View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Springer - Publisher Connector

Zhang and Xiao *Journal of Inequalities and Applications* (2015) 2015:181
DOI 10.1186/s13660-015-0707-3

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH

Open Access

# Optimal investment of a time-dependent renewal risk model with stochastic return

Jiesong Zhang[1,2*] and Qingxian Xiao[1]

*Correspondence:
j_s_zhang@126.com
[1]Business School, University of Shanghai for Science and Technology, Shanghai, 200093, China
[2]School of Mathematical Science, Huaibei Normal University, Huaibei, 235000, China

**Abstract**

Consider an insurance company which is allowed to invest into a riskless and a risky asset under a constant mix strategy. The total claim amount is modeled by a non-standard renewal risk model with dependence between the claim size and the inter-arrival time introduced by a Farlie-Gumbel-Morgenstern copula. The price of the risky asset is described by an exponential Lévy process. Based on some known results, the uniform asymptotic estimate for ruin probability with investment strategy is obtained with regularly varying tailed claims. Applying the asymptotic formula, we provide an approximation of the optimal investment strategy to maximize the expected terminal wealth subject to a risk constraint on the Value-at-Risk, which is defined with respect to finite-time discounted net loss. A numerical example is illustrated for the results, which demonstrates that big dependence parameter is advantageous for the insurer. We explain the reason by some inequalities.

**Keywords:** dependence; optimal portfolio; Lévy process; asymptotics; Value-at-Risk (VaR)

## 1 Introduction

The renewal risk model has been playing a fundamental role in classical and modern risk theory since it was introduced by Sparre Andersen in 1957. In this framework, the successive claims $\{X_k, k \geq 1\}$ form a sequence of independent identically distributed (i.i.d.) random variables (r.v.s), and their inter-arrival times $\{\xi_k, k \geq 1\}$ form another sequence of i.i.d. r.v.s, and the two sequences are mutually independent too. However, the independence assumption between the claim size $X$ and the inter-arrival time $\xi$ is too restrictive and sometimes unrealistic in many kinds of insurance. Here $X$ and $\xi$ denote the generic r.v.s of the claim sizes and inter-arrival times, respectively. In fact, for a line of business covering damages due to earthquakes, more quantities of damages are expected with a longer period between claims, since the claim amount (or the intensity of the catastrophe) and the time elapsed from the previous one are assumed to be positively related by the seismic gap hypothesis (see, *e.g.*, Nikoloulopoulos and Karlis [1] and Boudreault *et al.* [2]). Furthermore, the period between claims must be longer if the deductible of the insured increases, since some of the small claims will not be considered. During the last decade, initiated by Albrecher and Teugels [3], many scholars have started to propose some non-standard renewal risk models with various dependence structures. Among them, Albrecher and Teugels [3] considered that $(X, \xi)$ follows an arbitrary dependence structure based on a copula, and they derived explicit exponential estimates for

Springer

© 2015 Zhang and Xiao. This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

finite- and infinite-time ruin probabilities. Cossette *et al.* [4] assumed a generalized Farlie-Gumbel-Morgenstern (FGM) copula for $(X, \xi)$, and they derived the Laplace transform of the Gerber-Shiu discounted penalty function. Bargès *et al.* [5] derived the moments of the aggregate discounted claims with dependence introduced by a FGM copula and the counting process following a Poisson process. Asimit and Badescu [6] described the dependence structure via the conditional tail probability of $X$ given $\xi$, and they studied the tail behavior of discounted aggregate claims in the Cramér-Lundberg risk model in the presence of a constant force of interest and heavy-tailed claim sizes. Yong and Xiang [7] obtained an integro-differential equation for the expected discounted penalty function with a FGM copula between $X$ and $\xi$ and the distribution of $\xi$ being a sum of two independent exponential r.v.s.

In insurance practice, the economic result of an insurance company depends not only on insurance business, but also on how well the reserve is invested in market. The study of insurance risk models with stochastic return on investments has attracted considerable attention recently; see for example Yin and Wen [8] and references therein. Since many empirical investigations have illustrated that stock price processes have sudden downward (or upward) jumps which cannot be accounted for by a continuous exponential Brownian motion, to replace in the classical exponential Brownian motion the Wiener process by some general jump-diffusion or Lévy process is an important generalization. Considering that insurers are not allowed to invest all their wealth into risky assets, a riskless investment is usually also considered. These assumptions of investment portfolio are widely used in modern mathematical finance and actuarial science; see Klüppelberg and Kostadinova [9], Heyde and Wang [10], Kostadinova [11], among others.

In order to measure the integrate risk, Kostadinova [11] provided a definition of Value-at-Risk (VaR) based on infinite-time discounted net loss. Note that this definition does not depend on the initial capital and the time, the solution to the optimization problem there is also independent of the time period. In reality, insurers may pay more attention to their future finite-time risks, especially in the investment period, for example, one year, and can rearrange the investment strategy. So, a definition of VaR based on finite-time discounted net loss is a more realistic risk measurement.

Motivated by the considerations above, this paper uses a continuous time-dependent renewal risk model for insurance business. The joint distribution of $(X, \xi)$ is based on the classical FGM copula, which is defined by

$$C(u, v) = uv + \rho uv(1 - u)(1 - v), \tag{1}$$

for every $(u, v)$ in $[0, 1]^2$, in which $\rho \in (-1, 1)$ is called the dependence parameter.

