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Entanglement entropy of excited states in conformal perturbation theory and the Einstein equation

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ABSTRACT: For a conformal field theory (CFT) deformed by a relevant operator, the entanglement entropy of a ball-shaped region may be computed as a perturbative expansion in the coupling. A similar perturbative expansion exists for excited states near the vacuum. Using these expansions, this work investigates the behavior of excited state entanglement entropies of small, ball-shaped regions. The motivation for these calculations is Jacobson's recent work on the equivalence of the Einstein equation and the hypothesis of maximal vacuum entropy [[arXiv:1505.04753](https://arxiv.org/abs/1505.04753)], which relies on a conjecture stating that the behavior of these entropies is sufficiently similar to a CFT. In addition to the expected type of terms which scale with the ball radius as R^d , the entanglement entropy calculation gives rise to terms scaling as $R^{2\Delta}$, where Δ is the dimension of the deforming operator. When $\Delta \leq \frac{d}{2}$, the latter terms dominate the former, and suggest that a modification to the conjecture is needed.

KEYWORDS: Conformal and W Symmetry, Models of Quantum Gravity, Gauge-gravity correspondence

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1 Introduction

Entanglement entropy is a quantity with many profound and surprising connections to spacetime geometry, and is suspected to play an important role in a complete description of quantum gravity. It has featured prominently explanations of the origin of black hole entropy [1–7], stemming from the similarity between the area law for the Bekenstein-Hawking entropy and the area law for entanglement entropy. In holographic theories, the entanglement entropy of the CFT is intimately related to the bulk geometry by virtue of the Ryu-Takayanagi (RT) formula [8, 9] and its covariant generalization [10], which state that the entropy is dual to the area of an extremal surface in the bulk. These connections motivate the compelling idea that spacetime geometry and its dynamics may emerge from the entanglement structure of quantum fields. This “geometry from entanglement”

program has recently found a concrete realization in holography, where the bulk linearized Einstein equations were shown to follow from the RT formula [11–13].

Another recent development is a proposal by Jacobson [14], which builds upon his original derivation of the Einstein equation as a thermodynamic equation of state [15]. In this new work, he postulates that the local quantum gravity vacuum is an equilibrium state, in the sense that it is a state of maximal entanglement entropy. It is then demonstrated that this hypothesis is equivalent to the Einstein equation. Entanglement entropy is the key object relating the geometrical quantities on the one hand to the stress-energy of matter fields on the other. The connection between entanglement entropy and geometry stems from the area law, which arises since the entropy is dominated by modes near the entangling surface [6]. On the other hand, the entanglement entropy relates to matter stress-energy through the modular Hamiltonian, which, for a ball-shaped region in a CFT vacuum, is constructed from the stress-energy tensor.

The ability to express the modular Hamiltonian of a ball in terms of a simple integral of the stress tensor is special to a CFT. Extending the argument for the equivalence between Einstein’s equations and maximal vacuum entanglement to non-conformal fields requires taking the ball to be much smaller than any length scale appearing in the field theory. Since the theory will flow to an ultraviolet (UV) fixed point at short length scales, one expects to recover CFT behavior in this limit. Jacobson made a conjecture about the form of the entanglement entropy for excited states in small spherical regions that allowed the argument to go through. The purpose of the present paper is to check this conjecture using conformal perturbation theory (see also [16] for alternative ideas for checking the conjecture).

In this work, we will consider a CFT deformed by a relevant operator \mathcal{O} of dimension Δ , and examine the entanglement entropy for a class of excited states formed by a path integral over Euclidean space. The entanglement entropy in this case may be evaluated using recently developed perturbative techniques [17–22] which express the entropy in terms of correlation functions, and notably do not rely on the replica trick [23, 24]. In particular, one knows from the expansion in [17, 19] that the first correction to the CFT entanglement entropy comes from the $\mathcal{O}\mathcal{O}$ two-point function and the $K\mathcal{O}\mathcal{O}$ three point function, where K is the CFT vacuum modular Hamiltonian. However, those works did not account for the noncommutativity of the density matrix perturbation $\delta\rho$ with the original density matrix ρ_0 , so the results cannot be directly applied to find the finite change in entanglement entropy between the perturbed theory excited state and the CFT ground state.¹ Instead, we will apply the technique developed by Faulkner [21] to compute these finite changes to the entanglement entropy, which we review in section 2.2. The result for the change in entanglement entropy between the excited state and vacuum is

$$\delta S = \frac{2\pi\Omega_{d-2}}{d^2-1} \left[R^d \left(\delta\langle T_{00}^g \rangle - \frac{1}{2\Delta-d} \delta\langle T^g \rangle \right) - R^{2\Delta} \langle \mathcal{O} \rangle_g \delta\langle \mathcal{O} \rangle \frac{\Delta\Gamma(\frac{d}{2} + \frac{3}{2})\Gamma(\Delta - \frac{d}{2} + 1)}{(2\Delta-d)^2\Gamma(\Delta + \frac{3}{2})} \right], \tag{1.1}$$

¹However, references [19, 20] are able to reproduce universal logarithmic divergences when they are present.

which holds to first order in the variation of the state and for $\Delta \neq \frac{d}{2}$. Here, $\Omega_{d-2} = \frac{2\pi^{\frac{d}{2}-\frac{1}{2}}}{\Gamma(\frac{d}{2}-\frac{1}{2})}$ is the volume of the unit $(d-2)$ -sphere, R is the radius of the ball, $T_{\mu\nu}^g$ is the stress tensor of the deformed theory with trace T^g , $\langle \mathcal{O} \rangle_g$ stands for the vacuum expectation value of \mathcal{O} , and the δ refers to the change in each quantity relative to the vacuum value.

The case $\Delta = \frac{d}{2}$ requires special attention, since the above expression degenerates at that value of Δ . The result for $\Delta = \frac{d}{2}$ is

$$\delta S = 2\pi \frac{\Omega_{d-2}}{d^2-1} R^d \left[\delta \langle T_{00}^g \rangle + \delta \langle T^g \rangle \left(\frac{2}{d} - \frac{1}{2} H_{\frac{d+1}{2}} + \log \frac{\mu R}{2} \right) - \frac{d}{2} \langle \mathcal{O} \rangle_g \delta \langle \mathcal{O} \rangle \right], \quad (1.2)$$

where $H_{\frac{d+1}{2}}$ is a harmonic number, defined for the integers by $H_n = \sum_{k=1}^n \frac{1}{k}$ and for arbitrary values of n by $H_n = \gamma_E + \psi_0(n+1)$ with γ_E the Euler-Mascheroni constant, and $\psi_0(x) = \frac{d}{dx} \log \Gamma(x)$ the digamma function. This result depends on a renormalization scale μ which arises due to an ambiguity in defining a renormalized value for the vev $\langle \mathcal{O} \rangle_g$. The above result only superficially depends on μ , since this dependence cancels between the $\log \frac{\mu R}{2}$ and $\langle \mathcal{O} \rangle_g$ terms. These results agree with recent holographic calculations [25], and this work therefore establishes that those results extend beyond holography.

In both equations (1.1) and (1.2), the first terms scaling as R^d take the form required for Jacobson’s argument. However, when $\Delta \leq \frac{d}{2}$, the terms scaling as $R^{2\Delta}$ or $R^d \log R$ dominate over this term in the small R limit. This leads to some tension with the argument for the equivalence of the Einstein equation and the hypothesis of maximal vacuum entanglement. We revisit this point in section 5.1 and suggest some possible resolutions to this issue.

Before presenting the calculations leading to equations (1.1) and (1.2), we briefly review Jacobson’s argument in section 2.1, where we describe in more detail the form of the variation of the entanglement entropy that would be needed for the derivation of the Einstein equation to go through. We also provide a review of Faulkner’s method for calculating entanglement entropy in section 2.2, since it will be used heavily in the sequel. Section 3 describes the type of excited states considered in this paper, including an important discussion of the issue of UV divergences in operator expectation values. Following this, we present the derivation of the above result to first order in $\delta \langle \mathcal{O} \rangle$ in section 4. Finally, we discuss the implications of these results for the Einstein equation derivation and avenues for further research in section 5.

2 Background

2.1 Einstein equation from entanglement equilibrium

This section provides a brief overview of Jacobson’s argument for the equivalence of the Einstein equation and the maximal vacuum entanglement hypothesis [14]. The hypothesis states that the entropy of a small geodesic ball is maximal in a vacuum configuration of quantum fields coupled to gravity, i.e. the vacuum is an equilibrium state. This implies that as the state is varied at fixed volume away from vacuum, the change in the entropy must be zero at first order in the variation. In order for this to be possible, the entropy increase

of the matter fields must be compensated by an entropy decrease due to the variation of the geometry. Demanding that these two contributions to the entanglement entropy cancel leads directly to the Einstein equation.

Consider the simultaneous variations of the metric and the state of the quantum fields, $(\delta g_{ab}, \delta \rho)$. The metric variation induces a change δA in the surface area of the geodesic ball, relative to the surface area of a ball with the same volume in the unperturbed metric. Due to the area law, this leads to a proportional change δS_{UV} in the entanglement entropy

$$\delta S_{UV} = \eta \delta A. \tag{2.1}$$

Normally, the constant η is divergent and regularization dependent; however, one further assumes that quantum gravitational effects render it finite and universal. For small enough balls, the area variation is expressible in terms of the 00-component of the Einstein tensor at the center of the ball. Allowing for the background geometry from which the variation is taken to be any maximally symmetric space, with Einstein tensor $G_{ab}^{MSS} = -\Lambda g_{ab}$, (2.1) becomes [14]

$$\delta S_{UV} = -\eta \frac{\Omega_{d-2} R^d}{d^2 - 1} (G_{00} + \Lambda g_{00}). \tag{2.2}$$

The variation of the quantum state produces the compensating contribution to the entropy. At first order in $\delta \rho$, this is given by the change in the modular Hamiltonian K ,

$$\delta S_{IR} = 2\pi \delta \langle K \rangle, \tag{2.3}$$

where K is related to ρ_0 , the reduced density matrix of the vacuum restricted to the ball, via

$$\rho_0 = e^{-2\pi K} / Z, \tag{2.4}$$

with the partition function Z providing the normalization. Generically, K is a complicated, nonlocal operator; however, in the case of a ball-shaped region of a CFT, it is given by a simple integral of the energy density over the ball [26, 27],

$$K = \int_{\Sigma} d\Sigma^a \zeta^b T_{ab} = \int_{\Sigma} d\Omega_{d-2} dr r^{d-2} \left(\frac{R^2 - r^2}{2R} \right) T_{00}. \tag{2.5}$$

In this equation, ζ^a is the conformal Killing vector in Minkowski space² that fixes the boundary $\partial\Sigma$ of the ball. With the standard Minkowski time $t = x^0$ and spatial radial coordinate r , it is given by

$$\zeta = \left(\frac{R^2 - r^2 - t^2}{2R} \right) \partial_t - \frac{rt}{R} \partial_r. \tag{2.6}$$

If R is taken small enough such that $\langle T_{00} \rangle$ is approximately constant throughout the ball, equation (2.3) becomes

$$\delta S_{IR} = 2\pi \frac{\Omega_{d-2} R^d}{d^2 - 1} \delta \langle T_{00} \rangle. \tag{2.7}$$

²The conformal Killing vector is different for a general maximally symmetric space [25]. However, the Minkowski space vector is sufficient as long as $R^2 \ll \Lambda^{-1}$.

The assumption of vacuum equilibrium states that $\delta S_{\text{tot}} = \delta S_{\text{UV}} + \delta S_{\text{IR}} = 0$, and this requirement, along with the expressions (2.2) and (2.7), leads to the relation

$$G_{00} + \Lambda g_{00} = \frac{2\pi}{\eta} \delta \langle T_{00} \rangle, \tag{2.8}$$

which is recognizable as a component of the Einstein equation with $G_N = \frac{1}{4\eta}$. Requiring that this hold for all Lorentz frames and at each spacetime point leads to the full tensorial equation, and conservation of T_{ab} and the Bianchi identity imply that $\Lambda(x)$ is a constant.

The expression of δS_{IR} in (2.7) is special to a CFT, and cannot be expected to hold for more general field theories. However, it is enough if, in the small R limit, it takes the following form

$$\delta S_{\text{IR}} = 2\pi \frac{\Omega_{d-2} R^d}{d^2 - 1} (\delta \langle T_{00} \rangle + C g_{00}). \tag{2.9}$$

Here, C is some scalar function of spacetime, formed from expectation values of operators in the quantum theory. With this form of δS_{IR} , the requirement that δS_{tot} vanish in all Lorentz frames and at all points now leads to the tensor equation

$$G_{ab} + \Lambda g_{ab} = \frac{2\pi}{\eta} (\delta \langle T_{ab} \rangle + C g_{ab}). \tag{2.10}$$

Stress tensor conservation and the Bianchi identity now impose that $\frac{2\pi}{\eta} C(x) = \Lambda(x) + \Lambda_0$, and once again the Einstein equation with a cosmological constant is recovered.

The purpose of the present paper is to evaluate δS_{IR} appearing in equation (2.9) in a CFT deformed by a relevant operator of dimension Δ . It is crucial in the above derivation that C transform as a scalar under a change of Lorentz frame. As long as this requirement is met, complicated dependence on the state or operators in the theory is allowed. In the simplest case, C would be given by the variation of some scalar operator expectation value, $C = \delta \langle X \rangle$, with X independent of the quantum state, since such an object has trivial transformation properties under Lorentz boosts. We find this to be the case for the first order state variations we considered; however, the operator X has the peculiar feature that it depends explicitly on the radius of the ball. The constant C is found to have a term scaling with the ball size as $R^{2\Delta-d}$ (or $\log R$ when $\Delta = \frac{d}{2}$), and when $\Delta \leq \frac{d}{2}$, this term dominates over the stress tensor term as $R \rightarrow 0$. Furthermore, as pointed out in [25], even in the CFT where the first order variation of the entanglement entropy vanishes, the second order piece contains the same type of term scaling as $R^{2\Delta-d}$, which again dominates for small R . This leads to the conclusion that the local curvature scale $\Lambda(x)$ must be allowed to depend on R . This proposed resolution will be discussed further in section 5.1.

2.2 Entanglement entropy of balls in conformal perturbation theory

Checking the conjecture (2.9) requires a method for calculating the entanglement entropy of balls in a non-conformal theory. Faulkner has recently shown how to perform this calculation in a CFT deformed by a relevant operator, $\int f(x) \mathcal{O}(x)$ [21]. This deformation may be split into two parts, $f(x) = g(x) + \lambda(x)$, where the coupling $g(x)$ represents the deformation of the theory away from a CFT, while the function $\lambda(x)$ produces a variation

of the state away from vacuum. The change in entanglement relative to the CFT vacuum will then organize into a double expansion in g and λ ,

$$\delta S = S_g + S_\lambda + S_{g^2} + S_{g\lambda} + S_{\lambda^2} + \dots \tag{2.11}$$

The terms in this expansion that are $O(\lambda^1)$ and any order in g are the ones relevant for δS_{IR} in equation (2.9). Terms that are $O(\lambda^0)$ are part of the vacuum entanglement entropy of the deformed theory, and hence are not of interest for the present analysis. Higher order in λ terms may also be relevant, especially in the case that the $O(\lambda^1)$ piece vanishes, which occurs, for example, in a CFT.

We begin with the Euclidean path integral representations of the reduced density matrices in the ball Σ for the CFT vacuum ρ_0 and for the deformed theory excited state $\rho = \rho_0 + \delta\rho$. The matrix elements of the vacuum density matrix are

$$\langle \phi_- | \rho_0 | \phi_+ \rangle = \frac{1}{Z} \int_{\substack{\phi(\Sigma_+) = \phi_+ \\ \phi(\Sigma_-) = \phi_-}} \mathcal{D}\phi e^{-I_0}. \tag{2.12}$$

Here, the integral is over all fields satisfying the boundary conditions $\phi = \phi_+$ on one side of the surface Σ , and $\phi = \phi_-$ on the other side. The partition function Z is represented by an unconstrained path integral,

$$Z = \int \mathcal{D}\phi e^{-I_0}. \tag{2.13}$$

It is useful to think of the path integral (2.12) as evolution along an angular variable θ from the Σ_+ surface at $\theta = 0$ to the Σ_- surface at $\theta = 2\pi$ [28–30]. When this evolution follows the flow of the conformal Killing vector (2.6) (analytically continued to Euclidean space), it is generated by the conserved Hamiltonian K from equation (2.5). This leads to the operator expression for ρ_0 given in equation (2.4).

