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Fixed point results for $\{\alpha,\xi\}$-expansive locally contractive mappings

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Abstract
We recall the concepts of $\{\alpha,\xi\}$-contractive and α-admissible mappings on complete metric spaces to state related fixed point theorems. In this paper, we obtain some fixed point results for $\{\alpha,\xi\}$-expansive locally contractive mappings in complete metric spaces. The contractiveness of the mapping is only on a closed ball instead of the whole space. Our results unify, generalize, and complement various well-known comparable results in the literature.
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Keywords: expansive mapping; α-admissible; fixed point; closed ball

1 Introduction and preliminaries

The main revolution in the existence theory of many linear and nonlinear operators happened after the Banach contraction principle [1]. After the emergence of this principle many researchers put their efforts into studying the existence and solutions for nonlinear equations (algebraic, differential, and integral), a system of linear (nonlinear) equations and convergence of many computational methods. The Banach contraction gave us many important theories like variational inequalities, optimization theory, and many computational theories. Due to the wide importance of the Banach contraction, many authors generalized it in several directions [2–17]. Wang *et al.* in [18] defined expansion mappings in the form of the following theorem.

Theorem 1 [18] *Let (X,d) be a complete metric space. If F is a self-mapping on X and if there exists a constant $k>1$ such that*

$$d(Fx,Fy)\geq kd(x,y)$$

for all $x,y\in X$ and F is onto, then F has a unique fixed point in X.

On the other hand, Samet *et al.* in [19] introduced the concepts of $(\alpha\text{-}\psi)$-contractive and α-admissible mappings in complete metric spaces. They also proved a fixed point theorem for $(\alpha\text{-}\psi)$-contractive mappings in complete metric spaces using the concept of α-admissible mapping.

Let us denote by Ψ the family of non-decreasing functions $\psi:[0,+\infty)\to[0,+\infty)$ such that $\sum_{n=1}^{\infty}\psi^n(t)<+\infty$ for all $t>0$, where ψ^n is the nth iterate of ψ.

The following lemma can easily be deduced.

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Lemma 2 *If $\psi \in \Psi$, then $\psi(t) < t$ for all $t > 0$.*

Let us consider the following example.

Example 3 Let $\psi_1, \psi_2 : [0,+\infty) \to [0,+\infty)$ be defined in the following way:

$$\psi_1(t) = \frac{1}{3}t$$

and

$$\psi_2(t) = \begin{cases} \frac{1}{4}t, & \text{if } 0 \le t < 1, \\ \frac{1}{5}t, & \text{if } t \ge 1. \end{cases}$$

It is clear that $\psi_1, \psi_2 \in \Psi$. Moreover, note that ψ_1, ψ_2 are examples of continuous and discontinuous functions in Ψ.

In [19] Samet *et al.* defined the notion of α-admissible and (α-ψ)-contractive type mappings as follows.

Definition 4 Let F be a self-mapping on X and $\alpha : X \times X \to [0,+\infty)$ be a function. Then F is called α-admissible mapping if

$$\alpha(x,y) \ge 1 \quad \text{implies} \quad \alpha(Fx,Fy) \ge 1, \quad \forall x,y \in X.$$

Theorem 5 [19] *Let (X,d) be a complete metric space and F be α-admissible mapping. Assume that there exists $\psi \in \Psi$ such that*

$$\alpha(x,y)d(Fx,Fy) \le \psi\bigl(d(x,y)\bigr), \quad \forall x,y \in X \tag{1.1}$$

and suppose that:

(i) *there exists $x_0 \in X$ such that $\alpha(x_0,Fx_0) \ge 1$;*
(ii) *either F is continuous or for any sequence $\{x_n\}$ in X with $\alpha(x_n,x_{n+1}) \ge 1$, for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, we have $\alpha(x_n,x) \ge 1$, for all $n \in \mathbb{N} \cup \{0\}$.*

Then F has a fixed point.

Definition 6 A function F which is α-admissible and satisfying inequality (1.1) is called an (α-ψ)-contractive mapping.