For the investment, we assume a market consisting of a bond with a constant interest rate and some stock. Their respective prices follow the equations

$$Y_0(t) = e^{\delta t} \quad \text{and} \quad Y_1(t) = e^{L(t)}, \quad t \geq 0. \tag{2}$$

The constant $\delta > 0$ is the riskless interest rate. The process $\{L(t), t \geq 0\}$ is a Lévy process.

For the risk measurement, we use a concept of VaR based on finite-time discounted net loss, which is described as follows. Let $t > 0$ be a fixed-time horizon, $V_\theta^*(t)$ be the maximum discounted net loss process during the investment period, and then the risk measure VaR is defined as

$$\mathrm{VaR}_p\big(V_\theta^*(t)\big) = \inf\big\{x \in \mathbb{R} : \mathrm{P}\big(V_\theta^*(t) > x\big) \le p\big\}, \tag{3}$$

where $p \in (0,1)$ is some (typically small) probability.

The use of $\mathrm{VaR}_p(V_\theta^*(t))$ as a risk measure is explained by the fact that the insurer can prevent the maximum loss from exceeding this quantity in finite-time horizon with a sufficiently high probability $1 - p$. It could also be understood as the minimal initial capital required.

This paper mainly focuses on maximizing the expected terminal wealth of the insurer under the risk constraint of VaR as defined in (3). For this goal, the distribution function of $V_\theta^*(t)$ is needed. As it is hard or even impossible to obtain a closed-form expression for this distribution (equivalently, the function of ruin probability), we are interested in an approximation of it. So, the asymptotic estimate for the ruin probability is also discussed.

The paper is organized as follows. In Section 2 we model the integrated risk process consisting of underwriting risks and investment risks. To obtain analytical investment strategy, uniform asymptotic estimate for the ruin probability is given in Section 3. Section 4 characterizes the optimal investment model and the optimal strategy is provided. A numerical example and the impact of dependence parameter are illustrated in Section 5. The final Section 6 concludes the paper.

## 2 The integrated risk model

We first characterize the underwriting process without investment by a continuous time-dependent renewal risk model.

The claim amounts $\{X_k, k \ge 1\}$ during the time interval $[0,t]$ form a sequence of i.i.d. r.v.s with common distribution function $F = 1 - \overline{F}$, the inter-claim times $\{\xi_k, k \ge 1\}$ form another i.i.d. r.v.s with common distribution function $G = 1 - \overline{G}$. The dependence structure of $(X, \xi)$ is described as in (1). The arrival times of the successive claims $\{\tau_n = \sum_{k=1}^n \xi_k, n \ge 1\}$ constitute a renewal counting process

$$N(t) = \sum_{n=1}^{\infty} \mathrm{I}_{\{\tau_n \le t\}} = \sup\{n \ge 1 : \tau_n \le t\}, \quad t \ge 0,$$

where $\mathrm{I}_{\{\cdot\}}$ is the indicator function.

The renewal function of the renewal counting process $\{N(t), t \ge 0\}$ is defined as

$$\lambda_t = E\big[N(t)\big] = \sum_{n=1}^{\infty} \mathrm{P}(\tau_n \le t), \quad t \ge 0.$$

In particular, if $\{N(t), t \ge 0\}$ follows a Poisson distribution with intensity $\lambda > 0$, then $\lambda_t = \lambda t$; and if $\tau_1$ is distributed with $\Gamma(2, \lambda)$, then $\lambda_t = \lambda t/2 - (1 - e^{-2\lambda t})/4$. More explicit forms of $\lambda_t$ for different renewal counting processes can be found in Asmussen and Albrecher [12]. In this paper, the renewal function $\lambda_t$ is assumed to be differentiable and satisfy $\lambda_t' > 0$ and $\lambda_t < \infty$ for any finite $t$. The cumulative distribution functions and probability density functions of the claim arrival times $\tau_n$ is denoted by $G_n$ and $g_n$, respectively, for $n = 1, 2, \ldots$ .

Let $x > 0$ be the insurer's initial capital and a deterministic bounded function $c(t) \ge 0$ be the density of income payments from premium at time $t$. Then the insurer's surplus process is described as

$$U(t) = x + \int_0^t c(s)\,\mathrm{d}s - S(t),$$

where $S(t) = \sum_{n=1}^{N(t)} X_n$ denotes the aggregate claims by time $t$.

Now we assume that the insurer makes risk-free investment with a constant riskless interest and risky investment, modeled by an exponential Lévy process, as described in (2). Suppose $(\gamma, \sigma^2, \nu)$ be the characteristic triplet of the Lévy process $L(t)$ with characteristic exponent $\breve{\Psi}$, *i.e.* $E[e^{isL(t)}] = e^{t\breve{\Psi}(s)}$, $s \in \mathbb{R}$, and $t \geq 0$. Then $\breve{\Psi}$ has the Lévy-Khintchine representation

$$\breve{\Psi}(s) = is\gamma - \frac{\sigma^2}{2}s^2 + \int_{\mathbb{R}} \bigl(e^{isx} - 1 - isx\mathrm{I}_{\{|x|\leq 1\}}\bigr)\nu(\mathrm{d}x), \quad s \in \mathbb{R},$$

where $\gamma \in \mathbb{R}$, $\sigma \geq 0$, and the Lévy measure $\nu$ satisfies $\nu(0) = 0$ and $\int_{\mathbb{R}}(x^2 \wedge 1)\nu(\mathrm{d}x) < \infty$.