The path integral representation for ρ is given in a similar manner,

$$\langle \phi_- | \rho | \phi_+ \rangle = \frac{1}{N} \int_{\substack{\phi(\Sigma_+) = \phi_+ \\ \phi(\Sigma_-) = \phi_-}} \mathcal{D}\phi e^{-I_0 - \int f\mathcal{O}} \tag{2.14}$$

$$= \frac{1}{Z + \delta Z} \int_{\substack{\phi(\Sigma_+) = \phi_+ \\ \phi(\Sigma_-) = \phi_-}} \mathcal{D}\phi e^{-I_0} \left(1 - \int f\mathcal{O} + \frac{1}{2} \iint f\mathcal{O}f\mathcal{O} - \dots \right). \tag{2.15}$$

Again viewing this path integral as an evolution from Σ_+ to Σ_- , with evolution operator $\rho_0 = e^{-2\pi K}/Z$, we can extract the operator expression of $\delta\rho = \rho - \rho_0$,

$$\delta\rho = -\rho_0 \int f\mathcal{O} + \frac{1}{2}\rho_0 \iint T\{f\mathcal{O}f\mathcal{O}\} - \dots - \text{traces}, \tag{2.16}$$

where $T\{\}$ denotes angular ordering in θ . The “-traces” terms in this expression arise from δZ in (2.15). These terms ensure that ρ is normalized, or equivalently

$$\text{Tr}(\delta\rho) = 0. \tag{2.17}$$

We suppress writing these terms explicitly since they will play no role in the remainder of this work.

Using these expressions for ρ_0 and $\delta\rho$, we can now develop the perturbative expansion of the entanglement entropy,

$$S = -\text{Tr} \rho \log \rho. \tag{2.18}$$

It is useful when expanding out the logarithm to write this in terms of the resolvent integral,³

$$S = \int_0^\infty d\beta \left[\text{Tr} \left(\frac{\rho}{\rho + \beta} \right) - \frac{1}{1 + \beta} \right] \tag{2.19}$$

$$= S_0 + \text{Tr} \int_0^\infty d\beta \frac{\beta}{\rho_0 + \beta} \left[\delta\rho \frac{1}{\rho_0 + \beta} - \delta\rho \frac{1}{\rho_0 + \beta} \delta\rho \frac{1}{\rho_0 + \beta} + \dots \right]. \tag{2.20}$$

The first order term in $\delta\rho$ is straightforward to evaluate. Using the cyclicity of the trace and equation (2.17), the β integral is readily evaluated, and applying (2.4) one finds

$$\delta S^{(1)} = 2\pi \text{Tr}(\delta\rho K) = 2\pi\delta\langle K \rangle. \tag{2.21}$$

Note when $\delta\rho$ is a first order variation, this is simply the first law of entanglement entropy [32] (see also [33]).

The second order piece of (2.20) is more involved, and much of reference [21] is devoted to evaluating this term. The surprising result is that this term may be written holographically as the flux through an emergent AdS-Rindler horizon of a conserved energy-momentum current for a scalar field⁴ (see figure 1). The bulk scalar field ϕ satisfies the free Klein-Gordon equation in AdS with mass $m^2 = \Delta(\Delta - d)$, as is familiar from the usual holographic dictionary [34]. The specific AdS-Rindler horizon that is used is the one with a bifurcation surface that asymptotes near the boundary to the entangling surface $\partial\Sigma$ in the CFT. This result holds for *any* CFT, including those which are not normally considered holographic.

We now describe the bulk calculation in more detail. Poincaré coordinates are used in the bulk, where the metric takes the form

$$ds^2 = \frac{1}{z^2} (-dt^2 + dz^2 + dr^2 + r^2 d\Omega_{d-2}^2). \tag{2.22}$$

The coordinates (t, r, Ω_i) match onto the Minkowski coordinates of the CFT at the conformal boundary $z = 0$. The conformal Killing vector ζ^a of the CFT, defined in equation (2.6), extends to a Killing vector in the bulk,

$$\xi = \left(\frac{R^2 - t^2 - z^2 - r^2}{2R} \right) \partial_t - \frac{t}{R} (z\partial_z + r\partial_r). \tag{2.23}$$

The Killing horizon \mathcal{H}^+ of ξ^a defines the inner boundary of the AdS-Rindler patch for $t > 0$, and sits at

$$r^2 + z^2 = (R - t)^2. \tag{2.24}$$

³One can also expand the logarithm using the Baker-Campbell-Hausdorff formula, see e.g. [31].

⁴Reference [21] further showed that this is equivalent to the Ryu-Takayanagi prescription for calculating the entanglement entropy [8, 9], using an argument similar to the one employed in [12] deriving the bulk linearized Einstein equation from the Ryu-Takayanagi formula.

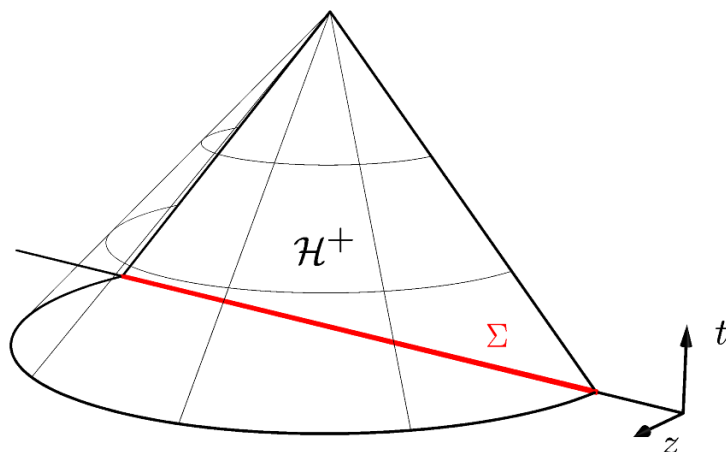


Figure 1. Bulk AdS-Rindler horizon \mathcal{H}^+ . The horizon extends from the bifurcation surface in the bulk at $t = 0$ along the cone to the tip at $z = 0, t = R$. The ball-shaped surface Σ in the boundary CFT shares a boundary with the bifurcation surface at $t = z = 0$.

The contribution of the second order piece of (2.20) to the entanglement entropy is

$$\delta S^{(2)} = -2\pi \int_{\mathcal{H}^+} d\Sigma^a \xi^b T_{ab}^B, \tag{2.25}$$

where the integral is over the horizon to the future of the bifurcation surface at $t = 0$. The surface element on the horizon is $d\Sigma^a = \xi^a d\chi dS$, where χ is a parameter for ξ^a satisfying $\xi^a \nabla_a \chi = 1$, and dS is the area element in the transverse space. T_{ab}^B is the stress tensor of a scalar field ϕ satisfying the Klein-Gordon equation,

$$\nabla_c \nabla^c \phi - \Delta(\Delta - d)\phi = 0. \tag{2.26}$$

Explicitly, the stress tensor is

$$T_{ab}^B = \nabla_a \phi \nabla_b \phi - \frac{1}{2}(\Delta(\Delta - d)\phi^2 + \nabla_c \phi \nabla^c \phi) g_{ab}, \tag{2.27}$$

which may be rewritten when ϕ satisfies the field equation (2.26) as

$$T_{ab}^B = \nabla_a \phi \nabla_b \phi - \frac{1}{4} g_{ab} \nabla_c \nabla^c \phi^2. \tag{2.28}$$

The boundary conditions for ϕ come about from its defining integral,

$$\phi(x_B) = \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}} \Gamma(\Delta - \frac{d}{2})} \int_{C(\delta)} d\tau \int d^{d-1} \vec{x} \frac{z^\Delta f(\tau, \vec{x})}{(z^2 + (\tau - it_B)^2 + (\vec{x} - \vec{x}_B)^2)^\Delta}, \tag{2.29}$$

where $x_B = (t_B, z, \vec{x}_B)$ are the real-time bulk coordinates, and (τ, \vec{x}) are coordinates on the boundary Euclidean section. The normalization of this field arises from a particular choice of the normalization for the $\mathcal{O}\mathcal{O}$ two-point function,

$$\langle \mathcal{O}(x) \mathcal{O}(0) \rangle = \frac{c_\Delta}{x^{2\Delta}}, \quad c_\Delta = \frac{(2\Delta - d)\Gamma(\Delta)}{\pi^{\frac{d}{2}} \Gamma(\Delta - \frac{d}{2})}, \tag{2.30}$$

which is chosen so that the relationship (2.31) holds. Note that sending $c_\Delta \rightarrow \alpha^2 c_\Delta$ multiplies ϕ by a single factor of α . The integrand in (2.29) has branch points at $\tau = i(t_B \pm \sqrt{z^2 + (\vec{x} - \vec{x}_B)^2})$, and the branch cuts extend along the imaginary axis to $\pm i\infty$. The notation $C(\delta)$ on the τ integral refers to the τ contour prescription, which must lie along the real axis and be cut off near 0 at $\tau = \pm\delta$. This can lead to a divergence in δ when the contour is close to the branch point (which can occur when $t_B \sim \sqrt{z^2 + (\vec{x} - \vec{x}_b)^2}$), and this ultimately cancels against a divergence in $\langle T_{00}\mathcal{O}\mathcal{O} \rangle$ from $\delta S^{(1)}$. More details about these divergences and the origin of this contour and branch prescription can be found in [21].

From equation (2.29), one can now read off the boundary conditions as $z \rightarrow 0$. The solution should be regular in the bulk, growing at most like $z^{d-\Delta}$ for large z if $f(\tau, \vec{x})$ is bounded. On the Euclidean section $t_B = 0$, it behaves for $z \rightarrow 0$ as

$$\phi \rightarrow f(0, \vec{x}_B)z^{d-\Delta} + \beta(0, \vec{x}_B)z^\Delta, \tag{2.31}$$

where the function β may be determined by the integral (2.29), but also may be fixed by demanding regularity of the solution in the bulk. This is consistent with the usual holographic dictionary [35, 36], where f corresponds to the coupling, and β is related to $\langle \mathcal{O} \rangle$ by⁵

$$\beta(x) = \frac{-1}{2\Delta - d} \langle \mathcal{O}(x) \rangle. \tag{2.32}$$

This formula follows from defining the renormalized expectation value $\langle \mathcal{O} \rangle$ using a holographically renormalized two-point function,

$$\langle \mathcal{O}(0)\mathcal{O}(x) \rangle^{z,\text{ren.}} = \frac{c_\Delta}{(z^2 + x^2)^\Delta} - (2\Delta - d)z^{d-2\Delta}\delta^d(x). \tag{2.33}$$

The δ function in this formula subtracts off the divergence near $x = 0$.⁶ Using the renormalized two-point function, the expectation value of \mathcal{O} at first order in f is

$$\langle \mathcal{O}(x) \rangle = - \int d^d y f(y) \langle \mathcal{O}(x)\mathcal{O}(y) \rangle^{z,\text{ren.}}, \tag{2.34}$$

and by comparing this formula to (2.29) at small values z and $t_B = 0$, one arrives at equation (2.32).

In real times beyond $t_B > z$, $\phi(x_B)$ has only a z^Δ component near $z = 0$. The integral effectively shuts off the coupling f in real times. This follows from the use of a Euclidean path integral to define the state; other real-time behavior may be achievable using the Schwinger-Keldysh formalism. When $t_B \sim z$, there are divergences associated with switching off the coupling in real times, and these are regulated with the $C(\delta)$ contour prescription.

Returning to the flux equation (2.25), since ξ^a is a Killing vector, this integral defines a conserved quantity, and may be evaluated on any other surface homologous to \mathcal{H}^+ . The

⁵The minus sign appearing here is due to the source in the generating functional being $-\int f\mathcal{O}$ as opposed to $\int f\mathcal{O}$.

⁶Additional subleading divergences are present when $\Delta \geq \frac{d}{2} + 1$, which involve subtractions proportional to derivatives of the δ -function.

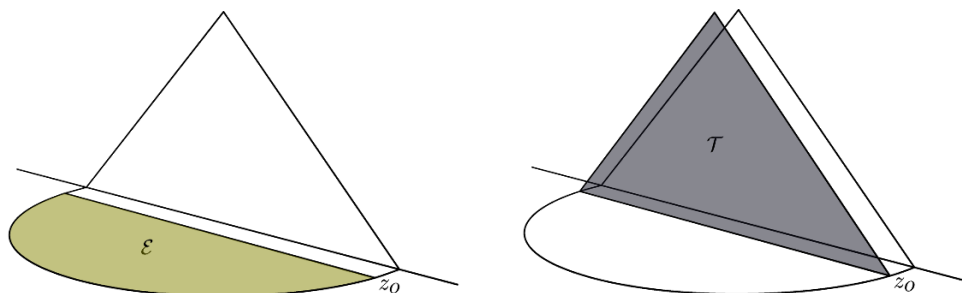


Figure 2. \mathcal{E} and \mathcal{T} surfaces over which the flux integrals (2.35) and (2.36) are computed.

choice which is most tractable is to push the surface down to $t_B = 0$, where the Euclidean AdS solution can be used to evaluate the stress tensor. The $t_B = 0$ surface \mathcal{E} covers the region between the horizon and $z = z_0$, where it must be cut off to avoid a divergence in the integral. To remain homologous to \mathcal{H}^+ , this must be supplemented by a timelike surface \mathcal{T} at the cutoff $z = z_0$ which extends upward to connect back with \mathcal{H}^+ . In the limit $z_0 \rightarrow 0$, the surface \mathcal{T} approaches the domain of dependence $D^+(\Sigma)$ of the ball-shaped region in the CFT (see figure 2). Finally, there will be a contribution from a region along the original surface \mathcal{H}^+ between z_0 and 0, but in the limit $z_0 \rightarrow 0$, the contribution to the integral from this surface will vanish.⁷

Using equation (2.28), the integral on the surface \mathcal{E} can be written out more explicitly:

$$\begin{aligned}
 & -2\pi \int_{\mathcal{E}} d\Sigma^a \xi^b T_{ab}^B \\
 & = 2\pi \int d\Omega_{d-2} \int_{z_0}^R \frac{dz}{z^{d-1}} \int_0^{\sqrt{R^2-z^2}} dr r^{d-2} \left[\frac{R^2 - r^2 - z^2}{2R} \right] \left[(\partial_\tau \phi)^2 - \frac{\nabla_E^2 \phi^2}{4z^2} \right]. \quad (2.35)
 \end{aligned}$$

This formula uses the solution on the Euclidean section in the bulk, with Euclidean time $\tau_B = it_B$. This is acceptable on the $t_B = 0$ surface since the stress tensor there satisfies $T_{\tau\tau}^B = -T_{tt}^B$. The Laplacian ∇_E^2 is hence the Euclidean AdS Laplacian. The \mathcal{T} surface integral is

$$\begin{aligned}
 & 2\pi \int_{\mathcal{T}} d\Sigma^a \xi^b T_{ab}^B \quad (2.36) \\
 & = \frac{2\pi}{z_0^{d-1}} \int d\Omega_{d-2} \int_0^R dt \int_0^{R-t} dr r^{d-2} \left\{ \left[\frac{R^2 - r^2 - t^2}{2R} \right] \partial_z \phi \partial_t \phi - \frac{z_0 t}{R} \left[(\partial_z \phi)^2 - \frac{\nabla^2 \phi^2}{4z_0^2} \right] \right\}.
 \end{aligned}$$

Here, note that the limits of integration have been set to coincide with $D^+(\Sigma)$, which is acceptable when taking $z_0 \rightarrow 0$.

