curve in section 6. Finally we conclude in section 7. The details of the calculations are relegated to appendices.

## 2 The "open" monodromy matrix and the quasimomenta

In this section we define the spectral curve for open strings from the logarithmic derivative of the eigenvalues of the boundary monodromy matrix.

### 2.1 Monodromy matrix

The boundary monodromy matrix is the analogue of the periodic monodromy matrix and generates an infinite family of conserved charges.

#### 2.1.1 Flat connections and integrability

Classically the open superstring on $AdS_5 \times S^5$ coupled to the $Y = 0$ brane is described with the help of the Green-Schwartz sigma model (GS$\sigma$M) taking values in su$(2, 2|4)$:[1]

$$S = -g \int d\tau d\sigma [\gamma^{\alpha\beta} \mathrm{str}(A^{(2)}_\alpha A^{(2)}_\beta) + k\epsilon^{\alpha\beta} \mathrm{str}(A^{(1)}_\alpha A^{(3)}_\beta)], \quad k = \pm 1 \tag{2.1}$$

where $A^{(i)}$ denote the various $\mathbb{Z}_4$ components of the Maurer-Cartan one form $A$:

$$A = -g^{-1}dg = \sum_{i=0}^{3} A^{(i)}, \qquad g \in \mathrm{SU}(2, 2|4) \tag{2.2}$$

In contrast to the closed string (periodic) case this sigma model has some non trivial but consistent and integrability preserving boundaries [19]. The integrability of the model is guaranteed by the existence of the "moving frame" flat connection, $L_\alpha$ ($\alpha = \tau,\ \sigma$):

$$L_\alpha = \mathbf{l}_0 A^{(0)}_\alpha + \mathbf{l}_1 A^{(2)}_\alpha + \mathbf{l}_2 \gamma_{\alpha\beta}\epsilon^{\beta\rho} A^{(2)}_\rho + \mathbf{l}_3 A^{(1)}_\alpha + \mathbf{l}_4 A^{(3)}_\alpha\,, \qquad \gamma_{\alpha\beta} = \mathrm{diag}(-1, 1) \tag{2.3}$$

[1] Our coupling constant is defined as $g = \frac{\sqrt{\lambda}}{4\pi}$ with $\lambda$ being the 't Hooft coupling.



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where the $\mathbf{l}_i$ parameters are obtained from requiring the equations of motions for $A^{(i)}$ to coincide with the conditions of vanishing curvature [5, 20]. They can be written in terms of a complex variable $\zeta$ as:

$$\mathbf{l}_0 = 1, \qquad \mathbf{l}_1 = \frac{1}{2}(\zeta^2 + \zeta^{-2}), \qquad \mathbf{l}_2 = -\frac{1}{2k}(\zeta^2 - \zeta^{-2}), \qquad \mathbf{l}_3 = \zeta, \qquad \mathbf{l}_4 = \zeta^{-1}. \tag{2.4}$$

The integrability preserving boundary conditions can be nicely formulated in terms of the "fixed" frame connection $l_\alpha$, which is the gauge transform of $L_\alpha$:

$$l_\alpha = gL_\alpha g^{-1} + \partial_\alpha g\, g^{-1} \tag{2.5}$$

and is correspondingly flat itself:

$$dl(\zeta) - l(\zeta) \wedge l(\zeta) = 0\ . \tag{2.6}$$

By introducing $\sum_{i=0}^{3} gA^{(i)}g^{-1} = \sum_{i=0}^{3} a^{(i)}$ the fixed frame $l_\alpha$ can be written as

$$l_\alpha = (\mathbf{l}_1 - 1)a^{(2)}_\alpha + \mathbf{l}_2\gamma_{\alpha\beta}\epsilon^{\beta\rho}a^{(2)}_\rho + (\mathbf{l}_3 - 1)a^{(1)}_\alpha + (\mathbf{l}_4 - 1)a^{(3)}_\alpha\ . \tag{2.7}$$

The integrability preserving boundary conditions are given by appropriate gluing conditions on this fixed frame flat connection ($l(\zeta) \equiv l_\tau(\zeta)d\tau + l_\sigma(\zeta)d\sigma$) as

$$l(\zeta) = \Omega(\bar{l}(\zeta^{-1})) \quad \text{at} \quad \sigma = 0,\ \pi, \tag{2.8}$$

where $\Omega$ is an involutive metric preserving automorphism ($\mathrm{str}(\Omega(A)\Omega(B)) = \mathrm{str}(AB)$, $\Omega^2 = 1$) and $\bar{l}(\zeta) \equiv l_\tau(\zeta)d\tau - l_\sigma(\zeta)d\sigma$. These conditions guarantee, that the boundary terms arising in the variation of the action as a result of the open ends just cancel.

### 2.1.2 Boundary monodromy matrix

Similarly to the closed string case the generator of the conserved quantities is described through the transport matrix

$$T(\sigma_2, \sigma_1, \zeta) = P\exp\left(\int_{\sigma_1}^{\sigma_2} d\sigma l_\sigma(\sigma, \zeta)\right) \tag{2.9}$$

In the periodic case, when $l(0,\zeta) = l(2\pi,\zeta)$, the generator is given by $\mathrm{Str}(T_\gamma(\zeta))$, where $T_\gamma$ (the "closed" monodromy matrix) is the transport matrix around the cylindrical world-sheet $T_\gamma(\zeta) = T(2\pi, 0, \zeta)$. In the presence of boundaries the authors of [19] define the open monodromy matrix as

$$T(\zeta) = U_0 T^{-1}(\pi, 0, \zeta^{-1})U_\pi T(\pi, 0, \zeta) \tag{2.10}$$

where $U_{0,\pi}$ are constant matrices ("classical reflection matrices") with $U^2_{0,\pi} = \pm 1$, and show that the supertrace of the monodromy matrix is time-independent,

$$\partial_\tau \mathrm{Str}(T(\zeta)) = 0 \quad \longleftrightarrow \quad U_i l_\tau(i,\zeta)U_i^{-1} = l_\tau(i, \zeta^{-1})\,, \qquad i = 0, \pi \tag{2.11}$$

provided the automorphism $\Omega_U(h) = U\, h\, U^{-1}$ has the appropriate properties eq. (2.8)).[2] Thus, in the case of integrable boundaries, we can think of $\mathrm{Str}(T(\zeta))$ as the classical version of the *double row transfer matrix*, which generate the conserved charges.

[2] For the giant graviton/$Y = 0$ brane the $U_{0,\pi}$ matrices are given explicitly in appendix A.



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### 2.1.3 Monodromy matrix and conserved charges

It is important to obtain the relation between the "open" monodromy matrix $T(\zeta)$ and the conserved global charges $Q$. To see this one expands $l_\sigma(\zeta)$ and $T(\zeta)$ around $\zeta=1$. Writing $\zeta = 1-w$ in (2.7) it is straightforward to show that $l_\sigma(1-w) = wJ^\tau/g + \dots$, where $J^\tau$ is the time-like component of the conserved $\partial_\alpha J^\alpha = 0$ global symmetry current [20] thus $T(\pi,0,1-w) = 1+w\tilde{Q}/g+\dots$ where $\tilde{Q} = \int_0^\pi d\sigma J^\tau$. In the presence of the boundaries only a part of $\tilde{Q}$ remains conserved, namely the components commuting with the $U$-s: $Q \in \tilde{Q}$, $[Q, U_{0,\pi}] = 0$. Using (2.10) one finds

$$.T(\zeta)|_{\zeta=1-w} = U_0U_\pi + \frac{w}{g}(U_0\tilde{Q}U_\pi + U_0U_\pi\tilde{Q}) + \dots = U_0U_\pi\left(1+\frac{w}{g}(U_\pi^{-1}\tilde{Q}U_\pi + \tilde{Q}) + \dots\right) \tag{2.12}$$

However $U_\pi^{-1}\tilde{Q}U_\pi$ is nothing but the image of $\tilde{Q}$ under the involutive automorhism of the algebra used to define the boundary conditions:

$$U_\pi^{-1}\tilde{Q}U_\pi = \Omega_{U_\pi^{-1}}(\tilde{Q})\,.$$

Since $\Omega^2_{U_\pi^{-1}} = 1$, $\Omega_{U_\pi^{-1}}$ splits the algebra into two eigensubspaces with eigenvalues $\pm 1$. The components belonging to the subspace with eigenvalue 1 constitute the conserved charges $Q$, while the components belonging to the subspace with $-1$ just cancel from eq. (2.12). Thus eventually eq. (2.12) becomes

$$T(\zeta)|_{\zeta=1-w} = U_0U_\pi\left(1+\frac{2w}{g}Q+\dots\right) \tag{2.13}$$

### 2.1.4 Symmetries of the monodromy matrix

The symmetry equations for $T(\zeta)$ are obtained by combining the transformation property of $L_\alpha(\zeta)$ under the $\mathbb{Z}_4$ automorphism [20]

$$\mathcal{K}L_\alpha(\zeta)^{\mathrm{ST}}\mathcal{K}^{-1} = -L_\alpha(i\zeta) \tag{2.14}$$

(where $\mathcal{K}$ is the $8\times 8$ matrix implementing the automorphism) and the $[U_0, g(0)] = 0$, $[U_\pi, g(\pi)] = 0$ properties [19] of the $U_{0,\pi}$ matrices.

First we relate $T(\zeta)$ to the analogous open monodromy matrix built with the aid of the $L_\alpha$ connection instead of $l_\alpha$: denoting $\tilde{T}(\pi,0,\zeta) = P\exp(\int\limits_0^\pi d\sigma L_\sigma(\zeta))$ we find

$$T(\zeta) = g(0)^{-1}\tilde{T}(\zeta)g(0), \quad \tilde{T}(\zeta) = U_0\tilde{T}^{-1}(\pi,0,\zeta^{-1})U_\pi\tilde{T}(\pi,0,\zeta)\,. \tag{2.15}$$

Then, since according to [19] $U_{0,\pi}$ also satisfy $\mathcal{K}^{-1}U_{0,\pi}\mathcal{K} = -U_{0,\pi}^{\mathrm{ST}}$, using also (2.14), one easily gets

$$\tilde{T}(i\zeta) = \mathcal{K}\Big(\tilde{T}^{-1}(\zeta)\Big)^{\mathrm{ST}}\mathcal{K}^{-1}\,, \tag{2.16}$$

It follows then that $T(\zeta)$ satisfies the symmetry equation

$$T(i\zeta) = \tilde{\mathcal{K}}(T^{-1}(\zeta))^{\mathrm{ST}}(\tilde{\mathcal{K}})^{-1}, \qquad \tilde{\mathcal{K}} = g(0)^{-1}\mathcal{K}(g(0)^{-1})^{\mathrm{ST}} \tag{2.17}$$

while the definition and $U^2_{0,\pi} = \pm 1$ guarantee that

$$T(\zeta^{-1}) = U_0T^{-1}(\zeta)U_0^{-1} \tag{2.18}$$

is also satisfied.

### 2.2 The spectral curve of quasi-momenta

In the following we define the spectral curve from the eigenvalues of the "open" monodromy matrix $T(\zeta)$.

#### 2.2.1 Quasi-momenta

The 4+4 eigenvalues $(\lambda_1, \dots \lambda_4|\mu_1, \dots \mu_4)$ of $T(\zeta)$ can be expressed in terms of the so called quasi-momenta of $S^5$ and $AdS_5$ as

$$\lambda_i = e^{-i\tilde{p}_i(\zeta)}, \qquad \mu_i = e^{-i\hat{p}_i(\zeta)}, \qquad i = 1, \dots 4 \tag{2.19}$$

Following from its definition $T(\zeta)$ depends analytically on $\zeta$ (apart from the points $\zeta = 0,\ \infty$), but this property is not necessarily inherited by the $\lambda_i$ $\mu_i$ eigenvalues. Just as in the closed string case [6] there are square root type singularities when two $\lambda$-s or two $\mu$-s coincide while at those points where eigenvalues having opposite gradings coincide both of them have first order poles. To obtain a single valued and analytic function on the entire complex plane with the exception of these singularities we define $Y(\zeta)$ - in analogy to the closed string case [6] — as

$$m(\zeta)Y(\zeta)m^{-1}(\zeta) = -i\zeta\frac{d}{d\zeta}\log(m(\zeta)T(\zeta)m^{-1}(\zeta)) \tag{2.20}$$

where $m(\zeta)$ diagonalizes $T(\zeta)$. This definition makes it possible to write[3]

$$Y(\zeta) = T^{-1}(\zeta)\Big(-i\zeta\frac{d}{d\zeta}T(\zeta) + [M(\zeta), T(\zeta)]\Big), \qquad M(\zeta) = -i\zeta m^{-1}\frac{d}{d\zeta}m\,. \tag{2.21}$$

The eigenvalues of $Y(\zeta)$ (that are the logarithmic derivatives of $\lambda_i$ and $\mu_i$) are determined by the zeroes and poles of its characteristic function

$$F(\tilde{y}(\zeta), \zeta) = 0, \qquad F(\hat{y}(\zeta), \zeta) = \infty, \qquad F(y, \zeta) = \frac{\tilde{P}(\zeta)}{\hat{P}(\zeta)}\mathrm{sdet}\Big(y - Y(\zeta)\Big). \tag{2.22}$$

(The polynomial prefactors are introduced to absorb the poles coming from $M(\zeta)$ without changing the curve see [6]). The symmetry equations of the "open" monodromy matrix (2.17), (2.18) can be converted into symmetry equations of $Y(\zeta)$ and $M(\zeta)$[4]

$$Y(i\zeta) = -\tilde{\mathcal{K}}Y(\zeta)^{\mathrm{ST}}\tilde{\mathcal{K}}^{-1}, \qquad M(i\zeta) = -\tilde{\mathcal{K}}M(\zeta)^{\mathrm{ST}}\tilde{\mathcal{K}}^{-1}, \tag{2.23}$$

$$Y(\zeta^{-1}) = \mathcal{N}Y(\zeta)\mathcal{N}^{-1}, \qquad M(\zeta^{-1}) = -U_0 M(\zeta)U_0^{-1}, \tag{2.24}$$

(with $\mathcal{N} = U_0 T(\zeta)$), where these equations imply that

$$F(y, i\zeta) = F(-y, \zeta), \qquad \text{and} \qquad F(y, \zeta^{-1}) = F(y, \zeta). \tag{2.25}$$

[3]This form shows that $Y(\zeta)$ contains only pole type singularities since $M(\zeta)$ has first order poles at the square root type branch points of $m(\zeta)$.