The so-called constant mix strategy is assumed, namely, at each instance of time an initially fixed fraction $\theta \in [0,1]$ of the wealth, is invested in the risky asset and a fraction $1-\theta$ is invested in the riskless asset; see, *e.g.*, Emmer and Klüppelberg [13], Kostadinova [11], Klüppelberg and Kostadinova [9], and Heyde and Wang [10]. Such strategy is dynamic in the sense that it requires a rebalancing of the portfolio at any moment of time depending on the corresponding price changes. The fraction $\theta$ is called the investment strategy.

For convenient to describe, denote by $\epsilon(L)$ the solution of the differential equation

$$\mathrm{d}Z(t) = Z(t-)\,\mathrm{d}L(t), \qquad Z(0) = 1,$$

where $L(t)$ is a Lévy process. Proposition 8.21 of Cont and Tankov [14] guarantees the existence and uniqueness of $\epsilon(L)$.

Next we follow the method used by Klüppelberg and Kostadinova [9] to introduce the integrated risk process (IRP). For an investment strategy $\theta \in [0,1]$, the IRP as the result of the insurance business and the net gains of the investment is defined as the solution to the stochastic differential equation

$$\mathrm{d}U_\theta(t) = c(t)\,\mathrm{d}t - \mathrm{d}S(t) + U_\theta(t-)\,\mathrm{d}\hat{L}_\theta(t), \quad t \geq 0, \qquad U_\theta(0) = x,$$

where $\mathrm{d}\hat{L}_\theta(t) = (1-\theta)\delta\mathrm{d}t + \theta\mathrm{d}\hat{L}(t)$, $\hat{L}(t)$ satisfies $\epsilon(\hat{L}(t)) = e^{L(t)}$.

Provided that the insurance and the investment process are independent, then, similar to Lemma 2.2 of Klüppelberg and Kostadinova [9], we can verify that

$$U_\theta(t) = \exp\bigl(L_\theta(t)\bigr)\left(x + \int_0^t c(s)\exp\bigl(-L_\theta(s)\bigr)\,\mathrm{d}s - \int_0^t \exp\bigl(-L_\theta(s)\bigr)\,\mathrm{d}S(s)\right),$$

where $L_\theta$ satisfies $\exp(L_\theta) = \epsilon(\hat{L}_\theta)$. In fact, according to Lemma 2.5 of Emmer and Klüppelberg [13], the $L_\theta(t)$ is also a Lévy process with characteristic triplet $(\gamma_\theta, \sigma_\theta^2, \nu_\theta)$ given by

$$\begin{aligned}\gamma_\theta &= \gamma\theta + (1-\theta)\left(\delta + \frac{\sigma^2}{2}\theta\right) \\ &\quad + \int_{\mathbb{R}} \bigl(\log\bigl(1+\theta\bigl(e^x-1\bigr)\bigr)\mathrm{I}_{\{|\log(1+\theta(e^x-1))|\leq 1\}} - \theta x\mathrm{I}_{\{|x|\leq 1\}}\bigr)\nu(\mathrm{d}x),\end{aligned}$$

$$\sigma_\theta^2 = \theta^2\sigma^2,$$
$$\nu_\theta(A) = \nu\bigl(\bigl\{x \in \mathbb{R} : \log\bigl(1 + \theta\bigl(e^x - 1\bigr)\bigr) \in A\bigr\}\bigr) \quad \text{for any Borel set } A \subset \mathbb{R}.$$

Define the Laplace exponent of $L$ and $L_\theta$ as

$$\phi(s) = \log E\bigl[e^{-sL(1)}\bigr] \quad \text{and} \quad \phi_\theta(s) = \log E\bigl[e^{-sL_\theta(1)}\bigr],$$

provided that they exist. From the proof of Lemma 4.1 of Klüppelberg and Kostadinova [9] we know $\phi_\theta(s) < \infty$ for all $\theta \in (0,1)$ and $s > 0$.

Lastly, following a long tradition in insurance, the discounted net loss process is defined as

$$V_\theta(t) = x - \exp\bigl(-L_\theta(t)\bigr)U_\theta(t) = \int_0^t \exp\bigl(-L_\theta(s)\bigr)\bigl(\mathrm{d}S(s) - c(s)\,\mathrm{d}s\bigr), \quad t \geq 0,$$

and the ruin probability of the time-dependent renewal risk model up to a finite time $t$ is as usual defined as

$$\psi_\theta(x,t) = \mathrm{P}\Bigl(\inf_{0<s\leq t} U_\theta(s) < 0 \Bigm| U_\theta(0) = x\Bigr), \quad t > 0.$$

## 3 Uniform asymptotic estimate for ruin probability

The non-standard renewal risk model with dependence structure and stochastic return is also considered by Li [15] and Fu and Ng [16]. They described the price process of the investment portfolio as an exponential Lévy process $\{e^{L(t)}, t \geq 0\}$, and the uniform asymptotics for ruin probability were obtained by restricting the claim-size distribution to different classes. Considering that the investment strategy is embedded in this paper, we first give the uniform asymptotic estimate for the ruin probability $\psi_\theta(x,t)$ when the claim-size distribution $F$ is regularly varying tailed, based on the fact that the corresponding stochastic process $L_\theta(t)$ is also a Lévy process and Corollary 2.2 of Li [15]. This estimate will be used for providing an approximation of the optimal investment strategy in Section 4.

