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Solvability of a q-fractional integral equation arising in the study of an epidemic model

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Abstract

We study the solvability of a functional equation involving q-fractional integrals. Such an equation arises in the study of the spread of an infectious disease that does not induce permanent immunity. Our method is based on the noncompactness measure argument in a Banach algebra and an extension of Darbo's fixed point theorem.

MSC: 62P10; 26A33; 45G10

Keywords: epidemic model; q-calculus; fractional order; measure of noncompactness

1 Introduction

In this paper, we deal with the solvability of the q-fractional integral equation

$$x(t) = \prod_{i=1}^{N} \left(f_i(t) + \frac{g_i(t, x(t))}{\Gamma_q(\alpha_i)} \int_0^t (t - qs)^{(\alpha_i - 1)} u_i(s, x(s)) \, d_q s \right), \quad t \in I, \tag{1.1}$$

where $I = [0, 1]$, $q \in (0, 1)$, $\alpha_i > 1$, $f_i : I \to \mathbb{R}$, and $g_i, u_i : I \times \mathbb{R} \to \mathbb{R}$ ($i = 1, \dots, N$).

For $N = 1$, $q = 1$, and $\alpha_1 = 1$, equation (1.1) arises in the study of the spread of an infectious disease that does not induce permanent immunity (see [1–3]).
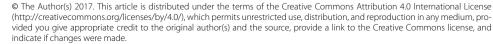
Equation (1.1) can be written as

$$x(t) = \prod_{i=1}^{N} \left(f_i(t) + g_i(t, x(t)) I_q^{\alpha_i} u_i(\cdot, x(\cdot))(t) \right), \quad t \in I,$$

where $I_q^{\alpha_i}$ is the q-fractional integral of order α_i defined by [4]

$$I_q^{\alpha_i} h(t) = \frac{1}{\Gamma_q(\alpha_i)} \int_0^t (t - qs)^{(\alpha_i - 1)} h(s) \, d_q s, \quad t \in I.$$

Via noncompactness measure argument in a Banach algebra, we provide sufficient conditions for the existence of at least one solution to equation (1.1). We also give an example in order to illustrate our existence result.

The concept of noncompactness measure was used by many authors in order to study the solvability of various classes of integral equations (see [5–15] and references therein). Very recently, Jleli, Mursaleen, and Samet [16] studied the solvability of a functional equation involving the q-fractional integral equation

$$x(t) = F\left(t, x(a(t)), \frac{f(t, x(t))}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} u(s, x(s)) \, d_q s\right), \quad t \in [0, 1].$$

Using the noncompactness measure tool, the authors obtained an existence result for such an equation. To the best of our knowledge, [16] is the only work dealing with the solvability of a q-fractional integral equation with noncompactness measure. In this paper, we use a different approach in order to establish our existence result. We first establish a fixed point theorem in a Banach algebra via a measure of noncompactness satisfying condition (m). Next, via the obtained fixed point result, we prove that, under certain conditions, equation (1.1) has at least one solution.

2 Preliminaries on quantum calculus

The concept of quantum calculus was introduced by Jackson [17, 18]. This subject is rich in history and has many applications (see [19–26]). In this section, we recall some basic facts on quantum calculus and present additional properties that will be used later. For more details, we refer to [21].

Let $q \in [0, \infty)\setminus\{1\}$. For $x \in \mathbb{R}$, we define the q-real number $[x]_q$ by

$$[x]_q = \frac{1 - q^x}{1 - q}.$$

The q-factorial of x is defined by

$$(x, q)_0 = 1, \qquad (x, q)_k = \prod_{i=0}^{k-1} (1 - xq^i), \quad k = 1, 2, \ldots, \infty.$$

For $(a, b) \in \mathbb{R}^2$, the q-analog of $(a - b)^k$ is defined by

$$(a - b)^{(0)} = 1, \qquad (a - b)^{(k)} = \prod_{i=0}^{k-1} (a - q^i b), \quad k = 1, 2, \ldots.$$

For $\beta \in \mathbb{R}$, $(a, b) \in \mathbb{R}^2$, and $a \geq 0$,

$$(a - b)^{(\beta)} = a^\beta \prod_{i=0}^{\infty} \frac{a - bq^i}{a - bq^{\beta+i}}.$$

Note that, for $b = 0$, we have

$$a^{(\beta)} = a^\beta.$$

Lemma 2.1 ([27]) *If $\beta > 0$ and $0 \leq a \leq b \leq t$, then*

$$(t - b)^{(\beta)} \leq (t - a)^{(\beta)}.$$

The q-gamma function is given by

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \quad x \notin \{0, -1, -2, \ldots\}.$$

We have the following property:

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x).$$

Let $f : [0, T] \to \mathbb{R}$ be a given function ($T > 0$). We define the q-integral of the function f by

$$I_q f(t) = \int_0^t f(s)\, d_q s = t(1-q) \sum_{n=0}^{\infty} f(tq^n) q^n, \quad t \in [0, T].$$

If $\eta \in [0, T]$, then

$$\int_\eta^T f(s)\, d_q s = \int_0^T f(s)\, d_q s - \int_0^\eta f(s)\, d_q s.$$

Lemma 2.2 *Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Then*

$$\left| \int_0^t f(s)\, d_q s \right| \le \int_0^t |f(s)|\, d_q s, \quad t \in [0,1].$$

Remark 2.3 If $0 \le t_1 \le t_2 \le 1$ and $f : [0,1] \to \mathbb{R}$ is a continuous function, then the inequality

$$\left| \int_{t_1}^{t_2} f(s)\, d_q s \right| \le \int_{t_1}^{t_2} |f(s)|\, d_q s$$

is not satisfied in general. As a counterexample, we refer to [21], p.12.

