Romaguera and Tirado *Fixed Point Theory and Applications* (2015) 2015:183
DOI 10.1186/s13663-015-0431-1

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

A characterization of Smyth complete quasi-metric spaces via Caristi's fixed point theorem

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Abstract
We obtain a quasi-metric generalization of Caristi's fixed point theorem for a kind of complete quasi-metric spaces. With the help of a suitable modification of its proof, we deduce a characterization of Smyth complete quasi-metric spaces which provides a quasi-metric generalization of the well-known characterization of metric completeness due to Kirk. Some illustrative examples are also given. As an application, we deduce a procedure which allows to easily show the existence of solution for the recurrence equation of certain algorithms.

MSC: 54H25; 47H10; 54E50; 68Q25

Keywords: fixed point; quasi-metric; complete; Smyth complete; algorithm; recurrence equation

1 Introduction and preliminaries

We start by recalling several notions and properties of the theory of quasi-metric spaces. Our basic references are [1] and [2].

By a quasi-metric on set X we mean a function $d : X \times X \to [0,\infty)$ such that for all $x, y, z \in X$: (i) $x = y \Leftrightarrow d(x,y) = d(y,x) = 0$; (ii) $d(x,z) \leq d(x,y) + d(y,z)$.

A quasi-metric space is a pair (X,d) such that X is a set and d is a quasi-metric on X.

Given a quasi-metric d on X, the function d^{-1} defined by $d^{-1}(x,y) = d(y,x)$ is also a quasi-metric on X, called the conjugate of d, and the function d^s defined by $d^s(x,y) = \max\{d(x,y), d^{-1}(x,y)\}$ is a metric on X.

Each quasi-metric d on X induces a T_0 topology τ_d on X which has as a base the family of open balls $\{B_d(x,r) : x \in X, \varepsilon > 0\}$, where $B_d(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

If τ_d is a T_1 topology on X, we say that (X,d) is a T_1 quasi-metric space.

Note that a quasi-metric space (X,d) is T_1 if and only if for each $x, y \in X$, condition $d(x,y) = 0$ implies $x = y$.

There exist many different notions of Cauchy net, Cauchy sequence and quasi-metric completeness in the literature (see, *e.g.*, [1–3]). For our purposes, here we will consider the following ones.

A net $(x_\alpha)_{\alpha\in\Lambda}$ in a quasi-metric space (X,d) is called left K-Cauchy if for each $\varepsilon > 0$ there is $\alpha_\varepsilon \in \Lambda$ such that $d(x_\alpha, x_\beta) < \varepsilon$ whenever $\alpha_\varepsilon \leq \alpha \leq \beta$. The notion of a left K-Cauchy sequence is defined in the obvious manner.

We say that a quasi-metric space (X,d) is complete if every left K-Cauchy net is convergent for $\tau_{d^{-1}}$, and say that it is sequentially complete if every left K-Cauchy sequence is convergent for $\tau_{d^{-1}}$. (Note that our notion of (sequential) completeness of (X,d) coincides with the usual notion of right K-(sequential) completeness of (X,d^{-1}).)

A quasi-metric space (X,d) is Smyth complete provided that every left K-Cauchy net in (X,d) is convergent for τ_{d^s} (compare Definition 8 in [4], [5], p.454, *etc.*).

The following well-known result is a consequence of Definition 8 and Theorem 9 in [4] (see also [6], p.323, [7], p.347).

Proposition 1 *A quasi-metric space* (X,d) *is Smyth complete if and only if every left* K-Cauchy *sequence in* (X,d) *is convergent for* τ_{d^s}.

The following implications are also known and easy to check:

$$\text{Smyth complete} \quad \Rightarrow \quad \text{complete} \quad \Rightarrow \quad \text{sequentially complete.}$$

However, the converse implications do not hold, in general. For instance, the Sorgenfrey quasi-metric space (see, *e.g.*, [5], p.463 or Example 1.1.6 in [1]) provides a distinguished example of a complete T_1 quasi-metric space which is not Smyth complete, while Stoltenberg presented in Example 2.4 of [8] an example of a sequentially complete T_1 quasi-metric space which is not complete.

On the other hand, Caristi proved in 1976 the following important and well-known generalization of the Banach contraction principle.

Theorem 1 ([9]) *Let* T *be a self-mapping of a complete metric space* (X,d). *If there is a lower semicontinuous function* $\varphi : X \to [0,\infty)$ *satisfying*

$$d(x,Tx) \leq \varphi(x) - \varphi(Tx)$$

for all $x \in X$, *then* T *has a fixed point in* X.

Kirk showed in [10] that the validity of Caristi's fixed point theorem in a metric space characterizes its completeness. More exactly, he proved the following.

Theorem 2 ([10]) *For a metric space* (X,d), *the following conditions are equivalent*:

(1) (X,d) *is complete.*
(2) *If* T *is a self-mapping of* X *such that there is a lower semicontinuous function* $\varphi : X \to [0,\infty)$ *satisfying* $d(x,Tx) \leq \varphi(x) - \varphi(Tx)$ *for all* $x \in X$, *then* T *has a fixed point in* X.

Extensions and generalizations of Theorems 1 and 2 to partial metric spaces, cone metric spaces, quasi-metric spaces and probabilistic metric spaces have been obtained by several authors (see, *e.g.*, [11–20]). In particular, Cobzaş ([15], Theorem 2.3) proved, among other interesting results, the following quasi-metric generalization of Caristi's fixed point theorem.