**Definition 1** A distribution function defined on $[0,\infty)$ is said to be regularly varying tailed with tail index $\alpha > 0$, denoted by $F \in \mathfrak{R}_{-\alpha}$, if $\overline{F}(x) > 0$ holds for all $x \geq 0$ and

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha},$$

holds for all $y > 0$.

If $F \in \mathfrak{R}_{-\alpha}$, then there is some slowly varying function $H(x)$ such that

$$\overline{F}(x) \sim x^{-\alpha}H(x), \quad x \to \infty.$$

Note that for two positive functions $a(x)$ and $b(x)$, the symbol $a(x) \sim b(x)$ means that $\lim_{x\to\infty} a(x)/b(x) = 1$. In fact, $\mathfrak{R}_{-\alpha}$ is a very important class of heavy-tailed distributions, which contains the Pareto, the inverse Gamma distributions *etc.*

The following lemmas will be used in the proofs of Theorems 1 and 2.

**Lemma 1** *Assume that* $0 < E[L(1)] < \infty$, *and* $\sigma > 0$ *or* $\nu_\theta((-\infty,0)) > 0$. *Then for any* $\theta \in (0,1]$,

(a) *There exists a unique positive* $\kappa = \kappa_\theta > 0$ *such that* $\phi_\theta(\kappa) = 0$.

(b) *Let* $\phi(-1) > \delta$. *For fixed* $\alpha > 0$ *the function* $\phi_\theta(\alpha)$ *is strictly convex in* $\theta$ *and* $\partial\phi_0(\alpha)/\partial\theta < 0$.

(c) *Let* $\phi(-1) > \delta$. *Then the function* $\kappa_\theta$ *is strictly decreasing in* $\theta$.

*Proof* See Lemma 4.1 in Klüppelberg and Kostadinova [9]. □

**Lemma 2** *Let* $\phi(-1) > \delta$. *Then there exists at most one positive solution to the equation* $\kappa_\theta = \alpha$ *with respect to* $\theta$. *When there is indeed a positive solution, denote it by* $\dot{\theta}$ *and define*

$$\bar{\theta} = \begin{cases} 0, & \alpha \geq \lim_{\theta\downarrow 0}\kappa_\theta, \\ \dot{\theta}, & \kappa_1 \leq \alpha < \lim_{\theta\downarrow 0}\kappa_\theta, \\ 1, & \alpha < \kappa_1. \end{cases}$$

*Then the condition* $0 < \alpha < \kappa_\theta$ *is equivalent to* $0 < \theta < \bar{\theta}$.

*Proof* It is immediate from Lemma 1. □

**Theorem 1** *Let* $F \in \mathfrak{R}_{-\alpha}$ *for some* $0 < \alpha < \infty$. *Assume that* $\phi(-1) > \delta$, $0 < E[L(1)] < \infty$, *and* $\sigma > 0$ *or* $\nu_\theta((-\infty,0)) > 0$. *Then, for any* $\theta \in (0,\bar{\theta})$,

$$\psi_\theta(x,t) \sim \overline{F}(x)\int_{0-}^{t} e^{s\phi_\theta(\alpha)}\,\mathrm{d}\lambda_s^* \tag{4}$$

*holds uniformly for* $t \in \Lambda := \{s : 0 < \lambda_s \leq \infty\}$, *where*

$$\lambda_s^* = \int_{0-}^{s} (1 + \lambda_{s-u})\bigl(1 + \rho\bigl(1 - 2\overline{G}(u)\bigr)\bigr)G(\mathrm{d}u).$$

**Remark 1** The condition $\phi(-1) > \delta$ guarantees that the expected value of the risky investment is larger than the riskless investment. The condition $\theta \in (0,\bar{\theta})$, *i.e.* $0 < \alpha < \kappa_\theta$, means that the extreme of insurance risk determines the tail behavior of the ultimate integrated risk for the discounted net loss process, other than the investment risks. For more explanations about these conditions, we refer the reader to Kostadinova [11], Klüppelberg and Kostadinova [9] and Tang *et al.* [17].

*Proof of Theorem* 1 According to the relation 2.9.1 in Nelsen [18] and the form of FGM copula $C(u,v)$, we know that

$$\begin{aligned} \mathrm{P}\bigl(X > x|\xi = t\bigr) &= 1 - \frac{\partial C(u,v)}{\partial u}\Big|_{u=G(t),v=F(x)} \\ &= \overline{F}(x)\bigl(1 + \rho F(x)\bigl(1 - 2\overline{G}(t)\bigr)\bigr) \\ &\sim \overline{F}(x)\bigl(1 + \rho\bigl(1 - 2\overline{G}(t)\bigr)\bigr) \end{aligned}$$

holds uniformly for $t \in \Lambda$. Denote $h(t) = 1 + \rho(1 - 2\overline{G}(t))$, then it is easy to see that

$$\mathrm{P}\bigl(X > x|\xi = t\bigr) \sim \mathrm{P}(X > x)h(t),$$

and the function $h(t)$ satisfies

$$0 < 1 - |\rho| \leq h(t) \leq 1 + |\rho| < 2.$$

Thus, both assumptions (A) and (B) of Li [15] hold.

By Lemma 1, we know that $\phi_\theta(0) = \phi_\theta(\kappa_\theta) = 0$, and then the fact $\phi_\theta(s) < 0$ for all $s \in (0, \kappa_\theta)$ follows from strict convexity of $\phi_\theta(s)$ in $s$ (for fixed $\theta$). Noting that $\theta \in (0, \bar{\theta})$, we also have $0 < \alpha < \kappa_\theta$ according to Lemma 2. Hence, there exists $\alpha^* > \alpha$ such that $\phi_\theta(\alpha^*) < 0$.