Let $f : [0,1] \to \mathbb{R}$ be a given function. The fractional q-integral of order $\alpha \ge 0$ of the function f is given by $I_q^0 f(t) = f(t)$ and

$$I_q^\alpha f(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} f(s)\, d q_s, \quad t \in [0,1], \alpha > 0.$$

Note that, for $\alpha = 1$, we have

$$I_q^1 f(t) = I_q f(t), \quad t \in [0,1].$$

For $f \equiv 1$,

$$I_q^\alpha 1(t) = \frac{1}{\Gamma_q(\alpha+1)} t^\alpha, \quad t \in [0,1].$$

3 A fixed point theorem in a Banach algebra via a measure of noncompactness satisfying condition (*m*)

In this section, we recall the axiomatic approach of noncompactness measure introduced by Banas and Goebel [10]. Next, we establish a fixed point theorem in a Banach algebra via a measure of noncompactness satisfying condition (*m*). This fixed point result plays an important rule in the proof of our existence result.

Let $(E, \cdot)$ be a Banach Algebra over $\mathbb{R}$ with respect to a certain norm $\| \cdot \|_E$. We denote by 0_E the zero vector of E. For $x \in E$ and $r > 0$, we denote by $B(x, r)$ the open ball in E of center x and radius r, that is,

$$B(x, r) = \big\{ y \in E : \|x - y\|_E < r \big\}.$$

We denote by $\mathcal{P}(E)$ the set of all nonempty subsets of E. If $M \in \mathcal{P}(E)$, then the symbol $\overline{M}$ denotes the closure of M. The symbol Conv(M) stands for the convex hull of M. For $(M, N) \in \mathcal{P}(E) \times \mathcal{P}(E)$ and $\alpha \in \mathbb{R}$, we denote

$$M + N = \big\{ x + y : (x, y) \in M \times N \big\}$$

and

$$\alpha M = \{ \alpha x : x \in M \}.$$

We denote by $\mathcal{P}_b(E)$ the set of all nonempty bounded subsets of E. For $M \in \mathcal{P}_b(E)$, we denote by $\|M\|$ the norm of M, that is,

$$\|M\| = \sup \big\{ \|x\|_E : x \in M \big\}.$$

We denote by $\mathcal{P}_{rc}(E)$ the set of all relatively compact subsets of E. For $M, N \in \mathcal{P}(E)$, we denote by MN the product set

$$MN = \big\{ x \cdot y : (x, y) \in M \times N \big\}.$$

In what follows, we recall the axiomatic approach of a measure of noncompactness introduced by Banas and Goebel [10].

Definition 3.1 Let $\mu : \mathcal{P}_b(E) \to [0, \infty)$ be a given mapping. We say that μ is a measure of noncompactness in E if it satisfies the following axioms:

(A1) The family $\ker \mu = \mu^{-1}(\{0\})$ is a subset of $\mathcal{P}_{rc}(E)$.

(A2) $(M, N) \in \mathcal{P}_b(E) \times \mathcal{P}_b(E), M \subset N \implies \mu(M) \le \mu(N)$.

(A3) $\mu(\overline{M}) = \mu(M), M \in \mathcal{P}_b(E)$.

(A4) $\mu(\text{Conv}(M)) = \mu(M), M \in \mathcal{P}_b(E)$.

(A5) $\mu(\lambda M + (1 - \lambda)N) \le \lambda \mu(M) + (1 - \lambda)\mu(N)$ for $\lambda \in [0, 1]$ and $(M, N) \in \mathcal{P}_b(E) \times \mathcal{P}_b(E)$.

(A6) If $\{M_n\}$ is a sequence of closed sets from $\mathcal{P}_b(E)$ such that $M_{n+1} \subset M_n$ for $n = 1, 2, \dots$ and if $\lim_{n \to \infty} \mu(M_n) = 0$, then the intersection set $M_\infty = \bigcap_{n=1}^{\infty} M_n$ is nonempty.

Now, let us recall the following important result, which is called Drabo's fixed point theorem [10, 28].

Lemma 3.2 *Let D be a nonempty, bounded, closed, and convex subset of E, and let $T : D \to D$ be a continuous mapping. Suppose that there exists a constant $k \in (0,1)$ such that*

$$\mu(TM) \leq k\mu(M)$$

for any nonempty subset M of D, where μ is a measure of noncompactness in E. Then T has at least one fixed point in D.

The following concept was introduced by Banas and Olszowy [13].