In [20], Shahi *et al.* complements the concept of (α-ξ)-contractive type mappings by considering χ as a family of non-decreasing continuous functions $\xi : [0,+\infty) \to [0,+\infty)$ with the following conditions:

(i) $\sum_{n=1}^{\infty} \xi^n(t) < +\infty$ for all $t > 0$, where ξ^n is the nth iterate of ξ;
(ii) $\xi(t) < t$ for all $t > 0$;
(iii) $\xi(t_1 + t_2) = \xi(t_1) + \xi(t_2)$ for all $t_1, t_2 \in [0,+\infty)$.

Remark 7 If $F : X \to X$ is an expansion mapping, then F is an (α-ξ)-expansive mapping, where $\alpha(x,y) = 1$, for all $x,y \in X$, and $\xi(t) = kt$, for all $t \ge 0$ and for some $k \in [0,1)$.

Theorem 8 [20] *Let (X,d) be a complete metric space and $F : X \to X$ be bijective mapping. Suppose there exist functions $\xi \in \chi$ and $\alpha : X \times X \to [0,+\infty)$ such that*

$$\xi\big(d(Fx,Fy)\big) \geq \alpha(x,y)d(x,y) \tag{1.2}$$

for all $x, y \in X$. Suppose the following assertions hold:

(i) *the F^{-1} is α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, F^{-1}x_0) \geq 1$;*
(iii) *either F is continuous, or a sequence $\{x_n\}$ in X converging to $x^* \in X$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, we have*

$$\alpha\big(F^{-1}x_n, F^{-1}x^*\big) \geq 1$$

for all $n \in \mathbb{N} \cup \{0\}$.

Then there exists a point x^* in X such that $x^* = Fx^*$.

Definition 9 A function F which is α-admissible and satisfying inequality (1.2) is called an $(\alpha\text{-}\xi)$-expansive contractive mapping.

For more details as regards $(\alpha\text{-}\psi)$ fixed point theory we refer the reader to [21–26].

In this paper, we use the concept of α-admissible to study fixed point theorems for expansive mappings satisfying $\{\alpha,\xi\}$-contractive conditions in a complete metric spaces. We also provide a non-trivial example to support our main result.

2 Main result

In the following main result, we prove the existence of the fixed point of the mapping satisfying an (α,ξ)-contractive condition on the closed ball. Also it is crucial in the sense that it requires the contractiveness of the mapping only on the closed ball instead of the whole space.

Definition 10 Let (X,d) be a complete metric space and $F : X \to X$ be given mappings. We say that F is an $\{\alpha,\xi\}$-expansive locally contractive mapping if there exists $x_0 \in X$, $r > 0$ and the functions $\xi \in \chi$ and $\alpha : X \times X \to [0,+\infty)$ are such that

$$\xi\big(d(Fx,Fy)\big) \geq \alpha(x,y)d(x,y) \tag{2.1}$$

for all $x, y \in \overline{B(x_0,r)}$. For $x_0 \in X$ and $0 < r \in \mathbb{R}$, let $\overline{B(x_0,r)} = \{x \in X : d(x,x_0) \leq r\}$ be a closed ball of radius r centered at x_0.

Theorem 11 *Let (X,d) be a complete metric space and $F : X \to X$ be an $\{\alpha,\xi\}$-expansive locally contractive and bijective mapping such that*

$$r \geq \sum_{i=0}^{j} \xi^i\big(d\big(x_0, F^{-1}x_0\big)\big) \quad \text{for all } j \in \mathbb{N}. \tag{2.2}$$

Suppose that the following assertions hold:

(i) *F^{-1} is α-admissible;*

(ii) $\alpha(x_0, F^{-1}x_0) \geq 1$;

(iii) *either F is continuous, or a sequence* $\{x_n\}$ *in* $\overline{B(x_0,r)}$ *converges to* $x^* \in \overline{B(x_0,r)}$ *and* $\alpha(x_n, x_{n+1}) \geq 1$, *for all* $n \in \mathbb{N} \cup \{0\}$, *and we have*

$$\alpha\left(F^{-1}x_n, F^{-1}x^*\right) \geq 1$$

for all $n \in \mathbb{N} \cup \{0\}$.

Then there exists a point x^* *in* $\overline{B(x_0,r)}$ *such that* $x^* = Fx^*$.

