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A local regularity of the complex Monge–Ampère equation

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Abstract We prove a local regularity (and a corresponding a priori estimate) for plurisubharmonic solutions of the nondegenerate complex Monge–Ampère equation assuming that their $W^{2,p}$-norm is under control for some $p > n(n-1)$. This condition is optimal. We use in particular some methods developed by Trudinger and an estimate for the complex Monge–Ampère equation due to Kołodziej.

1 Introduction

The aim of this note is to prove the following a priori estimate for the complex Monge–Ampère equation:

Theorem *Assume that* $p > n(n-1)$. *Let* $u \in W^{2,p}(\Omega)$ *(that is partial derivatives of* u *up to the second order are in* $L^p(\Omega)$*), where* Ω *is a domain in* $\mathbb{C}^n$*, be a plurisubharmonic solution of*

$$\det\left(u_{z_j\bar{z}_k}\right) = \psi > 0. \tag{1}$$

Assume that $\psi \in C^{1,1}(\Omega)$ *(that is* $\psi \in C^1(\Omega)$ *and the second partial derivatives of* ψ *are Lipschitz continuous). Then for* $\Omega' \Subset \Omega$ *we have*

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$$\sup_{\Omega'} \Delta u \leq C,$$

where C is a constant depending only on n, p, $\operatorname{dist}(\Omega', \partial\Omega)$, $\inf_\Omega \psi$, $||\psi||_{C^{1,1}(\Omega)}$ *and* $||\Delta u||_{L^p(\Omega)}$.

By a complex version of the Evans–Krylov theory (see e.g. [5] or [11]), once one has an upper bound for the Laplacian (and thus for mixed complex second derivatives) then also a $C^{2,\alpha}$-estimate follows. We thus get the following local regularity of plurisubharmonic solutions of (1)

$$u \in W^{2,p}_{loc} \quad \text{for some } p > n(n-1), \quad \psi \in C^\infty \Longrightarrow u \in C^\infty. \tag{2}$$

For $p > 2n(n-1)$ this (and the theorem) is a consequence of a general real theory from [13] (see [4]). For $p > n^2$ a similar a priori estimate for C^3-solutions (without a regularity result though) was recently shown in [7].

The main point about our result is that the condition $p > n(n-1)$ is essentially optimal. The fact that it is false for $p < n(n-1)$ follows from a complex counterpart of Pogorelov's example [10] from [4]: the function

$$u(z) = (1 + |z_1|^2)|z'|^{2-2/n},$$

where $z' = (z_2, \ldots, z_n)$, is in $W^{2,p}_{loc}$ if and only if $p < n(n-1)$, plurisubharmonic in $\mathbb{C}^n$, and satisfies

$$\det\left(u_{z_j\bar{z}_k}\right) = c_n(1 + |z_1|^2)^{n-2} \in C^\infty(\mathbb{C}^n)$$

(c_n is a constant depending only on n) in the weak sense of [2].

The corresponding estimates and regularity for the real Monge–Ampère equation can be found in [14].

The main tool in the proof of Theorem will be the following estimate of Kołodziej [8] (see also [9]): if a plurisubharmonic u with $u \geq 0$ on $\partial\Omega$ solves (1) (with ψ satisfying only $\psi \geq 0$) then for $q > 1$ we have

$$\sup_\Omega(-u) \leq C(q, n, \operatorname{diam}\Omega)||\psi||^{1/n}_{L^q(\Omega)}. \tag{3}$$

This result for $q = 2$ is due to Cheng and Yau (see [1,6]).

2 Proof of Theorem

By $C_1, C_2, \ldots$ we will denote possibly different positive constants depending only on the required quantities. Without loss of generality we may assume that $\Omega = B$ is the unit ball in $\mathbb{C}^n$ and that u is defined in some neighborhood of $\bar{B}$. We will use the notation $u_j = u_{z_j}$, $u_{\bar{j}} = u_{\bar{z}_j}$ and $\Delta u = \sum_j u_{j\bar{j}}$. As usual, by $(u^{i\bar{j}})$ we will denote the inverse transposed of $(u_{i\bar{j}})$.