Definition 3.3 Let μ be a measure of noncompactness in E. We say that μ satisfies condition (m) if

$$\mu(MN) \leq \|M\|\mu(N) + \|N\|\mu(M), \quad (M,N) \in \mathcal{P}_b(E) \times \mathcal{P}_b(E).$$

Lemma 3.4 *Let μ be a measure of noncompactness in E satisfying condition (m). Let $\{M_i\}_{i=1,\dots,q}$ be a finite sequence in $\mathcal{P}_b(E)$, $q \geq 2$. Then*

$$\mu\left(\prod_{i=1}^{q} M_i\right) \leq \sum_{i=1}^{q} \prod_{j=1, j\neq i}^{q} \|M_j\|\mu(M_i). \tag{3.1}$$

Proof We shall use the induction principle. For $q = 2$, (3.1) follows immediately from Definition 3.3. Suppose now that (3.1) is satisfied for some $q \geq 2$. We have to prove that

$$\mu\left(\prod_{i=1}^{q+1} M_i\right) \leq \sum_{i=1}^{q+1} \prod_{j=1, j\neq i}^{q+1} \|M_j\|\mu(M_i). \tag{3.2}$$

Using (3.1) and Definition 3.3, we have

$$\mu\left(\prod_{i=1}^{q+1} M_i\right) = \mu\left(\left(\prod_{i=1}^{q} M_i\right) M_{q+1}\right)$$

$$\leq \mu\left(\prod_{i=1}^{q} M_i\right)\|M_{q+1}\| + \mu(M_{q+1})\left\|\prod_{i=1}^{q} M_i\right\|$$

$$\leq \mu\left(\prod_{i=1}^{q} M_i\right)\|M_{q+1}\| + \mu(M_{q+1})\prod_{j=1}^{q} \|M_j\|$$

$$\leq \sum_{i=1}^{q} \prod_{j=1, j\neq i}^{q} \|M_j\|\mu(M_i)\|M_{q+1}\| + \mu(M_{q+1})\prod_{j=1}^{q} \|M_j\|$$

$$= \sum_{i=1}^{q} \prod_{j=1, j\neq i}^{q+1} \|M_j\|\mu(M_i)\|M_{q+1}\| + \mu(M_{q+1})\prod_{j=1}^{q} \|M_j\|$$

$$= \sum_{i=1}^{q+1} \prod_{j=1, j\neq i}^{q+1} \|M_j\|\mu(M_i).$$

Thus, we proved (3.2). Finally, (3.1) follows by the induction principle. □

Now, we deal with the fixed point problem

$$
\begin{cases}
\text{Find } x \in D \text{ such that} \\
x = \prod_{i=1}^{N} T_i x,
\end{cases}
\tag{3.3}
$$

where $D \in \mathcal{P}(E)$ and $T_i : D \to E$, $i = 1, \ldots, N$, $N \geq 1$, are given operators.

We have the following fixed point result.

Theorem 3.5 *Assume that D is nonempty, bounded, closed, and convex subset of the Banach algebra E. Assume also that the following conditions are satisfied:*

(i) *T_i is continuous, $i = 1, \ldots, N$.*

(ii) *$T_i D$ is bounded, $i = 1, \ldots, N$.*

(iii) *$TD \subset D$, where $Tx = \prod_{i=1}^{N} T_i x$.*

(iv) *There exists a finite sequence $\{k_i\}_{i=1}^{q} \subset (0, \infty)$ such that*

$$
\mu(T_i M) \leq k_i \mu(M), \quad i = 1, \ldots, N,
$$

for any nonempty subset M of D, where μ is a measure of noncompactness in E satisfying condition (m).

(v) *$\sum_{i=1}^{q} k_i \prod_{j=1, j \neq i}^{q} \| T_j D \| < 1$.*

Then problem (3.3) has at least one solution in D.

Proof Let M be nonempty subset of D. Using Lemma 3.4 and the considered assumptions, we obtain

$$
\mu(TM) = \mu\left(\prod_{i=1}^{N} T_i M \right) \leq \sum_{i=1}^{N} \prod_{j=1, j \neq i}^{N} \| T_j M \| \mu(T_i M)
$$

$$
\leq \sum_{i=1}^{N} k_i \prod_{j=1, j \neq i}^{N} \| T_j M \| \mu(M) \leq \left(\sum_{i=1}^{N} k_i \prod_{j=1, j \neq i}^{N} \| T_j D \| \right) \mu(M).
$$

Thus, we have proved that

$$
\mu(TM) \leq k \mu(M)
$$

for every nonempty subset M of D, where

$$
k = \sum_{i=1}^{N} k_i \prod_{j=1, j \neq i}^{N} \| T_j D \| < 1.
$$

The result follows by Lemma 3.2. □

Remark 3.6 For $N = 2$, Theorem 3.5 reduces to a fixed point theorem established in [13].

Remark 3.7 For $N = 1$, Theorem 3.5 reduces to Lemma 3.2.

4 Main result

In this section, we state and prove our main result concerning the existence of solutions to equation (1.1).