Theorem 3 ([15]) *Let T be a self-mapping of a sequentially complete T_1 quasi-metric space (X,d). If there is a function $\varphi : X \to [0,\infty)$ which is lower semicontinuous for $\tau_{d^{-1}}$ and satisfies*

$$d(x,Tx) \leq \varphi(x) - \varphi(Tx)$$

for all $x \in X$, then T has a fixed point in X.

Since complete and Smyth complete non-T_1 quasi-metric spaces provide efficient tools in several areas as asymmetric functional analysis, domain theory, theoretical computer science, complexity analysis of algorithms defined by recurrence equations, *etc.* (see, *e.g.*, [1, 4, 5, 7, 21, 22] and their references), it seems natural to discuss the question of generalizing Theorem 3 to (non-necessarily T_1) quasi-metric spaces. In this direction, we shall give an example of a sequentially complete quasi-metric space for which Theorem 3 does not hold. We shall show that, nevertheless, Theorem 3 remains valid for complete quasi-metric spaces. A suitable and slight modification of the proof of that result will be used to deduce a characterization of Smyth complete quasi-metric spaces which provides a generalization to the quasi-metric framework of Kirk's characterization of metric completeness. As an application, we obtain a procedure which allows to easily deduce the existence of solution for the recurrence equation of certain algorithms.

2 Results and examples

In order to simplify the terminology and the statements of our results, we shall use the following notions.

A self-mapping T of a quasi-metric space (X,d) will be called a d-Caristi mapping (resp. a d^s-Caristi mapping) on (X,d) if there is a function $\varphi : X \to [0,\infty)$ which is lower semicontinuous for $\tau_{d^{-1}}$ (resp. for τ_{d^s}) and satisfies $d(x,Tx) \leq \varphi(x) - \varphi(Tx)$ for all $x \in X$.

Clearly, every d-Caristi mapping is a d^s-Caristi mapping. The following example shows that the converse is not true in general.

Example 1 Let d be the quasi-metric on the set $\mathbb{N}$ of all positive integer numbers, given by $d(x,x) = 0$ for all $x \in \mathbb{N}$ and $d(x,y) = 1/x$ for all $x,y \in \mathbb{N}$ with $x \neq y$. Clearly $(\mathbb{N},d)$ is a T_1 quasi-metric space such that τ_d, and hence τ_{d^s} is the discrete topology on $\mathbb{N}$. Define $T : \mathbb{N} \to \mathbb{N}$ as $Tx = 2x$ for all $x \in \mathbb{N}$. Then $d(x,Tx) = 1/x = \varphi(x) - \varphi(Tx)$, where $\varphi : \mathbb{N} \to [0,\infty)$ is defined as $\varphi(x) = 2/x$ for all $x \in \mathbb{N}$. Since τ_{d^s} is the discrete topology on $\mathbb{N}$, φ is lower semicontinuous for τ_{d^s} and thus T is a d^s-Caristi mapping on $(\mathbb{N},d)$. Finally, suppose that T is also a d-Caristi mapping. Then there exists a function $\varphi : \mathbb{N} \to [0,\infty)$ which is lower semicontinuous for $\tau_{d^{-1}}$ and satisfies $d(x,2x) = 1/x \leq \varphi(x) - \varphi(2x)$ for all $x \in \mathbb{N}$. We easily deduce that $\varphi(1) \geq 1 + \varphi(2^x)$ for all $x \in \mathbb{N}$, which contradicts that φ is a lower semicontinuous function for $\tau_{d^{-1}}$ because the sequence $(2^n)_{n\in\mathbb{N}}$ converges to 1 for $\tau_{d^{-1}}$.

Our next example, based on Example 2.1 in [5], shows that condition T_1 cannot be removed in Theorem 3.

Example 2 Let $(\mathcal{A},d)$ be the non-T_1 quasi-metric space such that $\mathcal{A}$ is the family of all nonempty countable subsets of the set $\mathbb{R}$ of all real numbers, and d is the quasi-metric on $\mathcal{A}$ defined as $d(A,B) = 0$ if $A \subseteq B$, and $d(A,B) = 1$ otherwise. Let $(A_n)_{n\in\mathbb{N}}$ be a left K-Cauchy sequence in $(\mathcal{A}, d)$. Assume, without loss of generality, that $d(A_n, A_m) = 0$ whenever $n \leq m$, *i.e.*, $A_n \subseteq A_m$ whenever $n \leq m$. Since $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$ and $d(A_n, \bigcup_{n\in\mathbb{N}} A_n) = 0$ for all $n \in \mathbb{N}$, we deduce that $(\mathcal{A}, d)$ is sequentially complete. Now let

$$\Lambda = \{A \in \mathcal{A} : A \text{ is a nonempty finite subset of } \mathbb{R} \text{ consisting of irrational numbers}\}$$

ordered by inclusion. Then the net $(A)_{A\in\Lambda}$ is left K-Cauchy in $(\mathcal{A}, d)$ (see Example 2.1 in [5]) but it does not converge for $\tau_{d^{-1}}$ because the elements of $\mathcal{A}$ are countable subsets of $\mathbb{R}$. We conclude that $(\mathcal{A}, d)$ is not complete.

However, we have the following extension of Theorem 3 whose proof is based on a classical technique used by Kirk [10], which is inspired in the partial order of Brøndsted [23, 24].

