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Research Article

Convergence of Iterative Sequences for Fixed Point and Variational Inclusion Problems

Li Yu and Ma Liang

School of Management, University of Shanghai for Science and Technology, Shanghai 200093, China

Correspondence should be addressed to Li Yu, brucemath@139.com

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An iterative process is considered for finding a common element in the fixed point set of a strict pseudocontraction and in the zero set of a nonlinear mapping which is the sum of a maximal monotone operator and an inverse strongly monotone mapping. Strong convergence theorems of common elements are established in real Hilbert spaces.

1. Introduction and Preliminaries

Throughout this paper, we always assume that H is a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$.

Let C be a nonempty closed convex subset of H and $S : C \to C$ a nonlinear mapping. In this paper, we use $F(S)$ to denote the fixed point set of S. Recall that the mapping S is said to be nonexpansive if

$$\|Sx - Sy\| \le \|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

S is said to be κ-strictly pseudocontractive if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \le \|x - y\|^2 + \kappa\|(x - Sx) - (y - Sy)\|^2, \quad \forall x, y \in C. \tag{1.2}$$

The class of strictly pseudocontractive mappings was introduced by Browder and Petryshyn [1] in 1967. It is easy to see that every nonexpansive mapping is a 0-strictly pseudocontractive mapping.

Let $A : C \to H$ be a mapping. Recall that A is said to be monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C. \tag{1.3}$$

A is said to be inverse strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{1.4}$$

For such a case, A is also said to be α-inverse strongly monotone.

Let $M : H \to 2^H$ be a set-valued mapping. The set $D(M)$ defined by $D(M) = \{x \in H : Mx \neq \emptyset\}$ is said to be the domain of M. The set $R(M)$ defined by $R(M) = \bigcup_{x\in H} Mx$ is said to be the range of M. The set $G(M)$ defined by $G(M) = \{(x, y) \in H \times H : x \in D(M), y \in R(M)\}$ is said to be the graph of M.

Recall that M is said to be monotone if

$$\langle x - y, f - g\rangle > 0, \quad \forall (x, f), (y, g) \in G(M). \tag{1.5}$$

M is said to be maximal monotone if it is not properly contained in any other monotone operator. Equivalently, M is maximal monotone if $R(I + rM) = H$ for all $r > 0$. For a maximal monotone operator M on H and $r > 0$, we may define the single-valued resolvent $J_r = (I + rM)^{-1} : H \to D(M)$. It is known that J_r is firmly nonexpansive and $M^{-1}(0) = F(J_r)$.

Recall that the classical variational inequality problem is to find $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.6}$$

Denote by $\mathrm{VI}(C, A)$ of the solution set of (1.6). It is known that $x \in C$ is a solution to (1.6) if and only if x is a fixed point of the mapping $P_C(I - \lambda A)$, where $\lambda > 0$ is a constant and I is the identity mapping.

Recently, many authors considered the convergence of iterative sequences for the variational inequality (1.6) and fixed point problems of nonlinear mappings see, for example, [1–32].

In 2005, Iiduka and Takahashi [7] proved the following theorem.

Theorem IT. *Let C be a closed convex subset of a real Hilbert space H. Let A be an α-inverse-strongly monotone mapping of C into H, and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(C, A) \neq \emptyset$. Suppose that $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) S P_C(x_n - \lambda_n A x_n), \tag{1.7}$$

for every $n = 1, 2, \ldots$, where $\{\alpha_n\}$ is a sequence in $[0, 1)$ and $\{\lambda_n\}$ is a sequence in $[a, b]$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\{\lambda_n\} \in [a, b]$ for some a, b with $0 < a < b < 2\alpha$,

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty, \qquad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \qquad \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty, \tag{1.8}$$

then $\{x_n\}$ converges strongly to $P_{F(S)\cap VI(C,A)} x$.

In 2007, Y. Yao and J.-C. Yao [31] further obtained the following theorem.