Let $E = C(I; \mathbb{R})$ be the Banach space of all real-valued continuous functions in I equipped with the norm

$$\|u\|_E = \max\{|u(t)| : t \in I\}, \quad u \in E.$$

Clearly, E is a Banach algebra with respect to the operation $\cdot$ defined by

$$(u \cdot v)(t) = u(t)v(t), \quad t \in I, (u, v) \in E \times E.$$

Let $M \in \mathcal{P}_b(E)$. For $x \in M$ and $\varepsilon > 0$, set

$$\omega(x, \varepsilon) = \sup\{|x(t) - x(s)| : t, s \in I, |t - s| \leq \varepsilon\},$$
$$\omega(M, \varepsilon) = \sup\{\omega(x, \varepsilon) : x \in M\}.$$

It was proved in [10] that the mapping $\mu : \mathcal{P}_b(E) \to [0, \infty)$ defined by

$$\mu(M) = \lim_{\varepsilon \to 0^+} \omega(M, \varepsilon), \quad M \in \mathcal{P}_b(E), \tag{4.1}$$

is a measure of noncompactness in E. Moreover, μ satisfies condition (m) (see [13]).

Equation (1.1) can be written as

$$x = Tx = \prod_{i=1}^{N} T_i x, \tag{4.2}$$

where fo,r $i = 1, \ldots, N$,

$$(T_i x)(t) = f_i(t) + \frac{g_i(t, x(t))}{\Gamma_q(\alpha_i)} \int_0^t (t - qs)^{(\alpha_i - 1)} u_i(s, x(s)) \, d_q s, \quad t \in I.$$

We consider the following assumption:

(A1) For $i = 1, \ldots, N$, the functions $f_i : I \to \mathbb{R}$ and $g_i, u_i : I \times \mathbb{R} \to \mathbb{R}$ are continuous.

Lemma 4.1 *For every $i = 1, \ldots, N$, the operator $T_i : E \to E$ is well defined.*

Proof Fix $i \in \{1, \ldots, N\}$. We have just to prove that the operator

$$(S_i x)(t) = \int_0^t (t - qs)^{(\alpha_i - 1)} u_i(s, x(s)) \, d_q s, \quad t \in I, \tag{4.3}$$

maps E into it self. To do this, let us fix $x \in E$. For all $t \in I$, we have

$$(S_i x)(t) = t(1 - q) \sum_{n=0}^{\infty} q^n (t - q^{n+1} t)^{(\alpha_i - 1)} u_i(tq^n, x(tq^n))$$

$$= t^{\alpha_i}(1 - q) \sum_{n=0}^{\infty} q^n (1 - q^{n+1})^{(\alpha_i - 1)} u_i(tq^n, x(tq^n)).$$

Since $q^{n+1} \in (0,1)$, by Lemma 2.1 we have

$$\left(1 - q^{n+1}\right)^{(\alpha_i - 1)} \le (1 - 0)^{(\alpha_i - 1)} = 1.$$

Therefore, the continuity of u_i and the Weierstrass convergence theorem give us the desired result. $\qquad\square$

We suppose also that the following assumptions are satisfied:

(A2) For all $i = 1, \ldots, N$, there exist constants $C_i > 0$ and $d_i > 0$ such that

$$\left|g_i(t, x) - g_i(t, y)\right| \le C_i|x - y|^{d_i}, \quad (t, x, y) \in I \times I \times \mathbb{R}.$$

(A3) For all $i = 1, \ldots, N$, there exist a constant $e_i > 0$ and a nondecreasing continuous function $\varphi_i : [0, \infty) \to [0, \infty)$ such that

$$\left|u_i(t, x) - u_i(t, y)\right| \le \varphi_i\left(|x - y|^{e_i}\right), \quad (t, x, y) \in I \times I \times \mathbb{R}.$$

(A4) For all $i = 1, \ldots, N$, we have

$$u_i(t, 0) = 0, \quad t \in I.$$

(A5) There exists $r > 0$ such that

$$\|f_i\|_E + \frac{(C_i r^{d_i} + g_i^*)\varphi_i(r^{e_i})}{\Gamma_q(\alpha_i + 1)} \le r^{1/N},$$

where

$$g_i^* = \max\left\{\left|g_i(t, 0)\right| : t \in I\right\}.$$

Lemma 4.2 *Under Assumptions* (A1)-(A5), *for all $i = 1, \ldots, N$, we have*

$$T_i \overline{B(0_E, r)} \subset \overline{B\left(0_E, r^{1/N}\right)}.$$

Proof Let $i \in \{1, \ldots, N\}$ be fixed, and let $x \in \overline{B(0_E, r)}$. For all $t \in I$, we have

$$
\begin{aligned}
\left|(T_i x)(t)\right| &\le \left|f_i(t)\right| + \frac{|g_i(t, x(t))|}{\Gamma_q(\alpha_i)} \int_0^t (t - qs)^{(\alpha_i - 1)}\left|u_i\left(s, x(s)\right)\right| d_q s \\
&\le \|f_i\|_E + \frac{|g_i(t, x(t))|}{\Gamma_q(\alpha_i)} \int_0^t (t - qs)^{(\alpha_i - 1)}\left|u_i\left(s, x(s)\right)\right| d_q s \\
&\le \|f_i\|_E + \frac{|g_i(t, x(t)) - g_i(t, 0)| + |g_i(t, 0)|}{\Gamma_q(\alpha_i)} \int_0^t (t - qs)^{(\alpha_i - 1)}\left|u_i\left(s, x(s)\right)\right| d_q s.
\end{aligned}
$$