Proof Let x_0 be an arbitrary point in X. Define the sequence $\{x_n\}$ as follows:

$$x_n = Fx_{n+1}, \quad n \in \mathbb{N} \cup \{0\}. \tag{2.3}$$

By assumption $\alpha(x_0, F^{-1}x_0) \geq 1$ and as F^{-1} is α-admissible, we have

$$\alpha\left(F^{-1}x_0, F^{-1}x_1\right) \geq 1,$$

so we deduce that $\alpha(x_1, x_2) \geq 1$, which implies that

$$\alpha\left(F^{-1}x_1, F^{-1}x_2\right) \geq 1.$$

Using the same argument, we obtain $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Let us show that $x_n \in \overline{B(x_0,r)}$ for all $n \in \mathbb{N}$. Using inequality (2.2), we get

$$r \geq \sum_{i=0}^{j} \xi^i\left(d\left(x_0, F^{-1}x_0\right)\right).$$

It follows that $x_1 \in (\overline{B(x_0,r)})$. Let $x_2, \ldots, x_j \in \overline{B(x_0,r)}$, for some $j \in \mathbb{N}$. Now we prove that $x_{j+1} \in \overline{B(x_0,r)}$,

$$\begin{aligned} d(x_j, x_{j+1}) &\leq \xi\left(d(Fx_j, Fx_{j+1})\right) = \xi\left(d(x_{j-1}, x_j)\right) \\ &\leq \xi^2\left(d(Fx_{j-1}, Fx_j)\right) \\ &\leq \cdots \leq \xi^j\left(d(x_0, x_1)\right). \end{aligned} \tag{2.4}$$

Notice that $x_{j+1} \in \overline{B(x_0,r)}$, since

$$\begin{aligned} d(x_0, x_{j+1}) &= d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_j, x_{j+1}) \\ &\leq \sum_{i=0}^{j} \xi^i\left(d(x_0, x_1)\right) \\ &\leq r. \end{aligned}$$

Hence $x_n \in \overline{B(x_0,r)}$ and $x_n = Fx_{n+1}$, for all $n \in \mathbb{N} \cup \{0\}$. From the inequality (2.4), we have

$$d(x_n, x_{n+1}) \leq \xi^n\left(d(x_0, x_1)\right) \tag{2.5}$$

for all $n \in \mathbb{N} \cup \{0\}$. Now let $\varepsilon > 0$ and let $n(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{n \geq n(\varepsilon)} \xi^n\big(d(x_0, x_1)\big) < \varepsilon.$$

Then for $n, m \in \mathbb{N}$ with $m > n > n(\varepsilon)$ and using the triangular inequality, we obtain

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \leq \sum_{k=n}^{m-1} \xi^k\big(d(x_0, x_1)\big) \\ &\leq \sum_{n \geq n(\varepsilon)} \xi^k\big(d(x_0, x_1)\big) < \varepsilon. \end{aligned}$$

Thus we have proved that $\{x_n\}$ is a Cauchy sequence in $\overline{B(x_0, r)}$. Since (X, d) is a complete space, there exists $x^* \in \overline{B(x_0, r)}$ such that $x_n \to x^*$. From the continuity of F, it follows that $x_{n-1} = Fx_n \to Fx^*$ as $n \to +\infty$. By the uniqueness of the limit, we get $x^* = Fx^*$, that is, x^* is a fixed point of F. As $\{x_n\}$ is a sequence in X such that $x_n \to x^*$ and $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$. We have

$$\alpha\big(F^{-1}x_n, F^{-1}x^*\big) \geq 1, \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{2.6}$$

Utilizing the inequalities (2.1), (2.6), and the triangular inequality, we obtain

$$\begin{aligned} d\big(F^{-1}x^*, x^*\big) &\leq d\big(F^{-1}x^*, x_{n+1}\big) + d\big(x_{n+1}, x^*\big) \\ &= d\big(F^{-1}x^*, F^{-1}x_n\big) + d\big(x_{n+1}, x^*\big) \\ &\leq \alpha\big(F^{-1}x_n, F^{-1}x^*\big) d\big(F^{-1}x^*, F^{-1}x_n\big) + d\big(x_{n+1}, x^*\big) \\ &\leq \xi\big(d\big(x_n, x^*\big)\big) + d\big(x_{n+1}, x^*\big). \end{aligned}$$

