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Some new fixed point theorems for a mixed monotone maps in partially ordered metric spaces

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Abstract

In this paper, we prove some new fixed point theorems for a mixed monotone mapping under more generalized nonlinear contractive conditions in a metric space endowed with partial order. Our results generalize and improve several results due to the work of Gnana Bhaskar and Lakshmikantham.

Keywords: Coupled fixed point; Partially ordered set; Nonlinear contraction mapping; Monotone iterative technique

MSC 2010: 47H10

Introduction and preliminaries

The study of mixed monotone operators has been a matter of discussion since it was introduced in 1987, because it has not only important theoretical meaning but also wide applications in nonlinear differential and integral equations (see [1-14]). Recently, Gnana Bhaskar and Lakshmikantham investigated the existence of coupled fixed points and fixed points for a mixed monotone mapping under a weak linear contractive condition on partially ordered metric space (see [15]). The purpose of this paper is to study the existence of coupled fixed points and fixed points for a mixed monotone mapping on partially ordered metric space which satisfy the nonlinear contractive condition $(\Phi_i)(i = 1, 2)$ below. The results obtained in this paper generalize and improve the results corresponding to those obtained by Gnana Bhaskar and Lakshmikantham in [15].

Next, let us give some notations and definitions:

Let $(X, \le)$ be a partially ordered set, (X, d) be a metric space, and $R^+ = [0, +\infty)$.

Definition 1 ([15]). *Let $(X, \le)$ be a partially ordered set and $F : X \times X \longrightarrow X$. We say that F has the mixed monotone property if $F(x, y)$ is monotone nondecreasing in x and is monotone nonincreasing in y, that is, for any $x, y \in X$,*

$$x_1, x_2 \in X, x_1 \le x_2 \Rightarrow F(x_1, y) \le F(x_2, y)$$

and

$$y_1, y_2 \in X, y_1 \le y_2 \Rightarrow F(x, y_2) \le F(x, y_1).$$

Definition 2 ([15]). *We call an element $(x, y) \in X \times X$ a coupled fixed point of F if*

$$F(x, y) = x, \quad F(y, x) = y.$$

An element $x \in X$ is called fixed point of the F if $F(x, x) = x$.

Definition 3. *A function $\varphi : R^+ \times R^+ \longrightarrow R^+$ is said to have the property (Φ_1) if it satisfies the following conditions:*

(C_1). $\varphi(t_1, t_2) \ge \varphi(\bar{t}_1, \bar{t}_2)$ for $t_1 \ge \bar{t}_1 \ge 0$, $t_2 \ge \bar{t}_2 \ge 0$.
(C_2). $\lim\limits_{t \to +\infty} [t - \varphi(t, t)] = +\infty$.
(C_3). $\lim\limits_{n \to +\infty} \varphi^n(t, t) = 0$ for all $t > 0$, *where $\varphi^n(t, t)$ is the nth iteration of $\varphi(t, t)$.*

A function $\varphi : R^+ \times R^+ \longrightarrow R^+$ is said to have the property (Φ_2) if it satisfies the condition $(\Phi_1)(C_1)$ and $\sum\limits_{n=1}^{+\infty} \varphi^n(t, t) < +\infty$ for all $t > 0$, where $\varphi^n(t, t)$ is the nth iteration of $\varphi(t, t)$.

Lemma 1. *Let $\varphi : R^+ \times R^+ \longrightarrow R^+$ satisfies the condition (Φ_1). Then, the following conclusions hold:*

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(i). $\varphi(t,\ t) < t$ *for all* $t > 0$, *(ii).* $\lim_{t\to 0^+} \varphi(t,\ t) = 0$, and $\varphi(0,\ 0) = 0$.

Proof. (i). If the conclusion is not true, then there exists a $t_0 > 0$ such that

$$\varphi(t_0,\ t_0) \geq t_0.$$

By (Φ_1) and induction, it is easy to verify that

$$\varphi^n(t_0,\ t_0) \geq t_0\ (n = 0,\ 1,\ 2,\ \ldots).$$

From the above and $(\Phi_1)(C_3)$, we have

$$0 = \lim_{n\to+\infty} \varphi^n(t_0,\ t_0) \geq t_0 > 0,$$

which is a contradiction. Thus, (i) holds.