[4]Both $m(\zeta)$ and $m(\zeta^{-1})U_0$ diagonalize $T^{-1}(\zeta)$, so they must be related with some constant permutation matrix, $W$, as $m(\zeta) = Wm(\zeta^{-1})U_0$, which disappears from $M$.



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Therefore $F(y,\zeta)$ may depend analytically only on $y^2$, $y(\zeta^2+\zeta^{-2})$ and $\zeta^4+\zeta^{-4}$, and $y$ must be a function of $\zeta^2+\zeta^{-2}$. We introduce the variable

$$x = \frac{1+\zeta^2}{1-\zeta^2} \tag{2.26}$$

which is identical to the spectral parameter used in the closed string case [6] — and from now on we may think of (the eigenvalues of) the open monodromy matrix as being a function of $x$: $T(x)$. (Note that $\zeta \to 1/\zeta$ changes $x$ as $x \to -x$, while the $\zeta \to i\zeta$ map induces $x \to 1/x$).

### 2.2.2 Symmetries and analytical structure of the quasi-momenta

The symmetry equations (2.23), (2.24), (2.25) impose some restrictions on the quasimomenta $(\tilde{p}_i, \hat{p}_i)$. Since the $x \to 1/x$ $(\zeta \to i\zeta)$ inversion symmetry equations for $Y(\zeta)/F(y,\zeta)$ are the same as for the closed string case the restrictions they impose are also the same (with a minor difference):

$$\begin{aligned} \tilde{p}_{1,2}(x) &= -\tilde{p}_{2,1}(1/x), & \tilde{p}_{3,4}(x) &= -\tilde{p}_{4,3}(1/x), \\ \hat{p}_{1,2}(x) &= -\hat{p}_{2,1}(1/x), & \hat{p}_{3,4}(x) &= -\hat{p}_{4,3}(1/x), \end{aligned} \tag{2.27}$$

where the absence of winding in the $S^5$ component $\tilde{p}_i$ is the difference to the closed string case. On the other hand the $x \to -x$ $(\zeta \to 1/\zeta)$ reflection symmetry equations in (2.24), (2.25) require

$$\tilde{p}_i(-x) = -\tilde{p}_i(x), \quad \hat{p}_i(-x) = -\hat{p}_i(x), \quad i = 1, \ldots, 4. \tag{2.28}$$

These extra properties are the consequence of the boundaries and are not present for generic periodic states. Indeed, the quasimomenta $\hat{p}_i$ are related to the even eigenvalues $(y_i(-x) = y_i(x))$ of $Y(x)$ by $(x^2-1)\frac{dp_i}{dx} = y_i(x)$ since $\zeta\frac{d}{d\zeta} = (x^2-1)\frac{d}{dx}$ and we choose the integration constant to guarantee (2.25). Alternatively, the reflection symmetry of the quasimomenta can be directly obtained from (2.18).

The $l_\alpha$ (or $L_\alpha$) connection has singularities at $x = \pm 1$ $(\zeta = 0$ resp. $\zeta = \infty)$ and they imply simple poles for the quasimomenta. $l_\alpha$ is supertraceless since $l_\alpha \in \mathrm{psu}(2,2|4)$ while the Virasoro constraint — which is not modified by the presence of the boundary — forces its square also to be supertraceless. Combining these with the inversion and reflection symmetries synchronizes the various residues as:

$$\{\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4 | \hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4\} \sim \frac{x}{x^2-1}\{\alpha, \alpha, \beta, \beta | \alpha, \alpha, \beta, \beta\} \tag{2.29}$$

Finally we mention that the asymptotic behaviour in (2.13) can be converted into the $x \to \infty$ behaviour of the quasimomenta

$$\mathrm{diag}(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4 | \hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4) \sim \frac{2}{gx} iQ_{\mathrm{diag}} \tag{2.30}$$

where $Q_{\mathrm{diag}}$ is the sum of Cartan generators with eigenvalues characterising the solution. Note that they automatically commute with the diagonal $U$, eq. (A.7), of the $Y = 0$ brane.



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The $\{\tilde{p}_i(x)|\hat{p}_i(x)\}$ quasimomenta form an eight sheeted Riemann surface very similar to the closed string case, where the $\tilde{p}_i(x)$ and $\hat{p}_i(x)$ sheet functions are analytic almost everywhere. Apart from the single poles at $x = \pm 1$ where their residues are synchronized as in (2.29) they may have branch cuts with square root type end points connecting either two $\tilde{p}_i(x)$ or two $\hat{p}_i(x)$ sheets (corresponding to bosonic degrees of freedom) or they may have single poles existing simultaneously on a $\tilde{p}_i(x)$ and a $\hat{p}_j(x)$ sheet (corresponding to fermionic degrees of freedom). The important point is that these cuts and poles must respect the inversion and reflection symmetries: the (non invariant) generic ones come in fourfold multiplets to provide the representation of the symmetry.

## 3 Explicit quasimomenta for circular open strings

The metric on $AdS_5 \times S^5$ is given by

$$\begin{aligned} ds^2 =& -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho(d\alpha^2 + \sin^2\alpha d\Phi^2 + \cos^2\alpha d\phi^2) \\ &+ d\gamma^2 + \cos^2\gamma d\phi_1^2 + \sin^2\gamma(d\psi^2 + \cos^2\psi d\phi_2^2 + \sin^2\psi d\phi_3^2) \end{aligned} \tag{3.1}$$

while the global $X$, $Z$ and $Y$ coordinates of the $S^5$ are

$$X = \cos\gamma e^{i\phi_1} \qquad Z = \sin\gamma\cos\psi e^{i\phi_2} \qquad Y = \sin\gamma\sin\psi e^{i\phi_3} \tag{3.2}$$

The giant graviton corresponding to the $Z = 0$ brane ($\psi \equiv \pi/2$) is the $S^3$ described by

$$d\gamma^2 + \cos^2\gamma d\phi_1^2 + \sin^2\gamma d\phi_3^2 \tag{3.3}$$

while for the $Y = 0$ brane ($\psi \equiv 0$ or $\psi \equiv \pi$) it is the $S^3$ given by

$$d\gamma^2 + \cos^2\gamma d\phi_1^2 + \sin^2\gamma d\phi_2^2 \,. \tag{3.4}$$

These two $S^3$-s are of course obtained from each other by a rotation, however the open strings ending on them have different properties because the two $S^3$-s are aligned in a different way with respect to the ground state. In the following we will construct the explicit quasimomenta for $Y = 0$ brane set-up.

We first recall from [21], that a simple rotating spinning closed string solution when cut into "half" satisfies the boundary conditions

$$Y = 0, \qquad \partial_\sigma X = \partial_\sigma Z = 0 \tag{3.5}$$

appropriate for the $Y = 0$ brane, thus the half can be used as a rotating spinning open string solution. In the simplest case this solution just describes the open BMN string. We investigate the open monodromy matrix $T(x)$ for a class of solutions and obtain its eigenvalues explicitly not only for the BMN string but also for a subset of the rotating spinning strings.

This class of solutions is given by $\rho \equiv 0$ and the following $X$, $Y$ and $Z$

$$\gamma \equiv \frac{\pi}{2} \quad \leftrightarrow \quad X \equiv 0, \qquad Z = \cos(n\sigma)e^{iw\tau}, \qquad Y = \sin(n\sigma)e^{iw\tau} \qquad t = \kappa\tau \,, \tag{3.6}$$

where $\sigma$ is running in $(0,\pi)$ only, $n$ is an integer (we consider the case when it is an even integer), and the $\kappa$, $w$ constants are given as

$$w^2 = n^2 + \nu^2, \qquad \kappa^2 = \nu^2 + 2n^2 \tag{3.7}$$

in terms of $n$ and an arbitrary real constant $\nu$. (Note that the interior of the open string is away from the $Y = 0$ surface, only its endpoints move there). The energy and angular momenta of this open string solution are[5]

$$E = \frac{1}{2}\sqrt{\lambda}\kappa, \qquad J_Z = J_Y = \frac{1}{4}\sqrt{\lambda}\sqrt{n^2+\nu^2}\,. \tag{3.8}$$

The BMN string is obtained for $n = 0$, in this case $E$ and $J_{Z,Y}$ become proportional to each other.

The first step to construct the open monodromy matrix for this solution is to obtain the explicit form of the bosonic sectors current (which is, in fact the complete current since the solution has no fermionic component). We can do this in two ways: either we specialize the general expression in [19] to the present case or we construct it from the coset space representative appropriate for the solution: $g_{\rm sol} = e^{-P_0\kappa\tau}e^{P_8 w\tau}e^{J_{56}w\tau}e^{P_6 n\sigma}$. See appendix A for the explicit $P_i$ matrices. In either way one obtains

$$A_\tau^{(2)} = P_0\kappa + P_5 w\sin(n\sigma) + P_8 w\cos(n\sigma), \qquad A_\sigma^{(2)} = P_6 n, \qquad A_\sigma^{(0)} \equiv 0. \tag{3.9}$$

Through eq. (2.3)–(2.4) this leads to

$$L_\sigma = \frac{x^2+1}{x^2-1}P_6 n - \frac{2x}{x^2-1}\Big(P_0\kappa + P_5 w\sin(n\sigma) + P_8 w\cos(n\sigma)\Big)\,. \tag{3.10}$$

The relation between $T(x)$ and the analogous expression built with the aid of $L_\sigma$ instead of $l_\sigma$ is given in eq. (2.15), where $g(0) = g_{\rm sol}(0) = e^{-P_0\kappa\tau}e^{P_8 w\tau}e^{J_{56}w\tau}$. Since this is a similarity transformation as long as we are interested in the eigenvalues of $T(x)$ we may consider that of $\tilde{T}(x)$ instead.

### 3.1 The BMN string

For the BMN string, when $n = 0$, the situation is even simpler as

$$l_\sigma = -\frac{2x}{x^2-1}\nu(P_0 + P_8) \tag{3.11}$$

is independent of $\sigma$ thus $T(\pi,0,x)$ is readily obtained

$$T(\pi,0,x) = \exp\Big(\Omega(P_0+P_8)\Big), \qquad \Omega = -\frac{2\pi\nu x}{x^2-1}\,. \tag{3.12}$$

[5]Note that $L = J_Z + J_Y$ corresponds to the number of fields in the determinant type operators of $N = 4$ SYM. For example, in the operator such as $\mathcal{O} \sim \sum_k \epsilon_{i_1\ldots i_N}\epsilon^{j_1\ldots j_N}Y^{i_1}_{j_1}\ldots Y^{i_{N-1}}_{j_{N-1}}(Z^k\chi Z^{J-k})^{i_N}_{j_N}$, the complex scalar $Z$ and impurity $\chi$ correspond to the open strings configuration and the part of $Y$ products is the maximal giant graviton. In this example, the total number of $Z$ and $\chi$ fields is $L = J + 1$.



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This then leads through (2.10) and (A.7) to

$$T(\zeta) = (-)\begin{pmatrix} M & \\ & \mathrm{diag}(e^{i\Omega}, e^{i\Omega}, e^{-i\Omega}, e^{-i\Omega}) \end{pmatrix}, \quad M = \begin{pmatrix} \cos\Omega & 0 & \sin\Omega & 0 \\ 0 & \cos\Omega & 0 & -\sin\Omega \\ -\sin\Omega & 0 & \cos\Omega & 0 \\ 0 & \sin\Omega & 0 & \cos\Omega \end{pmatrix}. \tag{3.13}$$

Note that this open monodromy matrix is non diagonal even for the BMN string, since $M$ is non diagonal. Nevertheless $M$-s eigenvalues — two times $e^{i\Omega}$ and two times $e^{-i\Omega}$ — coincide with those in the lower right corner of $T(x)$. The quasimomenta for the BMN string as one reads off from (3.13) are[6]

$$\hat{p}_{1,2} = -\hat{p}_{3,4} = \tilde{p}_{1,2} = -\tilde{p}_{3,4} = \frac{2\pi\nu x}{x^2-1}, \tag{3.14}$$

### 3.2 Quasimomenta for the solutions with $n = 2N \neq 0$

As mentioned above for these solutions we determine the eigenvalues of the open monodromy matrix built from the $L_\sigma$ connection instead of $l_\sigma$

$$\tilde{T}(x) = U_0 \tilde{T}^{-1}(\pi, 0, -x) U_\pi \tilde{T}(\pi, 0, x), \qquad \tilde{T}(\pi, 0, x) = P\exp\left(\int_0^\pi d\sigma L_\sigma(x)\right), \tag{3.15}$$

since — according to eq. (2.15) — they are the same as that of $T(\zeta)$.