As $n \to \infty$, we can get $d(F^{-1}x^*, x^*) = 0$ by using the continuity of ξ. Therefore $F^{-1}x^* = x^*$. Then $Fx^* = F(F^{-1}x^*) = (FF^{-1})x^* = x^*$, hence the proof is completed. □

Example 12 Let $X = [0, +\infty)$ be endowed with the standard metric $d(x, y) = |x - y|$, for all $x, y \in X$. Define the mappings $F : X \to X$ and $\alpha : X \times X \to [0, +\infty)$ by

$$F(x) = \begin{cases} 2x, & \text{if } x \in [0, 1], \\ x + 5, & \text{otherwise} \end{cases}$$

and

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, 1], \\ \frac{5}{2}, & \text{otherwise.} \end{cases}$$

Then $\alpha(x, y) \geq 1$ for $x, y \in X$. Considering $x_0 = \frac{1}{2}$ and $r = \frac{1}{2}$, then $\overline{B(x_0, r)} = [0, 1]$. Clearly F is an α-ξ-contractive mapping with $\xi(t) = \frac{t}{2}$ as

$$\xi\big(d(Fx, Fy)\big) = \frac{d(Fx, Fy)}{2} = |x - y| = \alpha(x, y) d(x, y).$$

Now

$$\frac{1}{2} > \frac{1}{4} > \frac{1}{4}\sum_{i=0}^{n}\frac{1}{2^i} = \sum_{i=1}^{n}\xi^i\big(d\big(x_0, F^{-1}x_0\big)\big).$$

We prove that all the conditions of our main Theorem 11 are satisfied, only for $x, y \in \overline{B(x_0, r)}$. Now we prove that F^{-1} is α-admissible. Let $x, y \in X$ such that $\alpha(x, y) \geq 1$. This implies that $x \geq 1$ and $y \geq 1$. By the definitions of F^{-1} and α, by construction we have $\alpha(F^{-1}x, F^{-1}y) \geq 1$, since $x_0 = \frac{1}{2}$ and $F^{-1}x_0 = \frac{1}{4}$. Then by construction we have $\alpha(x_0, F^{-1}x_0) \geq 1$. Notice that F has fixed point 0. Now we prove that the contractive condition is not satisfied for $x, y \notin \overline{B(x_0, r)}$. We suppose $x = \frac{3}{2}$ and $y = 2$, then

$$\xi\big(d(Fx, Fy)\big) = \frac{d(Fx, Fy)}{2} = \frac{1}{4} \ngeq 5 = \alpha(x, y)d(x, y).$$

Now, to discuss the uniqueness of the fixed point deduced in Theorem 11, let us consider the following condition:

(P): For all $u, v \in \overline{B(x_0, r)}$, there exists $w \in \overline{B(x_0, r)}$ such that $\alpha(u, w) \geq 1$ and $\alpha(v, w) \geq 1$.

Then we get the following theorem.

Theorem 13 *Consider the same hypotheses of Theorem* 11, *together with condition* (P). *Then the obtained fixed point of F is unique.*

Proof From Theorem 11, the set of fixed points of F is non-empty. If u and v are two fixed points of F, that is, $Fu = u$ and $Fv = v$, then we can show that $u = v$. From the condition (P), there exists $w \in \overline{B(x_0, r)}$ such that $\alpha(u, w) \geq 1$ and $\alpha(v, w) \geq 1$. As F^{-1} is α-admissible, so we get

$$\alpha\big(u, F^{-1}w\big) \geq 1$$

and

$$\alpha\big(v, F^{-1}w\big) \geq 1$$

for all $n \in \mathbb{N} \cup \{0\}$. Therefore, by repeatedly applying the α-admissible property of F^{-1}, we get

$$\alpha\big(u, F^{-n}w\big) \geq 1 \tag{2.7}$$

and

$$\alpha\big(v, F^{-n}w\big) \geq 1 \tag{2.8}$$

for all $n \in \mathbb{N} \cup \{0\}$. Using the inequalities (2.1) and (2.7) and (2.8), we obtain