Theorem 4 *Every d-Caristi mapping on a complete quasi-metric space* (X, d) *has a fixed point in* X.

Proof Let (X, d) be a complete quasi-metric space and let $T : X \to X$ be a d-Caristi mapping on (X, d). Then there exists a function $\varphi : X \to [0, \infty)$ which is lower semicontinuous for $\tau_{d^{-1}}$ and satisfies

$$d(x, Tx) \leq \varphi(x) - \varphi(Tx)$$

for all $x \in X$. As in the classical metric case, define a binary relation $\preceq$ on X by

$$x \preceq y \quad \Longleftrightarrow \quad d(x, y) \leq \varphi(x) - \varphi(y)$$

for all $x, y \in X$. Clearly $\preceq$ is a partial order on X. Note also that $x \preceq Tx$ for all $x \in X$.

We shall prove that every (nonempty) linearly ordered subset of the partially ordered set $(X, \preceq)$ has an upper bound. Indeed, let A be a (nonempty) linearly ordered subset of X. We show that the net $(x_x)_{x\in A}$ is a left K-Cauchy net in (X, d) where we have defined $x_x := x$ for all $x \in A$. To this end, put $r = \inf_{x\in A} \varphi(x)$. Given an arbitrary $\varepsilon > 0$, choose $x \in A$ such that $\varphi(x) < r + \varepsilon$. Thus, for any $y, z \in A$ with $x \preceq y \preceq z$, we obtain

$$d(y, z) \leq \varphi(y) - \varphi(z) \leq \varphi(x) - \varphi(z) < r + \varepsilon - r = \varepsilon.$$

Consequently, $(x_x)_{x\in A}$ is a left K-Cauchy net in (X, d), and hence it converges, for $\tau_{d^{-1}}$, to some $p \in X$. Fix $x \in A$ and let $\varepsilon > 0$ be arbitrary. Then there is $y \in A$ such that $d(z, p) < \varepsilon$ and $\varphi(p) - \varphi(z) < \varepsilon$ whenever $z \in A$ and $y \preceq z$. Choose $z_0 \in A$ with $x \preceq z_0$ and $y \preceq z_0$. Hence

$$\begin{aligned} d(x, p) &\leq d(x, z_0) + d(z_0, p) < \varphi(x) - \varphi(z_0) + \varepsilon \\ &< \varphi(x) - \varphi(p) + 2\varepsilon. \end{aligned}$$

Since ε is arbitrary, we deduce that $d(x, p) \leq \varphi(x) - \varphi(p)$, *i.e.*, $x \preceq p$, so p is an upper bound of A. It follows from Zorn's lemma that $(X, \preceq)$ has a maximal element, say a. Since $a \preceq Ta$, we conclude that $a = Ta$, so a is a fixed point of T. The proof is finished. □

Of course, Caristi's fixed point theorem is a consequence of Theorem 4 when (X,d) is a metric space. Next we present two examples of complete quasi-metric spaces (X,d) with appropriate d-Caristi mappings, for which Caristi's fixed point theorem cannot be applied to the metric space (X,d^s).

Example 3 Let $X = \mathbb{N} \cup \{\infty\}$. Define a nonnegative real-valued function d on $X \times X$ by $d(\infty,\infty) = 0$, $d(x,y) = |1/x - 1/y|$ if $x,y \in \mathbb{N}$, $d(x,\infty) = 1/x$ and $d(\infty,x) = 1$ for all $x \in \mathbb{N}$. It is easily seen that (X,d) is a complete T_1 quasi-metric space (in fact, note $(X,\tau_{d^{-1}})$ is a compact topological space). Define $T : X \to X$ as $T\infty = \infty$, and $Tx = x^2$ for all $x \in \mathbb{N}$. Now define $\varphi : X \to [0,\infty)$ as $\varphi(\infty) = 0$, and $\varphi(x) = 1/x$ for all $x \in \mathbb{N}$. Then φ is clearly a lower semicontinuous function for $\tau_{d^{-1}}$. Since $d(\infty, T\infty) = d(1, T1) = 0$, and for every $x \in X\backslash\{1,\infty\}$,

$$d(x, Tx) = \frac{1}{x} - \frac{1}{x^2} = \varphi(x) - \varphi(Tx),$$

we conclude that T is a d-Caristi mapping on (X,d). Hence, we can apply Theorem 4 to this case. In fact, T has 1 and ∞ as fixed points. However, we cannot apply Caristi's fixed point theorem to the metric space (X,d^s) because it is not complete. Indeed, $(x)_{x\in\mathbb{N}}$ is a Cauchy sequence in (X,d^s) that does not converge for τ_{d^s}.

In the above example the metric space (X,d^s) is not complete. Now, we give an example of a complete quasi-metric space (X,d) where the metric space (X,d^s) is complete and there is a d-Caristi mapping on (X,d) which is not a Caristi mapping for the metric space (X,d^s).