We start with the matrix form of $L_\sigma$

$$L_\sigma = \frac{x^2+1}{x^2-1} P_6 n - \frac{2x}{x^2-1}\Big(P_0\kappa + P_5 w\sin(n\sigma) + P_8 w\cos(n\sigma)\Big) = \begin{pmatrix} H & 0 \\ 0 & K \end{pmatrix}, \tag{3.16}$$

where $H$ and $K$ are $4\times 4$ matrices. The $K$ matrix in the "$AdS_5$ corner" is diagonal

$$K = -\frac{2x}{x^2-1} P_0\kappa = -\frac{2x}{x^2-1}\kappa\frac{i}{2}\mathrm{diag}\,(1, 1, -1, -1), \tag{3.17}$$

thus the $AdS_5$ eigenvalues of $T(x)$ are the doubly degenerate $e^{\pm i\frac{2\pi\kappa x}{x^2-1}}$. This leads to the following $AdS_5$ quasimomenta

$$\hat{p}_{1,2} = -\hat{p}_{3,4} = \frac{2\pi\kappa x}{x^2-1} = \frac{x}{x^2-1}\frac{E}{g}. \tag{3.18}$$

The matrix $H$ in the "$S_5$ corner" of $L_\sigma$ is

$$H = \begin{pmatrix} 0 & -\tilde{b} \\ \tilde{b} & 0 \end{pmatrix}, \quad \tilde{b} = \tilde{\beta}\begin{pmatrix} xw(e^{in\sigma} + e^{-in\sigma}) & \hat{n} - xw(e^{in\sigma} - e^{-in\sigma}) \\ \hat{n} + xw(e^{in\sigma} - e^{-in\sigma}) & -xw(e^{in\sigma} + e^{-in\sigma}) \end{pmatrix} \tag{3.19}$$

where $\tilde{\beta} = \frac{1}{2(x^2-1)}$, $\hat{n} = n(x^2+1)$. The problem with this matrix is that it depends on a non trivial way on $\sigma$, which makes very complicated to compute its path ordered

[6] Note that these quasimomenta are identical to the ones of the closed BMN string.

exponential $\tilde{t}(\pi, 0, x) = P\exp(\int_0^\pi d\sigma H(x))$. We overcome this problem by recalling that this path ordered exponential is related to the solution of the (vector) differential equation $\partial_\sigma \psi = H\psi$ by $\psi(\sigma) = \tilde{t}(\sigma, 0, x)\psi(0)$. We solve this linear problem in appendix B and obtain the following $S^5$ quasimomenta:

$$\tilde{p}_1 = \frac{2\pi x}{x^2-1}\sqrt{\frac{n^2}{x^2}+w^2} = -\tilde{p}_4, \qquad \tilde{p}_2 = \frac{2\pi x}{x^2-1}\sqrt{n^2x^2+w^2} = -\tilde{p}_3. \tag{3.20}$$

(Note that these quasimomenta are identical to the "one cut" solutions in [7, 22] ). The set of quasimomenta given in eq. (3.18), (3.20) satisfies the requirements following from the inversion and reflection symmetries as well as the residue synchronization condition in a non trivial way.



## 4 Quasimomenta from the $Y$ system

For the periodic, closed string case the solutions of the AdS/CFT $Y$ and $T$ systems are obtained in the strong coupling scaling limit in [14, 15]. These limiting solutions can be compared to the result of semiclassical quantization based on the spectral curve, in particular, the classical quasimomenta could be described in terms of some conserved quantities and certain densities of Bethe roots (resolvent densities). In a recent paper [23] a conjecture is made that the $Y = 0$ brane is described by the same $Y$ and $T$ systems as the closed (periodic) case, only the asymptotic solutions and the analytic properties of the $Y$ and $T$ functions are different. Therefore, in this section, we repeat the procedure of [14] and [15] for the $Y = 0$ brane, i.e. we consider the strong coupling scaling limit of the asymptotic ($L \to \infty$) $Y$ and $T$ functions described in [23] and obtain the quasimomenta from the limiting solution.

### 4.1 Classical T-system for the $Y = 0$ brane

First, following [24], we consider the monodromy matrix $T(x)$ in appropriate unitary highest weight irreps $\Lambda$ of $SU(2,2|4)$ to describe the classical T-system, and denote their supertrace as $D_\Lambda = \mathrm{Str}_\Lambda T(x)$. The irreps having rectangular Young-tableaux $h_i = s + 2$, $i = 1, \ldots a$ (denoted as $[a, s]$) play a distinguished role, as they form a closed set under tensor multiplication: $[a, s] \otimes [a, s] = [a+1, s] \otimes [a-1, s] \oplus [a, s-1] \otimes [a, s+1]$. Evaluating this equation for the representatives of the monodromy matrix and taking the supertrace we find

$$D_{a,s}D_{a,s} = D_{a-1,s}D_{a+1,s} + D_{a,s-1}D_{a,s+1} \tag{4.1}$$

This classical equation is the strong coupling $g \to \infty$ limit of the quantum Hirota equation:

$$\mathbb{D}^+_{a,s}\mathbb{D}^-_{a,s} = \mathbb{D}_{a-1,s}\mathbb{D}_{a+1,s} + \mathbb{D}_{a,s-1}\mathbb{D}_{a,s+1} \tag{4.2}$$

where here and from now on $f^{[n]}(u) = f(u + n\frac{i}{2})$; $f^{\pm}(u) \equiv f^{[\pm 1]}(u)$. In this limit the left hand side of the latter equation contains no shift in the parameter $u = g(x + 1/x)$ (since $u \sim g$). Note that in the case of closed strings $T_{a,s} = \mathrm{Str}_{[a,s]}T_\gamma(x)$ satisfy the same equations (4.1). This gives further support to the conjecture made in [23].

The general solution of equations (4.1) (with appropriate "T hook" boundary conditions) is given in [15].

### 4.2 Asymptotic $Y$ and $T$ functions in the scaling limit

We collect here the asymptotic large $L \to \infty$ solutions of the $Y$ and $T$ (quantum Hirota eq. (4.2)) systems as given for the $Y = 0$ brane in [23].[7]

For each state in the theory there is a collection of $Y$ functions satisfying the $Y$ system relation

$$Y^+_{a,s} Y^-_{a,s} = \frac{(1+Y_{a,s-1})(1+Y_{a,s+1})}{(1+1/Y_{a-1,s})(1+1/Y_{a+1,s})} \tag{4.3}$$

where $a$ and $s$ are integers. Non-trivial $Y$ functions live on the "T-hook" in which either $a = 1$ or $s \in (-1, 0, 1)$ and $a$ positive. There are also two "exceptional" points $a = 2, s = \pm 2$.

This $Y$ system can be solved in terms of the $T$ system, whose elements in the boundary problem are denoted by $\mathbb{D}_{a,s}$:

$$Y_{a,s} = \frac{\mathbb{D}_{a,s-1}\mathbb{D}_{a,s+1}}{\mathbb{D}_{a-1,s}\mathbb{D}_{a+1,s}} \tag{4.4}$$

where $\mathbb{D}_{a,s}$ satisfies the quantum Hirota equations eq. (4.2), and live in a wider T-hook including the $a = 0, 2$ and $s = \pm 2$ lines, too.

#### 4.2.1 States in the $su(2)$ sector

Here we present the asymptotic solutions of these equations relevant for $Y = 0$ brane for states in the SU(2) subsector where the multiparticle states are composed of particles of $1\dot{1}$ type.[8] We analyze the general case afterwards.

The asymptotic transfer matrices $\mathbb{D}_{a,1}$ are generated by the generating functional

$$\begin{aligned} \mathcal{W}^{-1}_{su(2)} &= \left(1 - \mathcal{D}F\frac{\mathcal{R}^{(+)+}}{\mathcal{R}^{(-)+}}\mathcal{D}\right)\left(1-\mathcal{D}F\mathcal{D}\right)^{-1}\left(1-\mathcal{D}F\frac{u^+}{u^-}\mathcal{D}\right)^{-1}\left(1-\mathcal{D}F\frac{u^+}{u^-}\frac{\mathcal{B}^{(-)-}}{\mathcal{B}^{(+)-}}\mathcal{D}\right) \\ &= \sum_a (-1)^a \mathcal{D}^a \mathbb{D}_{a,1} \mathcal{D}^a \,, \end{aligned} \tag{4.5}$$

where $\mathcal{D} = e^{-\frac{i}{2}\partial_u}$, (and therefore $\mathcal{D}f = f^-\mathcal{D}$) and we normalized these transfer matrices similarly to the periodic case

$$F = \sqrt{\frac{Q^{[2]}(u)}{Q^{[-2]}(u)}\frac{u^-}{u^+}}\left(\frac{x^-}{x^+}\right)^{N+1+L}\left(\frac{\mathcal{R}^{(-)+}}{\mathcal{R}^{(+)+}}\right)\prod_{i=1}^{N}\sigma(p,p_i)\sigma(p_i,-p)\,. \tag{4.6}$$

The functions $\mathcal{B}^{(\pm)}, \mathcal{R}^{(\pm)}$ are defined as follows

$$\begin{aligned} \mathcal{R}^{(\pm)} &= \prod_{i=1}^{N}\left(x(p) - x^{\mp}(p_i)\right)\left(x(p)+x^{\pm}(p_i)\right)\,, \qquad & Q(u) &= \prod_{i=1}^{N}(u-u_i)(u+u_i) \\ \mathcal{B}^{(\pm)} &= \prod_{i=1}^{N}\left(\frac{1}{x(p)} - x^{\mp}(p_i)\right)\left(\frac{1}{x(p)}+x^{\pm}(p_i)\right), \qquad & x^{\pm}+\frac{1}{x^{\pm}} &= \frac{1}{g}\left(u \pm \frac{i}{2}\right)\,. \end{aligned} \tag{4.7}$$

The spectral parameter $u$ is related to the momentum as $u = \frac{1}{2}\cot(\frac{p}{2})\sqrt{1+16g^2\sin^2(\frac{p}{2})}$.

[7] These solutions are also called asymptotic $Y$ and $T$ functions.

[8] Here we denote the components of the fundamental representation of SU(2|2) by $a = 1 \ldots 4$, and to distinguish the two SU(2|2)-s in SU(2|2)$\times$SU(2|2) denote the components of the second SU(2|2) by $\dot{a}$. Thus the fundamental excitations can be classified by $a\dot{b}$, $a = 1, \ldots 4$, $\dot{b} = 1, \ldots 4$ and in the SU(2) subsector we consider only multiparticle states when all particles carry $a = 1 = \dot{a}$.

Note that this $N$ particle eigenvalue of the fundamental double row transfer matrix is similar to the $2N$ particle eigenvalue of the bulk transfer matrix where there is a "doubling" of particles: to every particle with $x_j$ there is a "reflected" one with $-x_j$. The presence of the $\frac{u^+}{u^-}$ factors is an extra modification that can be attributed to the boundary.

The $su(2)$ sector is symmetric $\mathbb{D}_{a,-1} = \mathbb{D}_{a,1}$ and asymptotically we have $\mathbb{D}_{a,0} = 1$, from which $\mathbb{D}_{a,\pm 2}$ can be calculated by the equations (4.2). $Y_{a,0}$ is given by the standard expression $Y_{a,0} = \frac{\mathbb{D}_{a,1}\mathbb{D}_{a,-1}}{\mathbb{D}_{a+1,0}\mathbb{D}_{a-1,0}} = \mathbb{D}_{a,1}\mathbb{D}_{a,-1}$. In the following we are interested in the scaling limit of $\mathbb{D}_{a,s}$, in particular whether it may be identified with $D_{a,s}$.

Now we consider the scaling strong coupling limit ($g \to \infty$) of the asymptotic large $L$ solution of the $Y = 0$ brane's $T$ system described above. In this limit the length $L$ and the number of particles $N$ (and also, if present, the number of auxiliary Bethe roots) go to infinity $L \sim N \sim g$. To describe this limit we introduce a new variable $z$ instead of $u$: $u = 2gz$ such that

$$x(z) = z + i\sqrt{1-z^2}, \qquad x_j = x^{\mathrm{ph}}(z_j) = z_j + \sqrt{z_j - 1}\sqrt{z_j + 1}, \tag{4.8}$$

and $x^{\pm}(z) = x(z \pm \frac{i}{4g})$, $\quad x_j^{\pm} = x^{\mathrm{ph}}(z_j \pm \frac{i}{4g})$. Treating $i/(4g)$ as a small parameter, after a straightforward computation one finds that the strong coupling limit of the various functions appearing in $\mathcal{W}_{su(2)}$ are

$$\begin{aligned}
&\frac{\mathcal{R}^{(+)+}}{\mathcal{R}^{(-)+}} \simeq f(z) = \exp\left[\frac{i}{g}(G_-(x) + G_+(x))\right]; \qquad && G_{\mp}(x) = \sum_{j=1}^{N} \frac{x_j^2}{x_j^2 - 1}\frac{1}{x \mp x_j} \\
&\frac{\mathcal{B}^{(-)-}}{\mathcal{B}^{(+)-}} \simeq \tilde{f}(z) = \exp\left[-\frac{i}{g}(G_-(1/x) + G_+(1/x))\right], && \frac{u^+}{u^-} \simeq h(z) = \exp\left(\frac{i}{2gz}\right)
\end{aligned} \tag{4.9}$$

and

$$F \simeq \Phi(z) = \exp\left[-\frac{L + 1 + \sum\limits_{j=1}^{N} E_j^{(1)}}{2g\sqrt{1-z^2}} - \frac{i}{4gz}\right], \tag{4.10}$$

where, at leading order, $E_j^{(1)} = \frac{x_j^2+1}{x_j^2-1}$ is the energy of the $j$-th fundamental particle. Here we used the AFS phase for the dressing factor $\sigma(p, p_i)$ in the strong coupling limit [25, 26] which is given as

$$\begin{aligned}
\log\sigma(z, x_j) &= \log\left(\frac{1 - \frac{1}{x^-(z)x_j^+}}{1 - \frac{1}{x^+(z)x_j^-}}\right) + 2ig(z_j - z)\log\left(\frac{x^-(z)x_j^- - 1}{x^+(z)x_j^- - 1}\frac{x^+(z)x_j^+ - 1}{x^-(z)x_j^+ - 1}\right) \\
&\simeq \frac{i(x(z) - x_j)}{g(-1 + x(z)^2)(-1 + x(z)x_j)(-1 + x_j^2)}
\end{aligned} \tag{4.11}$$

where the mirror variable is denoted by $z(p)$ and one can use $x^{\pm}(-p) = -x^{\mp}(p)$ for $\sigma(p_i, -p)$. Also, in the scaling limit the shifted spectral parameters $x^{\pm}$ becomes

$$x^{\pm}(z) = x(z) \pm \frac{i}{2g}\frac{x^2(z)}{x^2(z) - 1} + O(1/g^2), \tag{4.12}$$

as we expand $x^{\pm}(z) \simeq x(z) \pm \frac{i}{4g}\partial_z x(z)$.