(ii). By (i), it is easy to see that $\lim_{t\to 0^+} \varphi(t,\ t) = 0$ and $\varphi(0,\ 0) = 0$.

This completes the proof. □

Lemma 2. *Let* $\varphi : R^+ \times R^+ \longrightarrow R^+$ *satisfy the condition* (Φ_2). *Then, the conclusions of Lemma 1 hold.*

Proof. By the condition $\sum_{n=1}^{+\infty} \varphi^n(t,\ t) < +\infty$ for all $t > 0$, we have

$$\lim_{n\to+\infty} \varphi^n(t,\ t) = 0 \text{ for all } t > 0.$$

Thus, function φ satisfies $(\Phi_1)(C_1)$ and $(\Phi_1)(C_3)$.

By the same way as stated in Lemma 1, the rest can be proved.

This completes the proof. □

Definition 4. *The triple* $(X,\ d,\ \leq)$ *is called a partially ordered metric space if* $(X,\ \leq)$ *is a partially ordered set and* $(X,\ d)$ *is a metric space.*

The $(X,\ d,\ \leq)$ *is said to be complete partially ordered metric space if* $(X,\ d)$ *is a complete metric space.*

The $(X,\ d,\ \leq)$ *is said to have the property* $(I - D)$ *if it has the following properties:*

(i). If a nondecreasing sequence $\{x_n\} \to x$, *then* $x_n \leq x$, $\forall n$.

(ii). If a nonincreasing sequence $\{y_n\} \to y$, *then* $y_n \geq y$, $\forall n$.

Definition 5. *Let* $(X,\ d,\ \leq)$ *be a partially ordered metric space, the mapping* $F : X \times X \longrightarrow X$ *is called a nonlinear contraction mapping of type* $(\Phi_i)(i = 1,\ 2)$ *if there exists a function* $\varphi : R^+ \times R^+ \longrightarrow R^+$ *with the property* $(\Phi_i)(i = 1,\ 2)$ *such that*

$$d(F(x,\ y),\ F(u,\ v)) \leq \varphi(d(x,\ u),\ d(y,\ v))),\ \forall x \geq u,\ y \leq v.$$

Throughout this paper, assume that $(X,\ d,\ \leq)$ *is a complete partially ordered metric space.*

Main results

Theorem 1. *Let* $x_0\ ,y_0 \in X$ *and* $F : X \times X \longrightarrow X$ *be a continuous mixed monotone mapping such that*

$$x_0 \leq F(x_0,\ y_0),\ \ F(y_0,\ x_0) \leq y_0.$$

Assume that the following conditions hold:

(H_1).*Suppose that one of the following two conditions is satisfied:*

(a). *F is a nonlinear contraction mapping of type* (Φ_1).
(b). *F is a nonlinear contraction mapping of type* (Φ_2).

(H_2). *Suppose that one of the following two conditions is satisfied:*

(c). $x_0,\ y_0$ *in X are comparable.*
(d). *Every pair elements of X has an upper bound or a lower bound in X.*

Then, there exists $x^* \in X$ *such that* $x^* = F(x^*,\ x^*)$, *i.e.,* x^* *is a fixed point of mapping F. Moreover, the iterative sequences* $\{x_n\}$ *and* $\{y_n\}$ *given by*

$$x_n = F(x_{n-1},\ y_{n-1}) \text{ and } y_n = F(y_{n-1},\ x_{n-1}) \times (n = 1,\ 2,\ 3,\ \ldots) \tag{1}$$

converge to x^*, *i.e.,*

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x^*,$$

and

$$\begin{aligned} &x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots; \\ &y_0 \geq y_1 \geq y_2 \geq \ldots \geq y_n \geq \ldots. \end{aligned} \tag{2}$$

Further, if $x_0,\ y_0 \in X$ *are comparable, then*

$$\begin{cases} x_0\leq x_1\leq\ldots\leq x_n\leq\ldots\leq x^*\leq\ldots\leq y_n\leq\ldots\leq y_1\leq y_0, & \text{if } x_0\leq y_0; \\ y_0\leq y_1\leq\ldots\leq y_n\leq\ldots\leq x^*\leq\ldots\leq x_n\leq\ldots\leq x_1\leq x_0, & \text{if } y_0\leq x_0. \end{cases}$$

Proof. Using the same reasoning as in ([15], Theorem 2.1), we can obtain that (2), i.e., the sequences $\{x_n\}$ and $\{y_n\}$ are monotone. In the following, we will prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences.