Theorem YY. *Let C be a closed convex subset of a real Hilbert space H. Let A be an α-inverse-strongly monotone mapping of C into H, and let S be a nonexpansive mapping of C into itself such that $F(S) \cap \Omega \neq \emptyset$, where Ω denotes the set of solutions of a variational inequality for the α-inverse-strongly monotone mapping. Suppose that $x_1 = u \in C$ and $\{x_n\}, \{y_n\}$ are given by*

$$\begin{aligned} x_1 &= u \in C, \\ y_n &= P_C(x_n - \lambda_n A x_n), \\ x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n S P_C(I - \lambda_n A) y_n, \quad n \geq 1, \end{aligned} \tag{1.9}$$

where $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $[0,1]$ and $\{\lambda_n\}$ is a sequence in $[0, 2a]$. If $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2a$ and

(a) $\alpha_n + \beta_n + \gamma_n = 1$, *for all* $n \geq 1$,

(b) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(d) $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$,

then $\{x_n\}$ converges strongly to $P_{F(S)\cap\Omega} u$.

In this work, motivated by the above results, we consider the problem of finding a common element in the fixed point set of a strict pseudocontraction and in the zero set of a nonlinear mapping which is the sum of a maximal monotone operator and a inverse strongly monotone mapping. Strong convergence theorems of common elements are established in real Hilbert spaces. The results presented in this paper improve and extend the corresponding results announced by Iiduka and Takahashi [7] and Y. Yao and J.-C. Yao [31].

In order to prove our main results, we also need the following lemmas.

Lemma 1.1 (see [22]). *Let C be a nonempty closed convex subset of a Hilbert space H, $A : C \to H$ a mapping, and $M : H \to 2^H$ a maximal monotone mapping. Then,*

$$F(J_r(I - rA)) = (A + M)^{-1}(0), \quad \forall r > 0. \tag{1.10}$$

Lemma 1.2 (see [1]). *Let C be a nonempty closed convex subset of a real Hilbert space H and $S : C \to C$ a κ-strict pseudocontraction with a fixed point. Define $S : C \to C$ by $S_a x = ax + (1-a)Sx$ for each $x \in C$. If $a \in [\kappa, 1)$, then S_a is nonexpansive with $F(S_a) = F(S)$.*

Lemma 1.3 (see [25]). *Let C be a nonempty closed convex subset of a Hilbert space H and $S : C \to C$ a κ-strict pseudocontraction. Then,*

(a) *S is $((1+\kappa)/(1-\kappa))$-Lipschitz,*

(b) *$I - S$ is demi-closed, this is, if $\{x_n\}$ is a sequence in C with $x_n \rightharpoonup x$ and $x_n - Sx_n \to 0$, then $x \in F(S)$.*

Lemma 1.4 (see [28]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Hilbert space H, and let $\{\beta_n\}$ be a sequence in $(0,1)$ with*

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1. \tag{1.11}$$

Suppose that $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 1$ and

$$\limsup_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0. \tag{1.12}$$

Then, $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

Lemma 1.5 (see [29]). *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n, \tag{1.13}$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(a) $\sum_{n=1}^{\infty} \gamma_n = \infty$,

(b) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty}\alpha_n = 0$.

Lemma 1.6 (see [24]). *Let H be a Hilbert space and M a maximal monotone operator on H. Then, the following holds:*

$$\|J_r x - J_s x\|^2 \leq \frac{r-x}{r}\langle J_r x - J_s x, J_r x - x\rangle, \quad \forall s,t > 0, \ \ x \in H, \tag{1.14}$$

where $J_r = (I + rM)^{-1}$ and $J_s = (I + sM)^{-1}$.