Using Assumption (A2), we obtain

$$\left|g_i\left(t, x(t)\right) - g_i(t, 0)\right| + \left|g_i(t, 0)\right| \le C_i r^{d_i} + g_i^*,$$

By Assumptions (A3) and (A4) we have

$$\left| u_i\big(s, x(s)\big) \right| = \left| u_i\big(s, x(s)\big) - u_i(s, 0) \right| \leq \varphi_i\big(r^{e_i}\big).$$

Therefore, we obtain

$$
\begin{aligned}
\left| (T_i x)(t) \right| &\leq \|f_i\|_E + \frac{(C_i r^{d_i} + g_i^*)\varphi_i(r^{e_i})}{\Gamma_q(\alpha_i)} \int_0^t (t - qs)^{(\alpha_i - 1)} \, d_q s \\
&= \|f_i\|_E + \frac{(C_i r^{d_i} + g_i^*)\varphi_i(r^{e_i})}{\Gamma_q(\alpha_i + 1)} t^{\alpha_i} \\
&\leq \|f_i\|_E + \frac{(C_i r^{d_i} + g_i^*)\varphi_i(r^{e_i})}{\Gamma_q(\alpha_i + 1)}.
\end{aligned}
$$

By Assumption (A5) we get

$$\|T_i x\|_E \leq r^{1/N},$$

which proves the desired result. $\square$

Lemma 4.3 *Under Assumptions* (A1)-(A5), *for all* $i = 1, \ldots, N$,

$$T_i : \overline{B(0_E, r)} \to \overline{B\big(0_E, r^{1/N}\big)}$$

is continuous.

Proof Under the considered assumptions, we have just to prove that, for all $i = 1, \ldots, N$, the operator $S_i : \overline{B(0_E, r)} \to E$ defined by (4.3) is continuous. To do this, let us fix $i \in \{1, \ldots, N\}$ and $\varepsilon > 0$ such that $\|x - y\|_E \leq \varepsilon$, $(x, y) \in \overline{B(0_E, r)} \times \overline{B(0_E, r)}$. For all $t \in I$, we have

$$
\begin{aligned}
\left| (S_i x)(t) - (S_i y)(t) \right| &= \left| \int_0^t (t - qs)^{(\alpha_i - 1)} \big(u_i\big(s, x(s)\big) - u_i\big(s, y(s)\big) \big) \, d_q s \right| \\
&\leq \int_0^t (t - qs)^{(\alpha_i - 1)} \left| u_i\big(s, x(s)\big) - u_i\big(s, y(s)\big) \right| \, d_q s \\
&\leq \frac{u_r^i(\varepsilon) t^{\alpha_i}}{[\alpha_i]_q} \\
&\leq \frac{u_r^i(\varepsilon)}{[\alpha_i]_q},
\end{aligned}
$$

where

$$u_r^i(\varepsilon) = \sup\left\{ \left| u_i(t, x) - u_i(t, y) \right| : t \in I, x, y \in [-r, r], |x - y| \leq \varepsilon \right\}.$$

Therefore,

$$\|S_i x - S_i y\|_E \leq \frac{u_r^i(\varepsilon)}{[\alpha_i]_q}.$$

Passing to the limit as $\varepsilon \to 0$ and using the uniform continuity of u_i on the compact set $I \times [-r, r]$, we obtain

$$\lim_{\varepsilon \to 0} \frac{u_r^i(\varepsilon)}{[\alpha_i]_q} = 0,$$

which yields the desired result. $\qquad\square$

Next, we need the following additional assumption:

(A6) For all $i = 1, \ldots, N$, there exists a constant $\lambda_i > 0$ such that

$$t^{d_i} + \varphi_i(t^{e_i}) \leq \lambda_i t, \quad t \geq 0.$$

Lemma 4.4 *Under assumptions* (A1)-(A6), *for all* $i = 1, \ldots, N$, *we have*

$$\mu(T_i M) \leq k_i \mu(M)$$

for every nonempty subset M of $\overline{B(0_E, r)}$, *where*

$$k_i = \lambda_i \max\left\{ \frac{C_i \varphi_i(r^{e_i})}{\Gamma_q(\alpha_i + 1)}, \frac{(C_i r^{d_i} + g_i^*)}{\Gamma_q(\alpha_i)} \right\}.$$