For the expansion of $\mathcal{W}_{su(2)}^{-1}$ (or $\mathcal{W}_{su(2)}$), eq. (4.5), in the scaling limit it is important to emphasize that in this limit the operator $\mathcal{D}$ serves only as a formal expansion parameter since the $\pm i/(4g)$ shifts it generates become negligible. The limit of $\mathcal{W}_{su(2)}$ becomes

$$\mathcal{W}_{su(2)} \simeq \frac{(1-h(z)\Phi(z)\mathcal{D}^2)(1-\Phi(z)\mathcal{D}^2)}{(1-\Phi(z)h(z)\tilde{f}(z)\mathcal{D}^2)(1-\Phi(z)f(z)\mathcal{D}^2)} \equiv \tilde{\mathcal{W}}_{su(2)}\,, \tag{4.13}$$

and the scaling limit of $\mathbb{D}_{a,1} \simeq \tilde{D}_{a,1}$ is determined by $(\tilde{W}_{\mathrm{SU}(2)})^{-1} = \sum_a (-1)^a \tilde{D}_{a,1}\mathcal{D}^{2a}$. In the classical theory the generating function of the $\mathrm{SU}(2,2|4)$ (super)characters of the symmetric representations is

$$w_{4|4} = \frac{(1-y_1t)(1-y_2t)}{(1-x_1t)(1-x_2t)} \times \frac{(1-y_3t)(1-y_4t)}{(1-x_3t)(1-x_4t)} = \hat{W}^L(x_1,x_2;y_1,y_2)\hat{W}^R(x_3,x_4;y_3,y_4), \tag{4.14}$$

if $(x_1,\dots x_4|y_1,\dots y_4)$ denotes the set of eigenvalues of the group element and it is described in [15] how the characters $T_{a,s}(x_1,\dots x_4|y_1,\dots y_4)$ satisfying

$$T_{a,s}(x_1,\dots x_4|y_1,\dots y_4) = T_{a,-s}(1/x_4,\dots 1/x_1|1/y_4,\dots 1/y_1) \tag{4.15}$$

can be obtained from $w_{4|4}$. We want to use this machinery for the "open" monodromy matrix $T(x)$ to construct $D_{a,s}$ in the SU(2) subsector. To respect the $D_{a,s} = D_{a,-s}$ symmetry we assume $\lambda_1 = 1/\lambda_4$, $\lambda_2 = 1/\lambda_3$, $\mu_1 = 1/\mu_4$, $\mu_2 = 1/\mu_3$, and write

$$\hat{W}^L = \frac{(1-\mu_1t)(1-\mu_2t)}{(1-\lambda_1t)(1-\lambda_2t)} = \hat{W}^R\,, \tag{4.16}$$

with $(\hat{W}^L)^{-1} = \sum_a (-1)^a D_{a,1}t^a$. The key to identify $\hat{W}^L$ and $\tilde{W}_{\mathrm{SU}(2)}$ is to find a relation between the two formal expansion parameters $t$ and $\mathcal{D}$ that guarantees that $\tilde{D}_{a,1} = D_{a,1}$ Following [15] to this end we propose the relation $\mathcal{D}^2 = \Phi t$. With this choice one can do even more: by comparing (4.13) and (4.16) one can identify the $\lambda_1$, $\lambda_2$, $\mu_1$, $\mu_2$ eigenvalues of $T(x)$ with certain limiting functions

$$\begin{aligned}
\mu_1 &= \Phi = \exp\left[-\frac{L+1+\sum_{j=1}^{N} E_j^{(1)}}{2g\sqrt{1-z^2}} - \frac{i}{4gz}\right]; \quad \mu_2 = h\Phi = \exp\left[-\frac{L+1+\sum_{j=1}^{N} E_j^{(1)}}{2g\sqrt{1-z^2}} + \frac{i}{4gz}\right] \\
\lambda_1 &= \tilde{f}\Phi = \exp\left[-\frac{J}{2g\sqrt{1-z^2}} - \frac{i}{4gz} - \frac{i}{g}(H_-(1/x)+H_+(1/x))\right], \\
\lambda_2 &= hf\Phi = \exp\left[-\frac{J}{2g\sqrt{1-z^2}} + \frac{i}{4gz} + \frac{i}{g}(H_-(x)+H_+(x))\right]
\end{aligned} \tag{4.17}$$

where $H_\mp(x) = \sum_{j=1}^{N} \frac{x^2}{x^2-1}\frac{1}{x\mp x_j}$ and $J = L+1+N$. We use $\frac{1}{2g\sqrt{1-z^2}} = \frac{i}{g}\frac{x}{x^2-1}$ and $\sum_j E_j^{(1)} = N + 2\sum_j \frac{1}{x_j^2-1}$ (that was also exploited to obtain (4.17)) in eq. (2.19) to connect the eigenvalues of $T(x)$ with the quasi-momenta:

$$\begin{aligned}
\hat{p}_1(x) &= -\hat{p}_4(x) = -\hat{p}_2(1/x) = \hat{p}_3(1/x) = \frac{(J+2Q_2)x}{g(x^2-1)} + B(x) \\
\tilde{p}_1(x) &= -\tilde{p}_4(x) = -\tilde{p}_2(1/x) = \tilde{p}_3(1/x) = \frac{Jx}{g(x^2-1)} + B(x) + \frac{1}{g}(H_-(1/x)+H_+(1/x))
\end{aligned} \tag{4.18}$$



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where the boundary contribution is $B(x) = \frac{1}{2g}\frac{x}{x^2+1}$ and $\mathcal{Q}_2 = \sum_j \frac{1}{x_j^2-1}$. It is interesting to compare these quasi-momenta with the ones for the closed string case presented in [6] and in [15]: the (conserved) quantity $\mathcal{Q}_2$ is present in $\hat{p}_i$ just like in the closed string case, while the $\mathcal{Q}_1 = \sum_j \frac{x_j}{x_j^2-1}$ is absent from $\tilde{p}_i$. This can be understood by recalling the "doubling" of particles mentioned above: since to every particle with $x_j$ there is another one with $-x_j$ that is why $\mathcal{Q}_2$ appears with a factor of 2 while $\mathcal{Q}_1$ indeed cancels. Note that this argument also explains why the sums of $H_-$ and $H_+$ with various arguments appear in the quasi-momenta. $N$ (the number of particles) appears in the quasi-momenta as a result of working in the SU(2) grading — see the section on the duality transformation. Also, one can see that only the quasi-momenta $\tilde{p}_2$ and $\tilde{p}_3$, which correspond to $S^3 \subset S^5$, have resolvents corresponding to some particle excitations while the total set of quasimomenta satisfies all the symmetry and synchronization constraints. We also note that the boundary contribution, $B(x)$, gives a new pole structure in the Riemann surface as a quantum effect.

### 4.2.2 Generic states

Next we turn to the discussion of generic multiparticle states when the individual fundamental particles carry general labels $a\dot{b}$. As discussed in [23] and [27] to describe them one has to introduce (in addition to $x_i$) $2m_1^L$ $(2m_1^R)$ $y$ roots and $2m_2^L$ $(2m_2^R)$ $w$ roots in the $\mathrm{SU}(2|2)_L$ $(\mathrm{SU}(2|2)_R)$ eigenvalues of the corresponding double row transfer matrices. A new feature of the $Y=0$ brane's ABA is that there are only two types of auxiliary roots as opposed to the three types present in the closed string/bulk case, see [23] and [27]. To describe these roots we introduce

$$\begin{aligned} \mathcal{B}_1^L\mathcal{R}_3^L &= \prod_{j=1}^{m_1^L}\left(x(p)-y_j^L\right)\left(x(p)+y_j^L\right), \qquad \mathcal{R}_1^L\mathcal{B}_3^L = \prod_{j=1}^{m_1^L}\left(\frac{1}{x(p)}-y_j^L\right)\left(\frac{1}{x(p)}+y_j^L\right), \\ Q_2^L(u) &= \prod_{l=1}^{m_2^L}(u-w_l^L)(u+w_l^L)\,. \end{aligned} \tag{4.19}$$

The generating functional for the eigenvalues of the $\mathbb{D}_{a,1}$ double row transfer matrices in antisymmetric representations can be written in terms of these quantities as [23]

$$\begin{aligned} (\mathcal{W}^L_{su(2)})^{-1} &= \left(1-\mathcal{D}F^L\frac{\mathcal{R}^{(+)+}}{\mathcal{R}^{(-)+}}\frac{\mathcal{B}_1^{L-}\mathcal{R}_3^{L-}}{\mathcal{B}_1^{L+}\mathcal{R}_3^{L+}}\mathcal{D}\right)\left(1-\mathcal{D}F^L\frac{\mathcal{B}_1^{L-}\mathcal{R}_3^{L-}}{\mathcal{B}_1^{L+}\mathcal{R}_3^{L+}}\frac{Q_2^{L++}}{Q_2^L}\mathcal{D}\right)^{-1} \\ &\quad\times\left(1-\mathcal{D}F^L\frac{u^+}{u^-}\frac{\mathcal{R}_1^{L+}\mathcal{B}_3^{L+}}{\mathcal{R}_1^{L-}\mathcal{B}_3^{L-}}\frac{Q_2^{L--}}{Q_2^L}\mathcal{D}\right)^{-1}\left(1-\mathcal{D}F^L\frac{u^+}{u^-}\frac{\mathcal{B}^{(-)-}}{\mathcal{B}^{(+)-}}\frac{\mathcal{R}_1^{L+}\mathcal{B}_3^{L+}}{\mathcal{R}_1^{L-}\mathcal{B}_3^{L-}}\mathcal{D}\right) \\ &= \sum_a (-1)^a\mathcal{D}^a\mathbb{D}_{a,1}\mathcal{D}^a\,, \end{aligned} \tag{4.20}$$

where $su(2)$ refers to the fact that we are working in the $su(2)$ grading while $\mathbb{D}_{a,-1}$ is obtained by replacing every quantity here with upper index $L$ with the corresponding quantity with upper index $R$. Here

$$F^L = \sqrt{\frac{Q^{[2]}(u)}{Q^{[-2]}(u)}\frac{u^-}{u^+}}\left(\frac{x^-}{x^+}\right)^{N-m_1^L+1+L}\left(\frac{\mathcal{R}^{(-)+}}{\mathcal{R}^{(+)+}}\right)\prod_{i=1}^N\sigma(p,p_i)\sigma(p_i,-p)\,, \tag{4.21}$$



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and $F^R = F^L(m_1^L \to m_1^R)$. Thus, the two wings are not symmetric any more, $\mathbb{D}_{a,s} \neq \mathbb{D}_{a,-s}$, but $\mathbb{D}_{a,0} = 1$ and $Y_{a,0} = \frac{\mathbb{D}_{a,1}\mathbb{D}_{a,-1}}{\mathbb{D}_{a+1,0}\mathbb{D}_{a-1,0}} = \mathbb{D}_{a,1}\mathbb{D}_{a,-1}$.

In the scaling limit one finds that $F^{L,R} \simeq \Phi^{L,R}(z)$, where

$$\Phi^L(z) = \exp\left[-\frac{L - m_1^L + 1 + \sum\limits_{j=1}^{N} E_j^{(1)}}{2g\sqrt{1-z^2}} - \frac{i}{4gz}\right], \qquad \Phi^R(z) = \Phi^L(z)(m_1^L \to m_1^R). \tag{4.22}$$

To describe the scaling limit of the generating functionals $\mathcal{W}_{su(2)}^{L,R}$ we introduce the limiting functions

$$\begin{aligned}
\frac{\mathcal{R}_1^{L+}\mathcal{B}_3^{L+}}{\mathcal{R}_1^{L-}\mathcal{B}_3^{L-}} &\simeq \mathcal{P}^L(z) = \exp\left(\frac{i}{g}(K_-^L(1/x) + K_+^L(1/x))\right); \qquad K_\pm^L(x) = \sum_{i=1}^{m_1^L} \frac{x^2}{x^2-1}\frac{1}{x \pm y_i^L} \\
\frac{\mathcal{B}_1^{L-}\mathcal{R}_3^{L-}}{\mathcal{B}_1^{L+}\mathcal{R}_3^{L+}} &\simeq \mathcal{M}^L(z) = \exp\left(-\frac{i}{g}(K_-^L(x) + K_+^L(x))\right) \\
\frac{Q_2^{L--}}{Q_2^L} &\simeq q^L(z) = \exp\left(-\frac{i}{g}(V_-^L(x) + V_+^L(x) + V_-^L(1/x) + V_+^L(1/x))\right) \\
\frac{Q_2^{L++}}{Q_2^L} &\simeq (q^L(z))^{-1}; \qquad V_\pm^L(x) = \sum_{l=1}^{m_2^L} \frac{x^2}{x^2-1}\frac{1}{x \pm Y_l^L}
\end{aligned} \tag{4.23}$$

where $Y_l^L$ is defined as $Y_l^L = z_l^L + \sqrt{z_l^L - 1}\sqrt{z_l^L + 1}$, and $w_l^L = 2gz_l^L$. Using these functions the limiting expression of the generating functional becomes

$$\mathcal{W}_{su(2)}^L \simeq \frac{\left(1 - h(z)\Phi^L(z)\mathcal{P}^L(z)q^L(z)\mathcal{D}^2\right)\left(1 - \Phi^L(z)\mathcal{M}^L(z)(q^L(z))^{-1}\mathcal{D}^2\right)}{\left(1 - h(z)\Phi^L(z)\tilde{f}(z)\mathcal{P}^L(z)\mathcal{D}^2\right)\left(1 - f(z)\Phi^L(z)\mathcal{M}^L(z)\mathcal{D}^2\right)} \equiv \tilde{\mathcal{W}}_{su(2)}^L, \tag{4.24}$$

($\mathcal{W}_{su(2)}^R$ is obtained by replacing every quantity here with upper index $L$ with the corresponding quantity with upper index $R$).