If $(H_1)(a)$ holds, let

$$\begin{aligned} u_n &= d(x_0,\ x_{n+1}) = d(x_0,\ F(x_n,\ y_n)), \\ v_n &= d(y_0,\ y_{n+1}) = d(y_0,\ F(y_n,\ x_n))(n = 0,\ 1,\ 2,\ \ldots), \end{aligned}$$

$$h = \max\{u_0,\ v_0\} = \max\{d(x_0,\ F(x_0,\ y_0)),\ d(y_0,\ F(y_0,\ x_0))\}.$$

First, by the condition $(\Phi_1)(C_2)$, we know that there exists a positive number $c > h$ such that

$$t - \varphi(t,\ t) > h \text{ for all } t \geq c. \tag{3}$$

Now, we show that $u_n < c,\ v_n < c\ (n = 0,\ 1,\ 2,\ \ldots)$. If this is false, then there exists a nonnegative integer j such that

$$j = \min\{i : \max\{u_i,\ v_i\} \geq c\}.$$

By $\max\{u_0, v_0\} = h < c$, we know that j is a positive integer and $\max\{u_i, v_i\} < c$ $(i = 0, 1, 2, \ldots, j-1)$.

There are several possible cases which we need to consider.

Case 1. $u_j \geq c$ and $v_j < c$. If F is a nonlinear contraction mapping of type (Φ_1), we have

$$\begin{aligned} u_j = d(x_0, F(x_j, y_j)) &\leq d(x_0, F(x_0, y_0)) + d(F(x_0, y_0), \\ F(x_j, y_j)) &\leq h + d(F(x_j, y_j), F(x_0, y_0)) \\ &\leq h + \varphi(d(x_j, x_0), d(y_j, y_0)) \\ &= h + \varphi(d(x_0, F(x_{j-1}, y_{j-1})), \\ d(y_0, F(y_{j-1}, x_{j-1}))) &= h + \varphi(u_{j-1}, v_{j-1}) \\ &\leq h + \varphi(u_j, u_j), \end{aligned}$$

i.e., $u_j \geq c$ and $u_j - \varphi(u_j, u_j) \leq h$, which contradicts (3).

Case 2. $u_j < c$ and $v_j \geq c$. Using the same reasoning as in Case 1, we can obtain that

$$v_j \leq h + \varphi(v_j, v_j),$$

i.e., $v_j \geq c$ and $v_j - \varphi(v_j, v_j) \leq h$, which contradicts (3).

Case 3. $u_j \geq c$ and $v_j \geq c$. Without loss of generality, we can assume that $u_j \geq v_j \geq c$.

Thus, by Case 1, we know that

$$u_j \leq h + \varphi(u_{j-1}, v_{j-1}) \leq h + \varphi(u_j, u_j),$$

i.e., $u_j \geq c$ and $u_j - \varphi(u_j, u_j) \leq h$, which is in contradiction with (3).

From the above, it is easy to know that both sequences $\{u_n\}$ and $\{v_n\}$ are bounded. Now, we show that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences.

By (2), for any positive integer number p, we have

$$\begin{aligned} d(x_{n+p}, x_n) &= d(F(x_{n+p-1}, y_{n+p-1}), F(x_{n-1}, y_{n-1})) \\ &\leq \varphi(d(x_{n+p-1}, x_{n-1}), d(y_{n+p-1}, y_{n-1})) \\ &= \varphi(d(x_{n+p-1}, x_{n-1}), d(y_{n-1}, y_{n+p-1})). \end{aligned} \tag{4}$$

In the same way, we can get that

$$d(y_n, y_{n+p}) \leq \varphi(d(y_{n-1}, y_{n+p-1}), d(x_{n+p-1}, x_{n-1})). \tag{5}$$

Set $d(z_{n+p-1}, z_{n-1}) = \max\{d(x_{n+p-1}, x_{n-1}), d(y_{n-1}, y_{n+p-1})\}(n=1, 2, \ldots)$.