3 Producing excited states

This section describes the class of states that are formed from the Euclidean path integral prescription, and also discusses restrictions on the source function $f(x)$. One requirement

⁷This piece may become important in the limiting case $\Delta = \frac{d}{2} - 1$, which requires special attention. We will not consider this possibility further here.

is that the density matrix be Hermitian. For a density matrix constructed from a path integral as in (2.14), this translates to the condition that the deformed action $I_0 + \int f\mathcal{O}$ be reflection symmetric about the $\tau = 0$ surface on which the state is evaluated. When this is satisfied, ρ defines a pure state [37]. Since this imposes $f(\tau, \vec{x}) = f(-\tau, \vec{x})$, it gives the useful condition

$$\partial_\tau f(0, \vec{x}) = 0, \quad (3.1)$$

which simplifies the evaluation of the bulk integral (2.35).

Another condition on the state is that the stress tensor T_{ab}^g of the deformed theory and the operator \mathcal{O} have non-divergent expectation values, compared to the vacuum. These divergences are not independent, but are related to each other through Ward identities. The $\langle \mathcal{O} \rangle$ divergence is straightforward to evaluate,

$$\langle \mathcal{O}(0) \rangle = \frac{1}{N} \int \mathcal{D}\phi e^{-I_0} \left(1 - \int f\mathcal{O} + \dots \right) \mathcal{O}(0) \quad (3.2)$$

$$= - \int_{C(\delta)} d^d x f(x) \langle \mathcal{O}(0) \mathcal{O}(x) \rangle_0, \quad (3.3)$$

where the 0 subscript indicates a CFT vacuum correlation function. $C(\delta)$ refers to the regularization of this correlation function, which is a point-splitting cutoff for $|\tau| < \delta$. Note that δ is the same regulator appearing in the definition of the bulk scalar field, equation (2.29).

Only the change $\delta\langle \mathcal{O} \rangle$ in this correlation function relative to the deformed theory vacuum must be free of divergences. From the decomposition $f(x) = g(x) + \lambda(x)$, with $g(x)$ representing the deformation of the theory and $\lambda(x)$ the state deformation, one finds that the divergence in $\delta\langle \mathcal{O} \rangle$ comes from the coincident limit $x \rightarrow 0$. It can be extracted by expanding $\lambda(x)$ around $x = 0$. The leading divergence is then

$$\begin{aligned} \delta\langle \mathcal{O}(0) \rangle_{\text{div}} &= -\lambda(0) \int_{C(\delta)} d\tau \int d\Omega_{d-2} \int_0^\infty dr \frac{r^{d-2} c_\Delta}{(\tau^2 + r^2)^\Delta} \\ &= -\lambda(0) \frac{2\Gamma(\Delta - \frac{d}{2} + \frac{1}{2})}{\sqrt{\pi} \Gamma(\Delta - \frac{d}{2})} \delta^{d-2\Delta}. \end{aligned} \quad (3.4)$$

When $\Delta \geq \frac{d}{2}$, a divergence in $\delta\langle \mathcal{O} \rangle$ exists unless $\lambda(0) = 0$.⁸ Further, this must hold at every point on the $\tau = 0$ surface, which leads to the requirement that $\lambda(0, \vec{x}) = 0$. Additionally, there can be subleading divergences proportional to $\delta^{d-2\Delta+2n} \partial_\tau^{2n} \lambda(0, \vec{x})$ for all integers n where the δ exponent is negative or zero.⁹ Thus, the requirement on λ is that its first $2q$ τ -derivatives should vanish at $\tau = 0$, where

$$q = \left\lfloor \Delta - \frac{d}{2} \right\rfloor. \quad (3.5)$$

⁸When $\Delta = \frac{d}{2}$, after appropriately redefining c_Δ (see equation (4.37)), it becomes a $\log \delta$ divergence.

⁹Divergences proportional to the spatial derivative of λ are not present since the condition from the leading divergence already set these to zero.

We can also check that this condition leads to a finite value expectation value for the stress tensor, which for the deformed theory is

$$T_{ab}^g = \frac{2}{\sqrt{g}} \frac{\delta I}{\delta g^{ab}} = T_{ab}^0 - g \mathcal{O} g_{ab}, \tag{3.6}$$

where T_{ab}^0 is the stress tensor for the CFT. For the $T_{\tau\tau}^0$ component, the expectation value is

$$\langle T_{\tau\tau}^0(0) \rangle = \frac{1}{2} \iint_{C(\delta)} d^d x d^d y f(x) f(y) \langle T_{\tau\tau}^0(0) \mathcal{O}(x) \mathcal{O}(y) \rangle_0. \tag{3.7}$$

The divergence in this correlation function comes from $x, y \rightarrow 0$ simultaneously. It can be evaluated by expanding f around 0, and then employing Ward identities to relate it to the $\mathcal{O}\mathcal{O}$ two-point function (see, e.g. section C.2 of this paper or appendix D of [21]). The first order in λ piece, which gives $\delta \langle T_{\tau\tau}^0 \rangle$, is

$$\delta \langle T_{\tau\tau}^0 \rangle_{\text{div}} = -g \lambda(0) 2^{d-2\Delta} \frac{2\Gamma(\Delta - \frac{d}{2} + \frac{1}{2})}{\sqrt{\pi} \Gamma(\Delta - \frac{d}{2})} \delta^{d-2\Delta}. \tag{3.8}$$

The divergence in the actual energy density also receives a contribution from the \mathcal{O} divergence (3.4). Using (3.6), this is found to be

$$\delta \langle T_{\tau\tau}^g \rangle_{\text{div}} = -g \lambda(0) \frac{2\Gamma(\Delta - \frac{d}{2} + \frac{1}{2})}{\sqrt{\pi} \Gamma(\Delta - \frac{d}{2})} (2^{d-2\Delta} - 1) \delta^{d-2\Delta}. \tag{3.9}$$

As with the $\delta \langle \mathcal{O} \rangle$ divergence, requiring that $\lambda(0, \vec{x}) = 0$ ensures that the excited state has finite energy density.¹⁰ Subleading divergences and other components of T_{ab}^g can be evaluated in a similar way, and lead to the same requirements on λ as were found for the \mathcal{O} divergences.

4 Entanglement entropy calculation

Now we compute the change in entanglement entropy for the state formed by the path integral with the deformed action $I = I_0 + \int f \mathcal{O}$, with $f(x) = g(x) + \lambda(x)$ being a sum of the theory deformation g and the state deformation λ . The bulk term $\delta S^{(2)}$ in plays an important role in this case.¹¹ To evaluate this term, we need the solution for the scalar field in the bulk subject to the boundary conditions described in section 2.2. Since ϕ satisfies a linear field equation, so we may solve separately for the solution corresponding to g and the solution corresponding to λ . The function $g(x)$ is taken to be spatially constant, and either constant in Euclidean time or set to zero at some IR length scale L . Its solution is most readily found by directly evaluating the integral (2.29), and we will discuss it separately in each of the cases $\Delta > \frac{d}{2}$, $\Delta < \frac{d}{2}$ and $\Delta = \frac{d}{2}$ considered below.

¹⁰Curiously, the divergences in T_{ab}^g cancel without imposing $\lambda(0) = 0$ when $\Delta = \frac{d}{2}$.

¹¹A slightly simpler situation would be to consider the deformed action $I = I_0 + \int g \mathcal{O} + \int \lambda \mathcal{O}_s$, with $\Delta \neq \Delta_s$. Then $\delta S^{(2)}$ gives no contribution at first order in λ , since this term arises from the $\mathcal{O}\mathcal{O}_s$ two point function, which vanishes. However, in this case, the term at second order in λ would receive a contribution from $\delta S^{(2)}$, and it is computed in precisely the same way as described in this section. Hence we do not focus on this case where $\Delta \neq \Delta_s$.

The solution for $\lambda(x)$ takes the same form in all three cases, so we begin by describing it. On the Euclidean section in Poincaré coordinates, the field equation (2.26) is

$$\left[z^{d+1} \partial_z (z^{-d+1} \partial_z) + z^2 \left(\partial_\tau^2 + r^{-d+2} \partial_r (r^{d-2} \partial_r) + r^{-2} \nabla_{\Omega_{d-2}}^2 \right) \right] \phi - \Delta(\Delta - d)\phi = 0, \quad (4.1)$$

where $\nabla_{\Omega_{d-2}}^2$ denotes the Laplacian on the $(d-2)$ -sphere. Although one may consider arbitrary spatial dependence for the function $\lambda(x)$, the present calculation is concerned with the small ball limit, where the state may be taken uniform across the ball. We therefore restrict to $\lambda = \lambda(\tau)$. One can straightforwardly generalize to include corrections due to spatial dependence in λ , and these will produce terms suppressed in powers of R^2 .

Equation (4.1) may be solved by separation of variables. The τ dependence is given by $\cos(\omega\tau)$, since it must be τ -reflection symmetric. This leads to the equation for the z -dependence,

$$\partial_z^2 \phi - \frac{d-1}{z} \partial_z \phi - \left(\omega^2 + \frac{\Delta(\Delta-d)}{z^2} \right) \phi = 0. \quad (4.2)$$

This has modified Bessel functions as solutions, and regularity as $z \rightarrow \infty$ selects the solution proportional to $z^{\frac{d}{2}} K_\alpha(\omega z)$, with

$$\alpha = \frac{d}{2} - \Delta. \quad (4.3)$$

Hence, the final bulk solution is

$$\phi_\omega = \lambda_\omega \left(\frac{\omega}{2} \right)^{\Delta - \frac{d}{2}} \frac{2z^{\frac{d}{2}} K_\alpha(\omega z)}{\Gamma(\Delta - \frac{d}{2})} \cos \omega\tau, \quad (4.4)$$

where the normalization has been chosen so that the coefficient of $z^{d-\Delta}$ in the near-boundary expansion is

$$\lambda = \lambda_\omega \cos(\omega\tau). \quad (4.5)$$

A single frequency solution will not satisfy the requirement derived in section 3 that $\lambda(0, \vec{x})$ and its first $2q$ τ -derivatives vanish (where q was given in (3.5)). Instead, λ must be constructed from a wavepacket of several frequencies,

$$\lambda(\tau) = \int_0^\infty d\omega \lambda_\omega \cos(\omega\tau), \quad (4.6)$$

with Fourier components λ_ω satisfying

$$\int_0^\infty d\omega \omega^{2n} \lambda_\omega = 0 \quad (4.7)$$

for all nonnegative integers $n \leq q$. Finally, the coefficients λ_ω should fall off rapidly before ω becomes larger than R^{-1} , since such a state would be considered highly excited relative to the scale set by the ball size.

Using these solutions, we may proceed with the entanglement entropy calculation. The answer for $\Delta > \frac{d}{2}$ in section 4.1 comes from a simple application of the formula derived in [21]. In section 4.2 when considering $\Delta < \frac{d}{2}$, we must introduce a new element into the calculation to deal with IR divergences that arise. This is just a simple IR cutoff

in the theory deformation $g(x)$, which allows a finite answer to emerge, although a new set of divergences along the timelike surface \mathcal{T} must be shown to cancel. A similar story emerges in section 4.3 for $\Delta = \frac{d}{2}$, although extra care must be taken due to the presence of logarithms in the solutions.

4.1 $\Delta > \frac{d}{2}$

The full bulk scalar field separates into two parts,

$$\phi = \phi_0 + \phi_\omega, \tag{4.8}$$

with ϕ_ω from (4.4) describing the state deformation, while ϕ_0 corresponds to the theory deformation $g(x)$. Since no IR divergences arise at this order in perturbation theory when $\Delta > \frac{d}{2}$, we can take g to be constant everywhere. The solution in the bulk on the Euclidean section then takes the simple form

$$\phi_0 = gz^{d-\Delta}. \tag{4.9}$$

Given these two solutions, the bulk contribution to $\delta S^{(2)}$ may be computed using equation (2.35). Note that $\partial_\tau \phi = 0$ on the $\tau = 0$ surface, so we only need the $\nabla^2 \phi^2$ term in the integrand. Before evaluating this term, it is useful to expand ϕ_ω near $z = 0$. This expansion takes the form

$$\phi_\omega = \left[\lambda_\omega z^{d-\Delta} \sum_{n=0}^{\infty} a_n (\omega z)^{2n} + \beta_\omega z^\Delta \sum_{n=0}^{\infty} b_n (\omega z)^{2n} \right] \cos(\omega\tau), \tag{4.10}$$

where

$$\beta_\omega = \lambda_\omega \left(\frac{\omega}{2} \right)^{2\Delta-d} \frac{\Gamma(\frac{d}{2} - \Delta)}{\Gamma(\Delta - \frac{d}{2})}, \tag{4.11}$$

and the coefficients a_n and b_n are given in appendix A. The $O(\lambda^1)$ term in ϕ^2 is $2\phi_0\phi_\omega$, and this modifies the power series (4.10) by changing the leading powers to $z^{2(d-\Delta)}$ and z^d . The Laplacian in the bulk is

$$\nabla^2 = z^2 \partial_\tau^2 + z^{d+1} \partial_z (z^{-d+1} \partial_z). \tag{4.12}$$

Acting on the $\phi_0\phi_\omega$ series, the effect of the τ derivative is to multiply by $-\omega^2 z^2$, which shifts each term to one higher term in the series. The z derivatives do no change the power of z , but rather multiply each term by a constant, $2(d - \Delta + n)(d - 2\Delta + 2n)$ for the a_n series and $2n(d + 2n)$ for the b_n series (note in particular it annihilates the first term in the b_n series). After this is done, the series may be reorganized for $\tau = 0$ as

$$2\nabla^2 \phi_0 \phi_\omega = 2g\lambda_\omega z^{2(d-\Delta)} \sum_{n=0}^{\infty} c_n (\omega z)^{2n} + 2g\beta_\omega z^d \sum_{n=1}^{\infty} d_n (\omega z)^{2n}, \tag{4.13}$$

with the coefficients c_n and d_n computed in appendix A.

From this, we simply need to evaluate the integral (2.35) for each term in the series. For a given term of the form Az^η , the contribution to $\delta S^{(2)}$ is

$$\begin{aligned} \delta S_\eta^{(2)} &= -\frac{\pi}{2} \Omega_{d-2} \int_{z_0}^R \frac{dz}{z^{d+1}} \int_0^{\sqrt{R^2-z^2}} dr r^{d-2} \left[\frac{R^2 - r^2 - z^2}{2R} \right] Az^\eta \quad (4.14) \\ &= -A \frac{\pi \Omega_{d-2}}{4(d^2-1)} \left[R^\eta \frac{\Gamma(\frac{d}{2} + \frac{3}{2}) \Gamma(\frac{\eta}{2} - \frac{d}{2})}{\Gamma(\frac{\eta}{2} + \frac{3}{2})} + \frac{R^d z_0^{\eta-d} {}_2F_1\left(-\frac{d+1}{2}, \frac{\eta-d}{2}; \frac{\eta-d}{2} + 1; \frac{z_0^2}{R^2}\right)}{\frac{\eta}{2} - \frac{d}{2}} \right]. \quad (4.15) \end{aligned}$$

The second term in this expression contains a set of divergences at $z_0 \rightarrow 0$ for all values of $\eta < d$. These arise exclusively from the c_n series in (4.13). In general, the expansion of the hypergeometric function near $z_0 = 0$ can produce subleading divergences, which mix between different terms from the series (4.13). These divergences eventually must cancel against compensating divergences that arise from the \mathcal{T} surface integral in (2.36). Although we do not undertake a systematic study of these divergences, we may assume that they cancel out because the cutoff surface at z_0 was chosen arbitrarily, and the original integral (2.25) made no reference to it. Thus, we may simply discard these z_0 dependent divergences, and are left with only the first term in (4.15).¹²

There is another reason for discarding the z_0 divergences immediately: they only arise in states with divergent energy density. The coefficient of a term with a z_0 divergence is $2gc_n \omega^{2n} \lambda_\omega$. The final answer for the entanglement entropy will involve integrating over all values of ω . But the requirement of finite energy density (4.7) shows that all terms with $n \leq q$, corresponding to $\eta \leq 2d - 2\Delta + 2q$, will vanish from the final result. Given the definition of q in (3.5), these are precisely the terms in (4.15) that have divergences in z_0 . Note that since $\beta_\omega \propto \omega^{2\Delta-d}$, which is generically a non-integer power, the integral over ω will not vanish, so all the β_ω terms survive.