$$d\big(u, F^{-n}w\big) \leq \alpha\big(u, F^{-n}w\big)d\big(u, F^{-n}w\big) \leq \xi\big(d\big(u, F^{-n+1}w\big)\big)$$

for all $n \in \mathbb{N} \cup \{0\}$. Repeating the above inequality, we get

$$d\big(u, F^{-n}w\big) \leq \xi^n\big(d(u,w)\big) \tag{2.9}$$

for all $n \in \mathbb{N} \cup \{0\}$. Thus we have $F^{-n}w \to u$ as $n \to +\infty$. Using a similar technique to the above method, we obtain $F^{-n}w \to v$ as $n \to +\infty$. Now, the uniqueness of the limit of $F^{-n}w$ gives $u = v$. Hence the proof is completed. □

Now, we have the following result.

Theorem 14 *Let (X, d) be a complete metric space and let $F : X \to X$ be a bijective mapping. Suppose there exist functions $\xi \in \chi$ and $\alpha : X \times X \to [0, +\infty)$ such that*

$$\xi\big(d(Fx, Fy)\big) \geq \alpha(x,y)K(x,y), \quad \forall x, y \in X, \tag{2.10}$$

where

$$K(x,y) \in \big\{d(x, Fx), d(y, Fy)\big\}.$$

Suppose that the following assertions hold:

(i) *the F^{-1} is α-admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, F^{-1}x_0) \geq 1$*;
(iii) *either F is continuous, or a sequence $\{x_n\}$ in X converging to $x^* \in X$ and $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$, we have*

$$\alpha\big(F^{-1}x_n, F^{-1}x^*\big) \geq 1$$

for all $n \in \mathbb{N} \cup \{0\}$.

Then there exists a point x^* in X such that $x^* = Fx^*$.

Proof Let us define the sequence $\{x_n\}$ in X by

$$x_n = Fx_{n+1} n \in \mathbb{N} \cup \{0\},$$

where $x_0 \in X$ is chosen such that $\alpha(x_0, F^{-1}x_0) \geq 1$. Now, if $x_n = x_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, then n, x_n is a fixed point of F from the definition of $\{x_n\}$. Without loss of generality, we may assume that $x_n \neq x_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$. It is given that $\alpha(x_0, x_1) = \alpha(x_0, F^{-1}x_0) \geq 1$. Recalling that the F^{-1} is α-admissible, we have

$$\alpha(x_1, x_2) = \alpha\big(F^{-1}x_0, F^{-1}x_1\big) \geq 1.$$

Using mathematical induction, we obtain

$$\alpha(x_n, x_{n+1}) \geq 1 \tag{2.11}$$

for all $n \in \mathbb{N} \cup \{0\}$. Now, by (2.10) with $x = x_n$ and $y = x_{n+1}$, we obtain

$$K(x_n, x_{n+1}) \leq \alpha(x_n, x_{n+1})K(x_n, x_{n+1}) \leq \xi\big(d(Fx_n, Fx_{n+1})\big) = \xi\big(d(x_{n-1}, x_n)\big).$$

When $K(x_n, x_{n+1}) = d(Fx_n, x_n) = d(x_{n-1}, x_n)$, then we get a contradiction to the fact that $\xi(t) < t$. When $K(x_n, x_{n+1}) = d(Fx_{n+1}, x_{n+1}) = d(x_n, x_{n+1})$, then we get

$$d(x_n, x_{n+1}) \le \xi\big(d(x_{n-1}, x_n)\big) \tag{2.12}$$

for all $n \in \mathbb{N} \cup \{0\}$. Therefore, by repetition of the above inequality, we have

$$d(x_n, x_{n+1}) \le \xi\big(d(x_{n-1}, x_n)\big) \le \xi^2\big(d(x_{n-2}, x_{n-1})\big) \le \cdots \le \xi^n\big(d(x_0, x_1)\big). \tag{2.13}$$