Thus, by (4), (5), and $(\Phi_1)(C_1)$, we have

$$\begin{aligned} d(x_{n+p}, x_n) &\leq \varphi(d(z_{n+p-1}, z_{n-1}), d(z_{n-1}, z_{n+p-1})) \\ &\leq \ldots \\ &\leq \varphi^n(d(z_p, z_0), d(z_0, z_p)). \end{aligned}$$

In the same way, we can get that

$$d(y_n, y_{n+p}) \leq \varphi^n(d(z_p, z_0), d(z_0, z_p)).$$

Obviously, $d(z_0, z_p) \leq \max\{d(x_0, x_p), d(y_0, y_p)\} = \max\{u_{p-1}, v_{p-1}\}$.

Since $\{u_n\}$, $\{v_n\}$ are bounded sequences, there exists a real constant $M > 0$ such that $u_n \leq M$, $v_n \leq M$ $(n = 0, 1, 2, \ldots)$.

From the above and (Φ_1), we have

$$\begin{aligned} d(x_{n+p}, x_n) \leq \varphi^n(M, M) \to 0, \ d(y_n, y_{n+p}) \\ \leq \varphi^n(M, M) \to 0 (n \to +\infty). \end{aligned}$$

Therefore, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X.

If, on the other hand, $(H_1)(b)$ is satisfied, by (2) and (Φ_2), we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \\ &\leq \varphi(d(x_n, x_{n-1}), d(y_n, y_{n-1})) \\ &= \varphi(d(x_n, x_{n-1}), d(y_{n-1}, y_n)). \end{aligned} \tag{6}$$

In the same way, we can get that

$$d(y_n, y_{n+1}) \leq \varphi(d(y_{n-1}, y_n), d(x_n, x_{n-1})). \tag{7}$$

For each integer $n \geq 0$, define

$$a_n = d(x_{n+1}, x_n), \ b_n = d(y_n, y_{n+1}), \ c_n = \max\{a_n, b_n\}. \tag{8}$$

There are two possible cases which we need to consider.

Case 4. $c_0 = 0$. Note that $\max\{a_0, b_0\} = c_0 = 0$ implies that

$$\begin{aligned} d(F(x_0, y_0), x_0) &= d(x_1, x_0) = a_0 = 0, \\ d(y_0, F(y_0, x_0)) &= d(y_0, y_1) = b_0 = 0. \end{aligned}$$

Thus, we have that $x_0 = F(x_0, y_0)$ and $y_0 = F(y_0, x_0)$. It is easy to know by (1) that

$$x_n = x_0, \ y_n = y_0 (n = 1, 2, 3, \ldots).$$

Obviously, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X.

Case 5. $c_0 > 0$. From (6), (8), and (Φ_2), for any positive integer n, we have

$$\begin{aligned} a_n = d(x_{n+1}, x_n) &\leq \varphi(d(x_n, x_{n-1}), d(y_{n-1}, y_n)) \\ &= \varphi(a_{n-1}, b_{n-1}) \leq \varphi(c_{n-1}, c_{n-1}). \end{aligned}$$

In the same way, we can get that $b_n \leq \varphi(c_{n-1}, c_{n-1})$.

From the above and (8), we have

$$c_n \leq \varphi(c_{n-1}, c_{n-1}) \leq \ldots \leq \varphi^n(c_0, c_0)(n = 1, 2, \ldots)$$

Thus, by (Φ_2), we know that

$$\begin{aligned} d(x_{n+p}, x_n) &\leq \sum_{i=n}^{n+p-1} d(x_{i+1}, x_i) \\ &\leq \sum_{i=n}^{n+p-1} c_i \\ &\leq \sum_{i=n}^{n+p-1} \varphi^i(c_0, c_0) \\ &\leq \sum_{i=n}^{+\infty} \varphi^i(c_0, c_0) \to 0 \ (n \to +\infty). \end{aligned}$$

In the same way, we can get that

$$d(y_n, y_{n+p}) \leq \sum_{i=n}^{+\infty} \varphi^i(c_0, c_0) \to 0 \ (n \to +\infty).$$

From the above, we know that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X.