The resulting bulk contribution to the entanglement entropy at order λg is

$$\begin{aligned} \delta S_{\mathcal{E}, \lambda g}^{(2)} &= -\frac{g\pi^{\frac{d}{2}+\frac{1}{2}}}{4} \int_0^\infty d\omega \left[\lambda_\omega R^{2(d-\Delta)} \sum_{n=q+1}^\infty c_n \frac{\Gamma(\frac{d}{2} - \Delta + n)}{\Gamma(d - \Delta + \frac{3}{2} + n)} (\omega R)^{2n} \right. \\ &\quad \left. + \beta_\omega R^d \sum_{n=1}^\infty d_n \frac{\Gamma(\frac{d}{2} + n)}{\Gamma(\frac{d}{2} + \frac{3}{2} + n)} (\omega R)^{2n} \right]. \quad (4.16) \end{aligned}$$

This expression shows that the lowest order pieces scale as $R^{2(d-\Delta+q+1)}$ and R^{d+2} , which both become subleading with respect to the R^d scaling of the $\delta S^{(1)}$ piece for small ball size. Note that a similar technique could extend this result to spatially dependent $\lambda(x)$, and simply would amount to an additional series expansion.

One could perform a similar analysis for the $O(\lambda^2)$ contribution from $\delta S^{(2)}$. The series of $\nabla^2 \phi_\omega \phi_{\omega'}$ would organize into three series, with leading coefficients $\lambda_\omega \lambda_{\omega'} z^{2(d-\Delta)}$, $(\beta_\omega \lambda_{\omega'} + \lambda_\omega \beta_{\omega'}) z^d$, and $\beta_\omega \beta_{\omega'} z^{2\Delta}$. After integrating over ω and ω' , and noting which

¹²When $\eta = d + 2j$ for an integer j , there are subtleties related to the appearance of $\log z_0$ divergences. These cases arise when $\Delta = \frac{d}{2} + m$ with m an integer. We leave analyzing this case for future work.

terms vanish due to the requirement (4.7), one would find the leading contribution going as $\beta^2 R^{2\Delta}$. The precise value of this term is

$$\delta S_{\lambda^2}^{(2)} = -\frac{\pi\Omega_{d-2}}{d^2-1} R^{2\Delta} (\delta\langle\mathcal{O}\rangle)^2 \frac{\Delta\Gamma(\frac{d}{2} + \frac{3}{2})\Gamma(\Delta - \frac{d}{2} + 1)}{(2\Delta - d)^2\Gamma(\Delta + \frac{3}{2})}, \quad (4.17)$$

which is quite similar to the $R^{2\Delta}$ term in equation (1.1). This is again subleading when $\Delta > \frac{d}{2}$, but the same terms show up for $\Delta \leq \frac{d}{2}$ in sections 4.2 and 4.3, where they become the dominant contribution when R is taken small enough. The importance of these second order terms in the small R limit was first noted in [25].

The remaining pieces to calculate come from the integral over \mathcal{T} given by (2.36), and $\delta S^{(1)}$ in (2.21), which just depends on $\delta\langle T_{00}^0 \rangle$. When $\Delta > \frac{d}{2}$, the only contribution from the \mathcal{T} surface integral is near $t_B \sim z \rightarrow 0$. These terms were analyzed in appendix E of [21], and were found to give two types of contributions. The first were counter terms that cancel against the divergences in the bulk as well as the divergence in $\delta S^{(1)}$. Although subleading divergences were not analyzed, these can be expected to cancel in a predictable way. We also already argued that such terms are not relevant for the present analysis, due to the requirement of finite energy density. The second type of term is finite, and takes the form

$$\delta S_{\mathcal{T},\text{finite}}^{(2)} = -2\pi\Delta \int_{\Sigma} \zeta^t g \beta. \quad (4.18)$$

The relation between β and $\delta\langle\mathcal{O}\rangle$ identified in (2.32) implies from equation (4.11),

$$\delta\langle\mathcal{O}\rangle = \lambda_{\omega} \frac{2\Gamma(\frac{d}{2} - \Delta + 1)}{\Gamma(\Delta - \frac{d}{2})} \left(\frac{\omega}{2}\right)^{2\Delta-d}, \quad (4.19)$$

and assuming the ball is small enough so that this expectation value may be considered constant, (4.18) evaluates to

$$\delta S_{\mathcal{T},\text{finite}}^{(2)} = 2\pi \frac{\Omega_{d-2} R^d}{d^2-1} \left[\frac{\Delta}{2\Delta-d} g \delta\langle\mathcal{O}\rangle \right]. \quad (4.20)$$

Similarly, taking $\delta\langle T_{00}^0 \rangle$ to be constant over the ball, the final contribution is the variation of the modular Hamiltonian piece, given by

$$\delta S^{(1)} = 2\pi \int_{\Sigma} \zeta^t \delta\langle T_{00}^0 \rangle = 2\pi \frac{\Omega_{d-2} R^d}{d^2-1} \delta\langle T_{00}^0 \rangle. \quad (4.21)$$

Before writing the final answer, it is useful to write $\delta\langle\mathcal{O}\rangle$ in terms of the trace of the stress tensor of the deformed theory, T^g . The two are related by the dilatation Ward identity, which gives [38]

$$\delta\langle T^g \rangle = (\Delta - d)g\delta\langle\mathcal{O}\rangle. \quad (4.22)$$

Then, using the definition of the deformed theory's stress tensor (3.6) and summing up the contributions (4.16), (4.20), and (4.21), the total variation of the entanglement entropy at $O(\lambda^1 g^1)$ is

$$\delta S_{\lambda g} = 2\pi \frac{\Omega_{d-2} R^d}{d^2-1} \left[\delta\langle T_{00}^g \rangle - \frac{1}{2\Delta-d} \delta\langle T^g \rangle \right] + \delta S_{\mathcal{E},\lambda g}^{(2)}. \quad (4.23)$$

Since $\delta S_{\mathcal{E},\lambda g}^{(2)}$ is subleading, this matches the result (1.1) quoted in the introduction, apart from the $R^{2\Delta}$ term, which is not present because we have arranged for the renormalized vev $\langle \mathcal{O} \rangle_g$ to vanish. However, as noted in equation (4.17), we do find such a term at second order in λ .

4.2 $\Delta < \frac{d}{2}$

Extending the above calculation to $\Delta < \frac{d}{2}$ requires the introduction of one novel element: a modification of the coupling $g(x)$ to include an IR cutoff. It is straightforward to see why this regulator is needed. The perturbative calculation of the entanglement entropy involves integrals of the two point correlator over all of space, schematically of the form

$$\int d^d x g(x) \langle \mathcal{O}(0) \mathcal{O}(x) \rangle_0 = \int d^d x \frac{c_{\Delta} g(x)}{x^{2\Delta}}. \quad (4.24)$$

If this is cut off at a large distance L , the integral scales as $L^{d-2\Delta}$ (or $\log L$ for $\Delta = \frac{d}{2}$) when the coupling $g(x)$ is constant. This clearly diverges for $\Delta \leq \frac{d}{2}$.

The usual story with IR divergences is that resumming the higher order terms remedies the divergence, effectively imposing an IR cut off. Presumably this cut off is set by the scale of the coupling $L_{\text{eff}} \sim g^{\frac{1}{\Delta-d}}$, but since it arises from higher order correlation functions, it may also depend on the details of the underlying CFT. Although it may still be possible to compute these IR effects in perturbation theory [39–41], this goes beyond the techniques employed in the present work. However, if we work on length scales small compared to the IR scale, it is possible to capture the qualitative behavior by simply putting in an IR cut off by hand (see [42] for a related approach). We implement this IR cutoff by setting the coupling $g(x)$ to zero when $|\tau| \geq L$.¹³ We may then express the final answer in terms of the vev $\langle \mathcal{O} \rangle_g$, which implicitly depends on the IR regulator L .

The bulk term $\delta S^{(2)}$ involves a new set of divergences from the \mathcal{T} surface integral that were not present in the original calculation for $\Delta > \frac{d}{2}$ [21]. To compute these divergences and show that they cancel, we will need the real time behavior of the bulk scalar fields, in addition to its behavior at $t = 0$. These are described in appendix B.1. The important features are that ϕ_0 on the $t = 0$ surface takes the form

$$\phi_0 = -\frac{\langle \mathcal{O} \rangle_g}{2\Delta - d} z^{\Delta} + g z^{d-\Delta}, \quad (4.25)$$

and the vev $\langle \mathcal{O} \rangle_g$ is determined in terms of the IR cutoff L by

$$\langle \mathcal{O} \rangle_g = 2g L^{d-2\Delta} \frac{\Gamma(\Delta - \frac{d}{2} + \frac{1}{2})}{\sqrt{\pi} \Gamma(\Delta - \frac{d}{2})}. \quad (4.26)$$

For $t > 0$, the time-dependent solution is given by

$$\phi_0 = -\frac{\langle \mathcal{O} \rangle_g}{2\Delta - d} z^{\Delta} + g z^{d-\Delta} F(t/z), \quad (4.27)$$

¹³This will work only for $\Delta > \frac{d}{2} - \frac{1}{2}$. For lower operator dimensions, a stronger regulator is needed, such as a cutoff in the radial direction, but the only effect this should have is to change the value of $\langle \mathcal{O} \rangle_g$.

where the function F is defined in equation (B.7). To compute the divergences along \mathcal{T} , the form of this function is needed in the region $t \gg z$, where it simply becomes

$$F(t/z) \xrightarrow{t \gg z} B \left(\frac{t}{z} \right)^{d-2\Delta}, \quad (4.28)$$

with the proportionality constant B given in equation (B.8). The field ϕ_ω behaves similarly as long as $\omega^{-1} \gg z, t$. In particular, it has the same form as ϕ_0 in equations (4.25) and (4.27), but with g replaced by λ_ω , and $\langle \mathcal{O} \rangle_g$ replaced with $\delta \langle \mathcal{O} \rangle$, given by

$$\delta \langle \mathcal{O} \rangle = \lambda_\omega \frac{2\Gamma(\frac{d}{2} - \Delta + 1)}{\Gamma(\Delta - \frac{d}{2})} \left(\frac{\omega}{2} \right)^{2\Delta-d}, \quad (4.29)$$

which is the same relation as for $\Delta > \frac{d}{2}$, equation (4.19).

Armed with these solutions, we can proceed to calculate $\delta S^{(2)}$. In this calculation, the contribution from the timelike surface \mathcal{T} now has a novel role. Before, when $\Delta > \frac{d}{2}$, the integral from this surface died off as $z \rightarrow 0$ in the region $t_B > z$, and hence the integral there did not need to be evaluated. For $\Delta < \frac{d}{2}$, rather than dying off, this integral is now leads to divergences as $z \rightarrow 0$. These divergences either cancel among themselves, or cancel against divergences coming from bulk Euclidean surface \mathcal{E} , so that a finite answer is obtained in the end. These new counterterm divergences seem to be related to the alternate quantization in holography [25, 35], which invokes a different set of boundary counterterms when defining the bulk AdS action. It would be interesting to explore this relation further.

At first order in g and λ , three types of terms will appear, proportional to each of $\langle \mathcal{O} \rangle_g \delta \langle \mathcal{O} \rangle$, $(g\delta \langle \mathcal{O} \rangle + \lambda(0)\langle \mathcal{O} \rangle_g)$, or $g\lambda(0)$. Here, we allow $\lambda(0) \neq 0$ because there are no UV divergences arising in the energy density or \mathcal{O} expectation values when $\Delta < \frac{d}{2}$. The descriptions of the contribution from each of these terms are given below, and the details of the surface integrals over \mathcal{E} and \mathcal{T} are contained in appendix C.1.

The $\langle \mathcal{O} \rangle_g \delta \langle \mathcal{O} \rangle$ term has both a finite and a divergent piece coming from the integral over \mathcal{E} (see equation (C.2)). This divergence is canceled by the \mathcal{T} integral in the region $t_B \gg z_0$. This is interesting since it differs from the $\Delta > \frac{d}{2}$ case, where the bulk divergence was canceled by the \mathcal{T} integral in the region $t_B \lesssim z_0$. The final finite contribution from this term is

$$\delta S_{\mathcal{E},1}^{(2)} = -2\pi \langle \mathcal{O} \rangle_g \delta \langle \mathcal{O} \rangle \frac{\Omega_{d-2}}{d^2-1} R^{2\Delta} \frac{\Delta \Gamma(\frac{d}{2} + \frac{3}{2}) \Gamma(\Delta - \frac{d}{2} + 1)}{(2\Delta - d)^2 \Gamma(\Delta + \frac{3}{2})}. \quad (4.30)$$

It is worth noting that we can perform the exact same calculation with $\langle \mathcal{O} \rangle_g \delta \langle \mathcal{O} \rangle$ replaced by $\frac{1}{2} \delta \langle \mathcal{O} \rangle^2$ to compute the second order in λ change in entanglement entropy. The value found in this case agrees with holographic results [25].

The $g\delta \langle \mathcal{O} \rangle + \lambda(0)\langle \mathcal{O} \rangle_g$ term receives no contribution from the \mathcal{E} surface at leading order since this term in ϕ^2 scales as z^d in the bulk, and the z -derivatives in the Laplacian ∇^2 annihilate such a term. The surface \mathcal{T} produces a finite term, plus a collection of divergent terms from both regions $t \sim z$ and $t \gg z$, which cancel among themselves. The finite term is given by

$$\delta S_{\mathcal{T},2}^{(2)} = 2\pi \frac{\Omega_{d-2} R^d \Delta}{(d^2-1)(2\Delta-d)} (g\delta \langle \mathcal{O} \rangle + \lambda(0)\langle \mathcal{O} \rangle_g), \quad (4.31)$$

which is exactly analogous to the term (4.20) found for the case $\Delta > \frac{d}{2}$.

Finally, the term with coefficient $\lambda(0)g$ produces subleading terms, scaling as $R^{2(d-\Delta+n)}$ for positive integers n . Since these terms are subleading, we do not focus on them further. In this case, it must also be shown that the divergences appearing in the \mathcal{T} cancel amongst themselves, since no divergences arise from the \mathcal{E} integral. The calculations in appendix C.1 verify that this indeed occurs.

We are now able to write down the final answer for the change in entanglement entropy for $\Delta < \frac{d}{2}$. The contribution from $\delta S^{(1)}$ is exactly the same as the $\Delta > \frac{d}{2}$ case, and is given by (4.21). Following the same steps that led to equation (4.23), the contributions from the finite piece of $\delta S_{\mathcal{E},1}^{(2)}$ in (C.2) and $\delta S_{\mathcal{T},2}^{(2)}$ in (C.8) combine with $\delta S^{(1)}$ to give

$$\delta S_{\lambda g} = \frac{2\pi\Omega_{d-2}}{d^2-1} \left[R^d \left(\langle T_{00}^g \rangle - \frac{1}{2\Delta-d} \langle T^g \rangle \right) - R^{2\Delta} \langle \mathcal{O} \rangle_g \delta \langle \mathcal{O} \rangle \frac{\Delta \Gamma(\frac{d}{2} + \frac{3}{2}) \Gamma(\Delta - \frac{d}{2} + 1)}{(2\Delta-d)^2 \Gamma(\Delta + \frac{3}{2})} \right], \quad (4.32)$$

where we have set $\lambda(0) = 0$ for simplicity and to match the expression for $\Delta > \frac{d}{2}$, which required $\lambda(0) = 0$.