We will first prove Theorem assuming that u is in C^4. Differentiating (1) w.r.t. z_p and $\bar{z}_p$ we will get

$$u^{i\bar{j}}u_{i\bar{j}p} = (\log\psi)_p$$

and

$$u^{i\bar{j}}u_{i\bar{j}p\bar{p}} = (\log\psi)_{p\bar{p}} + u^{i\bar{l}}u^{k\bar{j}}u_{k\bar{l}\bar{p}}u_{i\bar{j}p}.$$

Therefore

$$u^{i\bar{j}}(\Delta u)_{i\bar{j}} \geq \Delta(\log\psi). \tag{4}$$

We will now use an idea from [12]. For some $\alpha, \beta \geq 2$ to be determined later set

$$w := \eta(\Delta u)^\alpha,$$

where

$$\eta(z) := (1 - |z|^2)^\beta$$

Then

$$w_i = \eta_i(\Delta u)^\alpha + \alpha\eta(\Delta u)^{\alpha-1}(\Delta u)_i$$

and

$$\begin{aligned} u^{i\bar{j}}w_{i\bar{j}} &= \alpha\eta(\Delta u)^{\alpha-1}u^{i\bar{j}}(\Delta u)_{i\bar{j}} + \alpha(\alpha-1)\eta(\Delta u)^{\alpha-2}u^{i\bar{j}}(\Delta u)_i(\Delta u)_{\bar{j}} \\ &\quad + 2\alpha(\Delta u)^{\alpha-1}\mathrm{Re}\left(u^{i\bar{j}}\eta_i(\Delta u)_{\bar{j}}\right) + (\Delta u)^\alpha u^{i\bar{j}}\eta_{i\bar{j}}. \end{aligned}$$

By (4) and the Schwarz inequality for $t > 0$

$$\begin{aligned} u^{i\bar{j}}w_{i\bar{j}} \geq\ & \alpha\eta(\Delta u)^{\alpha-1}\Delta(\log\psi) + \alpha(\alpha-1)\eta(\Delta u)^{\alpha-2}u^{i\bar{j}}(\Delta u)_i(\Delta u)_{\bar{j}} \\ & - t\alpha(\Delta u)^{\alpha-1}u^{i\bar{j}}(\Delta u)_i(\Delta u)_{\bar{j}} - \frac{1}{t}\alpha(\Delta u)^{\alpha-1}u^{i\bar{j}}\eta_i\eta_{\bar{j}} + (\Delta u)^\alpha u^{i\bar{j}}\eta_{i\bar{j}}. \end{aligned}$$

Therefore with $t = (\alpha-1)\eta/\Delta u$ we get

$$u^{i\bar{j}}w_{i\bar{j}} \geq \alpha\eta(\Delta u)^{\alpha-1}\Delta(\log\psi) + (\Delta u)^\alpha u^{i\bar{j}}\left(\eta_{i\bar{j}} - \frac{\alpha}{\alpha-1}\frac{\eta_i\eta_{\bar{j}}}{\eta}\right).$$

We now have

$$\begin{aligned} \eta_i &= -\beta z_i\eta^{1-1/\beta} \\ \eta_{i\bar{j}} &= -\beta\delta_{i\bar{j}}\eta^{1-1/\beta} + \beta(\beta-1)\bar{z}_i z_j\eta^{1-2/\beta} \end{aligned}$$

and thus

$$\left|\eta_{i\bar{j}}\right|, \quad \left|\frac{\eta_i \eta_{\bar{j}}}{\eta}\right| \le C(\beta)\eta^{1-2/\beta}.$$

We will get

$$u^{i\bar{j}} w_{i\bar{j}} \ge -C_1(\Delta u)^{\alpha-1} - C_2 w^{1-2/\beta}(\Delta u)^{2\alpha/\beta} \sum_{i,j} |u^{i\bar{j}}|.$$

