On Expansion of a Solution of General Nonautonomous Polynomial Differential Equation

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Abstract We give a recursive formula for an expansion of a solution of a general nonautonomous polynomial differential equation. The formula is given on the algebraic level with the use of a shuffle product. This approach minimizes the number of integrations on each order of expansion. Using combinatorics of trees, we estimate the radius of convergence of the expansion.

Keywords Polynomial differential equation · Generalized Abel equation · Riccati equation · Shuffle product · Combinatorics of trees

Mathematics Subject Classifications (2010) 17B80 · 34A05 · 34A26

1 Introduction

Consider a nonautonomous polynomial differential equation, known also as a generalized Abel differential equation:

$$\dot{x}(t) = u_0(t) + u_1(t)x(t) + \cdots + u_i(t)x^i(t) + \cdots + u_n(t)x^n(t),$$
$$x(0) = x_0, \tag{1.1}$$

with a solution $x : [0, T] \to \mathbb{R}$ on a small segment of the reals. In this class of differential equation, we have the following: for $n = 1$, the linear equation with a well-known formula for the general solution; for $n = 2$, the Riccati equation well known both for theoretical [3, 7, 17, 18] and practical (see [4] and references therein at the beginning of Section 4) reasons; for $n = 3$, the Abel differential equation of the first kind studied theoretically [15] and for practical reasons ([12, 14, 20] and references in [6, 15]); and for $n > 3$, the generalized Abel differential equations [1, 2]. Assuming $X_i = x^i \frac{\partial}{\partial x}$, for $i = 0, \ldots, n$, are differential

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vector fields on $\mathbb{R}$, one can, following Fliess [9] (see also [10, 13], and [11] with references therein), expand the solution of the equation in terms of iterated integrals

$$\int_0^t \int_0^{t_k} \cdots \int_0^{t_2} u_{i_k}(t_k) \cdots u_{i_1}(t_1)\, dt_1 \ldots dt_{k-1} dt_k$$

and iterated differential operators $X_{i_1} \cdots X_{i_k}$ acting on the identity function $h(x) = x$ and evaluated at the zero point. In this approach, one does not use a specific form of the vector fields, i.e., the fact that they are of polynomial type. In this article, we show another approach to expanding a solution of the above equation in terms of iterated integrals with the use of an important feature of Chen's iterated integral mapping, namely, that it is a shuffle algebra homomorphism (see the comment after formula (2.4)). In fact, with the use of Chen's mapping (2.4), we will be able to consider a purely algebraic problem instead of an analytic one. More precisely, assuming that the solution of Eq. (1.1) can be expanded in terms of iterated integrals, we show that an algebraic equation must be satisfied in the space of nonommutative series on $n+1$ letters. It will be easy to show the existence of the solution of the algebraic equation by a recursive formula of its homogeneous parts. Chen's mapping gives us the expansion of the solution of the initial problem as we state in Theorem 1. This is done in Section 2. Then, in Section 3, by counting elements of a class of trees in two different ways, we show convergence of the defined expansion of $x(t)$ for small times—this is stated in Theorem 2 (in Section 2).

As an application of this general approach, we consider, in Section 4, the cases of the linear equation (i.e., $n = 1$), the Riccati equation ($n = 2$), and the one where there are only two nonvanishing summands. In the first case, we reestablish a well-known formula for the general solution, and in the second case, we deduce convergence of the series defining coordinates of the second kind connected with $\mathfrak{a}_1$-type involutive distribution [16].

Finally, in Section 5, we compare the Chen-Fliess approach with the one given in this article. It occurs that in the latter case, the number of integrals to compute grows significantly slower with the order of approximation than that in the first case.

2 Existence and Convergence of an Expansion

Let $n \in \mathbb{N},$[1] $T > 0$, and let $u_0, \ldots, u_n : [0, T] \to \mathbb{R}$ be measurable and bounded (by a constant $M \in \mathbb{R}$) functions. Consider a nonautonomous polynomial differential equation

$$\dot{x}(t) = u_0(t) + \binom{n}{1} u_1(t)x(t) + \cdots + \binom{n}{i} u_i(t)x^i(t) + \cdots + u_n(t)x^n(t),$$
$$x(0) = 0. \tag{2.1}$$

Two comments are in order. Firstly, the Newton symbols occurring in the above formula are for convenience reasons—without these constants, it would be harder to estimate the radius of convergence of a defined series. Secondly, we assume that the initial value equals zero. This is without loss of generality in a sense that with the linear change of variables

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[1]Throughout the article, we assume $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.

$x \to x - x_0$, we can transform the equation with the initial value equals x_0 to another equation with different u_i's.

Integrating both sides of the equation, we get an integral equation:

$$x(t) = \int_0^t u_0(s) + \binom{n}{1} u_1(s)x(s) + \cdots + u_n(s)x^n(s) \, ds. \tag{2.2}$$

By Caratheory's theorem for a small $\epsilon > 0$, there exists an absolutely continuous solution $\hat{x} : [0, \epsilon] \to \mathbb{R}$ of the initial Eq. (2.1) in a sense that $\hat{x}$ satisfies the integral Eq. (2.2) for $t \in [0, \epsilon]$. We want to express the solution $\hat{x}$ by means of iterated integrals of products of u_i's. In order to do this, we introduce some algebraic tools.

