

Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2009, Article ID 515709, 9 pages
doi:10.1155/2009/515709

Research Article

Inequalities for the Polar Derivative of a Polynomial

M. Bidkham, M. Shakeri, and M. Eshaghi Gordji

Department of Mathematics, Faculty of Natural Sciences, Semnan University, Semnan 35195-363, Iran

Correspondence should be addressed to M. Bidkham, mdbidkham@gmail.com

Received 11 August 2009; Accepted 30 November 2009

Recommended by Narendra Kumar Govil

Let $p(z)$ be a polynomial of degree n and for any real or complex number α , and let $D_\alpha p(z) = np(z) + (\alpha - z)p'(z)$ denote the polar derivative of the polynomial $p(z)$ with respect to α . In this paper, we obtain new results concerning the maximum modulus of a polar derivative of a polynomial with restricted zeros. Our results generalize as well as improve upon some well-known polynomial inequalities.

Copyright © 2009 M. Bidkham et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction and Statement of Results

If $p(z)$ is a polynomial of degree n , then it is well known that

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|. \quad (1.1)$$

The above inequality, which is an immediate consequence of Bernstein's inequality applied to the derivative of a trigonometric polynomial, is best possible with equality holding if and only if $p(z)$ has all its zeros at the origin. If $p(z) \neq 0$ in $|z| < 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|. \quad (1.2)$$

Inequality (1.2) was conjectured by Erdős and later proved by Lax [1]. If the polynomial $p(z)$ of degree n has all its zeros in $|z| < 1$, then it was proved by Turán [2] that

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)|. \quad (1.3)$$

Inequality (1.2) was generalized by Malik [3] who proved that if $p(z) \neq 0$ in $|z| < k$, $k \geq 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |p(z)|. \quad (1.4)$$

For the class of polynomials having all its zeros in $|z| \leq k$, $k \geq 1$, Govil [4] proved that

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+k^n} \max_{|z|=1} |p(z)|. \quad (1.5)$$

Inequality (1.5) is sharp and equality holds for $p(z) = z^n + k^n$. By considering a more general class of polynomials $p(z) = a_0 + \sum_{v=t}^n a_v z^v$, $1 \leq t \leq n$, not vanishing in $|z| < k$, $k \geq 1$, Gardner et al. [5] proved that

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+s_0} \left\{ \max_{|z|=1} |p(z)| - m \right\}, \quad (1.6)$$

where $m = \min_{|z|=k} |p(z)|$ and $s_0 = k^{t+1} \{ ((t/n)(|a_t|/(|a_0| - m))k^{t-1} + 1) / ((t/n)(|a_t|/(|a_0| - m))k^{t+1} + 1) \}$.

Let $D_\alpha\{p(z)\}$ denote the polar derivative of the polynomial $p(z)$ of degree n with respect to the point α . Then

$$D_\alpha\{p(z)\} = np(z) + (\alpha - z)p'(z). \quad (1.7)$$

The polynomial $D_\alpha\{p(z)\}$ is of degree at most $n - 1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha \rightarrow \infty} \left[\frac{D_\alpha\{p(z)\}}{\alpha} \right] = p'(z). \quad (1.8)$$

As an extension of (1.5), it was shown by Aziz and Rather [6] that if $p(z)$ has all its zeros in $|z| \leq k$, $k \geq 1$, then for $|\alpha| \geq k$,

$$\max_{|z|=1} |D_\alpha p(z)| \geq n \left(\frac{|\alpha| - k}{1 + k^n} \right) \max_{|z|=1} |p(z)|. \quad (1.9)$$

Inequality (1.9) was later sharpened by Dewan and Upadhye [7], who proved the following theorem.

Theorem A. Let $p(z)$ be a polynomial of degree n having all its zeros in $|z| \leq k, k \geq 1$, then for $|\alpha| \geq k$,

$$\max_{|z|=1} |D_\alpha p(z)| \geq n(|\alpha| - k) \left\{ \frac{1}{1 + k^n} \max_{|z|=1} |p(z)| + \frac{1}{2k^n} \left(\frac{k^n - 1}{k^n + 1} \right) m \right\}, \tag{1.10}$$

where $m = \min_{|z|=k} |p(z)|$.

Recently, Dewan et al. [8] extended inequality (1.6) to the polar derivative of a polynomial and obtained the following result.