Example 4 As in Example 3, let $X = \mathbb{N} \cup \{\infty\}$. Define a nonnegative real-valued function d on $X \times X$ by $d(x,y) = 0$ if $x \le y$, and $d(x,y) = y$ if $y < x$ (here, $\le$ denotes the usual order on X). It is routine to check that (X,d) is a complete quasi-metric space (note that every net in X converges to ∞ for $\tau_{d^{-1}}$). Define $T : X \to X$ as $Tx = x+1$ for all $x \in \mathbb{N}$ and $T\infty = \infty$. Then $d(x, Tx) = 0$ for all $x \in X$, so that T is trivially a d-Caristi mapping on (X,d). Hence, we can apply Theorem 4. Finally, suppose that there exists a lower semicontinuous function, for τ_{d^s}, $\varphi : X \to [0,\infty)$, such that $d^s(x, Tx) \le \varphi(x) - \varphi(Tx)$ for all $x \in X$. Then

$$d^s(x,x+1) = x+1 \le \varphi(x) - \varphi(x+1)$$

for all $x \in \mathbb{N}$. We deduce that $\varphi(1) = \infty$, a contradiction. Hence, we cannot apply the classical Caristi fixed point theorem in this case.

Observe that the aforementioned example of Stoltenberg and Example 4 (or Example 3) above show that Theorems 3 and 4 are independent of each other.

Although we do not know whether the converse of Theorem 4 holds, *i.e.*, if Kirk's theorem can be generalized to complete quasi-metric spaces, we are going to show that it is possible to obtain such a generalization for Smyth complete quasi-metric spaces. To this end, the following essentially well-known fact (see, *e.g.*, Proposition 1.2.4 in [1]) will be useful.

Proposition 2 *Let $(x_n)_{n\in\mathbb{N}}$ be a left K-Cauchy sequence in a quasi-metric space (X,d). If $(x_n)_{n\in\mathbb{N}}$ has a subsequence convergent to $x \in X$ for τ_{d^s}, then $(x_n)_{n\in\mathbb{N}}$ converges to x for τ_{d^s}.*

Theorem 5 *A quasi-metric space (X,d) is Smyth complete if and only if every d^s-Caristi mapping on (X,d) has a fixed point in X.*

Proof Suppose that (X,d) is a Smyth complete quasi-metric space, and let T be a d^s-Caristi mapping on (X,d). Then there exists a function $\varphi : X \to [0,\infty)$ which is lower semicontinuous for τ_{d^s} and satisfies $d(x,Tx) \leq \varphi(x) - \varphi(Tx)$ for all $x \in X$. Exactly as in the proof of Theorem 4, we construct a left K-Cauchy net in (X,d), which converges for τ_{d^s} to an element $p \in X$ by Smyth completeness of (X,d). Finally, we deduce that p is a fixed point of T again as in the proof of Theorem 4 and taking into account that φ is now lower semicontinuous for τ_{d^s}.

Conversely, it will be enough to prove, by Proposition 1, that every left K-Cauchy sequence in (X,d) converges for τ_{d^s}. Assume the contrary. Then there exists a left K-Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in (X,d) which is not convergent for τ_{d^s}. For each $k \in \mathbb{N}$, there exists $n_k \geq k$ such that $d(x_{n_k},x_n) < 2^{-(k+1)}$ for all $n \geq n_k$. Therefore $d(x_{n_k},x_{n_{k+1}}) < 2^{-(k+1)}$ for all $k \in \mathbb{N}$. Put $y_k := x_{n_k}$ for all $k \in \mathbb{N}$. Then, by Proposition 2, we can suppose, without loss of generality, that $y_k \neq y_j$ whenever $k \neq j$, and that the sequence $\{y_k : k \in \mathbb{N}\}$ does not have any convergent subsequence for τ_{d^s}.

We want to show that the self-mapping T of X given by $Ty_k = y_{k+1}$ for all $k \in \mathbb{N}$, and $Tx = y_1$ for all $x \notin \{y_k : k \in \mathbb{N}\}$, is a d^s-Caristi mapping. To this end, construct a function $\varphi : X \to [0,\infty)$ as follows: $\varphi(y_k) = 2^{-k}$ for all $k \in \mathbb{N}$, and $\varphi(x) = d^s(x,y_1) + 1/2$ whenever $x \notin \{y_k : k \in \mathbb{N}\}$. Since, for each $k \in \mathbb{N}$, $\varphi(y_k) < \varphi(x)$ whenever $x \notin \{y_k : k \in \mathbb{N}\}$, and the function $x \to d^s(x,y_1)$ is continuous for τ_{d^s}, we immediately deduce that φ is lower semicontinuous for τ_{d^s}. Moreover, we have

$$d(y_k, Ty_k) = d(y_k, y_{k+1}) < 2^{-(k+1)} = \varphi(y_k) - \varphi(Ty_k)$$

for all $k \in \mathbb{N}$, and

$$d(x, Tx) = d(x, y_1) \leq d^s(x, y_1) = \varphi(x) - \varphi(Tx)$$

for all $x \notin \{y_k : k \in \mathbb{N}\}$, so T is a d^s-Caristi mapping on (X,d). However, T has no fixed point. This contradiction concludes the proof. □

As in the metric case, we are going to deduce a multivalued version of Theorem 5.

Given a quasi-metric space (X,d), we denote by $\mathcal{P}_0(X)$ the collection of all nonempty subsets of X. A multivalued mapping $T : X \to \mathcal{P}_0(X)$ will be called d^s-Caristi on (X,d) if there is a function $\varphi : X \to [0,\infty)$ which is lower semicontinuous for τ_{d^s} and satisfies the following condition: For each $x \in X$, there exists $y_x \in Tx$ such that $d(x,y_x) \leq \varphi(x) - \varphi(y_x)$.

As usual, we say that a point $z \in X$ is a fixed point of $T : X \to \mathcal{P}_0(X)$ if $z \in Tz$.