2. Main Results

Theorem 2.1. *Let H be a real Hilbert space H and C a nonempty close and convex subset of H. Let $M : H \to 2^H$ and $W : H \to 2^H$ two maximal monotone operators such that $D(M) \subset C$ and $D(W) \subset C$, respectively. Let $S : C \to C$ be a κ-strict pseudocontraction, $A : C \to H$ an α-inverse strongly monotone mapping, and $B : C \to H$ a β-inverse strongly monotone mapping. Assume that $\digamma := F(S) \cap (A+M)^{-1}(0) \cap (B+W)^{-1}(0) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{gathered} x_1 \in C, \\ y_n = J_{s_n}(x_n - s_n B x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n (\delta_n J_{r_n}(y_n - r_n A y_n) + (1-\delta_n) S J_{r_n}(y_n - r_n A y_n)), \quad \forall n \geq 1, \end{gathered} \tag{2.1}$$

where $u \in C$ is a fixed element, $J_{r_n} = (I + r_n M)^{-1}$ and $J_{s_n} = (I + s_n W)^{-1}$, $\{r_n\}$ is a sequence in $(0, 2\alpha)$, $\{s_n\}$ is a sequence in $(0, 2\beta)$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ are sequences in $[0,1]$.

Assume that the following restrictions are satisfied:

(a) $0 < a \leq r_n \leq b < 2\alpha$, $\lim_{n\to\infty}(r_n - r_{n+1}) = 0$,

(b) $0 < c \leq s_n \leq d < 2\beta_n$, $\lim_{n\to\infty}(s_n - s_{n+1}) = 0$,

(c) $0 \leq \kappa \leq \delta_n < e < 1$, $\lim_{n\to\infty}(\delta_n - \delta_{n+1}) = 0$,

(d) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(e) $0 < \liminf_{n\to\infty}\beta_n \leq \liminf_{n\to\infty}\beta_n < 1$.

Then, the sequence $\{x_n\}$ *converges strongly to* $q = P_{\mathcal{F}}u$.

Proof. The proof is split into five steps.

Step 1. Show that $\{x_n\}$ is bounded.

Note that $(I - r_nA)$ and $(I - s_nB)$ are nonexpansive for each fixed $n \geq 1$. Indeed, we see from the restriction (a) that

$$\begin{aligned} \left\|(I - r_nA)x - (I - r_nA)y\right\|^2 &= \left\|x - y\right\|^2 - 2r_n\langle x - y, Ax - Ay\rangle + r_n^2\left\|Ax - Ay\right\|^2 \\ &\leq \left\|x - y\right\|^2 - r_n(2\alpha - r_n)\left\|Ax - Ay\right\|^2 \\ &\leq \left\|x - y\right\|^2, \quad \forall x, y \in C. \end{aligned} \tag{2.2}$$

This shows that $(I - r_nA)$ is nonexpansive for each fixed $n \geq 1$, so is $(I - s_nB)$. Put

$$S_nx = \delta_nx + (1 - \delta_n)Sx, \quad \forall x \in C. \tag{2.3}$$

In view of the restriction (c), we obtain from Lemma 1.2 that S_n is a nonexpansive mapping with $F(S_n) = F(S)$ for each fixed $n \geq 1$. Fixing $p \in \mathcal{F}$ and since J_{r_n} and $I - r_nA$ are nonexpansive, we see that

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n\|u - p\| + \beta_n\|x_n - p\| + \gamma_n\|S_nJ_{r_n}(y_n - r_nAy_n) - p\| \\ &\leq \alpha_n\|u - p\| + \beta_n\|x_n - p\| + \gamma_n\|J_{r_n}(y_n - r_nAy_n) - p\| \\ &\leq \alpha_n\|u - p\| + \beta_n\|x_n - p\| + \gamma_n\|y_n - p\| \\ &\leq \alpha_n\|u - p\| + (1 - \alpha_n)\|x_n - p\|. \end{aligned} \tag{2.4}$$

By mathematical inductions, we see that $\{x_n\}$ is bounded and so is $\{y_n\}$. This completes Step 1.

Step 2. Show that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$.