Now if we want to use the procedure of [15] for $T(x)$ to construct $D_{a,s}$ in the generic case, then, in the lack of the $D_{a,s} = D_{a,-s}$ symmetry, we write [15]

$$\hat{W}^L = \frac{(1-\mu_1 t^L)(1-\mu_2 t^L)}{(1-\lambda_1 t^L)(1-\lambda_2 t^L)}, \qquad \hat{W}^R = \frac{(1-t^R/\mu_4)(1-t^R/\mu_3)}{(1-t^R/\lambda_4)(1-t^R/\lambda_3)}, \tag{4.25}$$

where the $t^L$ and $t^R$ formal expansion parameters are related to $\mathcal{D}^2$:

$$\mathcal{D}^2 = \Phi^L t^L, \qquad \mathcal{D}^2 = \Phi^R t^R, \tag{4.26}$$

These lead to the following quasi-momenta

$$\begin{aligned}
\hat{p}_1(x) = -\hat{p}_2(1/x) &= \frac{(J - m_1^L + 2\mathcal{Q}_2)x}{g(x^2-1)} + B(x) - \frac{1}{g}\sum_{\epsilon=\pm}(K_\epsilon^L(1/x) - V_\epsilon^L(x) - V_\epsilon^L(1/x)) \\
\hat{p}_3(x) = -\hat{p}_4(1/x) &= -\frac{(J - m_1^R + 2\mathcal{Q}_2)x}{g(x^2-1)} + B(x) - \frac{1}{g}\sum_{\epsilon=\pm}(K_\epsilon^R(x) - V_\epsilon^R(x) - V_\epsilon^R(1/x))
\end{aligned}$$

$$\tilde{p}_1(x) = -\tilde{p}_2(1/x) = \frac{(J-m_1^L)x}{g(x^2-1)} + B(x) + \frac{1}{g}\sum_{\epsilon=\pm}(H_\epsilon(1/x) - K_\epsilon^L(1/x))$$

$$\tilde{p}_3(x) = -\tilde{p}_4(1/x) = -\frac{(J-m_1^R)x}{g(x^2-1)} + B(x) + \frac{1}{g}\sum_{\epsilon=\pm}(H_\epsilon(x) - K_\epsilon^R(x)) \tag{4.27}$$

The root configurations represented by the resolvents can condense into cuts in the scaling limit. We will consider square roots and logarithmic cuts in section 6. We also notice that the boundary contribution $B(x)$, is present for any state and gives new pole structure in the Riemann surface for the spectral curve which can be interpreted as a boundary quantum effect.

### 4.2.3 Duality transformation

So far our considerations relied on the eigenvalue of the double row transfer matrix in the $su(2)$ sector. In the literature the $sl(2)$ sector is studied most frequently. The eigenvalues in these two sectors are connected by a duality transformation on the $y$ roots (see appendix C of [23]). Here we use this transformation to obtain the quasi-momenta corresponding to the $sl(2)$ grading.

In this transformation the $2m_1^A$ ($A = L, R$) $y$ roots ($y_i^A$ and $-y_i^A$) are exchanged for $2\tilde{m}_1^A$ dual roots $\tilde{y}$ while the $w$ roots are not changed ($\tilde{m}_2^A = m_2^A$). The number of dual roots is determined by the relation $\tilde{m}_1^A = N + 2m_2^A - m_1^A$ and we introduce the $\tilde{\mathcal{M}}_A$ and $\tilde{\mathcal{P}}_A$ resolvents of the dual $y$ roots in analogy with $\mathcal{M}_A$ and $\mathcal{P}_A$. Computing the scaling limit of equations (C.6) and (C.7) of [23] yields

$$\mathcal{M}_A = \exp\left(-\frac{i2m_2^A x}{g(x^2-1)}\right)\tilde{\mathcal{M}}_A^{-1}\exp\left(\frac{i}{g}\sum_{\epsilon=\pm}H_\epsilon(1/x)\right)q_A, \tag{4.28}$$

$$\mathcal{P}_A = \exp\left(-\frac{i2m_2^A x}{g(x^2-1)}\right)\tilde{\mathcal{P}}_A^{-1}\exp\left(-\frac{i}{g}\sum_{\epsilon=\pm}H_\epsilon(x)\right)q_A^{-1}, \tag{4.29}$$

while the dual version of $\Phi^A(z)$ is obtained as $\tilde{\Phi}^A(z) = \Phi^A(z)\ [m_1^A \to N + 2m_2^A - \tilde{m}_1^A]$. The $\lambda_i$, $\mu_i$ eigenvalues in the $sl(2)$ grading are obtained from the previous ones in the $su(2)$ grading by replacing $\Phi^A$ with $\tilde{\Phi}^A$ and also substituting eq. (4.28) and eq. (4.29).

The simplest situation is when we consider $N$ fundamental particles of $3\dot{3}$ type with no auxiliary (dual) roots $\tilde{m}_1^A = 0 = m_2^A$. Note that this requires a non vanishing $m_1^A$, in fact $m_1^A = N$, but eq.s (4.28)–(4.29) simplify and eventually one finds the quasi-momenta

$$\hat{p}_1(x) = -\hat{p}_2(1/x) = -\hat{p}_4(x) = \hat{p}_3(1/x) = \frac{(L+1+2\sum_j \frac{1}{x_j^2-1})x}{g(x^2-1)} + B(x) + \frac{1}{g}\sum_{\epsilon=\pm}H_\epsilon(1/x)$$

$$\tilde{p}_1(x) = -\tilde{p}_2(1/x) = -\tilde{p}_4(x) = \tilde{p}_3(1/x) = \frac{(L+1)x}{g(x^2-1)} + B(x) \tag{4.30}$$

Note that $N$, the number of particles, disappeared from the quasimomenta, and also the non vanishing resolvent densities moved from the $S^5$ components of the quasimomenta to the $AdS_5$ ones.

## 5 Quasimomenta from the all-loop boundary Bethe equations

In this section we derive the quasi-momenta (4.27) from the scaling limit of the all-loop boundary Bethe equations which were constructed for the $Y = 0$ brane set-up in [27, 28]. We start with the $su(2)$ sector.

### 5.1 $su(2)$ sector

In the $su(2)$ sector we consider $N$ particles, with rapidities $u_j$ but without any polarization, in a finite volume $L$, satisfying the $Y = 0$ brane boundary conditions on both ends. The only Bethe Ansatz equation in terms of $Y_{1,0}$ function reads as

$$Y_{1,0}(u_j) = \mathbb{D}_{1,1}(u_j)\mathbb{D}_{1,-1}(u_j) = \left(\frac{x_j^-}{x_j^+}\right)^{2L} \frac{Q^{[2]}(u_j)u_j^-}{Q^{[-2]}(u_j)u_j^+} \prod_{i=1}^{N} \sigma^2(p_j, p_i)\sigma^2(p_i, -p_j) = -1 \quad (5.1)$$

If we take logarithm of (5.1) with $g \sim L \sim N \gg 1$, the scaling Bethe equation becomes

$$2\pi n = \frac{2Jx}{g(x^2-1)} - 2B(x) - \frac{2}{g} \sum_{\epsilon=\pm} H_\epsilon(x) \quad (5.2)$$

where we used the boundary contribution $B(x)$ and defined $J = L+N$. In the $su(2)$ sector there is two-sheeted Riemann surface corresponding to $p(x)$ and $-p(x)$. These sheets are connected through cuts where Bethe roots with given mode numbers condense. The Bethe equation relates the quasimomenta

$$p(x) = \frac{Jx}{g(x^2-1)} - B(x) - \frac{1}{g} \sum_{\epsilon=\pm} H_\epsilon(x) \quad (5.3)$$

on the two sides of the cut as

$$p(x+i0) + p(x-i0) = 2\pi n \quad (5.4)$$

where $x$ belongs to the cut joining two sheets with mode number $n$.

### 5.2 Generic case

In the generic case we have not only the massive Bethe equation (5.1) but also the magnonic ones coming from the regularity of the double row transfer matrix, $\mathbb{D}_{1,1}$, at the auxiliary root positions:

$$\left.\frac{\mathcal{R}^{(+)+}Q_2^L}{\mathcal{R}^{(-)+}Q_2^{L++}}\right|_{x^+=\pm y_j^L} = 1 = \left.\frac{\mathcal{B}^{(-)-}Q_2^L}{\mathcal{B}^{(+)-}Q_2^{L--}}\right|_{x^-=\pm 1/y_j^L} ; \quad \left.\frac{Q_1^{L-}Q_3^{L-}Q_2^{++}u^-}{Q_1^{L+}Q_3^{L+}Q_2^{--}u^+}\right|_{u=\pm w_j^L} = -1 \quad (5.5)$$

It is more natural to rewrite these equations into a manifestly psu$(2,2|4)$ covariant way. In so doing we relabel the roots as

$$\begin{aligned} &x_j \longleftrightarrow x_{4,j} , \quad y_j^L \longleftrightarrow \frac{1}{x_{1,j}} , \quad y_{K_1+j}^L \longleftrightarrow x_{3,j} , \quad w_j^L \longleftrightarrow x_{2,j} , \quad K_2 \longleftrightarrow m_2^L \\ &N \longleftrightarrow K_4 , \quad y_j^R \longleftrightarrow x_{5,j} , \quad y_{K_5+j}^R \longleftrightarrow \frac{1}{x_{7,j}} , \quad w_j^R \longleftrightarrow x_{6,j} , \quad K_6 \longleftrightarrow m_2^R \end{aligned} \quad (5.6)$$

where we split the roots $y_j^{L/R}$ according to their absolute value, thus $m_1^L = K_1 + K_3$ and $m_1^R = K_5 + K_7$. In the following we analyze the scaling limit

$$g \sim u_a \sim K_a \sim L \gg 1, \quad a = 1, 2, \ldots, 7, \tag{5.7}$$

In this scaling limit, using the Riemann surfaces structure of the closed string, the Bethe equations can be written as:[9]

$$\begin{aligned}
2\pi n_{\tilde{2}\tilde{3}} &= \frac{2Jx}{g(x^2-1)} - 2B(x) - \frac{1}{g}\sum_{\epsilon=\pm}\left(2H^4_\epsilon(x) - H^7_\epsilon(1/x) - H^1_\epsilon(1/x) - H^3_\epsilon(x) - H^5_\epsilon(x)\right)\\
2\pi n_{\hat{1}\tilde{1}} &= -\frac{2G_4'(0)x}{g(x^2-1)} + \frac{1}{g}\sum_{\epsilon=\pm}\left(H^2_\epsilon(x) + H^2_\epsilon(1/x) - H^4_\epsilon(1/x)\right)\\
2\pi n_{\tilde{4}\hat{4}} &= -\frac{2G_4'(0)x}{g(x^2-1)} + \frac{1}{g}\sum_{\epsilon=\pm}\left(H^6_\epsilon(x) + H^6_\epsilon(1/x) - H^4_\epsilon(1/x)\right)\\
2\pi n_{\tilde{2}\hat{2}} &= \frac{2G_4'(0)x}{g(x^2-1)} - \frac{1}{g}\sum_{\epsilon=\pm}\left(H^4_\epsilon(x) - H^2_\epsilon(x) - H^2_\epsilon(1/x)\right)\\
2\pi n_{\hat{3}\tilde{3}} &= \frac{2G_4'(0)x}{g(x^2-1)} - \frac{1}{g}\sum_{\epsilon=\pm}\left(H^4_\epsilon(x) + H^6_\epsilon(x) + H^6_\epsilon(1/x)\right)\\
2\pi n_{\hat{1}\hat{2}} &= 2B(x) + \frac{1}{g}\sum_{\epsilon=\pm}\left(2H^2_\epsilon(x) + 2H^2_\epsilon(1/x) - H^1_\epsilon(x) - H^1_\epsilon(1/x) - H^3_\epsilon(x) - H^3_\epsilon(1/x)\right)\\
2\pi n_{\hat{3}\hat{4}} &= 2B(x) + \frac{1}{g}\sum_{\epsilon=\pm}\left(2H^6_\epsilon(x) + 2H^6_\epsilon(1/x) - H^5_\epsilon(x) - H^5_\epsilon(1/x) - H^7_\epsilon(x) - H^7_\epsilon(1/x)\right)
\end{aligned} \tag{5.8}$$

where we used $H^i_\pm(x) = \sum_{j=1}^{K_i} \frac{x^2}{x^2-1}\frac{1}{x \pm x_{i,j}}$, $G_4(x) = \sum_{j=1}^{K_4} \frac{x^2_{4,j}}{x^2_{4,j}-1}\frac{1}{x \pm x_{4,j}}$ and defined $J$ as $J = L + K_4 + \frac{K_1-K_3+K_7-K_5}{2} + 1$. These Bethe equations correspond to differences between the various quasi-momenta:

$$\begin{aligned}
\hat{p}_1(x) = -\hat{p}_2(1/x) &= +\frac{(J_L+2Q_2)x}{g(x^2-1)} + B(x) - \frac{1}{g}\sum_{\epsilon=\pm}\left(H^1_\epsilon(x) - H^2_\epsilon(x) - H^2_\epsilon(1/x) + H^3_\epsilon(1/x)\right)\\
\hat{p}_3(x) = -\hat{p}_4(1/x) &= -\frac{(J_R+2Q_2)x}{g(x^2-1)} + B(x) - \frac{1}{g}\sum_{\epsilon=\pm}\left(H^5_\epsilon(x) - H^6_\epsilon(x) - H^6_\epsilon(1/x) + H^7_\epsilon(1/x)\right)\\
\tilde{p}_1(x) = -\tilde{p}_2(1/x) &= +\frac{J_L x}{g(x^2-1)} + B(x) - \frac{1}{g}\sum_{\epsilon=\pm}\left(H^1_\epsilon(x) + H^3_\epsilon(1/x) - H^4_\epsilon(1/x)\right)\\
\tilde{p}_3(x) = -\tilde{p}_4(1/x) &= -\frac{J_R x}{g(x^2-1)} + B(x) + \frac{1}{g}\sum_{\epsilon=\pm}\left(H^4_\epsilon(x) - H^5_\epsilon(x) - H^7_\epsilon(1/x)\right)
\end{aligned} \tag{5.9}$$

where we defined $Q_2 = -G_4'(0)$, $J_L = L+1+K_4+K_1-K_3$ and $J_R = L+1+K_4-K_5+K_7$ such that $J_L + J_R = 2J$. We also used that

$$\sum_{j=1}^{K_i} \frac{x^2}{x^2-1}\frac{1}{x \pm \frac{1}{x_{i,j}}} = \frac{K_i x}{x^2-1} + H^i_\pm(1/x). \tag{5.10}$$

[9] Actually, the existence of the $Y = 0$ giant graviton breaks the residual symmetry $\mathrm{SU}(2|2)^2$ to $\mathrm{SU}(1|2)^2$ by the term $B(x)$.