Corollary *A quasi-metric space (X,d) is Smyth complete if and only if every d^s-Caristi multivalued mapping on (X,d) has a fixed point.*

Proof Suppose that (X,d) is Smyth complete, and let $T : X \to \mathcal{P}_0(X)$ be a d^s-Caristi multivalued mapping. Then there is a function $\varphi : X \to [0,\infty)$ which is lower semicontinuous for τ_{d^s} and satisfies that for each $x \in X$ there exists $y_x \in Tx$ such that $d(x,y_x) \leq \varphi(x) - \varphi(y_x)$. Define a self-mapping f on X as follows: $fx = y_x$ for all $x \in X$. Obviously f is a d^s-Caristi mapping on (X,d), so, by Theorem 5, there is $z \in X$ such that $z = fz$. Therefore $z = y_z$. Since $y_z \in Tz$, we conclude that z is a fixed point of T.

Conversely, suppose that every d^s-Caristi multivalued mapping on (X,d) has a fixed point. Then every d^s-Caristi mapping on (X,d) has a fixed point, so (X,d) is Smyth complete by Theorem 5. □

Note that if (X,d) is a quasi-metric space and T is a self-mapping of X such that $d(x, Tx) = 0$ for all $x \in X$, then T is a d^s-Caristi mapping on (X,d). If, in addition, (X,d) is Smyth complete, then T has a fixed point by Theorem 5. Our next example illustrates this situation.

Example 5 Let Σ be a nonempty alphabet. Denote by Σ^∞ the set of all finite and infinite words (sequences) over Σ, and denote by ϕ the empty word. For each $x,y \in \Sigma^\infty$, we define $x \sqcap y$ as the longest common prefix of x and y, and for each $x \in \Sigma^\infty$, we denote by $\ell(x)$ the length of x. Then $\ell(x) \in [1,\infty]$ whenever $x \neq \phi$ and $\ell(\phi) = 0$. Now, for each $x,y \in \Sigma^\infty$, let $d(x,y) = 0$ if x is a prefix of y, and $d(x,y) = 2^{-\ell(x\sqcap y)}$ otherwise. Then d is a quasi-metric on Σ^∞ [6, 25]. In fact, the quasi-metric space (Σ^∞,d) is Smyth complete [5], Example 3.1. Define $T : \Sigma^\infty \to \Sigma^\infty$ as follows: For each $x \in \Sigma^\infty$, Tx is an element of Σ^∞ such that x is a prefix of Tx with $\ell(Tx) = \ell(x) + 1$. Then $d(x, Tx) = 0$ for all $x \in \Sigma^\infty$. By Theorem 5, T has a fixed point. In fact, $Tx = x$ if and only if $\ell(x) = \infty$.

Observe that if (X,d) is a non-Smyth complete quasi-metric space such that (X,d^s) is complete, we can apply Caristi's fixed point theorem to (X,d^s). However, by Theorem 5, there exists a d^s-Caristi mapping on (X,d) without fixed point. We conclude this section with an example illustrating this fact.

Example 6 Let d be the quasi-metric on $\mathbb{R}$ given by $d(x,y) = y - x$ if $x \leq y$, and $d(x,y) = 1$ if $x > y$. Then $(\mathbb{R},d)$ is the Sorgenfrey quasi-metric space. Since $d^s(x,y) \geq 1$ for all $x,y \in \mathbb{R}$ with $x \neq y$, we deduce that the metric space $(\mathbb{R},d^s)$ is complete and τ_{d^s} is the discrete topology on $\mathbb{R}$. As we indicated in Section 1, $(\mathbb{R},d)$ is not Smyth complete (indeed, note that the sequence $((n-1)/n)_{n\in\mathbb{N}}$ is left K-Cauchy but it does not converge for τ_{d^s}). Define $T : \mathbb{R} \to \mathbb{R}$ as $Tx = 0$ for all $x > 0$, $T0 = -1$, and $Tx = x/2$ for all $x < 0$. Although T has no fixed point, we show that it is a d^s-Caristi mapping on $(\mathbb{R},d)$. To this end, define $\varphi : \mathbb{R} \to [0,\infty)$ as $\varphi(x) = 3$ for all $x > 0$, $\varphi(0) = 2$, and $\varphi(x) = -x$ for all $x < 0$. Obviously φ is lower semicontinuous for τ_{d^s}. Moreover, for $x > 0$, we obtain

$$d(x, Tx) = d(x, 0) = 1 = \varphi(x) - \varphi(Tx).$$

For $x = 0$, we obtain

$$d(x, Tx) = d(0, -1) = 1 = \varphi(x) - \varphi(Tx),$$

and for $x < 0$,

$$d(x, Tx) = d\left(x, \frac{x}{2}\right) = -\frac{x}{2} = \varphi(x) - \varphi(Tx).$$

Hence T is a d^s-Caristi mapping on (X,d) without fixed point. Finally, observe that for $x = -1$ one has

$$d^s(x, Tx) = 1 > \frac{1}{2} = \varphi(x) - \varphi(Tx).$$

3 An application

In this section we shall apply Theorem 5 to obtaining a general fixed point theorem in the setting of the complexity space, from which we shall deduce, in a unified and fast way, the existence of solution for a large class of algorithms defined by recurrence equations that includes Hanoi, Largetwo (average case), and Quicksort (worst case), (see, *e.g.*, [26] for a detailed study of these algorithms).