With the above preliminarily results, and since $L_\theta(t)$ is again a Lévy process, we know that all the conditions in Corollary 2.1 and Corollary 2.2 of Li [15] are satisfied. If Corollary 2.2 of Li [15] is applied, we can get

$$\psi_\theta(x,t) \sim \overline{F}(x) \int_{0-}^{t} e^{-s\phi_\theta(\alpha)} \, \mathrm{d}\lambda_s^*.$$

However, when checking the result of Corollary 2.2 therein, we found a typo. In (2.6) of Li [15], the integrand term $e^{-s\phi(\alpha)}$ should actually be $e^{s\phi(\alpha)}$, which can be seen from the first line of the proof of Lemma 3.9 there. That is to say, under the conditions of Corollary 2.2 of Li [15], uniformly for $t \in \Lambda$

$$\psi(x,t) \sim \overline{F}(x) \int_{0-}^{t} e^{s\phi(\alpha)} \, \mathrm{d}\lambda_s^*.$$

In fact, without consideration of the dependence, this result is just Theorem 3.1 of Tang *et al.* [17], and the right-hand side of the relation (2.2) in Heyde and Wang [10] can also be denoted by a similar formula, $\overline{F}(x) \int_0^T e^{s\psi_\theta(\alpha)} \, \mathrm{d}(\lambda s)$. Consequently, the uniform asymptotic estimate (4) is obtained. □

**Remark 2** As mentioned in the proof of Theorem 1, if $\rho = 0$, *i.e.* the dependence between the claim size $X$ and the inter-arrival time $\xi$ is not considered, Theorem 1 keeps consistent with Theorem 3.1 in Tang *et al.* [17]; if $\lambda_t = \lambda t$, Theorem 1 keeps consistent with Theorem 2.1 in Heyde and Wang [10] for the case $F \in \mathfrak{R}_{-\alpha}$. Thus Theorem 1 partly extends these two results. When compared with Corollary 2.1 of Fu and Ng [16] and Theorem 2.1 of Li [15], it is not difficult to find that the investment strategy is embedded here. Moreover, the conditions here are easier to check and the expression of the asymptotic estimate is more explicit, though the class of heavy-tailed distribution and dependence structure are more specific.

To analyze the impact of dependence parameter $\rho$ on the underwriting risks and investment strategies in the following conveniently, here we first show the impact of $\rho$ on the renewal function $\lambda_s^*$, which is the mean function of a delayed renewal counting process (see Li [15] for more details). The conclusion will be used in Section 5 for explaining the numerical results.

**Proposition 1** *The renewal function $\lambda_s^*$ decreases in the dependence parameter $\rho$.*

*Proof* Note that

$$\int_{0-}^{s} \left|(1 + \lambda_{s-u})\left(1 - 2\overline{G}(u)\right)\right| G(\mathrm{d}u) \leq \int_{0-}^{s} (1 + \lambda_{s-u}) G(\mathrm{d}u) \leq 1 + \lambda_s < \infty.$$

Thus, according to the Lebesgue dominated convergence theorem, we have

$$\frac{\partial\lambda_s^*}{\partial\rho} = \int_{0-}^{s} (1+\lambda_{s-u})\big(-1+2\overline{G}(u)\big)\mathrm{d}\overline{G}(u).$$

Let $u^*$ be the solution to the equation $\overline{G}(u) = 0.5$. If $s \le u^*$, it is easy to see that $\frac{\partial\lambda_s^*}{\partial\rho} \le 0$ since $-1+2\overline{G}(u) \ge 0$. If $s > u^*$, we have

$$\begin{aligned}
\frac{\partial\lambda_s^*}{\partial\rho} &= \int_{0-}^{u^*} (1+\lambda_{s-u})\big(2\overline{G}(u)-1\big)\mathrm{d}\overline{G}(u) + \int_{u^*}^{s} (1+\lambda_{s-u})\big(2\overline{G}(u)-1\big)\,\mathrm{d}\overline{G}(u) \\
&\le (1+\lambda_{s-u^*})\int_{0-}^{s} \big(2\overline{G}(u)-1\big)\,\mathrm{d}\overline{G}(u) \\
&= -(1+\lambda_{s-u^*})\overline{G}(s)G(s) \\
&\le 0.
\end{aligned}$$

Hence, $\frac{\partial\lambda_s^*}{\partial\rho} \le 0$ for any $0 < s < \infty$. □

## 4 Optimal investment

In this section, we discuss how the insurer adjusts his investment portfolio to maximize the expected terminal wealth under a risk constraint on VaR. An approximation of the optimal investment strategy is provided.

### 4.1 Optimal investment model

Considering a fixed-time horizon $t > 0$, the maximum loss process during the investment period is defined as $V_\theta^*(t) = \sup_{0\le s\le t} V_\theta(s)$ and the risk measurement of VaR is defined as in (3).

We consider the regime that the insurance business has more influence on the reserve process than the investment, *i.e.* the condition $\theta \in (0,\bar{\theta})$ is assumed. In fact, there exists an upper bound for the fraction of risky investment of insurance companies in many nations. Accordingly, the optimization model is constructed as

$$\max_{\theta\in(0,\bar{\theta})} E\big[U_\theta(t)\big] \quad \text{subject to} \quad \mathrm{VaR}_p\big(V_\theta^*(t)\big) \le C, \tag{5}$$

for a given constraint $C > 0$.