Then, one can easily check that specific differences between two quasi-momenta give the all-loop boundary Bethe equations in the scaling limit (5.8). Note that quasi-momenta for left wings and for right wings have different dependence on the numbers of Bethe roots.

We explained the analytic properties of quasimomenta for open strings attached to $Y = 0$ brane in section 2. Such analytic properties are related to physical information on conserved charges. Here, we can read off the same properties from the quasimomenta. First, let us investigate the synchronization of the residues at $x = \pm 1$. For example, the residues of $\hat{p}_{1,2}$ at $x = 1$ become

$$\hat{p}_{1,2} \simeq +\frac{(J_L + 2Q_2)}{2g} - \frac{1}{2g}\left(\sum_{j=1}^{K_1}\left(\frac{1}{1-x_{1,j}} + \frac{1}{1+x_{1,j}}\right) - \sum_{j=1}^{K_3}\left(\frac{1}{1-x_{3,j}} + \frac{1}{1+x_{3,j}}\right)\right),$$

which is equivalent to that of $\hat{p}_{3,4}$ at $x = 1$

$$\hat{p}_{3,4} \simeq -\frac{(J_R + 2Q_2)}{2g} - \frac{1}{2g}\left(\sum_{j=1}^{K_5}\left(\frac{1}{1-x_{5,j}} + \frac{1}{1+x_{5,j}}\right) - \sum_{j=1}^{K_7}\left(\frac{1}{1-x_{7,j}} + \frac{1}{1+x_{7,j}}\right)\right),$$

as a result of (5.8). Also, the inversion symmetry between each quasi-momenta and the reflection symmetry can be easily checked. Note that the absence of winding in the inversion symmetry corresponds to the absence of the 1st conserved charge $Q_1$ in the Bethe equation. Also, the reflection symmetry is expected as our quasi-momenta have the doubling nature.[10] Last but not least, the large $x$ asymptotics of the quasi-momenta are given in terms of the conserved charges as (for details see appendix C):

$$\lim_{x\to\infty}\begin{pmatrix} \hat{p}_1(x) & \hat{p}_2(x) \\ \hat{p}_3(x) & \hat{p}_4(x) \\ \tilde{p}_1(x) & \tilde{p}_2(x) \\ \tilde{p}_3(x) & \tilde{p}_4(x) \end{pmatrix} \simeq \frac{1}{gx}\begin{pmatrix} \Delta - S_1 + S_2 & \Delta + S_1 - S_2 \\ -\Delta - S_1 - S_2 & -\Delta + S_1 + S_2 \\ J_1 + J_2 - J_3 & J_1 - J_2 + J_3 \\ -J_1 + J_2 + J_3 & -J_1 - J_2 - J_3 \end{pmatrix} \tag{5.11}$$

Compared with the closed string case, we observe that only the prefactor has doubled, i.e. each charge comes with a factor two. But, it doesn't mean that open strings have doubled charges: all doubling nature is just the effect of the double monodromy matrix and the definition of the quasimomenta. Observe that due to the $Y = 0$ brane boundary condition we still had three angular momenta $J_{1,2,3}$ , spins $S_{1,2}$ and energy $\Delta$ coming from the $S^5$ and $AdS_5$ isometries, like in the closed strings case, ie. without any $D$-branes.

## 6 Quasiclassical fluctuations of open string solutions

One advantage of the algebraic curve formalism is that one can efficiently compute semi-classical contributions to conserved charges from the quasimomenta by exploiting their analytic properties. In this section, we will treat two kinds of open string solutions — the BMN string and the boundary giant magnon.[11]

[10] Therefore, symmetric distributions of Bethe roots on complex plane are suitable solutions for Bethe equation.

[11] Since the energy and angular momentum charges scale as $\sqrt{\lambda}$ for these string solutions, the $B(x)$ term can be ignored in the following.



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### 6.1 BMN string

Having constructed the quasimomenta for some simple classical string solutions we want to consider the quantum fluctuations around them. This also makes possible to compare the quasi momenta obtained from the $Y$ system/ABA and the ones that describe the quasiclassical fluctuations. To this end we follow the procedure summarized for the closed string case in [22]. In terms of the algebraic curve it means that we add some microscopic cuts — i.e. some finite number of poles — to the quasimomenta of the classical solution. These additional pole terms must satisfy several requirements that follow from the general equations for the cuts on the Riemann surface. These requirements fix their form completely and also determine the shift in the energy corresponding to the fluctuations. We focus mainly on the points in the procedure that are different from the closed string case.

The quasimomenta for the BMN string are given in (3.14) and to describe the quantum fluctuations in all components we make the substitution $p(x) \to p(x) + \delta p(x)$. If the microscopic cut (i.e. the pole) is shared by the sheets $i$ and $j$ then its location $x_n^{ij}$ is determined in leading order by the equation

$$p_i(x_n^{ij}) - p_j(x_n^{ij}) = 2\pi n, \qquad |x_n^{ij}| > 1\,, \tag{6.1}$$

and we denote by $N_n^{ij}$ the number of excitations with mode number $n$ between $i$ and $j$ (we also define $N^{ij} = \sum\limits_n N_n^{ij}$). In our case the non vanishing $x_n^{ij}$-s are independent of $i, j$ and depend only on the mode number $n$

$$x_n^{ij} \to x_n = \frac{1}{n}(\nu + \sqrt{n^2+\nu^2}). \tag{6.2}$$

The quasimomenta $p(x)+\delta p(x)$ should be analytical on the $x$ plane and satisfy the following requirements

- must have poles at $x_n^{ij}$ with residues $\pm\frac{1}{g}\alpha(x_n^{ij})N_n^{ij}$ (where $\alpha(x) = \frac{x^2}{x^2-1}$).
- obeying the $x \to 1/x$ (inversion) and the $x \to -x$ (reflection) symmetry properties

$$\tilde{p}_{1,2}(x) = -\tilde{p}_{2,1}(1/x), \quad \tilde{p}_{3,4}(x) = -\tilde{p}_{4,3}(1/x), \quad \tilde{p}_i(-x) = -\tilde{p}_i(x), \quad i=1,\ldots,4\,, \tag{6.3}$$
$$\hat{p}_{1,2}(x) = -\hat{p}_{2,1}(1/x), \quad \hat{p}_{3,4}(x) = -\hat{p}_{4,3}(1/x), \quad \hat{p}_i(-x) = -\hat{p}_i(x), \quad i=1,\ldots,4\,. \tag{6.4}$$

- the residues at $x = \pm 1$ should coincide for $\hat{p}_1, \hat{p}_2, \tilde{p}_1, \tilde{p}_2$ and for $\hat{p}_3, \hat{p}_4, \tilde{p}_3, \tilde{p}_4$
- introducing the notation $\sum\limits_i \equiv \sum\limits_{i=\hat{3}\hat{4}\tilde{3}\tilde{4}}$ and $\sum\limits_k \equiv \sum\limits_{k=\hat{1}\hat{2}\tilde{1}\tilde{2}}$ the large $x$ asymptotics of $\delta p(x)$ should be given by

$$\begin{pmatrix} \delta\hat{p}_1 \\ \delta\hat{p}_2 \\ \delta\hat{p}_3 \\ \delta\hat{p}_4 \end{pmatrix} \sim \frac{1}{xg} \begin{pmatrix} \delta\Delta + 2\sum\limits_i N^{\hat{1}i} \\ \delta\Delta + 2\sum\limits_i N^{\hat{2}i} \\ -\delta\Delta - 2\sum\limits_k N^{k\hat{3}} \\ -\delta\Delta - 2\sum\limits_k N^{k\hat{4}} \end{pmatrix} \qquad \begin{pmatrix} \delta\tilde{p}_1 \\ \delta\tilde{p}_2 \\ \delta\tilde{p}_3 \\ \delta\tilde{p}_4 \end{pmatrix} \sim \frac{1}{xg} \begin{pmatrix} -2\sum\limits_i N^{\tilde{1}i} \\ -2\sum\limits_i N^{\tilde{2}i} \\ 2\sum\limits_k N^{k\tilde{3}} \\ 2\sum\limits_k N^{k\tilde{4}} \end{pmatrix} \tag{6.5}$$



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The $x \to -x$ symmetry properties, the absence of winding in $\tilde{p}$ and the appearance of the factors of two in front of the sums in the asymptotic expressions are the new features of these requirements when compared to the closed string case, while $\delta\Delta$ determines in the same way the energy

$$E = \delta\Delta + \text{excitation numbers} \tag{6.6}$$

The $\delta p_i$ for the BMN case are obtained by enforcing the $x \to -x$ symmetry i.e. by writing

$$g \cdot \delta\hat{p}_2(x) = \hat{\alpha}\frac{2x}{x^2-1} + \sum_{i,n}\left(\frac{\alpha(x_n^{\hat{2}i})N_n^{\hat{2}i}}{x - x_n^{\hat{2}i}} + \frac{\alpha(x_n^{\hat{2}i})N_n^{\hat{2}i}}{x + x_n^{\hat{2}i}} + \frac{\alpha(x_n^{\hat{1}i})N_n^{\hat{1}i}}{1/x - x_n^{\hat{1}i}} + \frac{\alpha(x_n^{\hat{1}i})N_n^{\hat{1}i}}{1/x + x_n^{\hat{1}i}}\right) \tag{6.7}$$

$$g \cdot \delta\hat{p}_3(x) = \hat{\beta}\frac{2x}{x^2-1} + \sum_{k,n}\left(\frac{\alpha(x_n^{\hat{3}k})N_n^{\hat{3}k}}{x - x_n^{\hat{3}k}} + \frac{\alpha(x_n^{\hat{3}k})N_n^{\hat{3}k}}{x + x_n^{\hat{3}k}} + \frac{\alpha(x_n^{\hat{4}k})N_n^{\hat{4}k}}{1/x - x_n^{\hat{4}k}} + \frac{\alpha(x_n^{\hat{4}k})N_n^{\hat{4}k}}{1/x + x_n^{\hat{4}k}}\right) \tag{6.8}$$

while $\delta\tilde{p}_2$ is the same as $\delta\hat{p}_2$ with the substitutions $\hat{1}\hat{2} \to \tilde{1}\tilde{2}$ plus changing the signs in front of the sums ($\delta\tilde{p}_3$ is also obtained from $\delta\hat{p}_3$ by the substitutions $\hat{3}\hat{4} \to \tilde{3}\tilde{4}$ and changing the signs of the sums). The additional components of $\delta p$ are obtained by exploiting the inversion symmetry: $\delta\hat{p}_1(x) = -\delta\hat{p}_2(1/x)$, etc. These expressions reveal several interesting features: they contain only two unknown parameters ($\hat{\alpha}$ and $\hat{\beta}$) since the reflection symmetry allows no $x$ independent constant terms. This symmetry also doubled the poles: to every pole at $x_n^{ij}$ there is another one at $-x_n^{ij}$, furthermore the residues of these poles must be the same. The last sums in these expressions (and especially their signs) are introduced to guarantee that after exploiting the inversion symmetry we obtain the expected pole terms. The appearance of poles at $-x_n^{ij}$ in the Ansatz for $\delta p_i$ may be understood also by realising that they also solve eq. (6.1) but with $n \to -n$ on the right hand side. This means that together with the microscopic cut corresponding to the integer $n$ in (6.1) we also have a cut corresponding to $-n$, i.e. reflection symmetry doubles the cuts (in a similar way as inversion symmetry does it).

Matching the asymptotic behaviour of these $\delta p_i$-s to the one in (6.5) determines all the unknown parameters. Indeed from the asymptotics of $\delta\hat{p}_1/\delta\hat{p}_2$ (respectively $\delta\hat{p}_3/\delta\hat{p}_4$) we find

$$\delta\Delta = 2\hat{\alpha} + \sum_{n,i}\frac{\sqrt{\nu^2+n^2}-\nu}{\nu}(N_n^{\hat{1}i} + N_n^{\hat{2}i}), \quad -\delta\Delta = 2\hat{\beta} - \sum_{n,k}\frac{\sqrt{\nu^2+n^2}-\nu}{\nu}(N_n^{\hat{3}k} + N_n^{\hat{4}k}), \tag{6.9}$$

while from the asymptotics of $\delta\tilde{p}_1/\delta\tilde{p}_2$ (respectively $\delta\tilde{p}_3/\delta\tilde{p}_4$) it follows that

$$0 = 2\hat{\alpha} - \sum_{n,i}\frac{\sqrt{\nu^2+n^2}-\nu}{\nu}(N_n^{\tilde{1}i} + N_n^{\tilde{2}i}), \qquad 0 = 2\hat{\beta} + \sum_{n,k}\frac{\sqrt{\nu^2+n^2}-\nu}{\nu}(N_n^{\tilde{3}k} + N_n^{\tilde{4}k}). \tag{6.10}$$

A remarkable property of these equations that they determine $\delta\Delta$, $\hat{\alpha}$ and $\hat{\beta}$ without imposing any condition on the excitation numbers $N_n^{ij}$ since by their definition $\sum_i(N_n^{\hat{1}i} + N_n^{\hat{2}i} + N_n^{\tilde{1}i} + N_n^{\tilde{2}i}) = \sum_k(N_n^{\hat{3}k} + N_n^{\hat{4}k} + N_n^{\tilde{3}k} + N_n^{\tilde{4}k})$. We emphasize this because the analogous equations in the closed string case have a solution only if the excitation numbers satisfy a condition, which turns out to be the level matching condition. For open strings there is no level matching condition thus on physical grounds we expect that there is also no condition on the excitation numbers. The consistent expression for $\delta\Delta$ is

$$\delta\Delta = \sum_n \frac{\sqrt{\nu^2+n^2}-\nu}{\nu} \sum_i (N_n^{\hat{1}i} + N_n^{\hat{2}i} + N_n^{\tilde{1}i} + N_n^{\tilde{2}i}), \tag{6.11}$$

where we can see the BMN frequency for open strings attached to the maximal giant graviton [29].