Since X is a complete metric space, there exist $x^*, y^* \in X$ such that

$$\lim_{n\to\infty} x_n = x^*, \ \lim_{n\to\infty} y_n = y^*. \tag{9}$$

Thus, letting $n \longrightarrow \infty$ in (1) and by (9) and continuity of the mapping F, we have

$$x^* = F(x^*, y^*), \ y^* = F(y^*, x^*).$$

Next, we prove that $x^* = y^*$, i.e., $x^* = F(x^*, x^*)$.

If $(H_2)(c)$ holds, without loss of generality, we assume that $x_0 \leq y_0$.

There are two possible cases which we need to consider.

Case 6. $x_0 = y_0$, set $x^* = x_0 = y_0$ and $x_n = y_n = x^*(n = 1, 2, 3, \ldots)$, it is easy to verify that the conclusions of Theorem 1 hold.

Case 7. $x_0 < y_0$, then $d(x_0, y_0) > 0$. It is easy to know from the proof of Theorem 2.6 in [15] that

$$x_n \leq y_n (n = 1, 2, 3, \ldots). \tag{10}$$

Thus, by (10) and $(\Phi_i)(i = 1, 2)$, we have

$$\begin{aligned} d(y_n, x_n) &= (F(y_{n-1}, x_{n-1}), F(x_{n-1}, y_{n-1})) \\ &\leq \varphi(d(y_{n-1}, x_{n-1}), d(x_{n-1}, y_{n-1})) \\ &= \varphi(d(y_{n-1}, x_{n-1}), d(y_{n-1}, x_{n-1})) \\ &\leq \varphi^2(d(y_{n-2}, x_{n-2}), d(y_{n-2}, x_{n-2})) \\ &\leq \ldots\ldots \\ &\leq \varphi^n(d(y_0, x_0), d(y_0, x_0)) \to 0 (n \to \infty), \end{aligned}$$

i.e., $\lim_{n\to\infty} d(y_n, x_n) = 0$.

From the above and $\lim_{n\to\infty} x_n = x^*, \ \lim_{n\to\infty} y_n = y^*$, we have

$$\begin{aligned} d(x^*, y^*) \leq d(x^*, x_n) &+ d(x_n, y_n) \\ &+ d(y_n, y^*) \to 0 (n \to \infty). \end{aligned}$$

Therefore, $d(x^*, y^*) = 0$, i.e., $y^* = x^*$. Thus, $x^* = F(x^*, x^*)$. Similarly, if $x_0 > y_0$, then it is possible to show $x_n \geq y_n$ for all n and that $y^* = x^*$ and $x^* = F(x^*, x^*)$. If, on the other hand, $(H_2)(d)$ is satisfied, there are two possible cases which we need to consider.

Case 8. If x^* is comparable to y^*, then

$$d(x^*, y^*) = d(F(x^*, y^*), F(y^*, x^*)) \leq \varphi(d(x^*, y^*), d(x^*, y^*)).$$

From the above and Lemma 1 or Lemma 2, it is easy to know that $d(x^*, y^*) = 0$, i.e., $y^* = x^*$, and the conclusions of Theorem 1 hold.

Case 9. If x^* is not comparable to y^*, then there exists an upper bound or a lower bound of x^* and y^*. Without loss of generality, we assume that there exists a $z \in X$ such that

$$x^* \leq z, \ y^* \leq z. \tag{11}$$

From the proof of Theorem 2.5 in [15], we know that

$$\begin{cases} F^n(x^*, y^*) \leq F^n(z, y^*), \ F^n(y^*, x^*) \leq F^n(z, x^*), \\ F^n(x^*, y^*) \geq F^n(x^*, z), \ F^n(y^*, x^*) \geq F^n(y^*, z), \\ (n = 1, 2, 3, \ldots), \end{cases} \tag{12}$$

and

$$\begin{aligned} d(x^*, y^*) \leq\ & d(F(F^n(x^*, y^*), F^n(y^*, x^*)), F(F^n(x^*, z), F^n(z, x^*))) \\ &+ d(F(F^n(z, x^*), F^n(x^*, z)), F(F^n(x^*, z), F^n(z, x^*))) \\ &+ d(F(F^n(z, x^*), F^n(x^*, z)), F(F^n(y^*, x^*), F^n(x^*, y^*))). \end{aligned} \tag{13}$$