Theorem B. If $p(z) = a_0 + \sum_{v=t}^n a_v z^v, 1 \leq t \leq n$, is a polynomial of degree n having no zeros in $|z| < k, k \geq 1$, then for $|\alpha| \geq 1$,

$$\max_{|z|=1} |D_\alpha p(z)| \leq \frac{n}{1 + s_0} \left\{ (|\alpha| + s_0) \max_{|z|=1} |p(z)| - (|\alpha| - 1)m \right\}, \tag{1.11}$$

where $m = \min_{|z|=k} |p(z)|$ and $s_0 = k^{t+1} \{ ((t/n)(|a_t|/(|a_0| - m))k^{t-1} + 1) / ((t/n)(|a_t|/(|a_0| - m))k^{t+1} + 1) \}$.

In this paper, we will first generalize Theorem A as well as improve upon the bound obtained in inequality (1.10) by involving some of the coefficients of $p(z)$. More precisely, we prove the following.

Theorem 1.1. If $p(z) = \sum_{i=0}^n a_i z^i$ is a polynomial of degree $n \geq 3$ having all its zeros in $|z| \leq k, k \geq 1$, then for $|\alpha| \geq k$,

$$\begin{aligned} & \max_{|z|=1} |D_\alpha p(z)| \\ & \geq n(|\alpha| - k) \left[\frac{1}{k^n + 1} \max_{|z|=1} |p(z)| + \frac{k^n - 1}{2k^n(k^n + 1)} m \right. \\ & \quad + \frac{2|a_{n-1}|}{k(k^n + 1)(n + 1)} \left(\frac{k^n - 1}{n} - (k - 1) \right) \\ & \quad \left. + \frac{2|a_{n-2}|}{(k^n + 1)k^2} \left(\left(\frac{(k^n - 1) - n(k - 1)}{n(n - 1)} \right) - \left(\frac{(k^{n-2} - 1) - (n - 2)(k - 1)}{(n - 2)(n - 3)} \right) \right) \right] \\ & \quad + \frac{1}{k^{n-1}} \left[\frac{k^{n-1} - 1}{n - 1} - \frac{k^{n-3} - 1}{n - 3} \right] |(n - 1)a_1 + 2\alpha a_2| \\ & \quad + \frac{2}{k^{n-1}} \left(\frac{k^{n-1} - 1}{n + 1} \right) |na_0 + \alpha a_1| + n \frac{(|\alpha| + k)}{2k^n} m \end{aligned} \tag{1.12}$$

for $n > 3$ and

$$\begin{aligned} \max_{|z|=1} |D_\alpha p(z)| &\geq n(|\alpha| - k) \left[\frac{1}{k^n + 1} \max_{|z|=1} |p(z)| + \frac{k^n - 1}{2k^3(k^n + 1)} m \right. \\ &\quad + \frac{2|a_{n-1}|}{k(k^n + 1)(n + 1)} \left(\frac{k^n - 1}{n} - (k - 1) \right) \\ &\quad \left. + \frac{2k^{n-5}|a_{n-2}|}{(k^n + 1)} \left(\frac{(k - 1)^n}{n(n - 1)} \right) \right] \quad (1.13) \\ &\quad + \frac{k - 1}{2k^2} ((k + 1)|na_0 + \alpha a_1| + (k - 1)|(n - 1)a_1 + 2\alpha a_2) \\ &\quad + n \frac{(|\alpha| + k)}{2k^3} m \end{aligned}$$

for $n = 3$, where $m = \min_{|z|=k} |p(z)|$.

Now it is easy to verify that if $k \geq 1$, then $(k^n - 1)/n - (k - 1) \geq 0$, $[(k^{n-1} - 1)/(n - 1) - (k^{n-3} - 1)/(n - 3)] \geq 0$ and $[(((k^n - 1) - n(k - 1))/n(n - 1)) - (((k^{n-2} - 1) - (n - 2)(k - 1))/(n - 2)(n - 3))] \geq 0$ for $n > 3$. Hence for polynomial of degree $n \geq 3$, Theorem 1.1 is a refinement of Theorem A.

Dividing both sides of inequalities (1.12) and (1.13) by $|\alpha|$ and letting $|\alpha| \rightarrow \infty$, we get the following result.