Under the condition $U_\theta(0) = x$, we have

$$\psi_\theta(x,t) = \mathrm{P}\Big(\inf_{0<s\le t} U_\theta(s) < 0\Big) = \mathrm{P}\Big(\sup_{0<s\le t} V_\theta(s) > x\Big) = \mathrm{P}\big(V_\theta^*(t) > x\big).$$

Hence, the optimization problem (5) is equivalent to

$$\max_{\theta\in(0,\bar{\theta})} E\big[U_\theta(t)\big] \quad \text{subject to} \quad \inf\big\{x \in \mathbb{R} : \psi_\theta(x,t) \le p\big\} \le C. \tag{6}$$

### 4.2 The optimal strategies

Since there is usually no closed-form expression for $\psi_\theta(x,t)$ available, we use its approximation given by Theorem 1 to solve the optimal strategies. For this purpose, the instantaneous average claim rate is assumed to be smaller than the instantaneous premium rate, *i.e.*

$$c(t) > \mathrm{d}E\big[S(t)\big]/\mathrm{d}t. \tag{7}$$

**Remark 3** Equation (7) can be interpreted as the safety loading condition. If the underwriting process is characterized as the classical compound Poisson model, this relation is equivalent to $c > \lambda\mu$, where $c$ is the constant premium rate, $\lambda$ is the claim intensity, and $\mu$ is the mean of individual claim. In the case of renewal risk model, (7) is equivalent to $c \geq \mu\lambda_s'$ (see (3.14) in Tang *et al.* [17]). If the dependence structure between the claim size and the inter-arrival time as described in (1) is considered and $N(t)$ follows a homogeneous Poisson process, according to Bargès *et al.* [5], we know that (7) is equivalent to $c(t) > \lambda\mu + \rho\lambda(\mu' - \mu)e^{-2\lambda t}$, where $\mu' = \int_0^\infty (\overline{F}(x))^2\,\mathrm{d}x < \mu$. For this case, note that the instantaneous average claim rate is decreasing in $\rho$.

**Theorem 2** *Let the conditions of Theorem* 1 *and the safety loading condition* (7) *hold. Then, for any given* $t > 0$ *and sufficiently large* $x$, *the approximation of the optimal investment strategy is*

$$\tilde{\theta}^* = \sup\left\{\theta \in (0,\bar{\theta}) : p\left(\int_{0-}^t e^{s\phi_\theta(\alpha)}\,\mathrm{d}\lambda_s^*\right)^{-1} \geq \overline{F}(C)\right\}. \tag{8}$$

*Further, denote* $\Upsilon(\theta) = \int_{0-}^t e^{s\phi_\theta(\alpha)}\,\mathrm{d}\lambda_s^*$ *and suppose that* $\{\theta : p/\Upsilon(\theta) \geq \overline{F}(C)\} \neq \emptyset$ (*otherwise no solution exists*). *Then we have*

$$\tilde{\theta}^* = \begin{cases} \hat{\theta}, & p/\Upsilon(\bar{\theta}) < \overline{F}(C) < p/\Upsilon(0) \text{ or } \overline{F}(C) > \max(p/\Upsilon(0), p/\Upsilon(\bar{\theta})), \\ \bar{\theta}, & \text{otherwise}, \end{cases} \tag{9}$$

*where* $\hat{\theta}$ *is the bigger root of the equation* $\Upsilon(\theta) - p/\overline{F}(C) = 0$.

*Proof* According to the total probability formula,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}E\big[S(t)\big] &= \frac{\mathrm{d}}{\mathrm{d}t}\left(E\left[\sum_{n=1}^\infty X_n \mathrm{I}_{\{\tau_n \leq t\}}\right]\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{n=1}^\infty \int_0^t E[X_n|\tau_n = u]\,\mathrm{d}G_n(u)\right) \\ &= \sum_{n=1}^\infty E[X_n|\tau_n = t]g_n(t). \end{aligned}$$

Thus, by the independence of $S(t)$ and $L_\theta(t)$ and the stationary increment property of $L_\theta(t)$, we have

$$\begin{aligned} E\left[\int_0^t e^{L_\theta(t)-L_\theta(s)}\,\mathrm{d}S(s)\right] &= \sum_{n=1}^\infty E\big[e^{L_\theta(t)-L_\theta(s)} X_n \mathrm{I}_{\{\tau_n \leq t\}}\big] \\ &= \int_0^t E\big[e^{L_\theta(t)-L_\theta(s)}\big] \sum_{n=1}^\infty E[X_n|\tau_n = s]g_n(s)\,\mathrm{d}s \\ &= \int_0^t E\big[e^{L_\theta(t-s)}\big]\left(\frac{\mathrm{d}}{\mathrm{d}s}E\big[S(s)\big]\right)\mathrm{d}s. \end{aligned}$$

Hence,

$$E\big[U_\theta(t)\big] = xE\big[e^{L_\theta(t)}\big] + \int_0^t E\big[e^{L_\theta(t-s)}\big]\bigg(c(s) - \frac{\mathrm{d}}{\mathrm{d}s}E\big[S(s)\big]\bigg)\,\mathrm{d}s.$$