By induction, it is easy to show from (11) and mixed monotone property of F that

$$F^n(z, x^*) \geq F^n(x^*, z)(n = 1, 2, 3, \ldots). \tag{14}$$

Set $a_n = \max\{d(F^n(x^*, y^*), (F^n(x^*, z), d(F^n(y^*, x^*), (F^n(z, x^*))(n = 1, 2, 3, \ldots);$

$$M = \max\{d(x^*, F(x^*, z)), d(z, x^*), d(z, y^*)\}.$$

Obviously, $M > 0$.

Thus, by (11), (12), (13), and (14) and Lemma 1 (with respect to Lemma 2), we have

$$\begin{aligned} & d(F(F^n(z,x^*),\ F^n(x^*,\ z)),\ F(F^n(x^*,\ z),\ F^n(z,\ x^*))) \\ & \quad \leq \varphi(d(F^n(z,\ x^*),\ F^n(x^*,\ z)), d(F^n(x^*,\ z),\ F^n(z,\ x^*))) \\ & \quad \leq \varphi(d(F(F^{n-1}(z,\ x^*),\ F^{n-1}(x^*,\ z)), d(F(F^{n-1}(x^*,\ z), \\ & \qquad F^{n-1}(z,\ x^*)),\ d(F(F^{n-1}(x^*,\ z),\ F^{n-1}(z,\ x^*)), \\ & \qquad F(F^{n-1}(x^*,\ z))) \\ & \quad \leq \varphi^2(d(F^{n-1}(z,\ x^*),\ F^{n-1}(x^*,\ z)),\ d(F^{n-1}(z,\ x^*), \\ & \qquad F^{n-1}(x^*,\ z))) \\ & \quad \leq \cdots\cdots \\ & \quad \leq \varphi^{n+1}(d(z,\ x^*),\ d(z,\ x^*)) \\ & \quad \leq \varphi^{n+1}(M,\ M) \to 0 (n \to \infty); \end{aligned} \tag{15}$$

$$\begin{aligned} & d(F(F^n(x^*,y^*),\ F^n(y^*,\ x^*)),\ F(F^n(x^*,\ z),\ F^n(z,\ x^*))) \\ & \quad \leq \varphi(d(F^n(x^*,\ y^*),\ F^n(x^*,\ z)),\ d(F^n(y^*,\ x^*), \\ & \qquad F^n(z,\ x^*))) \\ & \quad \leq \varphi(a_n,\ a_n) \\ & \quad \leq \cdots\cdots \\ & \quad \leq \varphi^n(a_1,\ a_1) \\ & \quad \leq \varphi^{n+1}(M,\ M) \to 0 (n \to \infty). \end{aligned} \tag{16}$$

In the same way, we can get that

$$\begin{aligned} & d(F(F^n(z,\ x^*),\ F^n(x^*,\ z)),\ F(F^n(y^*,\ x^*),\ F^n(x^*,\ y^*))) \\ & \quad \leq \varphi^{n+1}(M,\ M) \to 0 (n \to \infty). \end{aligned} \tag{17}$$

Thus, by (15), (16), (17), and (13), we have that $d(x^*,\ y^*) = 0$, i.e., $y^* = x^*$.