### 6.2 Boundary giant magnon

In the bulk the classical giant magnon solution corresponds to a logarithmic cut in complex plane of Bethe roots with $H(x) = -i\log\frac{x-X^+}{x-X^-}$ [30]. As the algebraic curve description for open string is built on symmetric roots configurations, we propose the quasi-momenta for the classical boundary giant magnon as

$$\begin{aligned}
\hat{p}_1 &= \hat{p}_2 = -\hat{p}_3 = -\hat{p}_4 = \frac{2\Delta}{g}\frac{x}{x^2-1}, \\
\tilde{p}_2 &= \frac{2\Delta}{g}\frac{x}{x^2-1} - i\log\frac{x-X^+}{x-X^-} + i\log\frac{x+X^+}{x+X^-} = -\tilde{p}_3, \\
\tilde{p}_1 &= \frac{2\Delta}{g}\frac{x}{x^2-1} - i\log\frac{x-1/X^-}{x-1/X^+} + i\log\frac{x+1/X^-}{x+1/X^+} = -\tilde{p}_4,
\end{aligned} \tag{6.12}$$

where we replaced the doubled resolvent $H^4_\pm(x)$ in the generic quasimomenta (5.9) with two symmetric logarithmic cuts between $\pm X^+$ and $\pm X^-$. Please note that we don't introduce any twist factor in the quasi-momenta unlike in the periodic case. Then, by (6.12), the inversion and reflection symmetries between the quasi-momenta are automatically satisfied. The dispersion relation of the boundary giant magnon is obtained by the large $x$ asymptotics (5.11) and $e^{ip} = \frac{X^+}{X^-}$ and is given explicitly as[12]

$$\Delta - J_1 = \sqrt{J_2^2 + 16g^2\sin^2\frac{p}{2}} \equiv \epsilon(p). \tag{6.13}$$

Now, let us compute the semi-classical correction for the boundary giant magnon from the algebraic curve. The quasimomenta for boundary giant magnon can be thought as that of a periodic two-magnon state with the following constraint:

$$X_2^\pm = -X_1^\mp \equiv -X^\mp \tag{6.14}$$

where the twist factors are naturally cancelled since $p_1 = -p_2$. Then, we can take the multi-magnon computation by Hatsuda and Suzuki [31] and carefully impose the constraint (6.14). All $\delta p_{\hat{i}}$ would be equivalent to those of giant magnon with the periodic

[12] We'll confine to the simple boundary giant magnon with $J_2 = 1$ and $L = J_1$.

boundary condition. Only the $\delta p_{\tilde{i}}$ are different because we have to consider the effects of the simple poles at $x = -X^{\pm}$ and $x = -1/X^{\pm}$. For example, one can express $\delta p_{\tilde{1}}$ and $\delta p_{\tilde{3}}$ as

$$\begin{aligned}
\delta p_{\tilde{1}} &= \frac{Ax+B}{x^2-1} - \sum_{n,j=\tilde{3}\tilde{4}\hat{3}\hat{4}} \left[ \frac{N_n^{\tilde{1}j}\alpha(x_n^{\tilde{1}j})}{x-x_n^{\tilde{1}j}} - \frac{N_n^{\tilde{2}j}\alpha(x_n^{\tilde{2}j})}{1/x-x_n^{\tilde{2}j}} + \frac{N_n^{\tilde{1}j}\alpha(x_n^{\tilde{1}j})}{x+x_n^{\tilde{1}j}} - \frac{N_n^{\tilde{2}j}\alpha(x_n^{\tilde{2}j})}{1/x+x_n^{\tilde{2}j}} \right] \\
&\quad + \sum_{\beta=\pm} \left( \frac{A^\beta}{X^\beta - 1/x} - \frac{A^\beta}{X^\beta + 1/x} \right), \\
\delta p_{\tilde{3}} &= -\frac{Cx+D}{x^2-1} + \sum_{n,j=\tilde{1}\tilde{2}\hat{1}\hat{2}} \left[ \frac{N_n^{j\tilde{3}}\alpha(x_n^{j\tilde{3}})}{x-x_n^{j\tilde{2}}} - \frac{N_n^{j\tilde{4}}\alpha(x_n^{j\tilde{4}})}{1/x-x_n^{j\tilde{1}}} + \frac{N_n^{j\tilde{3}}\alpha(x_n^{j\tilde{3}})}{x+x_n^{j\tilde{2}}} - \frac{N_n^{j\tilde{4}}\alpha(x_n^{j\tilde{4}})}{1/x+x_n^{j\tilde{1}}} \right] \\
&\quad - \sum_{\beta=\pm} \left( \frac{A^\beta}{x - X^\beta} + \frac{A^\beta}{x + X^\beta} \right),
\end{aligned}$$

with unknown $A$, $B$, $C$, $D$ and $A^\beta$. Then, the reflection symmetry yields $B = D = 0$ and the inversion symmetries between quasimomenta and the large $x$ asymptotic conditions give a set of equations between the unknown coefficients as in [31]. In the periodic case one cannot determine the exact form of fluctuation frequencies of multi-magnon states within the spectral curve method, nevertheless, one can exactly evaluate the one-loop correction to the energy by using the saddle point approximation [31, 32]. This happens for the case of boundary giant magnon, too. We claim that the leading quantum correction to the energy of the boundary giant magnon has the following form:

$$\delta\epsilon_{1-\text{loop}} = \int \frac{dx}{2\pi i} \partial_x \Omega\,(x) \sum_{(ij)} (-1)^{F_{ij}}\, e^{-i(p_i - p_j)} \tag{6.15}$$

Here, $(ij)$ means all polarization pairs and the fluctuation frequency $\Omega\,(x)$ is given as

$$\Omega\,(x) = \frac{2}{x^2-1}\left(1 - \frac{X^- + X^+}{X^-X^+ + 1} x\right), \tag{6.16}$$

where we used the symmetric additional poles like $x = x_n^{ij}$ near $X_1^{\pm}$ and $x = -x_n^{ij}$ near $X_2^{\pm}$ and it corresponds to $\alpha_1 = \alpha_2 = \frac{1}{2}$ in the notation of [31].

Then, one can compare (6.15) to the boundary Lüscher's $F$-term formula to check the semiclassical result. The boundary Lüscher's $F$-term formula is given by

$$\delta E^{\mathrm{F}} = -\int_0^{\frac{\omega_1}{2}} \frac{dz}{2\pi} (\partial_z \tilde{p}(z)) \mathbb{S}_{ia}^{jb}\left(\frac{\omega}{2}+z, u\right) \mathbb{R}_j^k\left(\frac{\omega}{2}+z\right) \mathbb{S}_{lb}^{ka}\left(\frac{\omega}{2}-z, u\right) \mathbb{C}^{l\bar{l}} \mathbb{R}_{\bar{l}}^{\bar{i}}\left(\frac{\omega}{2}-z\right) \mathbb{C}_{\bar{i}i} e^{-2\tilde{\epsilon}L}, \tag{6.17}$$

where we used the expression of [33] for fundamental virtual particle with $Q = 1$ since all other mirror boundstate contributions with $Q > 1$ are suppressed in the strong coupling limit.[13] One can rewrite (6.17) to a form more appropriate to our problem as

$$\delta E^{\mathrm{F}} = -\int \frac{dq}{2\pi} \left(1 - \frac{\varepsilon'(p)}{\varepsilon'(q^*)}\right) e^{-2iq^*L} S_0(q^*, p) S_0(p, -q^*) f(q^*, p)^2 \tag{6.18}$$

[13] $\mathbb{C}$ is the charge conjugation matrix.

where $f(q^*,p)$ is a function determined by the $S$-matrix elements between the physical particle and virtual (mirror) particles:[14]

$$f(q^*,p) = 2a_1(q^*,p)a_1(p,-q^*) + a_2(q^*,p)a_2(p,-q^*) - 2a_6(q^*,p)a_5(p,-q^*) \tag{6.19}$$

Here, $q$ and $q^*$ are separately the energy and momenta of the virtual particles and they satisfy the on-shell relation $q^2+\epsilon^2(q^*)=0$. The functions appearing in (6.18) are given as follows [35]:

$$\begin{aligned}
S_0(p_1,p_2) &= \frac{x^+_{p_2}-x^-_{p_1}}{x^-_{p_2}-x^+_{p_1}}\frac{1-\frac{1}{x^-_{p_2}x^+_{p_1}}}{1-\frac{1}{x^+_{p_2}x^-_{p_1}}}\sigma^2(p_1,p_2)\\
a_1(p_1,p_2) &= \frac{x^-_{p_2}-x^+_{p_1}}{x^+_{p_2}-x^-_{p_1}}\sqrt{\frac{x^+_{p_2}}{x^-_{p_2}}}\sqrt{\frac{x^-_{p_1}}{x^+_{p_1}}}\\
a_2(p_1,p_2) &= \frac{\left(x^-_{p_1}-x^+_{p_1}\right)\left(x^-_{p_2}-x^+_{p_2}\right)\left(x^-_{p_2}-x^+_{p_1}\right)}{\left(x^-_{p_1}-x^+_{p_2}\right)\left(x^-_{p_2}x^-_{p_1}-x^+_{p_2}x^+_{p_1}\right)}\sqrt{\frac{x^+_{p_2}}{x^-_{p_2}}}\sqrt{\frac{x^-_{p_1}}{x^+_{p_1}}}\\
a_5(p_1,p_2) &= \frac{x^-_{p_1}-x^-_{p_2}}{x^-_{p_1}-x^+_{p_2}}\sqrt{\frac{x^+_{p_2}}{x^-_{p_2}}}\\
a_6(p_1,p_2) &= \frac{x^+_{p_1}-x^+_{p_2}}{x^-_{p_1}-x^+_{p_2}}\sqrt{\frac{x^-_{p_1}}{x^+_{p_1}}}
\end{aligned} \tag{6.20}$$

We note that the contribution from the reflection matrix $\mathbb{R}$ (including $\sigma(q^*,-q^*)$) is cancelled between left and right boundaries at the leading order. Then, one can straightforwardly check that (6.15) is equivalent to (6.18) as we have $x_{q^*}\simeq x$ and $x^\pm_p\equiv X^\pm$ in the scaling limit.[15] Finally, one can express the one-loop energy shift from (6.12) as:

$$\delta\epsilon_{1-\text{loop}} = \int\frac{dx}{\pi i}\frac{32x^3}{(x^2-1)^2}\frac{(X^+-X^-)^2 e^{\frac{-4ix\Delta}{g(x^2-1)}}}{(x-X^+)(x+X^-)(X^-x-1)(X^+x+1)} \tag{6.21}$$

Here, we omitted the second term in bracket of (6.16) as it is suppressed at the saddle point $x=i$.

## 7 Conclusions

In this paper we considered the generalization of the spectral curve from closed strings to open ones. We defined this spectral curve with the aid of the logarithms of the eigenvalues of the open monodromy matrix — the supertrace of which is the generator of conserved quantities. We showed that this definition makes possible to determine all the analytic properties of the spectral curve in the same way as in the case of closed strings and emphasized the consequences of the additional ("reflection") symmetry that is absent for closed strings.

We analyzed this spectral curve from different points of view in case of open strings attached to the $Y=0$ brane. First, from first principles, we determined the explicit form

[14] Note that we considered both of right-moving and left-moving of virtual particles.
[15] We used here $X^\pm=\frac{1}{X^\mp}$ because we consider the non-dyonic, simple boundary magnon.



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of the spectral curve for some simple classical open string solutions. Then, exploiting that for the $Y = 0$ brane both the ABA and the asymptotic $Y$ system solutions are available, we derived and characterized the curve as the appropriate scaling limit of these solutions. Finally we showed on two explicit examples how the spectrum of small fluctuations around a classical solution can be determined by appropriately modifying the well known procedure of the periodic case.

The consistent picture emerging from this series of investigations is that the quasimomenta of the open case are very similar to that of the closed string case. The differences — that mainly arise as a result of the reflection symmetry — are that the residues of the poles at $x = \pm 1$ appearing in the various quasimomenta are more tightly related to each other than in the periodic case and that the resolvent densities describing the various excitations come in the form of symmetric pairs (e.g. $H_-(x) + H_+(x)$ with poles at $x = x_j$ and $x = -x_j$, respectively). Also we found that the presence of the boundary gives a new pole, $B(x) = \frac{1}{2g}\frac{x}{x^2+1}$, to all of the quasimomenta as a quantum effect. See figure 1.

This boundary contribution is specific to the $Y = 0$ brane boundary conditions and shows up as a sub-leading quantum effect at the classical string regime while as a finite-size effect at one-loop gauge theory regime [48–50], see appendix D. By interpolating the quasimomenta from string theory to gauge theory, we can see that the physical poles at the imaginary coordinates $x = \pm i$ in $B(x)$ are unified into a single pole at $x = 0$ in gauge theory. Even though we ignored such a boundary contribution when we computed the semiclassical corrections to the BMN string and to the boundary giant magnon, the BMN calculation was consistent with the open string's pp-wave result, while the boundary giant magnon calculation with the Lüscher's leading $F$-term. In contrast, the boundary contribution is expected to show up as a subleading quantum effect and is important for short strings. It would be interesting to confirm this by computing the subleading corrections.

In this paper we derived the general properties of the boundary spectral curve and investigated it for open strings satisfying the $Y = 0$ brane boundary conditions. But ABA equations are also known for many other integrable boundary conditions [36–40] and it would be interesting to extend our analysis for those cases. Especially to calculate the spectral curve for the $q\bar{q}$ potential or for the vacuum expectation values of Wilson loops from first principles, since in [41–43] the authors used the Lax matrix instead of the monodromy matrix, while in [44–46] they analyzed the classical limit of the near BPS FiNLIE formulations to define such a curve.

Recently the quantum spectral curve was proposed in the periodic $AdS_5/CFT_4$ context [47]. As such a quantum curve has the entire information for the full quantum spectrum, constructing the quantum curve for the boundary problem would be an interesting direction for future research.