By Lemma 2.5 in Kostadinova [11], we know

$$E\big[e^{L_\theta(t)}\big] = \exp\big(t\big(\delta + \theta\big(\phi(-1) - \delta\big)\big)\big).$$

Therefore, if $\phi(-1) > \delta$, we know the mean function $E[e^{L_\theta(t)}]$ is increasing in $\theta$, and by (7), we know that the mean function $E[U_\theta(t)]$ is increasing in $\theta$. Consequently, the optimization (6) is equivalent to

$$\sup\big\{\theta \in (0,\bar{\theta}) : \inf\big\{x \in \mathbb{R} : \psi_\theta(x,t) \le p\big\} \le C\big\}. \tag{10}$$

Let $\theta^*$ be the exact solution of the optimization problem (10). Define

$$\tilde{\theta}^* = \sup\big\{\theta \in (0,\bar{\theta}) : \inf\big\{x \in \mathbb{R} : \tilde{\psi}_\theta(x,t) \le p\big\} \le C\big\},$$

where

$$\tilde{\psi}_\theta(x,t) = \overline{F}(x)\int_{0-}^t e^{s\phi_\theta(\alpha)}\,\mathrm{d}\lambda_s^*.$$

Then, by Theorem 1, we have $\psi_\theta(x,t) \sim \tilde{\psi}_\theta(x,t)$ uniformly for any $t \in \Lambda$. Thus, $\tilde{\theta}^*$ is an approximation of the exact solution $\theta^*$.

Next we analyze the value of $\tilde{\theta}^*$. Noting that $\lambda_s^*$ is strictly increasing in $s$, we know that $\int_{0-}^t e^{s\phi_\theta(\alpha)}\,\mathrm{d}\lambda_s^*$ is positive. By the fact that $\overline{F}(x)$ is decreasing in $x$, we have

$$\begin{aligned}\tilde{\theta}^* &= \sup\bigg\{\theta \in (0,\bar{\theta}) : \inf\bigg\{x \in \mathbb{R} : \overline{F}(x) \le p\bigg(\int_{0-}^t e^{s\phi_\theta(\alpha)}\,\mathrm{d}\lambda_s^*\bigg)^{-1}\bigg\} \le C\bigg\} \\ &= \sup\bigg\{\theta \in (0,\bar{\theta}) : p\bigg(\int_{0-}^t e^{s\phi_\theta(\alpha)}\,\mathrm{d}\lambda_s^*\bigg)^{-1} \ge \overline{F}(C)\bigg\}.\end{aligned}$$

Hence, (8) holds.

Further, by Lemma 1(b), the function $\phi_\theta(\alpha)$ is strictly convex in $\theta$ and satisfies $\partial\phi_0(\alpha)/\partial\theta < 0$. On the other hand, we have $\phi_0(\alpha) = -\delta\alpha < 0$. Thus, on the interval $(0,\bar{\theta})$, the function $\phi_\theta(\alpha)$ first decreases and then increases, and so is the function $\Upsilon(\theta)$. Therefore, the function $p/\Upsilon(\theta)$ first increases and then decreases. Consequently, (9) holds. □

## 5 Example

We consider an insurance company with Pareto claims, *i.e.*

$$\overline{F}(x) = \big(0.6/(0.6 + x)\big)^2, \quad x > 0.$$

Suppose that $N(t)$ is a homogeneous Poisson process with intensity $\lambda = 12$, then we know the instantaneous average rate of claims is $7.2 - 4.8e^{-24t}\rho$ from Remark 3. Then the instantaneous premium rate $c(t)$ is set to equal to $7.2 + 4.8e^{-24t}\rho$. Assume that the riskless

**Table 1 Optimal investment strategies and expected results with different dependence parameters**

| $\rho$ | $\tilde{\theta}^*$ | VaR | $E[U_\theta(t)]$ |
|---|---|---|---|
| –0.75 | 0.1805 | 64.0000 | 70.0677 |
| –0.50 | 0.2995 | 64.0000 | 71.9864 |
| –0.25 | 0.3534 | 64.0000 | 72.9035 |
| 0.00 | 0.3944 | 64.0000 | 73.6225 |
| 0.25 | 0.4286 | 64.0000 | 74.2372 |
| 0.50 | 0.4583 | 64.0000 | 74.7826 |
| 0.75 | 0.4849 | 64.0000 | 75.2806 |
| 0.85 | 0.4915 | 63.9526 | 75.4135 |
| 0.99 | 0.4915 | 63.7571 | 75.4460 |

interest rate $\delta = 0.05$ and the log returns of some stock are modeled by

$$L(t) = 0.15t + 0.2B(t) + \sum_{i=1}^{M(t)} \zeta_i,$$

where $\{B(t), t \geq 0\}$ is a standard Brownian motion, $M(t)$ is a homogeneous Poisson process with intensity 10, and the generic r.v. $\zeta$ follows $N(-0.01, 0.04)$.

Assume without loss of generality that the investment period is $[0, 1]$, and then we compute $\phi(-1) = 0.2705$, which is larger than the riskless interest rate. The numerical solution to the equation $\phi_\theta(2) = 0$ is $\bar{\theta} = 0.4915$, *i.e.* the upper bound of the fraction invested in risky asset.

Let $x = 64$ be the initial capital and $p = 0.1\%$ be the given probability, then the optimal strategies and expected results of investment with different dependence parameters are illustrated in Table 1.