Therefore, the conclusions of Theorem 1 hold. The proof of the Theorem 1 is complete. □

Remark 1. *In Theorem 1, if function* $\varphi : R^+ \times R^+ \longrightarrow R^+$ *is given by*

$$\varphi(t_1,\ t_2) = \frac{k}{2}(t_1 + t_2),\ t_1,\ t_2 \in R^+,$$

where $k \in [0,\ 1)$ *is a real constant.*

It is easy to verify that the function φ *has the property* $(\Phi_i)(i = 1,\ 2)$*, and mapping F satisfies all conditions of Theorem 2.5 and Theorem 2.6 in [15]. Thus, the conclusions of Theorem 2.5 and Theorem 2.6 in [15] hold.*

Therefore, our Theorem 1 improves and generalizes the Theorem 2.5 and Theorem 2.6 in [15].

From the proof of Theorem 1, it is easy to see that the following two theorems hold.

Theorem 2. *Let* $(X,\ d,\ \ \leq)$ *be a complete partially ordered metric space,* $x_0\ , y_0 \in X$ *and* $F : X \times X \longrightarrow X$ *be a mixed monotone mapping such that*

$$x_0 \leq F(x_0,\ y_0),\ F(y_0,\ x_0) \leq y_0$$

and condition (H_1) *is fulfilled; then, there exist* $x^*,\ y^* \in X$ *such that*

$$x^* = F(x^*,\ y^*) \text{ and } y^* = F(y^*,\ x^*).$$

Moreover, the iterative sequences $\{x_n\}$ *and* $\{y_n\}$ *given by (1) converge, respectively, to* x^* *and* y^**, and (2) holds.*

Remark 2. *Obviously, our Theorem 2 improves and generalizes the Theorem 2.1 in [15].*

Theorem 3. *Let* $(X,\ d,\ \ \leq)$ *be a complete partially ordered metric space having the property* $(I - D)$*,* $x_0\ , y_0 \in X$ *and* $F : X \times X \longrightarrow X$ *be a mixed monotone mapping such that*

$$x_0 \leq F(x_0,\ y_0),\ F(y_0,\ x_0) \leq y_0.$$

and condition (H_2) *is fulfilled; then, the conclusions of Theorem 2 hold.*

Remark 3. *Obviously, our Theorem 3 improves and generalizes the Theorem 2.2 in [15].*

Example

In this final section, we give an example to support our result.

Let $X = [-\frac{\pi}{12},\ \frac{\pi}{12}] \times [-\frac{\pi}{12},\ \frac{\pi}{12}]$ be the metric space endowed with the metric

$$\begin{aligned} & d(x,\ y) = |x_1 - y_1| + |x_2 - y_2|, \text{ for } x = (x_1,\ x_2), \\ & \qquad y = (y_1,\ y_2) \in X. \end{aligned}$$

Further, we endow the set X with the following partial order:

$$\begin{aligned} & \text{for } x = (x_1,\ x_2),\ y = (y_1,\ y_2) \in X, \\ & \quad x \leq y \Longleftrightarrow x_1 \leq y_1,\ x_2 \geq y_2. \end{aligned}$$

Obviously, $(X,\ d,\ \ \leq\ \ )$ is a complete partial ordered metric space.

Example 1. *Suppose that the mapping* $F : X \times X \longrightarrow X$ *is defined by*

$$F(x,\ y) = \frac{1}{4}(\frac{1}{24} + \sin 2(x_1 - x_2),\ \frac{1}{16} + \sin 2(y_1 - y_2)),$$

where $x = (x_1,\ x_2),\ y = (y_1,\ y_2) \in X$.

Then, there exists $x^* \in X$ *such that* $x^* = F(x^*,\ x^*)$*. Moreover, the iterative sequences* $\{x_n\}$ *and* $\{y_n\}$ *defined by*

$$\begin{aligned} & x_n = F(x_{n-1},\ y_{n-1}) \text{ and } y_n = F(y_{n-1},\ x_{n-1}) \\ & \qquad \times (n = 1,\ 2,\ 3,\ \ldots) \end{aligned}$$

converge to x^*, and

$$x_0 \le x_1 \le \ldots \le x_n \le \ldots \le x^* \le \ldots \le y_n \le \ldots \le y_1 \le y_0,$$

where $x_0 = (-\frac{\pi}{12},\ \frac{\pi}{12})$, $y_0 = (\frac{\pi}{12},\ -\frac{\pi}{12}) \in X$.

Proof. Obviously, $F : X \times X \longrightarrow X$ is a continuous mixed monotone mapping .