Corollary 1.2. *If $p(z) = \sum_{i=0}^n a_i z^i$ is a polynomial of degree $n \geq 3$ having all its zeros in $|z| \leq k$, $k \geq 1$, then*

$$\begin{aligned} \max_{|z|=1} |p'(z)| &\geq \frac{n}{k^n + 1} \left[\max_{|z|=1} |p(z)| + m + \frac{2}{k(n + 1)} \left(\frac{k^n - 1}{n} - (k - 1) \right) |a_{n-1}| \right. \\ &\quad \left. + \frac{2}{k^2} \left(\frac{(k^n - 1) - n(k - 1)}{n(n - 1)} - \frac{(k^{n-2} - 1) - (n - 2)(k - 1)}{(n - 2)(n - 3)} \right) |a_{n-2}| \right] \quad (1.14) \\ &\quad \times \frac{2(k^{n-1} - 1)}{k^{n-1}(n + 1)} |a_1| + \frac{2}{k^{n-1}} \left[\frac{k^{n-1} - 1}{n - 1} - \frac{k^{n-3} - 1}{n - 3} \right] |a_2| \end{aligned}$$

for $n > 3$ and

$$\begin{aligned} \max_{|z|=1} |p'(z)| &\geq \frac{n}{k^n + 1} \left[\max_{|z|=1} |p(z)| + m + \frac{2}{k(n + 1)} \left(\frac{k^n - 1}{n} - (k - 1) \right) |a_{n-1}| \right. \\ &\quad \left. + \frac{2}{k^2} \left(\frac{(k - 1)^n}{n(n - 1)} \right) |a_{n-2}| \right] + \frac{k - 1}{2k^2} ((k + 1)|a_1| + 2(k - 1)|a_2|) \quad (1.15) \end{aligned}$$

for $n = 3$, where $m = \min_{|z|=k} |p(z)|$.

These inequalities are sharp and equality holds for the polynomial $p(z) = z^n + k^n$.

If we take $k = 1$ in the previous Theorem, we get a result, which was proved by Aziz and Dawood [9].

Next we consider a class of polynomial having no zeros in $|z| < k$, where $k \geq 1$ and prove the following generalization of Theorem B.

Theorem 1.3. *If $p(z) = a_0 + \sum_{\nu=\mu}^n a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then for $0 < r \leq R \leq k$ and $|\alpha| \geq R$,*

$$\begin{aligned} \max_{|z|=R} |D_\alpha p(z)| \leq \frac{n}{1+s'_0} \left[\left(\frac{|\alpha|}{R} + s'_0 \right) \exp \left\{ n \int_r^R A_t dt \right\} \max_{|z|=r} |p(z)| \right. \\ \left. + \left(s'_0 + 1 - \left(\frac{|\alpha|}{R} + s'_0 \right) \exp \left\{ n \int_r^R A_t dt \right\} \right) m \right], \end{aligned} \quad (1.16)$$

where

$$\begin{aligned} A_t &= \frac{(\mu/n)(|a_\mu|/(|a_0| - m))k^{\mu+1}t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + (\mu/n)(|a_\mu|/(|a_0| - m))(k^{\mu+1}t^\mu + k^{2\mu}t)}, \\ s'_0 &= \left(\frac{k}{R} \right)^{\mu+1} \left\{ \frac{(\mu/n)(|a_\mu|Rk^{\mu-1}/(|a_0| - m)) + 1}{(\mu/n)(|a_\mu|k^{\mu+1}/(|a_0| - m)R) + 1} \right\}, \\ m &= \min_{|z|=k} |p(z)|. \end{aligned} \quad (1.17)$$

Remark 1.4. For $R = r = 1$ Theorem 1.3 reduces to Theorem B.

Remark 1.5. Dividing the two sides of (1.16) by $|\alpha|$ and letting $|\alpha| \rightarrow \infty$, we obtain a result of Chanam and Dewan [10].

2. Lemmas

For the proofs of these theorems we need the following lemmas.

Lemma 2.1. *If $p(z)$ has all its zeros in $|z| \leq 1$, then for every $|\alpha| \geq 1$,*

$$\max_{|z|=1} |D_\alpha p(z)| \geq \frac{n}{2} \left\{ (|\alpha| - 1) \max_{|z|=1} |p(z)| + (|\alpha| + 1)m \right\}, \quad (2.1)$$

where $m = \min_{|z|=1} |p(z)|$.

This lemma is due to Aziz and Rather [6].

Lemma 2.2. *If $p(z)$ is a polynomial of degree n , having all its zeros in $|z| \leq k$, where $k \geq 1$, then*

$$\max_{|z|=k} |p(z)| \geq \frac{2k^n}{1+k^n} \max_{|z|=1} |p(z)|. \quad (2.2)$$

Inequality (2.2) is best possible and equality holds for $p(z) = z^n + k^n$.