Let us recall that the so-called complexity space was introduced by Schellekens in [27] to the development of a topological foundation for the complexity analysis of algorithms and programs. Further contributions to the study of this space and its applications may be found in [7, 22, 28–30], *etc.*

The complexity space is the quasi-metric space $(\mathcal{C}, d_{\mathcal{C}})$, where

$$\mathcal{C} = \left\{ f : \mathbb{N} \to (0,\infty] : \sum_{n=1}^{\infty} 2^{-n} \frac{1}{f(n)} < \infty \right\},$$

and $d_{\mathcal{C}}$ is the quasi-metric on $\mathcal{C}$ given by

$$d_{\mathcal{C}}(f,g) = \sum_{n=1}^{\infty} 2^{-n} \max\left(\frac{1}{g(n)} - \frac{1}{f(n)}, 0 \right)$$

for all $f, g \in \mathcal{C}$. (We adopt the convention that $1/\infty = 0$.)

The set $\{f \in \mathcal{C} : f(n) < \infty \text{ for all } n \in \mathbb{N}\}$ is denoted by $\mathcal{C}_0$.

The elements of $\mathcal{C}$ are called complexity functions. According to Schellekens [27], p.540, given two complexity functions f and g, the numerical value $d_{\mathcal{C}}(f,g)$ (the complexity distance from f to g) can be interpreted as the relative progress made in lowering the complexity by replacing any program P with complexity function f by any program Q with complexity function g. Therefore, condition $d_{\mathcal{C}}(f,g) = 0$, with $f \neq g$, can be read as the program P is at least as efficient as the program Q because $d_{\mathcal{C}}(f,g) = 0$ if and only if $f(n) \leq g(n)$ for all $n \in \mathbb{N}$. Obviously, the metric $(d_{\mathcal{C}})^s$ is not able to give this information since in the case that $d_{\mathcal{C}}(f,g) = 0$, with $f \neq g$, we deduce that $d_{\mathcal{C}}(g,f) = (d_{\mathcal{C}})^s(f,g)$, and thus the last measure does not indicate that program is more efficient. However, we know that the program with complexity function f is more efficient than the one with complexity function g (see [27], p.541).

Now let c and a be positive real constants and $h \in \mathcal{C}_0$. Define

$$\mathcal{C}_{cah} = \left\{ f \in \mathcal{C} : f(1) = c \text{ and } f(n) \geq af(n-1) + h(n) \text{ for all } n \geq 2 \right\}.$$

Observe that $\mathcal{C}_{cah} \neq \emptyset$ since the complexity function f_1 defined by $f_1(1) = c$ and $f_1(n) = \infty$ for all $n \geq 2$ clearly belongs to $\mathcal{C}_{cah}$.

The restriction of the quasi-metric $d_{\mathcal{C}}$ to $\mathcal{C}_{cah}$ will be denoted by $d_{\mathcal{C}_{cah}}$.

The following auxiliary results will be useful in the proof of the main result of this section (Theorem 6 below).

Lemma 1 *Let* $(f_k)_{k\in\mathbb{N}}$ *be a sequence in* $\mathcal{C}$ *such that* $\lim_{k\to\infty}(d_{\mathcal{C}})^s(f,f_k) = 0$ *for some* $f \in \mathcal{C}$, *and let* $m \in \mathbb{N}$.

(a) *If* $f(m) < \infty$, *then* $f_k(m) < \infty$ *eventually, and* $\lim_{k\to\infty} f_k(m) = f(m)$.

(b) $f(m) = \infty$ *if and only if* $\lim_{k\to\infty} f_k(m) = \infty$.

Proof Since $\lim_{k\to\infty}(d_{\mathcal{C}})^s(f,f_k) = 0$, for each $\varepsilon > 0$, there is $k_\varepsilon \in \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} 2^{-n}\left|\frac{1}{f(n)} - \frac{1}{f_k(n)}\right| < \varepsilon$$

for all $k \geq k_\varepsilon$. In particular

$$2^{-m}\left|\frac{1}{f(m)} - \frac{1}{f_k(m)}\right| < \varepsilon \tag{1}$$

for all $k \geq k_\varepsilon$.

Suppose that $f(m) < \infty$. Taking $\varepsilon = 2^{-m}/f(m)$, it follows from (1) that $f_k(m) < \infty$ for all $k \geq k_\varepsilon$. Hence $\lim_{k\to\infty} f_k(m) = f(m)$ by Proposition 2 of [7]. Thus, we have shown (a).

If $f(m) = \infty$, relation (1) gives $2^{-m}/\varepsilon < f_k(m)$ for all $k \geq k_\varepsilon$. Since $\varepsilon > 0$ is chosen arbitrarily, we deduce that $\lim_{k\to\infty} f_k(m) = \infty$. Conversely, if $\lim_{k\to\infty} f_k(m) = \infty$, again it follows from (1) that $1/f(m) = 0$, *i.e.*, $f(m) = \infty$. Thus, we have shown (b). □

Lemma 2 ([22]) *The quasi-metric space* $(\mathcal{C}, d_{\mathcal{C}})$ *is Smyth complete.*

Lemma 3 *Let c and a be positive real constants and* $h \in \mathcal{C}_0$. *Then the quasi-metric space* $(\mathcal{C}_{cah}, d_{\mathcal{C}_{cah}})$ *is Smyth complete.*

Proof We first show that $\mathcal{C}_{cah}$ is a closed subset of the metric space $(\mathcal{C}, (d_{\mathcal{C}})^s)$. Indeed, let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{C}_{cah}$ and $f \in \mathcal{C}$ such that $\lim_{k\to\infty}(d_{\mathcal{C}})^s(f,f_k) = 0$. We shall show that $f(1) = c$ and $f(m) \geq af(m-1) + h(m)$ whenever $m \geq 2$.