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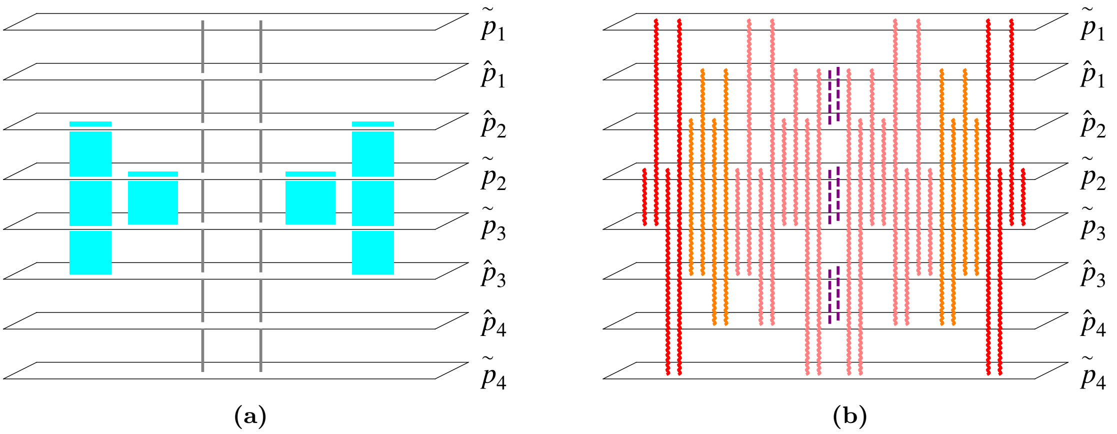


**Figure 1.** (a) General open string spectral curve is shown. It has poles at $x = \pm 1$ indicated by gray lines and cuts from various resolvents, which come in symmetric pairs due to the reflection symmetry. (b) Quasiclassical fluctuations for $Y = 0$ spectral curve are shown. The boundary condition results in poles connecting three pairs of surfaces at $x = \pm i$ (purple dashed lines). Open string fluctuations are related to additional poles connecting each Riemann sheets at $x = \pm x_n^{ij}$ in a symmetric way. Wavy lines with red, orange and pink color represent 16 polarizations: four $S^5$ modes, four $AdS_5$ and eight fermionic modes, respectively.

tions of the quasimomenta for periodic circular strings. Z.B. and L.P. were supported in part by OTKA 81461 and Z.B. also by a Lendület Grant. M.K. was partly supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) through the Center for Quantum Spacetime (CQUeST) of Sogang University with grant number 2005-0049409, Hungarian scholarship (type D) through the Balassi Institute and WCU Grant No. R32-2008-000-101300.

## A Transportation matrix and $U$ for the $Y = 0$ brane

We pointed out that $T(\zeta)$ can be defined from the transport matrix and the $U$ matrices. In [19] the $U$ matrix corresponding to the $Z = 0$ case was determined while here we need it for the $Y = 0$ one. We construct it below by solving all the necessary conditions listed in [19] rather then trying to rotate the result of [19] in an appropriate way.

To get the desired metric we use the coset element representative $g = g_{AdS_5} g_{S^5}$ with

$$g_{AdS_5} = e^{P_0 t} e^{-J_{13}\phi} e^{J_{24}\Phi} e^{-J_{14}\alpha} e^{P_1\rho}, \qquad g_{S^5} = e^{-J_{79}\phi_1} e^{P_8\phi_2} e^{J_{56}\phi_3} e^{P_6\psi} e^{P_7(\pi/2-\gamma)}. \tag{A.1}$$

The giant graviton corresponding to the $Y = 0$ brane is given by Dirichlet boundary conditions $\psi = 0$, $\rho = 0$ together with Neumann boundary conditions for the rest of the coordinates $\partial_\sigma\gamma = \partial_\sigma\phi_2 = \partial_\sigma\phi_1 = 0$. At the boundary, the bosonic sectors current, $A^{(2)}$, has the following world sheet components

$$A_\tau^{(2)} = P_0\partial_\tau t - P_7\partial_\tau\gamma + P_8 \sin\gamma\partial_\tau\phi_2 + P_9\cos\gamma\partial_\tau\phi_1, \quad A_\sigma^{(2)} = P_1\partial_\sigma\rho + P_6\sin\gamma\partial_\sigma\psi. \tag{A.2}$$

Therefore the natural Ansatz for the $U$ matrix is $U = aP_0 + bP_7P_8P_9$ with constant $a$, $b$ to be determined and plugging this into the conditions listed in [19] we found that up to normalization and relative sign

$$U = 2P_0 - i2^3 P_7 P_8 P_9. \tag{A.3}$$

In the paper we used the following conventions of $P_i$ matrices. The $so(4,1)$ generators $P_0, \ldots, P_4$ are described as

$$P_0, \ldots, P_4 = \begin{pmatrix} 0_{4\times4} & \\ & \frac{i}{2}\gamma^5, \frac{1}{2}\gamma^1, \ldots, \frac{1}{2}\gamma^4 \end{pmatrix} \tag{A.4}$$

while the $so(5)$ generators $P_5, \ldots, P_9$ as

$$P_5, \ldots, P_9 = \begin{pmatrix} \frac{i}{2}\gamma^1, \ldots, \frac{i}{2}\gamma^5 & \\ & 0_{4\times4} \end{pmatrix} \tag{A.5}$$

in terms of the $4 \times 4$ Dirac matrices $\gamma^i$ $(i = 1, \ldots 5)$: $\gamma^5 = \mathrm{diag}(1, 1, -1, -1)$

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \gamma^4 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \tag{A.6}$$

satisfying the Clifford algebra $\{\gamma^i, \gamma^j\} = 2\delta^{ij}$. Using these explicit expressions in (A.3) gives

$$U = i\ \mathrm{diag}(1, -1, 1, -1, 1, 1, -1, -1)\ . \tag{A.7}$$

## B Eigenvalues of $T(x)$ for circular strings with $n = 2N$

In this appendix we solve the differential equation $\partial_\sigma \psi = H\psi$ and determine the $S^5$ eigenvalues of $T(x)$. First, by a constant similarity transformation — we bring $H$ in (3.19) to the form $H \to \tilde{H} = \begin{pmatrix} i\tilde{b} & 0 \\ 0 & -i\tilde{b} \end{pmatrix}$ and solve

$$\partial_\sigma \psi = \pm i\tilde{b}\psi, \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{B.1}$$

Taking the upper sign we recognize that the equations can be solved by making the Ansatz[16]

$$\psi_1 - \psi_2 = Ae^{\alpha\sigma}, \qquad \psi_1 + \psi_2 = Be^{\gamma\sigma}, \tag{B.2}$$

if $\gamma - in = \alpha$ holds. Furthermore using this in the quadratic equation guaranteeing that we may have non trivial $A$ and $B$ determines $\gamma$ as

$$\gamma_{1,2} = \frac{i}{2}n \pm i\frac{x}{x^2 - 1}\sqrt{\frac{n^2}{x^2} + w^2}. \tag{B.3}$$

[16]We thank Romuald Janik for sharing his explicit calculation for the quasimomenta of the analogous circular string solution in the periodic case.



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Repeating this procedure with the lower sign in eq. (B.1) gives

$$\gamma_{3,4} = \frac{i}{2}n \pm i\frac{x}{x^2-1}\sqrt{n^2x^2+w^2} \tag{B.4}$$

As the eigenvalues of $\tilde{t}(\pi,0,x)$ can be obtained as the ratios $\psi(\pi)_j/\psi(0)_j$ in both cases we compute

$$\frac{\psi_1(\pi)}{\psi_1(0)} = e^{\gamma\pi}\frac{Ae^{-in\pi}+B}{A+B} = e^{\gamma\pi}, \qquad \frac{\psi_2(\pi)}{\psi_2(0)} = e^{\gamma\pi}\frac{B-Ae^{-in\pi}}{B-A} = e^{\gamma\pi}, \tag{B.5}$$

(In the last equality we exploited that $n=2N$). Since $\tilde{t}(\pi,0,-x)^{-1}$ has the same diagonal form as $\tilde{t}(\pi,0,x)$ eventually we find that $T(x)$'s eigenvalues in the $S^5$ corner are $-e^{2\pi\gamma_{1,2}}$ and $-e^{2\pi\gamma_{3,4}}$.

## C Bethe roots, Dynkin labels and conserved charges

In this appendix we analyze the asymptotics of the quasimomenta. By keeping the leading order term of the quasi-momenta, (5.9), in the large $x$ limit we find

$$\begin{aligned}
\hat{p}_1 &= \frac{J+Q_2-K_1+2K_2-K_3+K_4}{gx}, & \tilde{p}_1 &= \frac{J-K_1-K_3+K_4}{gx}\\
\hat{p}_2 &= \frac{J+Q_2+K_1-2K_2+K_3+K_4}{gx}, & \tilde{p}_2 &= \frac{J+K_1+K_3-K_4}{gx}\\
\hat{p}_3 &= -\frac{J+Q_2+K_4+K_5-2K_6+K_7}{gx}, & \tilde{p}_3 &= -\frac{J-K_4+K_5+K_7}{gx}\\
\hat{p}_4 &= -\frac{J+Q_2+K_4-K_5+2K_6-K_7}{gx}, & \tilde{p}_4 &= -\frac{J-K_4+K_5+K_7}{gx},
\end{aligned} \tag{C.1}$$

where $Q_2 = \delta\Delta$ is the second conserved charge — energy. We choose the gradings $\eta_1 = \eta_2 = 1$ and the unphysical hypercharge $B=0$, such that the numbers of the Bethe roots $K_j$ can be expressed in terms of the Dynkin labels of $SU(2,2)$ and $SU(4)$ , given by $[q_1,p,q_2]$ and $[s_1,r,s_2]$ as follows [9]:

$$\begin{aligned}
K_1 &= \frac{1}{2}J - \frac{1}{4}(2p+3q_1+q_2)\\
K_2 &= -\frac{1}{4}\left(2(r+Q_2)+3s_1+s_2+2p+3q_1+q_2\right)\\
K_3 &= -\frac{1}{2}J - \frac{1}{2}\left(2(r+Q_2)-s_1+s_2\right) - s_1 - \frac{1}{4}(2p-q_1+q_2) - q_1\\
K_4 &= -r - Q_2 - \frac{1}{2}(s_1+s_2+q_1+q_2) - p\\
K_5 &= -\frac{1}{2}J - \frac{1}{2}\left(2(r+Q_2)+s_1-s_2\right) - s_2 - \frac{1}{4}(2p+q_1-q_2) - q_2\\
K_6 &= -\frac{1}{4}\left(2(r+Q_2)+s_1+3s_2+2p+q_1+3q_2\right)\\
K_7 &= \frac{1}{2}J - \frac{1}{4}(2p+q_1+3q_2)
\end{aligned} \tag{C.2}$$

Since each Dynkin labels are related to the conserved charges as [6]

$$\begin{aligned}
q_1 &= J_2 - J_3, & p &= J_1 - J_2, & q_2 &= J_2 + J_3,\\
s_1 &= S_1 - S_2, & r &= -\Delta - S_1, & s_2 &= S_1 + S_2,
\end{aligned} \tag{C.3}$$

we finally obtain the large $x$ asymptotics (5.11) of our quasi-momenta.

## D Spectral curve from the one loop Bethe ansatz

The asymptotic limit of the one-loop BA equation for the open case was analyzed in [48, 49], while the spectral curve was proposed in this context in [50]. For completeness we summarize their findings.

The $Y$ function in the asymptotic limit simplifies to

$$Y_{1,0} = \left(\frac{u+i/2}{u-i/2}\right)^{2L} \frac{u-i/2}{u+i/2} \prod_{j=1}^{N} \frac{u-u_j+i}{u-u_j-i} \frac{u+u_j+i}{u+u_j-i}. \tag{D.1}$$

This $Y$ function is the same as the periodic one with particle content $(u_j, -u_j)$ in volume $2L$, with the exception of the factor $u^-/u^+$. This factor, when evaluated at $u_j$ in the BA equation (5.1), is responsible for removing the unwanted selfscattering piece. This will not be relevant in the scaling limit, in which $L \to \infty$ and roots scale as $u_j \propto L$. In this limit we reparametrize them as $u_j = Lx_j$, and expand the logarithm of the BA equations for large $L$:

$$\frac{1}{x_j} = \frac{1}{L} \sum_{k:k\neq j}^{N} \left(\frac{1}{x_j - x_k} + \frac{1}{x_j + x_k}\right) - 2\pi n_j \tag{D.2}$$

Clearly if $x_j$ is the solution of the equation with $n_j$ then $-x_j$ is a solution with $-n_j$. In the $L \to \infty$ limit roots condense on symmetric cuts localized around $\pm 1/(2\pi n_j)$. We introduce their densities and resolvents as

$$\rho(x) = \frac{1}{L}\sum_{k}^{N} \delta(x - x_j)\,; \qquad G(x) = \frac{1}{L}\sum_{k}^{N}\left(\frac{1}{x-x_j} + \frac{1}{x+x_j}\right) = \int_C dx' \frac{\rho(x')}{x-x'} \tag{D.3}$$

which are nonzero on cuts $C_\alpha^{\pm} = \pm(a_\alpha, b_\alpha)$ with $a > 0$ and $b > 0$. The resolvent is an analytic function on the complex plane with given cuts and has the asymptotics ($x \to \infty$):

$$G(x) = \frac{2\alpha}{x} + \dots\,; \qquad \int_C dx \rho(x) = 2\alpha = \frac{2N}{L} \tag{D.4}$$

The quasi momenta are related to the resolvent in a trivial way

$$p(x) = G(x) - \frac{1}{2x} \tag{D.5}$$

such that the BA equation takes the form

$$p(x+i0) + p(x-i0) = \pm 2\pi n_k\,; \qquad x \in C_k^{\pm} \tag{D.6}$$

Similarly to the periodic case $p(x) = \int^x dp$ is an Abelian integral for the meromorphic differential $dp$, which has two double poles at $x = 0$ and integer periods

$$\begin{aligned} 2\pi(n_k - n_j) &= p(x_k + i0) - p(x_j - i0) + p(x_k - i0) - p(x_j + i0) = \\ &= \int_{x_j - i0}^{x_k + i0} dp + \int_{x_j + i0}^{x_k - i0} dp = \oint_{B_{ij}} dp \end{aligned} \tag{D.7}$$



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on a hyperelliptic curve:

$$y^2 = \prod_{k}^{2N} (x - x_k)(x + x_k) \qquad C_k^+ = \{x_{2k}, x_{2k+1}^*\} \tag{D.8}$$

Comparing these results to the periodic case we observe that the only difference is that cuts appear in the classical limit in a symmetric way.



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