This lemma is according to Aziz [11].

Lemma 2.3. *If $p(z)$ is a polynomial of degree n , then for $R \geq 1$,*

$$\begin{aligned} \max_{|z|=R} |p(z)| &\leq R^n \max_{|z|=1} |p(z)| - \frac{2(R^n - 1)}{n + 2} |p(0)| \\ &\quad - \left[\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n - 2} \right] |p'(0)| \end{aligned} \quad (2.3)$$

if $n > 2$, and

$$\max_{|z|=R} |p(z)| \leq R^2 \max_{|z|=1} |p(z)| - \frac{(R-1)}{2} [(R+1)|p(0)| + (R-1)|p'(0)|] \quad (2.4)$$

if $n = 2$.

This lemma is according to Dewan et al. [12].

Lemma 2.4. *If $p(z)$ is a polynomial of degree $n \geq 3$ having no zeros in $|z| < 1$ and $m = \min_{|z|=1} |p(z)|$, then for $R \geq 1$,*

$$\begin{aligned} \max_{|z|=R} |p(z)| &\leq \left(\frac{R^n + 1}{2} \right) \max_{|z|=1} |p(z)| - \left(\frac{R^n - 1}{2} \right) m - |p'(0)| \frac{2}{(n+1)} \left[\frac{R^n - 1}{n} - (R-1) \right] \\ &\quad - |p''(0)| \left[\left(\frac{R^n - 1}{n(n-1)} \right) - \left(\frac{(R^{n-2} - 1) - (n-2)(R-1)}{(n-2)(n-3)} \right) \right] \end{aligned} \quad (2.5)$$

if $n > 3$, and

$$\begin{aligned} \max_{|z|=R} |p(z)| &\leq \left(\frac{R^n + 1}{2} \right) \max_{|z|=1} |p(z)| - \left(\frac{R^n - 1}{2} \right) m \\ &\quad - |p'(0)| \frac{2}{n+1} \left[\frac{R^n - 1}{n} - (R-1) \right] \\ &\quad - |p''(0)| \frac{(R-1)^n}{n(n-1)} \end{aligned} \quad (2.6)$$

if $n = 3$.

This result is according to Dewan et al. [13].

Lemma 2.5. *If $p(z) = a_0 + \sum_{\nu=\mu}^n a_\nu z^\nu$, $1 \leq \mu \leq n$ is a polynomial of degree n such that $p(z) \neq 0$ in $|z| < k$, $k > 0$, then for $0 < r \leq R \leq k$,*

$$\begin{aligned} \max_{|z|=R} |p(z)| \leq & \exp \left\{ n \int_r^R \frac{(\mu/n)(|a_\mu|/(|a_0| - m))k^{\mu+1}t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + (\mu/n)(|a_\mu|/(|a_0| - m))(k^{\mu+1}t^\mu + k^{2\mu}t)} dt \right\} \max_{|z|=r} |p(z)| \\ & + \left[1 - \exp \left\{ n \int_r^R \frac{(\mu/n)(|a_\mu|/(|a_0| - m))k^{\mu+1}t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + (\mu/n)(|a_\mu|/(|a_0| - m))(k^{\mu+1}t^\mu + k^{2\mu}t)} dt \right\} \right] m, \end{aligned} \tag{2.7}$$

where $m = \min_{|z|=k} |p(z)|$.

Lemma 2.5 is according to Chanam and Dewan [10].

3. Proof of the Theorems

Proof of Theorem 1.1. By hypothesis that the polynomial $p(z)$ has all its zeros in $|z| \leq k$, where $k \geq 1$, therefore all the zeros of the polynomial $G(z) = p(kz)$ lie in $|z| \leq 1$. Applying Lemma 2.1 to the polynomial $G(z)$ and noting that $|\alpha|/k \geq 1$, we get

$$\max_{|z|=1} |D_{\alpha/k} G(z)| \geq \frac{n}{2} \left[\left(\frac{|\alpha|}{k} - 1 \right) \max_{|z|=1} |G(z)| + \left(\frac{|\alpha|}{k} + 1 \right) \min_{|z|=1} |G(z)| \right], \tag{3.1}$$

that is,

$$\max_{|z|=k} |D_\alpha p(z)| \geq \frac{n}{2} \left[\left(\frac{|\alpha| - k}{k} \right) \max_{|z|=k} |p(z)| + \left(\frac{|\alpha| + k}{k} \right) m \right]. \tag{3.2}$$