To this end, we distinguish the following cases.

Case 1. $m = 1$. Then $f_k(1) = c$ for all $k \in \mathbb{N}$, so by Lemma 1(b), $f(1) < \infty$. Then $f(1) = c$ by Lemma 1(a).

Case 2. $m > 1$ and $f(m) = \infty$. Then $f(m) \geq af(m-1) + h(m)$, obviously.

Case 3. $m > 1$ and $f(m) < \infty$. Then, by Lemma 1(a), there is $k_0 \in \mathbb{N}$ such that $f_k(m) < \infty$ for all $k \geq k_0$, and $\lim_{k\to\infty} f_k(m) = f(m)$. From this equality and the fact that $f_k \in \mathcal{C}_{cah}$, we deduce the existence of $k_1 \geq k_0$ such that for each $k \geq k_1$,

$$1 + f(m) \geq f_k(m) \geq af_k(m-1) + h(m). \tag{2}$$

Consequently, $f_k(m-1) < \infty$ for all $k \geq k_1$, and by Lemma 1(b), $f(m-1) < \infty$ (otherwise, $\lim_k f_k(m-1) = \infty$, which contradicts (2)). Therefore, we also have $\lim_{k\to\infty} f_k(m-1) = f(m-1)$, by Lemma 1(a).

Now choose an arbitrary $\varepsilon > 0$. Then there exists $k_\varepsilon \in \mathbb{N}$ such that

$$\left|f_k(m-1) - f(m-1)\right| < \varepsilon \quad \text{and} \quad \left|f_k(m) - f(m)\right| < \varepsilon$$

for all $k \geq k_\varepsilon$. Hence

$$\varepsilon + f(m) > f_k(m) \geq af_k(m-1) + h(m) \geq a\big(f(m-1) + \varepsilon\big) + h(m)$$

for all $k \geq k_\varepsilon$. Thus $\varepsilon + f(m) > a(f(m-1) + \varepsilon) + h(m)$ for any $\varepsilon > 0$, so $f(m) \geq af(m-1) + h(m)$. Consequently, $f \in C_{cah}$, and hence $\mathcal{C}_{cah}$ is closed in the metric space $(\mathcal{C}, (d_{\mathcal{C}})^s)$. Then $(\mathcal{C}_{cah}, d_{\mathcal{C}_{cah}})$ is Smyth complete by Lemma 2. □

Theorem 6 *Let c and a be positive real constants with* $a \geq 1$, *let* $h \in \mathcal{C}_0$, *and let* Ψ *be the mapping on* $\mathcal{C}_{cah}$ *defined as*

$$\Psi(f)(n) = \begin{cases} c & \text{if } n = 1, \\ f(n-1) + h(n) & \text{if } n \geq 2. \end{cases} \tag{3}$$

Then the following hold:

(A) Ψ *is a self-mapping on* $\mathcal{C}_{cah}$.

(B) *For each* $f \in \mathcal{C}_{cah}$,

$$d_{\mathcal{C}_{cah}}(f, \Psi f) = \varphi(f) - \varphi(\Psi f),$$

where $\varphi : \mathcal{C}_{cah} \to [0, \infty)$ *is the lower semicontinuous function for* $\tau_{(d_{\mathcal{C}_{cah}})^s}$ *given by*

$$\varphi(f) = \frac{a+1}{2ac} - \sum_{n=1}^{\infty} 2^{-n} \frac{1}{f(n)}$$

for all $f \in \mathcal{C}_{cah}$.

(C) Ψ *has a fixed point in* $\mathcal{C}_{cah}$.

Proof (A) Let $f \in \mathcal{C}_{cah}$. Then $\Psi f(1) = c$ by definition of Ψ. We also have $\Psi f(2) = af(1) + h(2) = a\Psi f(1) + h(2)$.

Now let $n > 2$. Then

$$\begin{aligned} \Psi f(n) &= af(n-1) + h(n) \\ &\geq a\big[af(n-2) + h(n-1)\big] + h(n) \\ &= a\Psi f(n-1) + h(n). \end{aligned}$$

We conclude that $\Psi f \in \mathcal{C}_{cah}$.

(B) We first observe that, in fact, $\varphi(f) \geq 0$ for all $f \in \mathcal{C}_{cah}$. Indeed, since $a \geq 1$, we have $f(n) \geq f(n-1)$ for all $n \geq 2$, and thus $f(n) \geq f(2) \geq ac$ for all $n \geq 2$. Therefore

$$\begin{aligned} \sum_{n=1}^{\infty} 2^{-n} \frac{1}{f(n)} &= \frac{1}{2c} + \sum_{n=2}^{\infty} 2^{-n} \frac{1}{f(n)} \\ &\leq \frac{1}{2c} + \frac{1}{2ac} = \frac{a+1}{2ac}. \end{aligned}$$

Let now $f \in \mathcal{C}_{cah}$ and $(f_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{C}_{cah}$ such that $\lim_{k\to\infty}(d_{\mathcal{C}_{cah}})^s(f,f_k) = 0$. Since

$$\begin{aligned}\varphi(f) - \varphi(f_k) &= \sum_{n=1}^{\infty} 2^{-n}\frac{1}{f_k(n)} - \sum_{n=1}^{\infty} 2^{-n}\frac{1}{f(n)} \\ &\leq \sum_{n=1}^{\infty} 2^{-n}\left|\frac{1}{f_k(n)} - \frac{1}{f(n)}\right| \leq 2(d_{\mathcal{C}_{cah}})^s(f,f_k),\end{aligned}$$

we deduce that $\varphi(f) \leq \liminf_{k\to\infty}\varphi(f_k)$. Therefore φ is lower semicontinuous for $\tau_{(d_{\mathcal{C}_{cah}})^s}$.