Notice from Lemma 1.6 that

$$\begin{aligned} \|y_{n+1} - y_n\| &\le \|(x_{n+1} - s_{n+1}Bx_{n+1}) - (x_n - s_nBx_n)\| \\ &\quad + \|J_{s_{n+1}}(x_n - s_nBx_n) - J_{s_n}(x_n - s_nBx_n)\| \\ &\le \|x_{n+1} - x_n\| + |s_{n+1} - s_n|\|Bx_n\| \\ &\quad + \frac{|s_{n+1} - s_n|}{s_{n+1}}\|J_{s_{n+1}}(x_n - s_nBx_n) - (x_n - s_nBx_n)\| \\ &\le \|x_{n+1} - x_n\| + 2M_1|s_{n+1} - s_n|, \end{aligned} \tag{2.5}$$

where M_1 is an appropriate constant such that

$$M_1 = \max\left\{\sup_{n\ge1}\{\|Bx_n\|\}, \sup_{n\ge1}\left\{\frac{\|J_{s_{n+1}}(x_n - s_nBx_n) - (x_n - s_nBx_n)\|}{s_{n+1}}\right\}\right\}. \tag{2.6}$$

Put

$$z_n = J_{r_n}(y_n - r_nAy_n), \quad \forall n \ge 1. \tag{2.7}$$

In a similar way, we can obtain from Lemma 1.6 that

$$\begin{aligned} \|z_{n+1} - z_n\| &\le \|(y_{n+1} - r_{n+1}Ay_{n+1}) - (y_n - r_nAy_n)\| \\ &\quad + \|J_{r_{n+1}}(y_n - r_nAy_n) - J_{r_n}(y_n - r_nAy_n)\| \\ &\le \left\|y_{n+1} - -y_n\right\| + |r_{n+1} - r_n|\left\|Ay_n\right\| \\ &\quad + \frac{|r_{n+1} - r_n|}{r_{n+1}}\|J_{r_{n+1}}(y_n - r_nAy_n) - (y_n - r_nAy_n)\| \\ &\le \|y_{n+1} - y_n\| + 2M_2|r_{n+1} - r_n|, \end{aligned} \tag{2.8}$$

where M_2 is an appropriate constant such that

$$M_2 = \max\left\{\sup_{n\ge1}\{\|Ay_n\|\}\ , \sup_{n\ge1}\left\{\frac{\|J_{r_{n+1}}(y_n - r_nAx_n) - (y_n - r_nAy_n)\|}{r_{n+1}}\right\}\right\}. \tag{2.9}$$

Substituting (2.5) into (2.8) yields that

$$\|z_{n+1} - z_n\| \le \|x_{n+1} - x_n\| + M_3(|s_{n+1} - s_n| + |r_{n+1} - r_n|), \tag{2.10}$$

where M_3 is an appropriate constant such that

$$M_3 = \max\{2M_1, 2M_2\}. \tag{2.11}$$

It follows from (2.10) that

$$\begin{aligned}\|S_{n+1}z_{n+1} - S_n z_n\| &\le \|z_{n+1} - z_n\| + \|z_n - Sz_n\||\delta_n - \delta_{n+1}| \\ &\le \|x_{n+1} - x_n\| + M_4(|s_{n+1} - s_n| + |r_{n+1} - r_n| + |\delta_n - \delta_{n+1}|),\end{aligned} \tag{2.12}$$

where M_4 is an appropriate constant such that

$$M_4 = \max\left\{\sup_{n\ge1}\{\|z_n - Sz_n\|\}, M_3\right\}. \tag{2.13}$$

Put

$$l_n = \frac{x_{n+1} - \beta_n x_n}{1-\beta_n}, \quad \forall n \ge 1. \tag{2.14}$$

Note that

$$\begin{aligned}l_{n+1} - l_n &= \frac{\alpha_{n+1}u + \gamma_{n+1}S_{n+1}z_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n u + \gamma_n S_n z_n}{1-\beta_n} \\ &= \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right)u + \frac{\gamma_{n+1}}{1-\beta_{n+1}}(S_{n+1}z_{n+1} - S_n z_n) \\ &\quad + \left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\right)S_n z_n \\ &= \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right)(u - S_n z_n) + \frac{\gamma_{n+1}}{1-\beta_{n+1}}(S_{n+1}z_{n+1} - S_n z_n).\end{aligned} \tag{2.15}$$