Fix q with $1 < q < p/(n(n-1))$. Since $||\Delta u||_p$ (this way we will denote norms in $L^p(B)$) is under control, it follows that $||u_{i\bar{j}}||_p$ and $||u^{i\bar{j}}||_{p/(n-1)}$ are as well. It follows that for

$$\alpha = 1 + \frac{p}{qn}, \quad \beta = 2\left(1 + \frac{qn}{p}\right)$$

we have

$$\left\| \left(u^{i\bar{j}} w_{i\bar{j}}\right)_- \right\|_{qn} \le C_3 \left(1 + \left(\sup_B w\right)^{1-2/\beta}\right),$$

where $f_- := -\min(f, 0)$.

By [2] we can find continuous plurisubharmonic v vanishing on ∂B and such that

$$\det(v_{i\bar{j}}) = \left(\left(u^{i\bar{j}} w_{i\bar{j}}\right)_-\right)^n$$

(weakly). Essentially by an inequality between arithmetic and geometric means (see [3] how to extend it to the weak case) we have

$$\begin{aligned} u^{i\bar{j}} v_{i\bar{j}} &\ge n\left(\det\left(u^{i\bar{j}}\right)\right)^{1/n}\left(\det\left(v_{i\bar{j}}\right)\right)^{1/n} \\ &= n\psi^{-1/n}\left(u^{i\bar{j}} w_{i\bar{j}}\right)_- \\ &\ge -\frac{1}{C_4} u^{i\bar{j}} w_{i\bar{j}}. \end{aligned}$$

It follows that $w \le -C_4 v$ and by Kołodziej's inequality (3)

$$\begin{aligned} \sup_B w &\le C_5 || \det(v_{i\bar{j}}) ||_q^{1/n} \\ &= C_5 || \left(u^{i\bar{j}} w_{i\bar{j}}\right)_- ||_{qn} \\ &\le C_6 \left(1 + \left(\sup_B w\right)^{1-2/\beta}\right). \end{aligned}$$

Therefore $w \le C_7$ and the desired estimate follows if $u \in C^4$.

Now assume that the solution is just in $W^{2,p}$. Similarly to [2], instead of Δu we will consider for $\varepsilon > 0$ the following approximations to the Laplacian

$$T = T_\varepsilon u = \frac{n+1}{\varepsilon^2}(u_\varepsilon - u),$$

where

$$u_\varepsilon(z) = \frac{1}{\lambda(B(z,\varepsilon))} \int\limits_{B(z,\varepsilon)} u \, \mathrm{d}\lambda$$

and λ denotes the Lebesgue measure in $\mathbb{C}^n$. Since $T_\varepsilon u \to \Delta u$ weakly as $\varepsilon \to 0$, it is enough to show a uniform upper bound for T independent of ε.

By [2] we have

$$u^{i\bar{j}} u_{\varepsilon,i\bar{j}} \geq n\psi^{-1/n} \left(\det(u_{\varepsilon,i\bar{j}})\right)^{1/n} \geq n\psi^{-1/n} (\psi^{1/n})_\varepsilon$$

and thus, coupling this with $u^{i\bar{j}} u_{i\bar{j}} = n$, we obtain the following counterpart of (4)

$$u^{i\bar{j}} T_{i\bar{j}} \geq n\psi^{-1/n} T_\varepsilon(\psi^{1/n}) \geq -C_8.$$

Changing the definition of w to ηT^α (since u is plurisubharmonic, T is nonnegative, hence T^α is well defined) and repeating the previous computations we will get

$$u^{i\bar{j}} w_{i\bar{j}} \geq C_9 T^{\alpha-1} - C_{10} w^{1-2/\beta} T^{2\alpha/\beta} \sum_{i,j} \left| u^{i\bar{j}} \right|.$$

The rest of the proof is now the same as before.

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