Furthermore, for each $f \in \mathcal{C}_{cah}$, we have $f \geq \Psi f$, and hence

$$\begin{aligned}d_{\mathcal{C}_{cah}}(f,\Psi f) &= \sum_{n=1}^{\infty} 2^{-n}\max\left(\frac{1}{\Psi f(n)} - \frac{1}{f(n)}, 0\right) = \sum_{n=1}^{\infty} 2^{-n}\left(\frac{1}{\Psi f(n)} - \frac{1}{f(n)}\right) \\ &= \sum_{n=1}^{\infty} 2^{-n}\frac{1}{\Psi f(n)} - \sum_{n=1}^{\infty} 2^{-n}\frac{1}{f(n)} = \varphi(f) - \varphi(\Psi f).\end{aligned}$$

(C) From (B) we deduce that Ψ is a $(d_{\mathcal{C}_{cah}})^s$-Caristi mapping on $(\mathcal{C}_{cah}, d_{\mathcal{C}_{cah}})$. Then Ψ has a fixed point by Lemma 3 and Theorem 5. □

It follows from Theorem 6 that those algorithms defined by recurrence equations, whose associated functional is a mapping Ψ of type (3), admit a solution. We conclude the paper by applying this fact to deduce the existence of solution for the three algorithms mentioned at the beginning of this section.

Example 7 The algorithm Hanoi solves the celebrated Towers of Hanoi problem. The running time of computing of this algorithm is the solution of the recurrence equation $S : \mathbb{N} \to (0,\infty)$ given by

$$S(n) = \begin{cases} c & \text{if } n = 1, \\ 2S(n-1) + d & \text{if } n \geq 2, \end{cases}$$

with $c, d > 0$ (see, *e.g.*, [26]). The functional Ψ_S naturally associated to S is defined as

$$\Psi_S(f)(n) = \begin{cases} c & \text{if } n = 1, \\ 2f(n-1) + d & \text{if } n \geq 2. \end{cases}$$

Clearly Ψ_S is a mapping of type (3) for $a = 2$, and $h \in \mathcal{C}_0$ defined as $h(n) = d$ for all $n \in \mathbb{N}$. By Theorem 6, there exists $f_S \in \mathcal{C}_{cah}$ such that $f_S = \Psi f_S$. Hence f_S is a solution of the recurrence equation S.

Example 8 The algorithm Largetwo is a typical example of average case behavior whose running time of computing is the solution of the recurrence equation $S : \mathbb{N} \to (0,\infty)$ given by

$$S(n) = \begin{cases} c & \text{if } n = 1, \\ S(n-1) + 2 - 1/n & \text{if } n \geq 2, \end{cases}$$

with $c > 0$ (see, *e.g.*, [26]). The functional Ψ_S naturally associated to S is defined as

$$\Psi_S(f)(n) = \begin{cases} c & \text{if } n = 1, \\ f(n-1) + 2 - 1/n & \text{if } n \geq 2. \end{cases}$$

Clearly Ψ_S is a mapping of type (3) for $a = 1$, and $h \in \mathcal{C}_0$ defined as $h(n) = 2 - 1/n$ for all $n \in \mathbb{N}$. By Theorem 6, there exists $f_S \in \mathcal{C}_{cah}$ such that $f_S = \Psi f_S$. Hence f_S is a solution of the recurrence equation S.

Example 9 The running time of computing of the well-known algorithm Quicksort is, for the worst case, the solution of the recurrence equation $S : \mathbb{N} \to (0, \infty)$ given by

$$S(n) = \begin{cases} c & \text{if } n = 1, \\ S(n-1) + bn & \text{if } n \geq 2, \end{cases}$$

with $c, b > 0$ (see, *e.g.*, [26]). The functional Ψ_S naturally associated to S is defined as

$$\Psi_S(f)(n) = \begin{cases} c & \text{if } n = 1, \\ f(n-1) + bn & \text{if } n \geq 2. \end{cases}$$

Clearly Ψ_S is a mapping of type (3) for $a = 1$, and $h \in \mathcal{C}_0$ defined as $h(n) = bn$ for all $n \in \mathbb{N}$. By Theorem 6, there exists $f_S \in \mathcal{C}_{cah}$ such that $f_S = \Psi f_S$. Hence f_S is a solution of the recurrence equation S.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors contributed equally in writing this article. They read and approved the final manuscript.

Acknowledgements

The authors are grateful to the reviewers for several suggestions which have allowed to improve the first version of the paper. This research is supported by the Ministry of Economy and Competitiveness of Spain, Grant MTM2012-37894-C02-01.

Received: 4 May 2015 Accepted: 24 September 2015 Published online: 06 October 2015

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