It follows from (2.12) that

$$\begin{aligned}\|l_{n+1} - l_n\| &\le \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right|\|u - S_n z_n\| + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\|S_{n+1}z_{n+1} - S_n z_n\| \\ &\le \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right|\|u - S_n z_n\| + \|x_{n+1} - x_n\| \\ &\quad + M_4(|s_{n+1} - s_n| + |r_{n+1} - r_n| + |\delta_n - \delta_{n+1}|).\end{aligned} \tag{2.16}$$

This in turn implies from the restrictions (a)–(e) that

$$\limsup_{n\to\infty}(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \le 0. \tag{2.17}$$

From Lemma 1.4, we obtain that

$$\lim_{n\to\infty}\|l_n - x_n\| = 0. \tag{2.18}$$

Notice that

$$x_{n+1} - x_n = (1-\beta_n)(l_n - x_n). \tag{2.19}$$

It follows that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{2.20}$$

This completes Step 2.

Step 3. Show that $\|x_n - Sx_n\| \to 0$ as $n \to \infty$.

Since J_{r_n} and J_{s_n} are nonexpansive, we see that

$$\|z_n - p\|^2 \le \|x_n - p\|^2 - r_n(2\alpha - r_n)\|Ay_n - Ap\|^2, \tag{2.21}$$

$$\|y_n - p\|^2 \le \|x_n - p\|^2 - s_n(2\beta - s_n)\|Bx_n - Bp\|^2. \tag{2.22}$$

It follows from (2.21) that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \alpha_n\|u - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|S_n z_n - p\|^2\\ &\le \alpha_n\|u - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|z_n - p\|^2\\ &\le \alpha_n\|u - p\|^2 + \|x_n - p\|^2 - \gamma_n r_n(2\alpha - r_n)\|Ay_n - Ap\|^2.\end{aligned} \tag{2.23}$$

This in turn implies that

$$\gamma_n r_n(2\alpha - r_n)\|Ay_n - Ap\|^2 \le \alpha_n\|u - p\|^2 + \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|). \tag{2.24}$$

In view of (2.20), we see from the restrictions (a), (d), and (e) that

$$\lim_{n\to\infty}\|Ay_n - Ap\| = 0. \tag{2.25}$$

It follows from (2.22) that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \alpha_n\|u-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|S_nz_n-p\|^2\\ &\le \alpha_n\|u-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|J_{r_n}(y_n-r_nAy_n)-p\|^2\\ &\le \alpha_n\|u-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|y_n-p\|^2\\ &\le \alpha_n\|u-p\|^2+\|x_n-p\|^2-\gamma_ns_n(2\beta-s_n)\|Bx_n-Bp\|^2.\end{aligned} \tag{2.26}$$

This in turn implies that

$$\gamma_ns_n(2\beta-s_n)\|Bx_n-Bp\|^2\le \alpha_n\|u-p\|^2+\|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|). \tag{2.27}$$