4.3 $\Delta = \frac{d}{2}$

Similar to the $\Delta < \frac{d}{2}$ case, there are IR divergences that arise when $\Delta = \frac{d}{2}$. These are handled as before with an IR cutoff L , on which the final answer explicitly depends. A new feature arises, however, when expressing the answer in terms of $\langle \mathcal{O} \rangle_g$ rather than L : the appearance of a renormalization scale μ . The need for this renormalization scale can be seen by examining the expression for $\langle \mathcal{O} \rangle_g$, which depends on the $\mathcal{O}\mathcal{O}$ two-point function with $\Delta = \frac{d}{2}$:

$$\langle \mathcal{O} \rangle_g = - \int d^d x \frac{g c'_\Delta}{x^d} = -g c'_\Delta \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \int \frac{d\tau}{\tau}. \quad (4.33)$$

This has a logarithmic divergence near $x = 0$ which must be regulated. The UV-divergent piece can be extracted using the point-splitting cutoff for $|\tau| < \delta$; however, there is an ambiguity in identifying this divergence since the upper bound of this integral cannot be sent to ∞ . The appearance of the renormalization scale is related to matter conformal anomalies that exist for special values of Δ [38, 43, 44]. Thus we must impose an upper cutoff on the integral, which introduces the renormalization scale μ^{-1} . The divergent piece of $\langle \mathcal{O} \rangle_g$ is then

$$\langle \mathcal{O} \rangle_g^{\text{div.}} = g c'_\Delta \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} 2 \log \mu \delta. \quad (4.34)$$

Now we can determine the renormalized vev of \mathcal{O} , using the IR-regulated τ integral,

$$\langle \mathcal{O} \rangle_g^{\text{ren.}} = \langle \mathcal{O} \rangle_g - \langle \mathcal{O} \rangle_g^{\text{div.}} = - \int^L d\tau \int d^{d-1} x \frac{g c'_\Delta}{x^d} - g c'_\Delta \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} 2 \log \mu \delta \quad (4.35)$$

$$= -g c'_\Delta \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} 2 \log \mu L. \quad (4.36)$$

The final answer we derive for the entanglement entropy when $\Delta = \frac{d}{2}$ will depend on $\log L$ but not on explicitly μ or $\langle \mathcal{O} \rangle_g$. Only after rewriting it in terms of $\langle \mathcal{O} \rangle_g^{\text{ren.}}$ does the μ dependence appear.

One other small modification is necessary when $\Delta = \frac{d}{2}$. The normalization c_Δ for the $\mathcal{O}\mathcal{O}$ two point function defined in (2.30) has a double zero at $\Delta = \frac{d}{2}$ which must be removed. This is easily remedied by dividing by $(2\Delta - d)^2$ [35, 45], so that the new constant appearing in the two point function is

$$c'_\Delta = \frac{\Gamma(\Delta)}{2\pi^{\frac{d}{2}}\Gamma(\Delta - \frac{d}{2} + 1)} \xrightarrow{\Delta \rightarrow \frac{d}{2}} \frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}}. \quad (4.37)$$

This change affects the normalization of the bulk field ϕ by dividing by a single factor of $1/(2\Delta - d)$, so that

$$\phi(x_B) = \frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}} \int_{C(\delta)} d\tau \int d^{d-1}\vec{x} \frac{z^\Delta f(\tau, \vec{x})}{(z^2 + (\tau - it_B)^2 + (\vec{x} - \vec{x}_B)^2)^\Delta}. \quad (4.38)$$

These are all the components needed to proceed with the calculation of the entanglement entropy. As before, we solve for the bulk field ϕ_0 associated with a constant coupling g , set to zero for $|\tau| > L$. The ϕ_ω field associated with the state deformation $\lambda = \lambda_\omega \cos \omega\tau$ is again given by a modified Bessel function on the Euclidean section. Its form along the timelike surface \mathcal{T} is derived from the integral representation (4.38), and particular care must be taken in the region $t_B \sim z$, where a divergence in δ appears. Although this divergence is not present if we require $\lambda(0) = 0$, we analyze the terms that it produces for generality. This δ divergence is shown to cancel against a similar divergence in $\delta S^{(1)}$ related to the divergence in the $\langle T_{00}\mathcal{O}\mathcal{O} \rangle$ three-point function.

The full real-time solutions for ϕ_0 and ϕ_ω are given in appendix B.2. The ϕ_0 solution from equation (B.15) takes the form

$$\phi_0 = gz^{\frac{d}{2}} G(t_B/z, \delta/z, L/z), \quad (4.39)$$

with the function G defined in equation (B.16). The dependence of this function on δ is needed only in the region $t_B \sim z$; everywhere else it can safely be taken to zero. On the \mathcal{E} surface where $t_B = 0$, the solution in the limit $L \gg z$ is

$$\phi_0 = gz^{\frac{d}{2}} \log \frac{2L}{z} = -\langle \mathcal{O} \rangle_g^{\text{ren.}} - gz^{\frac{d}{2}} \log \frac{\mu z}{2}, \quad (4.40)$$

where the second equality uses the value of $\langle \mathcal{O} \rangle_g^{\text{ren.}}$ derived in (4.36). We also need ϕ_0 in the region $t_B \gg z$, given by

$$\phi_0 = gz^{\frac{d}{2}} \log \frac{L}{t_B}. \quad (4.41)$$

For ϕ_ω , the solution on the \mathcal{E} surface is still given by a modified Bessel function as in equation (4.4), but must be divided by $(2\Delta - d)$ according to our new normalization (4.38),

$$\phi_\omega = \lambda_\omega z^{\frac{d}{2}} K_0(\omega z) \xrightarrow{z \rightarrow 0} -\lambda_\omega z^{\frac{d}{2}} \left(\gamma_E + \log \frac{\omega z}{2} \right). \quad (4.42)$$

By writing the argument of the log term as in equation (4.40), one can read off the renormalized operator expectation value,

$$\delta\langle\mathcal{O}\rangle^{\text{ren.}} = \lambda_\omega \left(\gamma_E + \log \frac{\omega}{\mu} \right). \quad (4.43)$$

Beyond $t_B = 0$, as long as $\omega^{-1} \gg t_B$, the solution can be written in a similar form as (4.39). When $t_B \gg z$, this is given by

$$\phi_\omega = -\lambda_\omega z^{\frac{d}{2}} (\gamma_E + \log \omega t_B). \quad (4.44)$$

Now that we have the form of the solutions on the surfaces \mathcal{E} and \mathcal{T} , the entanglement calculation contains four parts. The first is the integral over \mathcal{E} , where a $\log z_0$ divergence appears. This cancels against a collection of divergences from the \mathcal{T} surface. The second part is the \mathcal{T} surface near $t_B \sim z$. This region produces more divergences in z_0 and δ , some of which cancel the bulk divergence. The third part is the integral over \mathcal{T} for $t_B \gg z$, which eliminates the remaining z_0 divergences. Finally, an additional divergence from the stress tensor in $\delta S^{(1)}$ cancels the δ divergence, producing a finite answer.

Appendix C.2 describes the details of these calculations. In the end, the contributions from equations (C.16), (C.12), (C.22), (C.32) and (C.41) combine together to give the following total change in entanglement entropy, at $O(\lambda^1 g^1)$,

$$\begin{aligned} \delta S_{\lambda_\omega g} = 2\pi \frac{\Omega_{d-2} R^d}{d^2 - 1} & \left\{ \delta\langle T_{00}^0 \rangle^{\text{ren.}} + g\lambda_\omega \left[\frac{d}{2} \log \left(\frac{2L}{R} \right) \left(\gamma_E + \log \frac{\omega R}{2} \right) \right. \right. \\ & \left. \left. + \frac{d}{4} H_{\frac{d+1}{2}} \left(\gamma_E + \log \frac{R^2 \omega}{4L} \right) - \log \mu R - \frac{1}{8} \left(H_{\frac{d+1}{2}}^{(2)} + H_{\frac{d+1}{2}} (H_{\frac{d+1}{2}} - 2) \right) \right] \right\}. \quad (4.45) \end{aligned}$$

This is the answer for a single frequency ω in the state deformation function $\lambda(x)$. Since $\lambda(0) \neq 0$, this result cannot be immediately interpreted as the entanglement entropy of an excited state, since the state has a divergent expectation value for \mathcal{O} .¹⁴ To get the entanglement entropy for an excited state, we should integrate over all frequencies, and use the fact that $\int d\omega \lambda_\omega = 0$. When this is done, all terms with no $\log \omega$ dependence drop out. Also, we no longer need to specify that operator expectation values are renormalized, since the change in expectation values between two states is finite and scheme-independent.

We would like to express the answer in terms of $\delta\langle\mathcal{O}\rangle$. By integrating equation (4.43) over all frequencies and using that $\lambda(0) = 0$, we find

$$\delta\langle\mathcal{O}\rangle = \int_0^\infty d\omega \lambda_\omega \log \omega. \quad (4.46)$$

With this, the total change in entanglement entropy for nonsingular states coming from integrating (4.45) over all frequencies is

$$\delta S_{\lambda g} = 2\pi \frac{\Omega_{d-2} R^d}{d^2 - 1} \left[\delta\langle T_{00}^0 \rangle + g \frac{d}{2} \delta\langle\mathcal{O}\rangle \left(\frac{1}{2} H_{\frac{d+1}{2}} + \log \frac{2L}{R} \right) \right]. \quad (4.47)$$

¹⁴However, viewing ω as an IR regulator, this equation can be adapted to express the change in *vacuum* entanglement entropy between a CFT and the deformed theory.

This can be expressed in terms of the deformed theory's stress tensor T_{00}^g and trace T^g using equations (3.6) and (4.22),

$$\delta S_{\lambda g} = 2\pi \frac{\Omega_{d-2} R^d}{d^2 - 1} \left[\delta \langle T_{00}^g \rangle + \delta \langle T^g \rangle \left(\frac{2}{d} - \frac{1}{2} H_{\frac{d+1}{2}} + \log \frac{R}{2L} \right) \right]. \quad (4.48)$$

Although the answer is scheme-independent in the sense that μ does not explicitly appear, there is a dependence on the IR cutoff L . This cutoff is related to the renormalized vev $\langle \mathcal{O} \rangle_g^{\text{ren.}}$ via (4.36), which does depend on the renormalization scheme. Thus the dependence on L in the above answer can be traded for $\langle \mathcal{O} \rangle_g^{\text{ren.}}$, at the cost of introducing (spurious) μ -dependence,

$$\delta S_{\lambda g} = 2\pi \frac{\Omega_{d-2} R^d}{d^2 - 1} \left[\delta \langle T_{00}^g \rangle + \delta \langle T^g \rangle \left(\frac{2}{d} - \frac{1}{2} H_{\frac{d+1}{2}} + \log \frac{\mu R}{2} \right) - \frac{d}{2} \langle \mathcal{O} \rangle_g \delta \langle \mathcal{O} \rangle \right], \quad (4.49)$$

which is the result quoted in the introduction, equation (1.2).

5 Discussion

The equivalence between the Einstein equation and maximum vacuum entanglement of small balls relies on a conjecture about the behavior of the entanglement entropy of excited states, equation (2.9). This work has sought to check the conjecture in CFTs deformed by a relevant operator. In doing so, we have derived new results on the behavior of excited state entanglement entropy in such theories, encapsulated by equations (1.1) and (1.2). These results agree with holographic calculations [25] that employ the Ryu-Takayanagi formula. Thus, this work extends those results to *any* CFT, including those which are not thought to have holographic duals.

For deforming operators of dimension $\Delta > \frac{d}{2}$ considered in section 4.1, the calculation is a straightforward application of Faulkner's method for computing entanglement entropies [21]. One subtlety in this case is the presence of UV divergences in $\delta \langle \mathcal{O} \rangle$ and $\delta \langle T_{00}^0 \rangle$ unless the state deformation function $\lambda(x)$ is chosen appropriately. As discussed in section 3, this translates to the condition that λ and sufficiently many of its τ -derivatives vanish on the $\tau = 0$ surface. When the entanglement entropy of the state is calculated, this condition implies that terms scaling with the ball radius as $R^{2(d-\Delta+n)}$, which are present for generic $\lambda(x)$, vanish, where n is a positive integer less than or equal to $\lfloor \Delta - \frac{d}{2} \rfloor$. As R approaches zero, these terms dominate over the energy density term, which scales as R^d . This shows that regularity of the state translates to the dominance of the modular Hamiltonian term in the small ball limit when $\Delta > \frac{d}{2}$. The subleading terms arising from this calculation are given in equation (4.16).

Section 4.2 then extends this result to operators of dimension $\Delta < \frac{d}{2}$. In this case, IR divergences present a novel facet to the calculation. To deal with these divergences, we impose an IR cutoff on the coupling $g(x)$ at scale L . A more complete treatment of the IR divergences would presumably involve resumming higher order contributions, which then would effectively impose an IR cutoff in the lower order terms. This cutoff should be of the order $L_{\text{eff.}} \sim g^{\frac{1}{\Delta-d}}$, but can depend on other details of the CFT, including

any large parameters that might be present. Note this nonanalytic dependence of the IR cutoff on the coupling signals nonperturbative effects are at play [46, 47]. After the IR cutoff is imposed, the calculation of the entanglement entropy proceeds as before. In the final answer, equation (1.1), the explicit dependence on the IR cutoff is traded for the renormalized vacuum expectation value $\langle \mathcal{O} \rangle_g$. This expression agrees with the holographic calculation to first order in $\delta \langle \mathcal{O} \rangle$ in the case that $\langle \mathcal{O} \rangle_g$ is nonzero [25].

Finally, the special case of $\Delta = \frac{d}{2}$ is addressed in section 4.3. Here, both UV and IR divergences arise, and these are dealt with in the same manner as the $\Delta > \frac{d}{2}$ and $\Delta < \frac{d}{2}$ cases. The answer before imposing that the state is nonsingular is given in equation (4.45), and it depends logarithmically on an arbitrary renormalization scale μ . This scale μ arises when renormalizing the stress tensor expectation value $\delta \langle T_{00}^0 \rangle$, as is typical of logarithmic UV divergences. Note that the dependence on μ in the final answer is only superficial, since the combination $\delta \langle T_{00}^0 \rangle^{\text{ren.}} - \log \mu R$ appearing there is independent of the choice of μ . Furthermore, for regular states, $\delta \langle T_{00}^0 \rangle$ is UV finite, and hence the answer may be written without reference to the renormalization scale as in (4.48), although it explicitly depends on the IR cutoff. In some cases, such as free field theories, the appropriate IR cutoff may be calculated exactly [25, 48, 49]. Re-expressing the answer in terms of $\langle \mathcal{O} \rangle_g$ instead of the IR cutoff, as in equation (1.2), re-introduces the renormalization scale μ , since the vev requires renormalization and hence is μ -dependent. Again, this dependence on μ is superficial; it cancels between $\langle \mathcal{O} \rangle_g$ and the $\log \frac{\mu R}{2}$ terms.

5.1 Implications for the Einstein equation

We now ask whether the results (1.1) and (1.2) are consistent with the conjectured form of the entanglement entropy variation (2.9). The answer appears to be yes, with the following caveat: the scalar function C explicitly depends on the ball size R . This comes about from the $R^{2\Delta}$ in equation (1.1), in which case C contains a piece scaling as $R^{2\Delta-d}$, and from the $R^d \log R$ term in (1.2), which gives C a $\log R$ term. When $\Delta \leq \frac{d}{2}$, these terms are the dominant component of the entanglement entropy variation when the ball size is taken to be small.

The question now shifts to whether R -dependence in the function C still allows the derivation of the Einstein equation to go through. As long as $C(R)$ transforms as a scalar under Lorentz boosts for fixed ball size R , the tensor equation (2.10) still follows from the conjectured form of the entanglement entropy variation (2.9) [14]. One then concludes from stress tensor conservation and the Bianchi identity that the curvature scale of the maximally symmetric space characterizing the local vacuum is dependent on the size of the ball, $\Lambda = \Lambda(x, R)$.¹⁵ There does not seem to be an immediate reason disallowing an R -dependent Λ .