It is easy to compute that

$$\begin{aligned} F(x_0,\ y_0) &= \frac{1}{4}(\ \frac{1}{24} - \sin\frac{\pi}{3}\ ,\ \frac{1}{16} + \sin\frac{\pi}{3}\ ) \\ &= (\ \frac{1-12\sqrt{3}}{96}\ ,\ \frac{1+8\sqrt{3}}{64}\ ) \\ &> (-\frac{\pi}{12},\ \frac{\pi}{12}\ ) = x_0, \end{aligned}$$

$$\begin{aligned} F(y_0,\ x_0) &= \frac{1}{4}(\ \frac{1}{24} + \sin\frac{\pi}{3}\ ,\ \frac{1}{16} + \sin\frac{\pi}{3}\ ) \\ &= (\ \frac{1+12\sqrt{3}}{96}\ ,\ \frac{1-8\sqrt{3}}{64}\ ) \\ &< (\frac{\pi}{12},\ -\frac{\pi}{12}) = y_0. \end{aligned}$$

For $x = (x_1,\ x_2)$, $y = (y_1,\ y_2)$, $u = (u_1,\ u_2)$, $v = (v_1,\ v_2) \in X$ satisfying $x \ge u$, $y \le v$, i.e., $x_1 \ge u_1$, $x_2 \le u_2$, $y_1 \le v_1$, $y_2 \ge v_2$, we have

$$\begin{aligned} d(F(x,\ y),\ F(u,\ v)) &= \frac{1}{4}[|\sin 2(x_1 - x_2) - \sin 2(u_1 - u_2)| \\ &\quad + |\sin 2(y_1 - y_2) - \sin 2(v_1 - v_2)|] \\ &= \frac{1}{2}[|\cos(x_1 - x_2 + u_1 - u_2) \\ &\quad \times \sin(x_1 - x_2 - u_1 + u_2)| \\ &\quad + |\cos(y_1 - y_2 + v_1 - v_2) \\ &\quad \times \sin(y_1 - y_2 - v_1 + v_2)|] \\ &\le \frac{1}{2}[\sin(x_1 - u_1 + u_2 - x_2) \\ &\quad + \sin(v_1 - y_1 + y_2 - v_2)] \\ &\le \sin\frac{x_1 - u_1 + u_2 - x_2 + v_1 - y_1 + y_2 - v_2}{2} \\ &= \sin\frac{1}{2}[d(x,\ u) + d(y,\ v)] \\ &\equiv \varphi(d(x,\ u)\ ,\ d(y,\ v)), \end{aligned}$$

where

$$\varphi(t_1\ , t_2) = \begin{cases} \sin\frac{t_1+t_2}{2}, & t_1\ ,\ t_2 \in [0\ ,\ \frac{\pi}{3}]\ , \\ \sin(\frac{\pi}{6} + \frac{t_2}{2}), & t_1 > \frac{\pi}{3},\ t_2 \in [0\ ,\ \frac{\pi}{3}]\ , \\ \sin(\frac{\pi}{6} + \frac{t_1}{2}), & t_2 > \frac{\pi}{3},\ t_1 \in [0\ ,\ \frac{\pi}{3}]\ , \\ \frac{\sqrt{3}}{2}, & t_1 > \frac{\pi}{3},\ t_2 > \frac{\pi}{3}. \end{cases}$$

Obviously,

$$\varphi(t\ , t) = \begin{cases} \sin t, & t \in [\,0\ ,\ \frac{\pi}{3}\,]\ , \\ \frac{\sqrt{3}}{2}, & t > \frac{\pi}{3}. \end{cases}$$

It is easy to know that, $\varphi^n(t,\ t) = \underbrace{\sin\sin\ldots\sin}_{n} t$, and $\lim_{n\to\infty} \varphi^n(t,\ t) = 0$, $\forall t \in R^+$.

From the above, we know that the mapping F satisfies all conditions of Theorem 1, it follows by Theorem 1 that our conclusion holds. The proof is complete. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

LZ, GS, GW, and HW contributed equally to each part of this work. All authors read and approved the final manuscript.

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