Given $\varepsilon > 0$ and let $n(\varepsilon) \in \mathbb{N}$ such that $\sum_{n \ge n(\varepsilon)} \xi^n(d(x_0, x_1)) < \varepsilon$. Let $n, m \in \mathbb{N}$ with $m > n > n(\varepsilon)$ and use the triangular inequality; we obtain

$$\begin{aligned} d(x_n, x_m) &\le \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \le \sum_{k=n}^{m-1} \xi^k\big(d(x_0, x_1)\big) \\ &\le \sum_{n \ge n(\varepsilon)} \xi^k\big(d(x_0, x_1)\big) < \varepsilon. \end{aligned}$$

Thus we proved that $\{x_n\}$ is a Cauchy sequence in X. As (X, d) is a complete metric space, there exists $x^* \in X$ such that $x_n \to x^*$. Suppose F is continuous, it follows that $x_{n-1} = Fx_n \to Fx^*$ as $n \to +\infty$. By the uniqueness of the limit, we get $x^* = Fx^*$, that is, x^* is a fixed point of F, since $\{x_n\}$ is a sequence in X such that $x_n \to x^*$ and $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N} \cup \{0\}$. So from the hypotheses, we have

$$\alpha\big(F^{-1}x_n, F^{-1}x^*\big) \ge 1 \tag{2.14}$$

for all $n \in \mathbb{N} \cup \{0\}$. Utilizing the inequalities (2.10), (2.14), and the triangular inequality, we obtain

$$K\big(F^{-1}x^*, F^{-1}x_n\big) \le \alpha\big(F^{-1}x_n, F^{-1}x^*\big)K\big(F^{-1}x^*, F^{-1}x_n\big) \le \xi\big(d(x_n, x^*)\big),$$

where

$$K\big(F^{-1}x^*, F^{-1}x_n\big) \in \big\{d\big(F^{-1}x^*, x^*\big), d\big(F^{-1}x_n, x_n\big)\big\}.$$

In any case, by taking the limit as $n \to \infty$, we get $d(F^{-1}x^*, x^*) = 0$. Therefore $F^{-1}x^* = x^*$. Thus, $Fx^* = F(F^{-1}x^*) = (FF^{-1})x^* = x^*$. Hence, F has a fixed point in X. □

Remark 15 The function F may have more than one fixed point.

Finally, we prove a Suzuki type-fixed point result for expansive mappings in which the continuity of the mapping is needed. However, it is still unknown whether the continuity is a necessary condition or not.

Theorem 16 *Let (X, d) be a complete metric space and let $F : X \to X$ be bijective mapping. Define a non-decreasing function $\theta : (1, +\infty) \to (1, 2)$ by $\theta(r) = 1 + \frac{1}{r}$. Assume that there exists $r > 1$ such that $\theta(r)d(x, Fx) \ge d(x, y)$ implies $d(Fx, Fy) \ge rd(x, y)$ for all $x, y \in X$. If F is a continuous function, there exists a point $x^* \in X$ such that $x^* = Fx^*$.*

Proof Let $x_0 \in X$. We define the sequence $\{x_n\}$ in X by

$$x_n = Fx_{n+1} n \in \mathbb{N} \cup \{0\}.$$

Since $\theta(r) > 1$, we get $d(x_{n+1}, x_n) < \theta(r) d(x_{n+1}, x_n) = \theta(r) d(x_{n+1}, Fx_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$. By the hypotheses

$$rd(x_{n+1}, x_n) \leq d(Fx_{n+1}, Fx_n) = d(x_n, x_{n-1})$$

for all $n \in \mathbb{N} \cup \{0\}$. This implies that

$$d(x_{n+1}, x_n) \leq ld(x_n, x_{n-1}) \leq \cdots \leq l^n d(x_1, x_0),$$

where $l = \frac{1}{r} < 1$. One can easily prove that $\{x_n\}$ is a Cauchy sequence. As X is a complete metric space, $\{x_n\}$ converges to some $x^* \in X$. Since F is a continuous function, we get

$$Fx^* = F\Big(\lim_{n \to \infty} x_{n+1}\Big) = \lim_{n \to \infty} F(x_{n+1}) = x^*.$$

Thus F has a fixed point, and hence the proof is completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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