There are two requirements on $\Lambda(R)$ for this to be a valid interpretation. First, Λ^{-1} should remain much larger than R^2 in order to justify using the flat space conformal Killing vector (2.6) for the CFT modular Hamiltonian, and also to justify keeping only the first order correction to the area due to curvature in equation (2.2). Since $C(R)$ is dominated

¹⁵This idea was proposed by Ted Jacobson, and I thank him for discussions regarding this point.

by the $R^{2\Delta}$ for $\Delta \leq \frac{d}{2}$ as $R \rightarrow 0$, it determines $\Lambda(R)$ by

$$\Lambda(R) = \frac{2\pi}{\eta} C \sim \ell_P^{d-2} \langle \mathcal{O} \rangle_g \delta \langle \mathcal{O} \rangle R^{2\Delta-d}. \tag{5.1}$$

The the requirement that $\Lambda(R)R^2 \ll 1$ becomes

$$\frac{R}{\ell_P} \ll \left(\frac{1}{\ell_P^{2\Delta} \langle \mathcal{O} \rangle_g \delta \langle \mathcal{O} \rangle} \right)^{\frac{1}{2\Delta-d+2}}. \tag{5.2}$$

Since $2\Delta - d + 2 \geq 0$ by the CFT unitarity bound for scalar operators, this inequality can always be satisfied by choosing R small enough. Furthermore, since $\langle \mathcal{O} \rangle_g \delta \langle \mathcal{O} \rangle$ should be small in Planck units, the right hand side of this inequality is large, and hence can be satisfied for $R \gg \ell_P$. A second requirement is that Λ remain sub-Planckian to justify using a semi-classical vacuum state when discussing the variations. This means $\Lambda(R)\ell_P^2 \ll 1$, which then implies

$$\frac{R}{\ell_P} \gg \left(\ell_P^{2\Delta} \langle \mathcal{O} \rangle_g \delta \langle \mathcal{O} \rangle \right)^{\frac{1}{d-2\Delta}}. \tag{5.3}$$

This now places a lower bound on the size of the ball for which the derivation is valid. However, the R -dependence in $\Lambda(R)$ is only significant when $d - 2\Delta$ is positive, and hence the right hand side of this inequality is small. Thus, there should be a wide range of R values where both (5.2) and (5.3) are satisfied. The implications of such an R -dependent local curvature scale merit further investigation. Perhaps it is related to a renormalization group flow of the cosmological constant [50].

A second, more speculative possibility is that the $R^{2\Delta}$ and $\log R$ terms are resummed due to higher order corrections into something that is subdominant in the $R \rightarrow 0$ limit. One reason for suspecting that this may occur is that the $R^{2\Delta}$ at second order in the state variation can dominate over the lower order R^d terms at small R , possibly hinting at a break down of perturbation theory.¹⁶ As a trivial example, suppose the $R^{2\Delta}$ term arose from a function of the form

$$\frac{R^d}{1 + (R/R_0)^{2\Delta-d}}. \tag{5.4}$$

Since $\Delta < \frac{d}{2}$, this behaves like $R^d - R^{2\Delta} R_0^{d-2\Delta}$ when $R \gg R_0$. However, about $R = 0$, it becomes

$$\frac{R^d}{1 + (R/R_0)^{2\Delta-d}} \xrightarrow{R \rightarrow 0} R_0^d \left(\frac{R}{R_0} \right)^{2(d-\Delta)}, \tag{5.5}$$

which is subleading with respect to a term scaling as R^d . Note however that something must determine the scale R_0 in this argument, and it is difficult to find a scale that is free of nonanalyticities in the coupling or operator expectation values. It would be interesting to analyze whether these sorts of nonperturbative effects play a role in the entanglement entropy calculation.

Finally, one may view the R dependence in Λ as evidence that the relation between maximal vacuum entanglement and the Einstein equation does not hold for some states. In

¹⁶However, reference [25] found that terms at third order in the state variation are subdominant to this term for small values of R .

fact, there is some evidence that the relationship must not hold for some states for which the entanglement entropy is not related to the energy density of the state. A particular example is a coherent state, which has no additional entanglement entropy relative to the vacuum despite possessing energy [51].

5.2 Future work

This work leads to several possibilities for future investigations. First is the question of how the entanglement entropy changes under a change of Lorentz frame. The equivalence between vacuum equilibrium and the Einstein equation rests crucially on the transformation properties of the quantity C appearing in equation (2.9). Only if it transforms as a scalar can it be absorbed in to the local curvature scale $\Lambda(x)$. The calculation in this work was done for a large class of states defined by a Euclidean path integral. For a boosted state, one could simply repeat the calculation using the Euclidean space relative to the boosted frame, and the same form of the answer would result. For states considered here that were stationary on time scales on the order R (since $\omega R \ll 1$), it seems plausible that the states constructed in the boosted Euclidean space contain the boosts of the original states. However, this point should be investigated more thoroughly. Another possibility for checking how the entanglement entropy changes under boosts is to use the techniques developed in [22], which provides perturbative methods for evaluating the change in entanglement entropy under a deformation of the region Σ . In particular, a formula is derived that applies to timelike deformations of the surface, and hence could be used to investigate the behavior under boosts.

Performing the calculation to the next order in perturbation theory would also provide new nontrivial checks on the conjecture, in addition to providing new insights for the general theory of perturbative entanglement entropy calculations. This has been done in holography [25], so it would be interesting to see if the holographic results continue to match for a general CFT. The entanglement entropy at the next order in perturbation theory depends on the $\mathcal{O}\mathcal{O}\mathcal{O}$ three point function [19]. One reason for suspecting that the holographic results will continue to work stems from the universal form of this three point function in CFTs. For scalar operators, it is completely fixed by conformal invariance up to an overall constant. Thus, up to this multiplicative constant, there is nothing in the calculation distinguishing between holographic and non-holographic theories. At higher order, one would eventually expect the holographic calculation to differ from the general case. For example, the four point function has much more freedom, depending on an arbitrary function of two conformally invariant cross-ratios. It is likely that universal statements about the entanglement entropy would be hard to make at that order.

The IR divergences when $\Delta \leq \frac{d}{2}$ were dealt with using an IR cutoff, which captures the qualitative behavior of the answer, but misses out on the precise details of how the coupling suppresses the IR region. It may be possible to improve on this calculation at scales above the IR scale using established techniques for handling IR divergences perturbatively [39–41], or by examining specific cases that are exactly solvable [39, 48, 49]. IR divergences continue to plague the calculations at higher order in perturbation theory. This can be

seen by examining the $\mathcal{O}\mathcal{O}\mathcal{O}$ three point function,

$$\iint d^d x_1 d^d x_2 \langle \mathcal{O}(0) \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle = \iint d^d x_1 d^d x_2 \frac{c}{|x_1|^\Delta |x_2|^\Delta |x_1 - x_2|^\Delta}. \quad (5.6)$$

By writing this in spherical coordinates, performing the angular integrals, and defining $u = \frac{r_2}{r_1}$, this may be written

$$c \Omega_{d-1} \Omega_{d-2} \pi \int_0^\infty du \int_0^\infty dr_1 r_1^{2d-3\Delta-1} u^{d-\Delta-1} (1+u)^{-\Delta} {}_2F_1\left(\frac{1}{2}, \frac{\Delta}{2}; 1; \frac{2u}{(1+u)^2}\right), \quad (5.7)$$

This is clearly seen to diverge in the IR region $r_1 \rightarrow \infty$ when $\Delta \leq \frac{2d}{3}$, so that some operators that produced IR finite results in the two-point function now produce IR divergences.

Finally, one may be interested in extending Jacobson's derivation to include higher order corrections to the Einstein equation. There are two possibilities for pursuing this. First, one may consider higher order in R^2 corrections to the entanglement entropy. On the geometrical side, this involves considering additional terms in the Riemann normal coordinate expansion of the metric about a point. This could also lead to deformations of the entangling surface $\partial\Sigma$, and these effects could be computed perturbatively using the techniques of [17, 19, 20, 22]. Additional corrections would come about in the computation of δS_{IR} from spatial variation of the state across the ball, as well as subleading contributions in the energy of the state. It may be interesting to see whether these expansions can be carried out further to compute the higher curvature corrections to Einstein's equation. Another approach would be to compute the Wald entropy associated with the ball [52–54], with additional corrections added to account for the nonzero extrinsic curvature of the surface [55]. This is the appropriate generalization of the area terms to the entanglement entropy when the gravitational theory contains higher curvature corrections. In this case, care has to be taken in order to determine what is held fixed during the variation.¹⁷ It would be interesting to derive the relationship implied by the maximal vacuum entanglement principle in this case.

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¹⁷I thank Rob Myers for suggesting this approach for incorporating higher curvature terms in the Einstein equation.

A Coefficients for the bulk expansion

This appendix lists the coefficients appearing in section 4.1 for the expansion of ϕ_ω and $\nabla^2\phi_0\phi_\omega$. Given its definition (4.4), the coefficients appearing in the expansion (4.10) follow straightforwardly from known expansions of the modified Bessel functions [56]:

$$a_n = \frac{\Gamma(\frac{d}{2} - \Delta + 1)}{4^n n! \Gamma(\frac{d}{2} - \Delta + n + 1)} \tag{A.1}$$

$$b_n = \frac{\Gamma(\Delta - \frac{d}{2} + 1)}{4^n n! \Gamma(\Delta - \frac{d}{2} + n + 1)}. \tag{A.2}$$

When acting with ∇^2 on the series $\phi_0\phi_\omega$, the τ and z derivatives mix adjacent terms in the series. The relation this gives is

$$c_n = 2(d - \Delta + n)(d - 2\Delta + 2n)a_n - a_{n-1}, \tag{A.3}$$

which, given the properties of the a_n , simplifies to

$$c_n = 2(d - \Delta)(d - 2\Delta + 2n)a_n. \tag{A.4}$$

Similarly, for the d_n series,

$$d_n = 2n(d + 2n)b_n - b_{n-1}, \tag{A.5}$$

which implies

$$d_n = 4n(d - \Delta)b_n. \tag{A.6}$$

B Real-time solutions for $\phi(x)$

B.1 $\Delta < \frac{d}{2}$

This appendix derives the real time behavior of the fields ϕ_0 and ϕ_ω . Starting with ϕ_0 , the coupling $g(x)$ is a constant g for $|\tau|$ less than the IR cutoff L , and zero otherwise. The bulk solution found by evaluating (2.29) is

$$\phi_0 = gz^{d-\Delta} \frac{\Gamma(\Delta - \frac{d}{2} + \frac{1}{2})}{\sqrt{\pi} \Gamma(\Delta - \frac{d}{2})} \left[\int_0^{L/z} dy (1 + (y - it_B/z)^2)^{\frac{d}{2} - \Delta - \frac{1}{2}} + \text{c.c.} \right] \tag{B.1}$$

$$= gz^{d-\Delta} \frac{\Gamma(\Delta - \frac{d}{2} + \frac{1}{2})}{\sqrt{\pi} \Gamma(\Delta - \frac{d}{2})} \left[\frac{L - it_B}{z} {}_2F_1 \left(\frac{1}{2}, \Delta - \frac{d}{2} + \frac{1}{2}; \frac{3}{2}; \frac{-(L - it_B)^2}{z^2} \right) + \frac{it_B}{z} {}_2F_1 \left(\frac{1}{2}, \Delta - \frac{d}{2} + \frac{1}{2}; \frac{3}{2}; \frac{t_B^2}{z^2} \right) + \text{c.c.} \right]. \tag{B.2}$$

Here, notice that no cut off near $y = 0$ was needed, since the $\mathcal{O}\mathcal{O}$ two point function has no UV divergences. However, one still has to be mindful of the branch prescription, which is appropriately handled by adding the complex conjugate as directed in the expressions above (denoted by “c.c.”). When $t_B > z$, the branch in the hypergeometric function along the real axis is dealt with by replacing $t_B \rightarrow t_B + i\delta$, and taking the $\delta \rightarrow 0$ limit.

This solution can be simplified in the two regimes of interest, namely on \mathcal{E} with $t_B = 0$ and on \mathcal{T} in the $z \rightarrow 0$ limit. In the first case, ϕ_0 reduces to

$$\phi_0|_{t_B=0} = gz^{d-\Delta} - z^\Delta \frac{gL^{d-2\Delta}\Gamma(\Delta - \frac{d}{2} + \frac{1}{2})}{\sqrt{\pi}\Gamma(\Delta - \frac{d}{2} + 1)} {}_2F_1\left(\Delta - \frac{d}{2}, \Delta - \frac{d}{2} + \frac{1}{2}; \Delta - \frac{d}{2} + 1; \frac{-z^2}{L^2}\right), \quad (\text{B.3})$$

and since we are assuming $R \ll L$, we only need this in the small z limit,

$$\phi_0 \rightarrow gz^{d-\Delta} - z^\Delta \frac{gL^{d-2\Delta}\Gamma(\Delta - \frac{d}{2} + \frac{1}{2})}{\sqrt{\pi}\Gamma(\Delta - \frac{d}{2} + 1)}. \quad (\text{B.4})$$

From this, one immediately reads off the vev of \mathcal{O} ,

$$\langle \mathcal{O} \rangle_g = 2gL^{d-2\Delta} \frac{\Gamma(\Delta - \frac{d}{2} + \frac{1}{2})}{\sqrt{\pi}\Gamma(\Delta - \frac{d}{2})}. \quad (\text{B.5})$$

The real time behavior near $z \rightarrow 0$ and with $t_B \ll L$ takes the form

$$\phi_0 = -\frac{\langle \mathcal{O} \rangle_g}{2\Delta - d} z^\Delta + gz^{d-\Delta} F(t_B/z), \quad (\text{B.6})$$

with

$$F(s) = \begin{cases} 1 & s < 1 \\ \frac{\sqrt{\pi}(s^2-1)^{\frac{d}{2}-\Delta+\frac{1}{2}}}{s\Gamma(\Delta-\frac{d}{2}+1)\Gamma(\frac{d}{2}-\Delta+\frac{1}{2})} {}_2F_1\left(1, \frac{1}{2}; \Delta - \frac{d}{2} + 1; \frac{1}{s^2}\right) & s > 1 \end{cases}. \quad (\text{B.7})$$

In particular, for large argument, this function behaves as

$$F(s \rightarrow \infty) = Bs^{d-2\Delta}; \quad B = \frac{\sqrt{\pi}}{\Gamma(\Delta - \frac{d}{2} + 1)\Gamma(\frac{d}{2} - \Delta + \frac{1}{2})}. \quad (\text{B.8})$$

We also need the solution for the field corresponding to the state deformation $\lambda(x)$. The oscillatory behavior for the choice (4.5) for this function serves to regulate the IR divergences, and hence no additional IR cutoff is needed. Thus the bulk solution on the Euclidean section (4.4) is still valid. The real time behavior of the solution is given by the following integral,

$$\phi_\omega = \lambda_\omega z^{d-\Delta} \frac{\Gamma(\Delta - \frac{d}{2} + \frac{1}{2})}{\sqrt{\pi}\Gamma(\Delta - \frac{d}{2})} \left[\int_0^\infty dy \cos(\omega zy) (1 + (y - it_B/z)^2)^{\frac{d}{2}-\Delta-\frac{1}{2}} + \text{c.c.} \right]. \quad (\text{B.9})$$