In view of (2.20), we see from the restrictions (a), (d), and (e) that

$$\lim_{n\to\infty}\|Bx_n-Bp\|=0. \tag{2.28}$$

Since J_{r_n} is firmly nonexpansive, we obtain that

$$\begin{aligned}\|z_n-p\|^2 &= \|J_{r_n}(y_n-r_nAy_n)-J_{r_n}(p-r_nAp)\|^2\\ &\le \langle z_n-p,(y_n-r_nAy_n)-(p-r_nAp)\rangle\\ &= \frac{1}{2}\Big(\|z_n-p\|^2+\|(y_n-r_nAy_n)-(p-r_nAp)\|^2\\ &\qquad -\|(z_n-p)-((y_n-r_nAy_n)-(p-r_nAp))\|^2\Big)\\ &\le \frac{1}{2}\Big(\|z_n-p\|^2+\|y_n-p\|^2-\|z_n-y_n+r_n(Ay_n-Ap)\|^2\Big)\\ &= \frac{1}{2}\Big(\|z_n-p\|^2+\|y_n-p\|^2-\|z_n-y_n\|^2-r_n^2\|Ay_n-Ap\|^2\\ &\qquad -2r_n\langle z_n-y_n,Ay_n-Ap\rangle\Big)\\ &\le \frac{1}{2}\Big(\|z_n-p\|^2+\|y_n-p\|^2-\|z_n-y_n\|^2+2r_n\|z_n-y_n\|\|Ay_n-Ap\|\Big)\\ &\le \frac{1}{2}\Big(\|z_n-p\|^2+\|x_n-p\|^2-\|z_n-y_n\|^2+2r_n\|z_n-y_n\|\|Ay_n-Ap\|\Big).\end{aligned} \tag{2.29}$$

This in turn implies that

$$\|z_n-p\|^2\le \|x_n-p\|^2-\|z_n-y_n\|^2+2r_n\|z_n-y_n\|\|Ay_n-Ap\|. \tag{2.30}$$

In a similar way, we can obtain that

$$\|y_n - p\|^2 \le \|x_n - p\|^2 - \|y_n - x_n\|^2 + 2s_n\|y_n - x_n\|\|Bx_n - Bp\|. \tag{2.31}$$

In view of (2.30), we see that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \alpha_n\|u - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|S_n z_n - p\|^2 \\ &\le \alpha_n\|u - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|z_n - p\|^2 \\ &\le \alpha_n\|u - p\|^2 + \|x_n - p\|^2 - \gamma_n\|z_n - y_n\|^2 + 2r_n\|z_n - y_n\|\|Ay_n - Ap\|.\end{aligned} \tag{2.32}$$

It follows that

$$\begin{aligned}\gamma_n\|z_n - y_n\|^2 \le{}& \alpha_n\|u - p\|^2 + \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|) \\ &+ 2r_n\|z_n - y_n\|\|Ay_n - Ap\|.\end{aligned} \tag{2.33}$$

In view of (2.25), we obtain from the restrictions (d) and (e) that

$$\lim_{n\to\infty}\|z_n - y_n\| = 0. \tag{2.34}$$

Notice from (2.31), we see that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \alpha_n\|u - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|S_n z_n - p\|^2 \\ &\le \alpha_n\|u - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|z_n - p\|^2 \\ &\le \alpha_n\|u - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|y_n - p\|^2 \\ &\le \alpha_n\|u - p\|^2 + \|x_n - p\|^2 - \gamma_n\|y_n - x_n\|^2 + 2s_n\|y_n - x_n\|\|Bx_n - Bp\|.\end{aligned} \tag{2.35}$$

It follows that

$$\begin{aligned}\gamma_n\|y_n - x_n\|^2 \le{}& \alpha_n\|u - p\|^2 + \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|) \\ &+ 2s_n\|y_n - x_n\|\|Bx_n - Bp\|.\end{aligned} \tag{2.36}$$

In view of (2.28), we obtain from the restrictions (d) and (e) that

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{2.37}$$

Combining (2.34) with (2.37) yields that

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{2.38}$$

Note that

$$x_{n+1} - x_n = \alpha_n(u - x_n) + \gamma_n(S_n z_n - x_n). \tag{2.39}$$

In view of (2.20), we see from the restriction (d) that

$$\lim_{n\to\infty}\|S_n z_n - x_n\| = 0. \tag{2.40}$$

Note that

$$Sz_n - x_n = \frac{S_n z_n - x_n}{1-\delta_n} + \frac{\delta_n(x_n - z_n)}{1-\delta_n}. \tag{2.41}$$

From (2.38) and (2.40), we get from the restriction (c) that

$$\lim_{n\to\infty}\|Sz_n - x_n\| = 0. \tag{2.42}$$

Notice that

$$\|Sx_n - x_n\| \le \|Sx_n - Sz_n\| + \|Sz_n - x_n\|. \tag{2.43}$$

In view of (2.38) and (2.42), we see from Lemma 1.3 that

$$\lim_{n\to\infty}\|Sx_n - x_n\| = 0. \tag{2.44}$$

This completes Step 3.