The polynomial $p(z)$ is of degree $n > 3$ and so $D_\alpha p(z)$ is the polynomial of degree $n - 1$, where $n - 1 > 2$, hence applying Lemma 2.3 to the polynomial $D_\alpha p(z)$, we get for $k \geq 1$

$$\begin{aligned} \max_{|z|=k} |D_\alpha p(z)| \leq & k^{n-1} \max_{|z|=1} |D_\alpha p(z)| - \frac{2(k^{n-1} - 1)}{n + 1} |na_0 + \alpha a_1| \\ & - \left[\frac{k^{n-1} - 1}{n - 1} - \frac{k^{n-3} - 1}{n - 3} \right] |(n - 1)a_1 + 2\alpha a_2|. \end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3), we get for $k \geq 1$

$$\begin{aligned} \max_{|z|=1} |D_\alpha p(z)| &\geq \frac{n}{2} \left[\left(\frac{|\alpha| - k}{k^n} \right) \max_{|z|=k} |p(z)| + \left(\frac{|\alpha| + k}{k^n} \right) m \right] \\ &\quad + \frac{2(k^{n-1} - 1)}{k^{n-1}(n+1)} |na_0 + \alpha a_1| \\ &\quad + \frac{1}{k^{n-1}} \left[\left(\frac{k^{n-1} - 1}{n-1} \right) - \left(\frac{k^{n-3} - 1}{n-3} \right) \right] |(n-1)a_1 + 2\alpha a_2|. \end{aligned} \quad (3.4)$$

Since the polynomial $p(z)$ has all zeros in $|z| \leq k$, $k \geq 1$, the polynomial $q(z) = z^n p(1/z)$ has no zero in $|z| < 1/k$, hence the polynomial $q(z/k)$ has all its zeros in $|z| \geq 1$, therefore on applying Lemma 2.4 to the polynomial $q(z/k)$, we get

$$\begin{aligned} \max_{|z|=k \geq 1} \left| q\left(\frac{z}{k}\right) \right| &\leq \left(\frac{k^n + 1}{2} \right) \max_{|z|=1} \left| q\left(\frac{z}{k}\right) \right| - \left(\frac{k^n - 1}{2} \right) \min_{|z|=1} \left| q\left(\frac{z}{k}\right) \right| \\ &\quad - \frac{2|a_{n-1}|}{(n+1)k} \left[\frac{k^n - 1}{n} - (k-1) \right] \\ &\quad - \frac{2|a_{n-2}|}{k^2} \left[\left(\frac{(k^n - 1) - n(k-1)}{n(n-1)} \right) - \left(\frac{(k^{n-2} - 1) - (n-2)(k-1)}{(n-2)(n-3)} \right) \right]. \end{aligned} \quad (3.5)$$

Since $\max_{|z|=1} |q(z/k)| = (1/k^n) \max_{|z|=k} |p(z)|$ (and similarly for the minima), (3.5) is equivalent to

$$\begin{aligned} \max_{|z|=k} |p(z)| &\geq \left(\frac{2k^n}{k^n + 1} \right) \max_{|z|=1} |p(z)| + \left(\frac{k^n - 1}{k^n + 1} \right) m \\ &\quad + \frac{4k^{n-1}|a_{n-1}|}{(k^n + 1)(n+1)} \left[\frac{k^n - 1}{n} - (k-1) \right] \\ &\quad + \frac{4k^{n-2}|a_{n-2}|}{k^n + 1} \left[\left(\frac{(k^n - 1) - n(k-1)}{n(n-1)} \right) - \left(\frac{(k^{n-2} - 1) - (n-2)(k-1)}{(n-2)(n-3)} \right) \right]. \end{aligned} \quad (3.6)$$

Combining (3.4) and (3.6) we get the desired result. This completes the proof of inequality (1.12). The proof of the Theorem in the case $n = 3$ follows along the same lines as the proof of (1.12) but instead of inequalities (2.3) and (2.5), we use inequalities (2.4) and (2.6), respectively. \square