To make further progress on this integral, we note that we only need the solution up to times $t_B \sim R \ll \omega^{-1}$. In this limit, the solution should not be sensitive to the details of the IR regulator. Therefore, the answer should be the same as for ϕ_0 in (B.6), the only difference being the numerical value for the operator expectation value. This behavior can be seen by breaking the integral into two regions, $(0, \frac{a}{z})$ and $(\frac{a}{z}, \infty)$, with $t_B \ll a \ll \omega^{-1}$. In the first region, the cosine can be set to 1 since its argument is small. The resulting integral is identical to (B.1), with L replaced by a . In the second region, the integration

variable y is large compared to 1 and t_B/z , so the integral reduces to

$$\lambda_\omega z^{d-\Delta} \frac{2\Gamma(\Delta - \frac{d}{2} + \frac{1}{2})}{\sqrt{\pi}\Gamma(\Delta - \frac{d}{2})} \int_{a/z}^\infty dy \cos(\omega zy) y^{d-2\Delta-1} \tag{B.10}$$

$$= \lambda_\omega z^\Delta \left(\frac{\omega}{2}\right)^{2\Delta-d} \frac{\Gamma(\frac{d}{2} - \Delta)}{\Gamma(\Delta - \frac{d}{2})} + \lambda_\omega z^{d-\Delta} \left(\frac{a}{z}\right)^{d-2\Delta} \frac{\Gamma(\Delta - \frac{d}{2} + \frac{1}{2})}{\sqrt{\pi}\Gamma(\Delta - \frac{d}{2} + 1)}, \tag{B.11}$$

valid for $a \ll \omega^{-1}$. The second term in this expression cancels against the same term appearing in the first integration region, effectively replacing it with the first term in (B.11). The final answer for the real time behavior of ϕ_ω near $z = 0$ is

$$\phi_\omega = -\frac{\delta\langle\mathcal{O}\rangle}{2\Delta - d} z^\Delta + \lambda_\omega z^{d-\Delta} F(t_B/z), \tag{B.12}$$

where we have identified $\delta\langle\mathcal{O}\rangle$ as

$$\delta\langle\mathcal{O}\rangle = \lambda_\omega \frac{2\Gamma(\frac{d}{2} - \Delta + 1)}{\Gamma(\Delta - \frac{d}{2})} \left(\frac{\omega}{2}\right)^{2\Delta-d}. \tag{B.13}$$

B.2 $\Delta = \frac{d}{2}$

Here we derive the real-time behavior of ϕ_0 and ϕ_ω when $\Delta = \frac{d}{2}$. We begin with ϕ_0 . The integral (4.38) can be evaluated, with τ -cutoffs at δ and L to give

$$\phi_0 = \frac{gz^{\frac{d}{2}}}{2} \left[\int_{\delta/z}^{L/z} dy (1 + (y - it_B/z)^2)^{-\frac{1}{2}} + \text{c.c.} \right] \tag{B.14}$$

$$= gz^{\frac{d}{2}} G(t_B/z, \delta/z, L/z), \tag{B.15}$$

where

$$G(s, \varepsilon, l) = \frac{1}{2} (\sinh^{-1}(l - is) - \sinh^{-1}(\varepsilon - is) + \text{c.c.}). \tag{B.16}$$

The δ -dependence in (B.15) is needed in the region $t_B \sim z$ where it is necessary for regularizing a divergence. Everywhere else the limit $\delta \rightarrow 0$ may be taken. Also, since we will need this solution in the regions where z and t_B are at most on the order of $R \ll L$, we often use the limiting form of this function taking $L \gg z, t_B$. In particular, on the surface \mathcal{E} with $t_B = 0$, it evaluates to

$$\phi_0 \rightarrow gz^{\frac{d}{2}} \log \frac{2L}{z}, \tag{B.17}$$

plus terms suppressed by $\frac{z^2}{L^2}$. It is useful to express this in terms of the renormalized vev for \mathcal{O} calculated in (4.36):

$$\phi_0 \rightarrow -\langle\mathcal{O}\rangle_g^{\text{ren.}} z^{\frac{d}{2}} - gz^{\frac{d}{2}} \log \frac{\mu z}{2}. \tag{B.18}$$

The log term in this expression is what would have resulted if we had cut the integral (B.14) off at μ^{-1} rather than L . Finally, it is also useful to have the form of the function (B.15) along \mathcal{T} , where $t_B \gg z$,

$$\phi_0 \rightarrow gz^{\frac{d}{2}} \log \frac{L}{t_B}. \tag{B.19}$$

At $t_B = 0$, the modified Bessel function solution for ϕ_ω is still valid, and the appropriate normalization is given in equation (4.42). We also need expressions for the behavior of ϕ_ω along the surface \mathcal{T} . When $t_B \ll \omega^{-1}$, the same arguments that led to equation (B.12) for $\Delta < \frac{d}{2}$ can be applied to the defining integral for ϕ_ω to show it takes the form

$$\phi_\omega = \beta_\omega z^{\frac{d}{2}} + \lambda_\omega z^{\frac{d}{2}} G(t_B/z, \delta/z, a/z); \quad \beta_\omega = -\gamma_E - \log \omega a, \quad (\text{B.20})$$

where a is the intermediate scale introduced in the integral, as in equation (B.10), and satisfies $t_B \ll a \ll \omega^{-1}$. Note that this answer does not actually depend on a since it will cancel between the log and G terms, but it is convenient to make this separation when evaluating the \mathcal{T} surface integrals in section C.2. From this, the form of ϕ_ω can be read off for $t_B \gg z$:

$$\phi_\omega \rightarrow -\lambda_\omega z^{\frac{d}{2}} (\gamma_E + \log \omega t_B). \quad (\text{B.21})$$

C Surface integrals

This appendix gives the details of the \mathcal{E} and \mathcal{T} surface integrals for $\Delta < \frac{d}{2}$ (section C.1) and for $\Delta = \frac{d}{2}$ (section C.2).

C.1 $\Delta < \frac{d}{2}$

Each integral in this case will be proportional to one of $\langle \mathcal{O} \rangle_g \delta \langle \mathcal{O} \rangle$, $(g \delta \langle \mathcal{O} \rangle + \lambda(0) \langle \mathcal{O} \rangle_g)$, or $\lambda(0)g$. In each case, we show explicitly that the possibly divergent terms coming from the $z_0 \rightarrow 0$ limit cancel, as they must to give an unambiguous answer.

1. $\langle \mathcal{O} \rangle_g \delta \langle \mathcal{O} \rangle$ term. This term arises from the piece of ϕ_0 and ϕ_ω that goes like $\frac{-z^\Delta}{2\Delta-d}$. In particular, it has no dependence on t_B anywhere. On the surface \mathcal{E} , since $\partial_\tau \phi = 0$, the integrand in (2.35) only depends on $\nabla^2 \phi^2$. Working to leading order in R means only keeping the z derivatives in the Laplacian. The term in this expression with coefficient $\langle \mathcal{O} \rangle_g \delta \langle \mathcal{O} \rangle$ is $\frac{2z^{2\Delta}}{(2\Delta-d)^2}$, and acting with the Laplacian on this gives $\frac{4\Delta z^{2\Delta}}{2\Delta-d}$. Then the \mathcal{E} integral is

$$\delta S_{\mathcal{E},1}^{(2)} = -2\pi \langle \mathcal{O} \rangle_g \delta \langle \mathcal{O} \rangle \frac{\Delta \Omega_{d-2}}{2\Delta-d} \int_{z_0}^R dz z^{2\Delta-d-1} \int_0^{\sqrt{R^2-z^2}} dr r^{d-2} \left[\frac{R^2 - r^2 - z^2}{2R} \right] \quad (\text{C.1})$$

$$= -2\pi \langle \mathcal{O} \rangle_g \delta \langle \mathcal{O} \rangle \frac{\Delta \Omega_{d-2}}{d^2-1} \left[R^{2\Delta} \frac{\Gamma(\frac{d}{2} + \frac{3}{2}) \Gamma(\Delta - \frac{d}{2} + 1)}{(2\Delta-d)^2 \Gamma(\Delta + \frac{3}{2})} - \frac{R^d z_0^{2\Delta-d}}{(2\Delta-d)^2} \right]. \quad (\text{C.2})$$

Note this consists of a finite term scaling as $R^{2\Delta}$ and a divergence in z_0 .

The divergence must cancel against the integral over \mathcal{T} , given by (2.36). Unlike the case $\Delta > \frac{d}{2}$, this integral has a vanishing contribution from the region $t_B \sim z$, but instead a divergent contribution from $t_B \gg z$. Again picking out the $\langle \mathcal{O} \rangle_g \delta \langle \mathcal{O} \rangle$ term in the

integrand (2.36), we find

$$\delta S_{\mathcal{T},1}^{(2)} = -2\pi \langle \mathcal{O} \rangle_g \delta \langle \mathcal{O} \rangle \frac{\Omega_{d-2} z_0^{-d+1}}{(2\Delta - d)^2} \int_0^R dt \int_0^{R-t} dr r^{d-2} \frac{t}{R} \left[2(\Delta z_0^{\Delta-1})^2 - \frac{\Delta}{z^2} (2\Delta - d) z^{2\Delta} \right] \quad (\text{C.3})$$

$$= -2\pi \langle \mathcal{O} \rangle_g \delta \langle \mathcal{O} \rangle \frac{\Delta \Omega_{d-2} R^d z_0^{2\Delta-d}}{(d^2 - 1)(2\Delta - d)^2}. \quad (\text{C.4})$$

Here, we see this cancels the divergence in (C.2), and thus we are left with only the finite term in that expression.

2. $g\delta\langle\mathcal{O}\rangle + \lambda(\mathbf{0})\langle\mathcal{O}\rangle_g$ term. On the surface \mathcal{E} , this term comes from the part of one field going like z^Δ , and the other going like $z^{d-\Delta}$. Hence, when we evaluate this term in $\nabla^2\phi^2$ for the bulk integral, we will be acting on a term proportional to z^d , which is annihilated by the Laplacian. So the bulk will only contribute terms that are subleading to R^d terms from $\delta S^{(1)}$. The calculation of these subleading terms would be similar to the calculation for in section 4.1, but we do not pursue this further here.

Instead, we examine the integral over \mathcal{T} , which can produce finite contributions. Along this surface, the fields are now time dependent, and hence all terms in equation (2.36) are important. We start by focusing on the terms involving time derivatives of ϕ . The z -derivative acts on the term going as $\frac{-z^\Delta}{2\Delta-d}$, and the t derivative on $z^{d-\Delta}F(t/z)$. To properly account for the behavior of F when $t \sim z$, it is useful to split the t integral into two regions, $(0, c)$ and (c, R) with $z \ll c \ll R$. In the first region this gives

$$-2\pi \frac{\Delta \Omega_{d-2}}{2\Delta - d} \int_0^c dt \int_0^R dr r^{d-2} \left(\frac{R^2 - r^2}{2R} \right) \partial_t F(t/z_0) = \frac{-2\pi \Delta \Omega_{d-2} R^d}{(2\Delta - d)(d^2 - 1)} F(t/z_0) \Big|_0^c. \quad (\text{C.5})$$

From (B.7), we see that $F(0) = 1$, and the value at $t = c$ can be read off using the asymptotic form for F in equation (B.8). This form is also useful for evaluating the integral in the second region, where the integral is

$$\begin{aligned} & \frac{-2\pi \Delta \Omega_{d-2} (d - 2\Delta)}{(2\Delta - d)} B z_0^{2\Delta-d} \int_c^R dt \int_0^{R-t} dr r^{d-2} \left(\frac{R^2 - r^2 - t^2}{2R} \right) t^{d-2\Delta-1} \\ &= \frac{-2\pi \Delta \Omega_{d-2}}{(2\Delta - d)} B z_0^{2\Delta-d} \left[R^{2(d-\Delta)} \frac{d\Gamma(d-1)\Gamma(d-2\Delta+2)}{\Gamma(2d-2\Delta+2)} - \frac{c^{d-2\Delta} R^d}{d^2 - 1} \right], \end{aligned} \quad (\text{C.6})$$

where this equality holds for $c \ll R$. The second term cancels the c -dependent term of (C.5), while the first term is a remaining divergence which must cancel against the other piece of the \mathcal{T} integral. This is the piece coming from the second bracketed expression in equation (2.36). This term receives no contribution from the region $t \sim z$, so we can evaluate it in the region $t \gg z$, using the asymptotic form for $F(t/z)$. Evaluating the derivatives in this expression (and recalling that only the z -derivatives in the Laplacian will produce a nonzero contribution at $z \rightarrow 0$), this leads to

$$\begin{aligned} & \frac{2\pi \Omega_{d-2}}{(2\Delta - d)} B z_0^{2\Delta-d} \int_0^R dt \int_0^{R-t} dr r^{d-2} \frac{d\Delta}{R} t^{d-2\Delta+1} \\ &= \frac{2\pi \Delta \Omega_{d-2}}{2\Delta - d} B z_0^{2\Delta-d} \frac{d\Gamma(d-1)\Gamma(d-2\Delta+2)}{\Gamma(2d-2\Delta+2)}, \end{aligned} \quad (\text{C.7})$$

which cancels the remaining term in (C.6).

Hence the only contribution remaining comes from (C.5) at $t = 0$, and gives

$$\delta S_{\mathcal{T},2}^{(2)} = \frac{2\pi\Omega_{d-2}R^d\Delta}{(d^2-1)(2\Delta-d)}(g\delta\langle\mathcal{O}\rangle + \lambda(0)\langle\mathcal{O}\rangle_g). \quad (\text{C.8})$$

3. $g\lambda(0)$ term. The final type of term arises when both fields behave as $z^{d-\Delta}F(t/z)$. The \mathcal{E} surface term will go like $R^{2(d-\Delta)}$, and hence will be subleading compared to the R^d terms. In fact, this calculation is essentially the same as the change in vacuum entanglement when deforming by a constant source, and the form of this term is given in equation (4.34) of [21] (although that calculation was originally performed only for $\Delta > \frac{d}{2}$). Also there is no divergence in z_0 in these terms.

On the other hand, the integral over \mathcal{T} does lead to potential divergences, but we will show that these all cancel out as expected. We may focus on the region $t \gg z$ since there is no contribution from $t \sim z$. Using the asymptotic form (B.8) for F , the part of the integral (2.36) involving t derivatives becomes

$$\begin{aligned} & 2\pi\Omega_{d-2}2\Delta(d-2\Delta)B^2z_0^{2\Delta-d}\int_0^R dt \int_0^{R-t} dr r^{d-2} \left(\frac{R^2-r^2-t^2}{2R}\right) t^{2d-4\Delta-1} \\ &= 2\pi\Omega_{d-2}B^2z_0^{2\Delta-d}R^{3d-4\Delta}\frac{\Delta d\Gamma(d-1)\Gamma(2d-4\Delta+2)}{\Gamma(3d-4\Delta+2)}. \end{aligned} \quad (\text{C.9})$$

Similarly, the second bracketed term in (2.36) evaluates to

$$\begin{aligned} & -2\pi\Omega_{d-2}\Delta dB^2z_0^{2\Delta-d}\int_0^R dt \int_0^{R-t} dr r^{d-2} \frac{t^{2d-4\Delta+1}}{R} \\ &= -2\pi\Omega_{d-2}B^2z_0^{2\Delta-d}R^{3d-4\Delta}\frac{\Delta d\Gamma(d-1)\Gamma(2d-4\Delta+2)}{\Gamma(3d-4\Delta+2)}, \end{aligned} \quad (\text{C.10})$$

perfectly canceling against (C.9). Hence, the \mathcal{T} surface integral gives no contribution, and the full $g\lambda(0)$ contribution, coming entirely from the \mathcal{E} surface, is subleading.

C.2 $\Delta = \frac{d}{2}$

Here we compute the surface integrals and divergence in $\delta S^{(1)}$ when $\Delta = \frac{d}{2}$. The calculation is divided into four parts: the \mathcal{E} surface integral, the \mathcal{T} surface integral for $t_B \sim z_0$, the \mathcal{T} surface integral for $t_B \gg z_0$, and the $\delta S^{(1)}$ divergence.