Step 4. Show that $\limsup_{n\to\infty}\langle u - q, x_n - q\rangle \le 0$, where $q = P_{\mathcal{F}}u$.

To show it, we may choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u - q, x_n - q\rangle = \limsup_{i\to\infty}\langle u - q, x_{n_i} - q\rangle. \tag{2.45}$$

Since $\{x_{n_i}\}$ is bounded, we can choose a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ converging weakly to $\hat{x}$. We may, without loss of generality, assume that $x_{n_i} \rightharpoonup \hat{x}$, where $\rightharpoonup$ denotes the weak convergence. Next, we prove that $\hat{x} \in \mathcal{F}$. In view of (2.44), we can conclude from Lemma 1.3 that $\hat{x} \in F(S)$ easily. Notice that

$$y_n - r_n A y_n \in z_n + r_n M z_n. \tag{2.46}$$

Let $\mu \in M\nu$. Since M is monotone, we have

$$\left\langle \frac{y_n - z_n}{r_n} - Ay_n - \mu, z_n - \nu \right\rangle \geq 0. \tag{2.47}$$

In view of the restriction (a), we see from (2.34) that

$$\langle -A\overline{x} - \mu, \overline{x} - \nu \rangle \geq 0. \tag{2.48}$$

This implies that $-A\overline{x} \in M\overline{x}$, that is, $\overline{x} \in (A + M)^{-1}(0)$. In similar way, we can obtain that $\overline{x} \in (B + W)^{-1}(0)$. This proves that $\overline{x} \in \mathcal{F}$. It follows from (2.45) that

$$\limsup_{n \to \infty} \langle u - q, x_n - q \rangle \leq 0. \tag{2.49}$$

This completes Step 4.

Step 5. Show that $x_n \to q$ as $n \to \infty$.

Notice that

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \alpha_n \langle u - q, x_{n+1} - q \rangle + \beta_n \langle x_n - q, x_{n+1} - q \rangle \\
&\quad + \gamma_n \langle S_n J_{r_n}(y_n - r_n A y_n) - q, x_{n+1} - q \rangle \\
&\leq \alpha_n \langle u - q, x_{n+1} - q \rangle + \frac{\beta_n}{2}\left(\|x_n - q\|^2 + \|x_{n+1} - q\|^2 \right) \\
&\quad + \frac{\gamma_n}{2}\left(\|S_n J_{r_n}(y_n - r_n A y_n) - q\|^2 + \|x_{n+1} - q\|^2 \right) \\
&\leq \alpha_n \langle u - q, x_{n+1} - q \rangle + \frac{\beta_n}{2}\left(\|x_n - q\|^2 + \|x_{n+1} - q\|^2 \right) \\
&\quad + \frac{\gamma_n}{2}\left(\|y_n - q\|^2 + \|x_{n+1} - q\|^2 \right) \\
&\leq \alpha_n \langle u - q, x_{n+1} - q \rangle + \frac{1 - \alpha_n}{2}\left(\|x_n - q\|^2 + \|x_{n+1} - q\|^2 \right).
\end{aligned} \tag{2.50}$$

This in turn implies that

$$\|x_{n+1} - q\|^2 \leq (1 - \alpha_n)\|x_n - q\|^2 + 2\alpha_n \langle u - q, x_{n+1} - q \rangle. \tag{2.51}$$

In view of (2.49), we conclude from Lemma 1.5 that

$$\lim_{n \to \infty} \|x_n - q\| = 0. \tag{2.52}$$

This completes Step 5. This whole proof is completed. □

If S is a nonexpansive mapping and $\delta_n = 0$, then Theorem 2.1 is reduced to the following.