Proof of Theorem 1.3. By hypothesis that the polynomial $p(z) = a_0 + \sum_{\nu=\mu}^n a_\nu z^\nu$, $1 \leq \mu \leq n$, has no zero in $|z| < k$, where $k \geq 1$, therefore the polynomial $F(z) = p(Rz)$ has no zero in $|z| \leq k/R$, where $k/R \geq 1$. Since $|\alpha/R| \geq 1$, using Theorem B we have

$$\max_{|z|=1} |D_{\alpha/R}[F(z)]| = \max_{|z|=R} |D_\alpha[p(z)]| \leq \frac{n}{1+s'_0} \left\{ \left(\frac{|\alpha|}{R} + s'_0 \right) \max_{|z|=R} |p(z)| - \left(\frac{|\alpha|}{R} - 1 \right) m \right\}, \quad (3.7)$$

where $m = \min_{|z|=k/R} |F(z)| = \min_{|z|=k} |p(z)|$ and

$$s'_0 = \left(\frac{k}{R}\right)^{\mu+1} \left\{ \frac{(\mu/n)(|a_\mu| R k^{\mu-1}/(|a_0| - m)) + 1}{(\mu/n)(|a_\mu| k^{\mu+1}/(|a_0| - m)R) + 1} \right\}. \quad (3.8)$$

Using Lemma 2.5 in the previous inequality, we get

$$\begin{aligned} & \max_{|z|=R} |D_\alpha p(z)| \\ & \leq \frac{n}{1+s'_0} \left[\left(\frac{|\alpha|}{R} + s'_0 \right) \exp \left\{ n \int_r^R \frac{(\mu/n)(|a_\mu|/(|a_0| - m)) k^{\mu+1} t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + (\mu/n)(|a_\mu|/(|a_0| - m))(k^{\mu+1} t^\mu + k^{2\mu} t)} dt \right\} \max_{|z|=r} |p(z)| \right. \\ & \quad \left. + \left(s'_0 + 1 - \left(\frac{|\alpha|}{R} + s'_0 \right) \right) \right. \\ & \quad \left. \times \exp \left\{ n \int_r^R \frac{(\mu/n)(|a_\mu|/(|a_0| - m)) k^{\mu+1} t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + (\mu/n)(|a_\mu|/(|a_0| - m))(k^{\mu+1} t^\mu + k^{2\mu} t)} dt \right\} \right] m. \end{aligned} \quad (3.9)$$

This completes the proof of the theorem. \square

References

- [1] P. D. Lax, "Proof of a conjecture of P. Erdős on the derivative of a polynomial," *Bulletin of the American Mathematical Society*, vol. 50, pp. 509–513, 1944.
- [2] P. Turán, "Über die Ableitung von polynomen," *Compositio Mathematica*, vol. 7, pp. 89–95, 1939.
- [3] M. A. Malik, "On the derivative of a polynomial," *Journal of the London Mathematical Society*, vol. 1, pp. 57–60, 1969.
- [4] N. K. Govil, "On the derivative of a polynomial," *Proceedings of the American Mathematical Society*, vol. 41, pp. 543–546, 1973.
- [5] R. B. Gardner, N. K. Govil, and A. Weems, "Some results concerning rate of growth of polynomials," *East Journal on Approximations*, vol. 10, no. 3, pp. 301–312, 2004.
- [6] A. Aziz and N. A. Rather, "A refinement of a theorem of Paul Turán concerning polynomials," *Mathematical Inequalities & Applications*, vol. 1, no. 2, pp. 231–238, 1998.
- [7] K. K. Dewan and C. M. Upadhye, "Inequalities for the polar derivative of a polynomial," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 9, no. 4, article 119, 9 pages, 2008.
- [8] K. K. Dewan, N. Singh, and A. Mir, "Extensions of some polynomial inequalities to the polar derivative," *Journal of Mathematical Analysis and Applications*, vol. 352, no. 2, pp. 807–815, 2009.
- [9] A. Aziz and Q. M. Dawood, "Inequalities for a polynomial and its derivative," *Journal of Approximation Theory*, vol. 54, no. 3, pp. 306–313, 1988.
- [10] B. Chanam and K. K. Dewan, "Inequalities for a polynomial and its derivative," *Journal of Mathematical Analysis and Applications*, vol. 336, no. 1, pp. 171–179, 2007.
- [11] A. Aziz, "Inequalities for the derivative of a polynomial," *Proceedings of the American Mathematical Society*, vol. 89, no. 2, pp. 259–266, 1983.
- [12] K. K. Dewan, J. Kaur, and A. Mir, "Inequalities for the derivative of a polynomial," *Journal of Mathematical Analysis and Applications*, vol. 269, no. 2, pp. 489–499, 2002.
- [13] K. K. Dewan, N. Singh, and A. Mir, "Growth of polynomials not vanishing inside a circle," *International Journal of Mathematical Analysis*, vol. 1, no. 9–12, pp. 529–538, 2007.