1. \mathcal{E} surface integral. Equation (2.35) shows that we need to compute the Laplacian acting on $(\phi_0 + \phi_\omega)^2$. At leading order, only the z -derivatives from the Laplacian contribute since the other derivatives are suppressed by a factor of z^2 . Using the bulk solutions found for ϕ_0 (B.17) and ϕ_ω (4.42), the \mathcal{E} surface integral at $O(\lambda^1 g^1)$ is

$$\begin{aligned} \delta S_{\mathcal{E}}^{(2)} &= -4\pi\Omega_{d-2}g\lambda_\omega \int_{z_0}^R \frac{dz}{z} \int_0^{\sqrt{R^2-z^2}} dr r^{d-2} \left[\frac{R^2-r^2-z^2}{8R}\right] \left[2 + d\gamma_E + d \log \frac{\omega z^2}{4L}\right] \\ &= -2\pi g\lambda_\omega \frac{\Omega_{d-2}R^d}{d^2-1} \int_{z_0/R}^1 \frac{dw}{w} (1-w^2)^{\frac{d+1}{2}} \left(1 + \frac{d}{2}\gamma_E + \frac{d}{2} \log \frac{w^2 R^2 \omega}{4L}\right). \end{aligned} \quad (\text{C.11})$$

The divergence in z_0 comes from w near zero, and so can be extracted by setting the $(1-w^2)$ term in the integrand to 1, its value at $w=0$. The divergent integral evaluates to

$$\delta S_{\mathcal{E},\text{div.}}^{(2)} = -2\pi g \lambda_\omega \frac{\Omega_{d-2} R^d}{d^2-1} \log\left(\frac{R}{z_0}\right) \left(1 + \frac{d}{2}\gamma_E + \frac{d}{2}\log\frac{\omega R z_0}{4L}\right), \quad (\text{C.12})$$

and the remaining finite piece with $z_0 \rightarrow 0$ is

$$\delta S_{\mathcal{E},\text{fin.}}^{(2)} = -2\pi g \lambda_\omega \frac{\Omega_{d-2} R^d}{d^2-1} \int_0^1 \frac{dw}{w} \left[(1-w^2)^{\frac{d+1}{2}} - 1\right] \left(1 + \frac{d}{2}\gamma_E + \frac{d}{2}\log w^2 \frac{R^2 \omega}{4L}\right). \quad (\text{C.13})$$

The following two identities are needed to evaluate this,

$$\int_0^1 \frac{dw}{w} \left[(1-w^2)^{\frac{d+1}{2}} - 1\right] = -\frac{1}{2} H_{\frac{d+1}{2}} \quad (\text{C.14})$$

$$\int_0^1 \frac{dw}{w} \left[(1-w^2)^{\frac{d+1}{2}} - 1\right] \log w = \frac{1}{8} \left(H_{\frac{d+1}{2}}^{(2)} + H_{\frac{d+1}{2}}^2\right), \quad (\text{C.15})$$

where the harmonic number H_n was defined below equation (1.2), and $H_n^{(2)}$ is a second order harmonic number, defined for the integers by $H_n^{(2)} = \sum_{k=1}^n \frac{1}{k^2}$, and for arbitrary complex n by $H_n^{(2)} = \frac{\pi^2}{6} - \psi_1(n+1)$, where $\psi_1 = \frac{d^2}{dx^2} \log \Gamma(x)$. With these, the finite piece (C.13) becomes

$$\delta S_{\mathcal{E},\text{fin.}}^{(2)} = 2\pi g \lambda_\omega \frac{\Omega_{d-2} R^d}{d^2-1} \left[\frac{d}{4} H_{\frac{d+1}{2}} \left(\gamma_E + \log \frac{\omega R^2}{4L}\right) - \frac{1}{8} \left(H_{\frac{d+1}{2}}^{(2)} + H_{\frac{d+1}{2}} (H_{\frac{d+1}{2}} - 2)\right) \right]. \quad (\text{C.16})$$

2. \mathcal{T} surface near $t_B \sim z$. This region contains several divergences in z_0 and δ . The specific range of t_B will be $t_B \in (0, c)$, with $z \ll c \ll R$. Only the first bracketed term in (2.36) contributes in this region, and using the general solutions for ϕ_0 and ϕ_ω from equations (B.15) and (B.20), it gives at $O(\lambda^1 g^1)$

$$\delta S_{\mathcal{T},\text{div.}}^{(2)} = 2\pi g \frac{\Omega_{d-2} R^d}{d^2-1} \int_0^c dt \left[\frac{d}{2} \partial_t (\lambda_\omega G_L G_a + \beta_\omega G_L) + \lambda_\omega z_0 (\partial_z G_L \partial_t G_a + \partial_z G_a \partial_t G_L) \right], \quad (\text{C.17})$$

having introduced the shorthand $G_L \equiv G(t/z_0, \delta/z_0, L/z_0)$ and similarly for G_a . The first term in this expression is a total derivative so can be integrated directly. The boundary term at $t=0$ is

$$2\pi g \lambda_\omega \frac{\Omega_{d-2} R^d}{d^2-1} \frac{d}{2} \log\left(\frac{2L}{z_0}\right) \left(\gamma_E + \log \frac{\omega z_0}{2}\right). \quad (\text{C.18})$$

At the other boundary $t=c \gg z_0$, the asymptotic formulas (B.21) and (B.19) produce the term

$$-2\pi g \lambda_\omega \frac{\Omega_{d-2} R^d}{d^2-1} \frac{d}{2} \log\left(\frac{L}{c}\right) (\gamma_E + \log \omega c). \quad (\text{C.19})$$

The remaining terms in (C.17) contain a divergence in δ , coming from $t \sim z$. To extract it, we focus specifically on the regions $(z_0 - u, z_0 + v)$ and $(z_0 + v, c)$, where $u, v \ll z$ and positive. It is straightforward to show that the integral over the region $(0, z_0 - u)$ is $O(\delta)$,

and so does not contribute when δ is sent to zero. The divergence in the $(z_0 - u, z_0 + v)$ region can be evaluated by taking a scaling limit with a change of variables, $t_B = z_0 + s\delta$, and expanding the integrand about $\delta = 0$. After also taking the limit $L/z_0, a/z_0 \rightarrow \infty$ in the integrand, the integral in this region becomes

$$-\lambda_\omega \int_{-u/\delta}^{v/\delta} ds \frac{s + \sqrt{1 + s^2}}{1 + s^2} \rightarrow -\lambda_\omega \log \frac{2v}{\delta}, \tag{C.20}$$

which holds for $u, v \gg \delta$. For the region $(z + v, c)$, we can take $\delta/z \rightarrow 0$ and $L/z, a/z \rightarrow \infty$, which produces the integral

$$2\lambda_\omega \int_{z_0+v}^c dt \left(\frac{1}{\sqrt{t^2 - z_0^2}} - \frac{t}{t^2 - z_0^2} \right) \rightarrow \lambda_\omega \log \frac{8v}{z_0}, \tag{C.21}$$

where we have taken the limits $c/z_0 \gg 1, v/z_0 \ll 1$.

The final collection of the four contributions (C.18), (C.19), (C.20) and (C.21) is

$$\delta S_{\mathcal{T}, \text{div.}}^{(2)} = 2\pi g \lambda_\omega \frac{\Omega_{d-2} R^d}{d^2 - 1} \left[\frac{d}{2} \log \left(\frac{2L}{z_0} \right) \left(\gamma_E + \log \frac{\omega z_0}{2} \right) - \frac{d}{2} \log \left(\frac{L}{c} \right) \left(\gamma_E + \log \omega c \right) + \log \frac{4\delta}{z_0} \right]. \tag{C.22}$$

3. \mathcal{T} surface for $t_B \gg z$. In this region, $t_B \gg z$, and we can use the asymptotic forms (B.19) and (B.21) for the fields ϕ_0 and ϕ_ω . We start with the first bracketed term in equation (2.36),

$$\delta S_{\mathcal{T},1}^{(2)} = 2\pi g \lambda_\omega \Omega_{d-2} \int_c^R dt \int_0^{R-t} dr r^{d-2} \left[\frac{R^2 - r^2 - t^2}{2R} \right] \frac{d}{2t} \left(\gamma_E + \log \frac{t^2 \omega}{L} \right) \tag{C.23}$$

$$= 2\pi g \lambda_\omega \frac{\Omega_{d-2} R^d}{d^2 - 1} \frac{d}{2} \int_{c/R}^1 \frac{ds}{s} (1-s)^d (1+ds) \left(\gamma_E + \log \frac{s^2 R^2 \omega}{L} \right). \tag{C.24}$$

The divergence in this integral comes from $s = 0$, so it can be separated out by setting $(1-s)^d(1+ds)$ to 1 (its value at $s = 0$), leading to

$$\int_{c/R}^1 \frac{ds}{s} \left(\gamma_E + \log \frac{s^2 R^2 \omega}{L} \right) = \log \left(\frac{R}{c} \right) \left(\gamma_E + \log \frac{cR\omega}{L} \right). \tag{C.25}$$

The remaining finite piece of the integral is

$$\int_0^1 \frac{ds}{s} \left[(1-s)^d (1+ds) - 1 \right] \left(\gamma_E + \log \frac{s^2 R^2 \omega}{L} \right). \tag{C.26}$$

Evaluation of this integral involves the following identities,

$$\int_0^1 \frac{ds}{s} \left[(1-s)^d (1+ds) - 1 \right] = 1 - H_{d+1}, \tag{C.27}$$

$$\int_0^1 \frac{ds}{s} \left[(1-s)^d (1+ds) - 1 \right] \log s = \frac{1}{2} \left(H_{d+1}^{(2)} + H_{d+1} (H_{d+1} - 2) \right), \tag{C.28}$$

where the harmonic numbers H_n and $H_n^{(2)}$ were defined below equations (1.2) and (C.15). Using these to compute (C.26), and combining the answer with equation (C.25) gives

$$\begin{aligned} \delta S_{\mathcal{T},1}^{(2)} = 2\pi g\lambda_\omega \frac{\Omega_{d-2}R^d}{d^2-1} \frac{d}{2} & \left[\log\left(\frac{R}{c}\right) \left(\gamma_E + \log\frac{cR\omega}{L} \right) \right. \\ & \left. - (H_{d+1}-1) \left(\gamma_E + \log\frac{R^2\omega}{L} \right) + H_{d+1}^{(2)} + H_{d+1}(H_{d+1}-2) \right]. \end{aligned} \quad (\text{C.29})$$

Finally, we compute the second bracketed term of (2.36). Only the z -derivatives in the Laplacian term $\nabla^2\phi^2$ contribute in the limit $z \rightarrow 0$. Since ϕ^2 scales as z^d , the z -derivatives in the Laplacian annihilate it, and hence this piece is zero. The integral then becomes

$$\begin{aligned} \delta S_{\mathcal{T},2}^{(2)} &= 2\pi g\lambda_\omega \Omega_{d-2} \left(\frac{d}{2}\right)^2 2 \int_0^R dt \int_0^{R-t} dr r^{d-2} \frac{t}{R} \log\left(\frac{L}{t}\right) (\gamma_E + \log\omega t) \quad (\text{C.30}) \\ &= 2\pi g\lambda_\omega \frac{\Omega_{d-2}R^d}{d^2-1} \frac{d}{2} \left[-H_{d+1}^{(2)} - H_{d+1}(H_{d+1}-2) + (H_{d+1}-1) \left(\gamma_E + \log\frac{R^2\omega}{L} \right) \right. \\ & \quad \left. - \log\left(\frac{R}{L}\right) (\gamma_E + \log R\omega) \right]. \end{aligned} \quad (\text{C.31})$$

The finite terms cancel against those appearing in (C.29), and the final combined result is

$$\delta S_{\mathcal{T},1+2}^{(2)} = 2\pi g\lambda_\omega \frac{\Omega_{d-2}R^d}{d^2-1} \frac{d}{2} \log\left(\frac{L}{c}\right) (\gamma_E + \log\omega c), \quad (\text{C.32})$$

which perfectly cancels the c -dependent terms in (C.22). Hence, no finite terms result from the integral along \mathcal{T} in the $t_B \gg z$ region.

4. $\delta S^{(1)}$ term. The final divergence in δ comes from the expectation value of the CFT stress tensor, in $\delta S^{(1)}$. At order $g\lambda_\omega$, this is given by

$$\delta \langle T_{00}^0(0) \rangle = - \int d^d x_a d^d x_b g\lambda_\omega(x_b) \langle T_{\tau\tau}^0(0) \mathcal{O}(x_a) \mathcal{O}(x_b) \rangle. \quad (\text{C.33})$$

The only divergence in this correlation function comes from when $x_a \rightarrow x_b \rightarrow 0$, and is logarithmic in the cutoff δ . As was the case for the logarithmic divergence in $\langle \mathcal{O} \rangle$, regulating this divergence involves introducing a renormalization scale μ that separates the divergence from the finite part of the correlation function. This is done by cutting off the τ integrals when $|\tau_a| \geq \mu^{-1}$ and $|\tau_b| \geq \mu^{-1}$.

The divergence comes from the leading piece in the expansion of $\lambda_\omega(x)$ about $x = 0$,

$$\delta \langle T_{\tau\tau}^0(0) \rangle_{\text{div.}} = g\lambda_\omega \int d^d x_a d^d x_b \langle T_{\tau\tau}^0(0) \mathcal{O}(x_a) \mathcal{O}(x_b) \rangle. \quad (\text{C.34})$$

This divergence can be evaluated using the same method described in appendix D of [21]. The translation invariance of the correlation function allows one to write it as an integral of the stress tensor averaged over the spatial volume,

$$g\lambda_\omega \frac{1}{V} \int d^{d-1} \vec{x} \int_{C(\delta,\mu)} d\tau_a \int_{C(\delta,\mu)} d\tau_b \int d\vec{x}_a d\vec{x}_b \langle T_{\tau\tau}^0(0, \vec{x}) \mathcal{O}(x_a) \mathcal{O}(x_b) \rangle. \quad (\text{C.35})$$

The stress tensor integrated over \vec{x} is now a conserved quantity, and so the surface of integration may be deformed away from $\tau = 0$. As long as it does not encounter the points τ_a or τ_b , the surface can be pushed to infinity, so that the correlation function vanishes. This is possible if τ_a and τ_b have the same sign. However, when τ_a and τ_b have opposite signs, one of them will be passed as the surface is pushed to infinity. This leads to a contribution from the operator insertion at that point, as dictated by the translation Ward identity. Let us choose to push past τ_a . For $\tau_a < 0$, the contribution from the operator insertion is

$$-g\lambda_\omega \frac{1}{V} \int d\vec{x} d\vec{x}_a d\vec{x}_b \int_\delta^\mu d\tau_b \int_{-\mu}^{-\delta} d\tau_a \partial_{\tau_a} \langle \mathcal{O}(x_a) \mathcal{O}(x_b) \rangle \delta(\vec{x} - \vec{x}_a) \quad (\text{C.36})$$

$$= -g\lambda_\omega c'_\Delta S_{d-2} \frac{\sqrt{\pi} \Gamma(\frac{d}{2} - \frac{1}{2})}{2\Gamma(\frac{d}{2})} \int_\delta^\mu d\tau_b \left[\frac{1}{\tau_b + \delta} - \frac{1}{\tau_b + \mu} \right] \quad (\text{C.37})$$

$$= -\frac{1}{2} g\lambda_\omega \log \frac{\mu}{4\delta}, \quad (\text{C.38})$$

where in this last equality we have taken $\mu \gg \delta$. It is straightforward to check that for $x_a^0 > 0$, you get the same contribution, so that the full divergent piece of the stress tensor is

$$\delta \langle T_{00}(\vec{x}) \rangle_{\text{div.}} = g\lambda_\omega \log \frac{\mu}{4\delta}. \quad (\text{C.39})$$

This then defines a renormalized stress tensor expectation value,

$$\delta \langle T_{00}(0) \rangle^{\text{ren.}} = \delta \langle T_{00}(0) \rangle - g\lambda_\omega \log \frac{\mu}{4\delta}. \quad (\text{C.40})$$

Finally, the contribution to $\delta S^{(1)}$ comes from integrating $\delta \langle T_{00}(\vec{x}) \rangle$ over the ball Σ according to equation (2.21). Since the stress tensor expectation value may be assumed constant over a small enough ball, the expression for $\delta S^{(1)}$ in terms of the renormalized stress tensor expectation value is

$$\delta S_{\lambda_g}^{(1)} = 2\pi \frac{\Omega_{d-2} R^d}{d^2 - 1} \left(\delta \langle T_{00}^0 \rangle^{\text{ren.}} + g\lambda_\omega \log \left(\frac{\mu}{4\delta} \right) \right). \quad (\text{C.41})$$

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