Proof Fix $i \in \{1, \ldots, N\}$. Let M be a nonempty subset of $\overline{B(0_E, r)}$, $\varepsilon > 0$, and $x \in M$. Let $(t_1, t_2) \in I \times I$ be such that $|t_1 - t_2| \leq \varepsilon$. Without loss of the generality, we may assume that $t_1 \geq t_2$. Therefore, we have

$$\left| (T_i x)(t_1) - (T_i x)(t_2) \right|$$

$$\leq \left| f_i(t_1) - f_i(t_2) \right| + \left| \frac{g_i(t_1, x(t_1))}{\Gamma_q(\alpha_i)} \int_0^{t_1} (t_1 - qs)^{(\alpha_i - 1)} u_i(s, x(s)) \, d_q s \right.$$

$$\left. - \frac{g_i(t_2, x(t_2))}{\Gamma_q(\alpha_i)} \int_0^{t_2} (t_2 - qs)^{(\alpha_i - 1)} u_i(s, x(s)) \, d_q s \right|$$

$$\leq \omega(f_i, \varepsilon) + \left| \frac{g_i(t_1, x(t_1))}{\Gamma_q(\alpha_i)} \int_0^{t_1} (t_1 - qs)^{(\alpha_i - 1)} u_i(s, x(s)) \, d_q s \right.$$

$$\left. - \frac{g_i(t_2, x(t_2))}{\Gamma_q(\alpha_i)} \int_0^{t_1} (t_1 - qs)^{(\alpha_i - 1)} u_i(s, x(s)) \, d_q s \right|$$

$$+ \left| \frac{g_i(t_2, x(t_2))}{\Gamma_q(\alpha_i)} \int_0^{t_1} (t_1 - qs)^{(\alpha_i - 1)} u_i(s, x(s)) \, d_q s \right.$$

$$\left. - \frac{g_i(t_2, x(t_2))}{\Gamma_q(\alpha_i)} \int_0^{t_2} (t_2 - qs)^{(\alpha_i - 1)} u_i(s, x(s)) \, d_q s \right|$$

$$\leq \omega(f_i, \varepsilon) + \frac{|g_i(t_1, x(t_1)) - g_i(t_2, x(t_2))|}{\Gamma_q(\alpha_i)} \int_0^{t_1} (t_1 - qs)^{(\alpha_i - 1)} \left| u_i(s, x(s)) \right| d_q s$$

$$+ \frac{|g_i(t_2, x(t_2))|}{\Gamma_q(\alpha_i)} \left| \int_0^{t_1} (t_1 - qs)^{(\alpha_i - 1)} u_i(s, x(s)) \, d_q s - \int_0^{t_2} (t_2 - qs)^{(\alpha_i - 1)} u_i(s, x(s)) \, d_q s \right|$$

$$:= \omega(f_i, \varepsilon) + (I) + (II),$$

where

$$\omega(f_i, \varepsilon) = \sup\{|f_i(t) - f_i(s)| : t, s \in I, |t - s| \leq \varepsilon\}.$$

Now, let us estimate (I) and (II).

• Estimate of (I). We have

$$\frac{1}{\Gamma_q(\alpha_i)} \int_0^{t_1} (t_1 - qs)^{(\alpha_i - 1)} |u_i(s, x(s))| \, d_q s \leq \frac{\varphi_i(r^{e_i})}{\Gamma_q(\alpha_i + 1)}.$$

On the other hand,

$$\begin{aligned}
|g_i(t_1, x(t_1)) - g_i(t_2, x(t_2))| &\leq |g_i(t_1, x(t_1)) - g_i(t_1, x(t_2))| + |g_i(t_1, x(t_2)) - g_i(t_2, x(t_2))| \\
&\leq C_i |x(t_1) - x(t_2)|^{d_i} + \omega(g_i, \varepsilon) \\
&\leq C_i \omega(x, \varepsilon)^{d_i} + \omega(g_i, \varepsilon),
\end{aligned}$$

where

$$\omega(g_i, \varepsilon) = \sup\{|g_i(t, x) - g_i(s, x)| : t, s \in I, |t - s| \leq \varepsilon, x \in [-r, r]\}.$$

Therefore,

$$(I) \leq \frac{(C_i \omega(x, \varepsilon)^{d_i} + \omega(g_i, \varepsilon)) \varphi_i(r^{e_i})}{\Gamma_q(\alpha_i + 1)}. \tag{4.4}$$

• Estimate of (II). First, observe that

$$|g_i(t_2, x(t_2))| \leq |g_i(t_2, x(t_2)) - g_i(t_2, 0)| + g_i^*.$$

Therefore, by Assumption (A2),

$$|g_i(t_2, x(t_2))| \leq C_i r^{d_i} + g_i^*.$$

Next, we have

$$\begin{aligned}
&\left| \int_0^{t_1} (t_1 - qs)^{(\alpha_i - 1)} u_i(s, x(s)) \, d_q s - \int_0^{t_2} (t_2 - qs)^{(\alpha_i - 1)} u_i(s, x(s)) \, d_q s \right| \\
&= (1 - q) \sum_{n=0}^{\infty} q^n (1 - q^{n+1})^{(\alpha_i - 1)} |t_1^{\alpha_i} u_i(q^n t_1, x(q^n t_1)) - t_2^{\alpha_i} u_i(q^n t_2, x(q^n t_2))|.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&|t_1^{\alpha_i} u_i(q^n t_1, x(q^n t_1)) - t_2^{\alpha_i} u_i(q^n t_2, x(q^n t_2))| \\
&\quad \leq t_1^{\alpha_i} |u_i(q^n t_1, x(q^n t_1)) - u_i(q^n t_1, x(q^n t_2))| \\
&\qquad + |t_1^{\alpha_i} u_i(q^n t_1, x(q^n t_2)) - t_2^{\alpha_i} u_i(q^n t_2, x(q^n t_2))|
\end{aligned}$$