Corollary 2.2. *Let H be a real Hilbert space H and C a nonempty close and convex subset of H. Let $M : H \to 2^H$ and $W : H \to 2^H$ be two maximal monotone operators such that $D(M) \subset C$ and $D(W) \subset C$, respectively. Let $S : C \to C$ be a nonexpansive mapping, $A : C \to H$ an α-inverse strongly monotone mapping and $B : C \to H$ a β-inverse strongly monotone mapping. Assume that $\mathcal{F} := F(S) \cap (A+M)^{-1}(0) \cap (B+W)^{-1}(0) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{gathered} x_1 \in C, \\ y_n = J_{s_n}(x_n - s_n B x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S J_{r_n}(y_n - r_n A y_n), \quad \forall n \geq 1, \end{gathered} \tag{2.53}$$

where $u \in C$ is a fixed element, $J_{r_n} = (I + r_n M)^{-1}$ and $J_{s_n} = (I + s_n W)^{-1}$, $\{r_n\}$ is a sequence in $(0, 2\alpha)$, $\{s_n\}$ is a sequence in $(0, 2\beta)$ and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$. Assume that the following restrictions are satisfied:

(a) $0 < a \leq r_n \leq b < 2\alpha$, $\lim_{n\to\infty}(r_n - r_{n+1}) = 0$,

(b) $0 < c \leq s_n \leq d < 2\beta_n$, $\lim_{n\to\infty}(s_n - s_{n+1}) = 0$,

(c) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(d) $0 < \liminf_{n\to\infty}\beta_n \leq \liminf_{n\to\infty}\beta_n < 1$.

Then, the sequence $\{x_n\}$ converges strongly to $q = P_{\mathcal{F}} u$.

Next, we consider the problem of finding common fixed points of three strict pseudocontractions.

Theorem 2.3. *Let C be a nonempty closed convex subset of a real Hilbert space H and P_C the metric projection from H onto C. Let $S : C \to C$ be a κ-strict pseudocontraction, $T_A : C \to H$ an α-strict pseudocontraction, and $B : C \to H$ a β-strict pseudocontraction. Assume that $\mathcal{F} := F(S) \cap F(T_A) \cap F(T_B) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{gathered} x_1 \in C, \\ z_n = (1 - s_n)x_n + s_n T_B x_n, \\ y_n = (1 - r_n)z_n + r_n T_A z_n \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n(\delta_n y_n + (1 - \delta_n)S y_n), \quad \forall n \geq 1, \end{gathered} \tag{2.54}$$

where $u \in C$ is a fixed element, $\{r_n\}$ is a sequence in $(0, 1-\alpha)$, $\{s_n\}$ is a sequence in $(0, 1-\beta)$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ are sequences in $[0,1]$. Assume that the following restrictions are satisfied

(a) $0 < a \leq r_n \leq b < 1 - \alpha$, $\lim_{n\to\infty}(r_n - r_{n+1}) = 0$,

(b) $0 < c \leq s_n \leq d < 1 - \beta_n$, $\lim_{n\to\infty}(s_n - s_{n+1}) = 0$,

(c) $0 \leq \kappa \leq \delta_n < e < 1$, $\lim_{n\to\infty}(\delta_n - \delta_{n+1}) = 0$,

(d) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(e) $0 < \liminf_{n\to\infty}\beta_n \le \liminf_{n\to\infty}\beta_n < 1$.

Then, the sequence $\{x_n\}$ converges strongly to $q = P_{\mathcal{F}}u$.

Proof. Putting $A = I - T_A$, we see that A is $((1-\alpha)/2)$-inverse strongly monotone. We also have $F(T_A) = \mathrm{VI}(C, A)$ and $P_C(x_n - r_nAx_n) = (1-r_n)x_n + r_nTx_n$. Putting $B = I - T_B$, we see that B is $(1-\beta)/2$-inverse strongly monotone. We also have $F(T_B) = \mathrm{VI}(C, B)$ and $P_C(x_n - s_nBx_n) = (1-s_n)x_n + s_nRu_n$. In view of Theorem 2.1, we can obtain the desired results immediately. □

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