From Table 1, it is easy to see that the fraction of risky investment increases as the dependence parameter goes up. The reason for this result is that the intensity and average rate of claims both decrease as the dependence parameter increases (see Proposition 1 and Remark 3). Thus the underwriting risks becomes smaller, more amount of wealth can be invested in risky asset. Meanwhile, the wealth at the end of the planning period gets more and the integrated risk equals the constraint $C = 64$ when the fraction of risky investment are within the interval of investment strategies. In fact, the uniform asymptotic for the ruin probability also equals the upper bound 0.1%. When the dependence parameter become so large that the optimal strategy exceeds the interval $(0, \bar{\theta})$, the fraction of risky investment $\tilde{\theta}^*$ is set to $\bar{\theta}$ according to the convention, and the expected terminal wealth is still on the rise. Therefore, the connection between claim size and inter-arrival time has a direct impact on the optimal investment strategy and the expected result. The larger the dependence parameter is, the more favorable it is to the insurance company.

## 6 Conclusion

This paper mainly concerns the problem about how the insurer adjust his investment portfolio to maximize the expected terminal wealth. The underwriting process is modeled by a non-standard renewal risk model with dependence introduced by a FGM copula and the price of the risky asset is characterized by an exponential Lévy process. Based on, but different from Li [15] and Fu and Ng [16], the integrated risk process is investigated with investment strategy and the uniform asymptotic estimate for ruin probability is provided with some conditions more convenient to verify. Considering that the insurers may be more attentive to their future finite-time risks, the infinite-time discounted net loss in the definition of VaR which is considered by Kostadinova [11] is replaced by a finite-time horizontal as the risk measure. Applying the asymptotic formula, an approximation of the optimal investment strategy depending on the dependence parameter and the time period is obtained and a numerical example is illustrated for the results. When investigating the impact of the dependence parameter, we find that the bigger the dependence parameter is, the more advantageous it is to the insurer. The reason for this result is explained by some inequalities.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally to the manuscript read and approved the final manuscript.

**Acknowledgements**
The authors thank the referees for pointing out some errors in a previous version and for several comments that have led to improvement of this work. The research is supported by the National Natural Science Foundation of China (NO. 11171221) and the Natural Science Foundation of Anhui Higher Education Institutions of China (NO. KJ2014B18).

Received: 12 November 2014 Accepted: 20 May 2015 Published online: 06 June 2015

**References**

1. Nikoloulopoulos, AK, Karlis, D: Fitting copulas to bivariate earthquake data: the seismic gap hypothesis revisited. Environmetrics **19**, 251-269 (2008)
2. Boudreault, M, Cossette, H, Landriault, D, Marceau, E: On a risk model with dependence between interclaim arrivals and claim sizes. Scand. Actuar. J. **5**, 265-285 (2006)
3. Albrecher, H, Teugels, J: Exponential behavior in the presence of dependence in risk theory. J. Appl. Probab. **43**, 265-285 (2006)
4. Cossette, H, Marceau, E, Marri, F: On the compound Poisson risk model with dependence based on a generalized Farilie-Gunmbel-Morgenstern copula. Insur. Math. Econ. **43**, 444-455 (2008)
5. Bargès, M, Cossette, H, Loisel, S, Marceau, E: On the moments of the aggregate discounted claims with dependence introduced by a FGM copula. ASTIN Bull. **41**, 215-238 (2011)
6. Asimit, AV, Badescu, AL: Extremes on the discounted aggregate claims in a time dependent risk model. Scand. Actuar. J. **2**, 93-104 (2010)
7. Yong, W, Xiang, H: Differential equations for ruin probability in a special risk model with FGM copula for the claim size and the inter-claim time. J. Inequal. Appl. **2012**, 156 (2012)
8. Yin, C, Wen, Y: An extension of Paulsen-Gjessing's risk model with stochastic return on investments. Insur. Math. Econ. **52**, 469-476 (2013)
9. Klüppelberg, C, Kostadinova, R: Integrated insurance risk models with exponential Lévy investment. Insur. Math. Econ. **42**, 560-577 (2008)
10. Heyde, CC, Wang, D: Finite-time ruin probability with an exponential Lévy process investment return and heavy-tailed claims. Insur. Math. Econ. **41**, 206-224 (2009)
11. Kostadinova, R: Optimal investment for insurers when the stock price follows an exponential Lévy process. Insur. Math. Econ. **41**, 250-263 (2007)
12. Asmussen, S, Albrecher, H: Ruin Probabilities. World Scientific, London (2010)
13. Emmer, S, Klüppelberg, C: Optimal portfolios when stock prices follow an exponential Lévy process. Finance Stoch. **8**, 17-44 (2004)
14. Cont, R, Tankov, P: Financial Modelling with Jump Processes. Chapman & Hall/CRC, Boca Raton (2004)
15. Li, J: Asymptotics in a time-dependent renewal risk model with stochastic return. J. Math. Anal. Appl. **387**, 1009-1023 (2012)
16. Fu, KA, Ng, CYA: Asymptotics for the ruin probability of a time-dependent renewal risk model with geometric Lévy process investment returns and dominatedly-varying-tailed claims. Insur. Math. Econ. **56**, 80-87 (2014)
17. Tang, Q, Wang, G, Yuen, KC: Uniform tail asymptotics for the stochastic present value of aggregate claims in the renewal risk model. Insur. Math. Econ. **46**, 362-370 (2010)
18. Nelsen, RB: An Introduction to Copulas. Springer, New York (2006)