$$\leq \varphi_i\big(\big|x\big(q^n t_1\big) - x\big(q^n t_2\big)\big|^{e_i}\big) + A_i(\varepsilon)$$

$$\leq \varphi_i\big(\omega(x,\varepsilon)^{e_i}\big) + A_i(\varepsilon),$$

where

$$A_i(\varepsilon) = \sup\big\{\big|H(\tau,s,x) - H\big(\tau',s',x\big)\big| : \tau,s,\tau',s' \in I, \big|\tau - \tau'\big| \leq \varepsilon, \big|s - s'\big| \leq \varepsilon, x \in [-r,r]\big\}$$

and

$$H(\tau,s,x) = \tau^{\alpha_i} u_i(s,x), \quad (\tau,s,x) \in I \times I \times \mathbb{R}.$$

As a consequence, we get

$$\left|\int_0^{t_1} (t_1 - qs)^{(\alpha_i - 1)} u_i\big(s,x(s)\big)\, d_q s - \int_0^{t_2} (t_2 - qs)^{(\alpha_i - 1)} u_i\big(s,x(s)\big)\, d_q s\right|$$

$$\leq \varphi_i\big(\omega(x,\varepsilon)^{e_i}\big) + A_i(\varepsilon).$$

Then

$$(II) \leq \frac{(C_i r^{d_i} + g_i^*)(\varphi_i(\omega(x,\varepsilon)^{e_i}) + A_i(\varepsilon))}{\Gamma_q(\alpha_i)}. \tag{4.5}$$

Now, combining (4.4) with (4.5), we get

$$\omega(T_i x, \varepsilon) \leq \omega(f_i,\varepsilon) + \frac{(C_i \omega(x,\varepsilon)^{d_i} + \omega(g_i,\varepsilon))\varphi_i(r^{e_i})}{\Gamma_q(\alpha_i + 1)} + \frac{(C_i r^{d_i} + g_i^*)(\varphi_i(\omega(x,\varepsilon)^{e_i}) + A_i(\varepsilon))}{\Gamma_q(\alpha_i)}.$$

Therefore,

$$\omega(T_i M, \varepsilon) \leq \omega(f_i,\varepsilon) + \frac{(C_i \omega(M,\varepsilon)^{d_i} + \omega(g_i,\varepsilon))\varphi_i(r^{e_i})}{\Gamma_q(\alpha_i + 1)}$$

$$+ \frac{(C_i r^{d_i} + g_i^*)(\varphi_i(\omega(M,\varepsilon)^{e_i}) + A_i(\varepsilon))}{\Gamma_q(\alpha_i)}.$$

Passing to the limit as $\varepsilon \to 0$, we obtain

$$\mu(T_i M) \leq \frac{C_i \mu(M)^{d_i} \varphi_i(r^{e_i})}{\Gamma_q(\alpha_i + 1)} + \frac{(C_i r^{d_i} + g_i^*)\varphi_i(\mu(M)^{e_i})}{\Gamma_q(\alpha_i)}.$$

Then

$$\mu(T_i M) \leq \max\left\{\frac{C_i \varphi_i(r^{e_i})}{\Gamma_q(\alpha_i + 1)}, \frac{(C_i r^{d_i} + g_i^*)}{\Gamma_q(\alpha_i)}\right\}\big(\mu(M)^{d_i} + \varphi_i\big(\mu(M)^{e_i}\big)\big).$$

By Assumption (A6) we obtain

$$\mu(T_i M) \leq \lambda_i \max\left\{\frac{C_i \varphi_i(r^{e_i})}{\Gamma_q(\alpha_i + 1)}, \frac{(C_i r^{d_i} + g_i^*)}{\Gamma_q(\alpha_i)}\right\}\mu(M),$$

which proves the desired result. □

Now, we are able to state and prove our main result.

Theorem 4.5 *Suppose that Assumptions* (A1)-(A6) *are satisfied. If*

$$r^{\frac{N-1}{N}} \sum_{i=1}^{N} \lambda_i \max\left\{ \frac{C_i \varphi_i(r^{e_i})}{\Gamma_q(\alpha_i + 1)}, \frac{(C_i r^{d_i} + g_i^*)}{\Gamma_q(\alpha_i)} \right\} < 1, \tag{4.6}$$

then equation (1.1) *has at least one solution* $x^* \in \overline{B(0_E, r)}$.

Proof Observe that $x \in \overline{B(0_E, r)}$ is a solution to equation (1.1) if and only if x is a solution to (3.3), where $D = \overline{B(0_E, r)}$, and T is given by (4.2). In order to prove our existence result, we have to check that all the assumptions of Theorem 3.5 are satisfied. By Lemma 4.3, for all $i = 1, \ldots, N$, $T_i : \overline{B(0_E, r)} \to E$ is a continuous operator. By Lemma 4.2, for all $i = 1, \ldots, N$, $T_i \overline{B(0_E, r)}$ is bounded. Moreover, for all $x \in \overline{B(0_E, r)}$, we have

$$\|Tx\|_E = \left\| \prod_{i=1}^{N} T_i x \right\|_E \leq \prod_{i=1}^{N} \|T_i x\|_E \leq \prod_{i=1}^{N} r^{1/N} = r.$$

Therefore, $T\overline{B(0_E, r)} \subset \overline{B(0_E, r)}$. By Lemma 4.4 condition (iv) of Theorem 3.5 is satisfied with

$$k_i = \lambda_i \max\left\{ \frac{C_i \varphi_i(r^{e_i})}{\Gamma_q(\alpha_i + 1)}, \frac{(C_i r^{d_i} + g_i^*)}{\Gamma_q(\alpha_i)} \right\}, \quad i = 1, \ldots, N.$$

On the other hand, from Lemma 4.2 and (4.6) we have

$$\sum_{i=1}^{q} k_i \prod_{j=1, j \neq i}^{q} \left\| T_j \overline{B(0_E, r)} \right\| \leq r^{\frac{N-1}{N}} \sum_{i=1}^{N} \lambda_i \max\left\{ \frac{C_i \varphi_i(r^{e_i})}{\Gamma_q(\alpha_i + 1)}, \frac{(C_i r^{d_i} + g_i^*)}{\Gamma_q(\alpha_i)} \right\} < 1.$$

Therefore, all conditions of Theorem 3.5 are satisfied, and the desired result follows. □

We end the paper with the following illustrative example.

Example Consider the functional equation

$$x(t) = \frac{t}{32} + \left[\frac{3}{2}\right]_{\frac{1}{2}} \left(\frac{t}{2} + \frac{x(t)}{4}\right) \int_0^t (t - s/2)^{(1/2)} \frac{x(s)}{(2 + s^2)} \, d_q s, \quad t \in [0, 1]. \tag{4.7}$$

Equation (4.7) is a a particular case of equation (1.1) with

$$N = 1, \qquad q = \frac{1}{2}, \qquad \alpha_1 = \frac{3}{2}, \qquad f_1(t) = \frac{t}{32},$$

$$g_1(t, x) = \left(\frac{t}{2} + \frac{x}{4}\right) \Gamma_{\frac{1}{2}}\left(\frac{5}{2}\right), \qquad u_1(t, x) = \frac{x}{2 + t^2}.$$

Obviously, the functions $f_1 : [0, 1] \to \mathbb{R}$ and $g_1, u_1 : [0, 1] \times \mathbb{R} \to \mathbb{R}$ are continuous. Then Assumption (A1) of Theorem 4.5 is satisfied. For all $(t, x, y) \in [0, 1] \times \mathbb{R} \times \mathbb{R}$, we have

$$\left| g_1(t, x) - g_1(t, y) \right| \leq \frac{\Gamma_{\frac{1}{2}}\left(\frac{5}{2}\right)}{4} |x - y|.$$

Therefore, Assumption (A2) is satisfied with

$$C_1 = \frac{\Gamma_{\frac{1}{2}}\left(\frac{5}{2}\right)}{4}, \qquad d_1 = 1.$$

For all $(t, x, y) \in [0,1] \times \mathbb{R} \times \mathbb{R}$, we have

$$\left| u_1(t, x) - u_1(t, y) \right| \leq \frac{1}{2} |x - y|.$$

Moreover, $u_1(t, 0) = 0$ for all $t \in [0,1]$. Therefore, Assumptions (A3) and (A4) are satisfied with

$$\varphi_1(t) = \frac{t}{2}, \qquad e_1 = 1.$$

Note that, in this case, we have

$$\|f_1\|_E = \frac{1}{32}, \qquad g_1^* = \frac{\Gamma_{\frac{1}{2}}\left(\frac{5}{2}\right)}{2}.$$

It is not difficult to check that Assumption (A5) is satisfied for every $r \geq \frac{1}{20}$. For every $t \geq 0$, we have

$$t^{d_1} + \varphi\left(t^{e_1}\right) = \frac{3}{2}t.$$

Then Assumption (A6) is satisfied with $\lambda_1 = \frac{3}{2}$. Moreover, for $r = \frac{1}{20}$, we have

$$\lambda_1 \max\left\{ \frac{C_1 \varphi_1(r^{e_1})}{\Gamma_q(\alpha_1 + 1)}, \frac{(C_1 r^{d_1} + g_1^*)}{\Gamma_q(\alpha_1)} \right\} = 0.99391166196 < 1.$$

By Theorem 4.5 we deduce that equation (4.7) admits at least one solution $x^* \in \overline{B(0_E, \frac{1}{20})}$.

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