To each function u_i, we assign a formal variable a_i, which we call a letter. The set of all letters $A = \{a_0, \ldots, a_n\}$ is called an alphabet. Juxtapositioning letters, we can obtain words of an arbitrary length $k \in \mathbb{N}$; the set of such words is denoted by $A_k^* = \{b_1 \cdots b_k | b_1, \ldots, b_k \in A\}$. For $k = 0$, the set $A_0^* = \{1\}$ contains only one—empty—word. The set of all words is denoted by $A^* = \bigcup_{k=0}^\infty A_k^*$. The juxtaposition gives rise to an associative, noncommutative product on the set of words $A^* \times A^* \ni (v, w) \to v \cdot w = vw \in A^*$ called the concatenation product; then the set A^* with the concatenation product and the neutral element $1 \in A^*$ is a free monoid generated by A. Taking $\mathbb{R}$-linear combination of words and bilinearly extending the concatenation product, we get the $\mathbb{R}$-algebras $\mathbb{R}\langle A \rangle$ of noncommutative polynomials on A and $\mathbb{R}\langle\langle A \rangle\rangle$ of noncommutative series on A. In both algebras, we can consider the bilinear product $\sqcup\!\sqcup : \mathbb{R}\langle\langle A \rangle\rangle \otimes \mathbb{R}\langle\langle A \rangle\rangle \to \mathbb{R}\langle\langle A \rangle\rangle$—the shuffle product—defined recursively for words by $1 \sqcup\!\sqcup w = w \sqcup\!\sqcup 1 = w$ for any $w \in A^*$, and

$$(vb) \sqcup\!\sqcup (wc) = (v \sqcup\!\sqcup (wc)) \cdot b + ((vb) \sqcup\!\sqcup w) \cdot c \tag{2.3}$$

for all $b, c \in A$, and $v, w \in A^*$. It is easy to see that the shuffle product is commutative; thus, with 1 as the neutral element, it gives rise to an additional commutative algebra structure on $\mathbb{R}\langle A \rangle$ and $\mathbb{R}\langle\langle A \rangle\rangle$. We will use both concatenation and shuffle products in our considerations. It is important to indicate the priority of the shuffle product over the concatenation product in all formulas of this article, so that $v \sqcup\!\sqcup w \cdot a$ always means $(v \sqcup\!\sqcup w) \cdot a$.

On $\mathbb{R}\langle\langle A \rangle\rangle$, we introduce a natural scalar product $(\cdot|\cdot) : \mathbb{R}\langle\langle A \rangle\rangle \times \mathbb{R}\langle\langle A \rangle\rangle \to \mathbb{R}$, which for elements $v, w \in A^* \times A^*$ is given by

$$(v|w) = \begin{cases} 1 & \text{for } v = w \\ 0 & \text{for } v \neq w \end{cases}.$$

For $S \in \mathbb{R}\langle\langle A \rangle\rangle$, let $S_k \in \mathbb{R}\langle A \rangle$ be the k-degree homogenous part of S, i.e.,

$$S_k = \sum_{v \in A_k^*} (S|v) \, v.$$

Clearly, $S = \sum_{k=0}^\infty S_k$.

Define the linear homomorphism $\Upsilon^t : \mathbb{R}\langle A \rangle \to \mathbb{R}$ by $\Upsilon^t(1) = 1$, and

$$A_k^* \ni v = a_{i_1} \cdots a_{i_k} \mapsto \Upsilon^t(v) := \int_0^t u_{i_k}(t_k) \int_0^{t_k} \cdots \int_0^{t_2} u_{i_1}(t_1) \, dt_1 \ldots dt_{k-1} dt_k.$$

Equivalently, the homomorphism can be defined recursively by

$$\Upsilon^t(va_i) := \int_0^t \Upsilon^s(v) \, u_i(s) \, ds \tag{2.4}$$

for any $v \in A^*$ and $a_i \in A$. Since u_i's are bounded, the definition is correct for all $t \geq 0$. One can check that Υ^t is in fact a shuffle algebra homomorphism, i.e., $\Upsilon^t(v \sqcup\!\sqcup w) =$

$\Upsilon^t(v)\,\Upsilon^t(w)$ (see [8, 19]) which is a crucial feature in what follows. For a general series $S \in \mathbb{R}\langle\langle A\rangle\rangle$, the homomorphism Υ^t is obviously not well defined since $\Upsilon^t(S)$ can be divergent. We restrict the definition of Υ^t to series $S \in \mathbb{R}\langle\langle A\rangle\rangle$ and times $t \geq 0$ for which the series

$$\sum_{k=0}^{\infty} \Upsilon^t(S_k)$$

is convergent.

Coming back to the initial problem, assume that there exists a series $\hat{S} \in \mathbb{R}\langle\langle A\rangle\rangle$ such that the solution $\hat{x}$ of Eq. (2.1) satisfies $\hat{x}(t) = \Upsilon^t(\hat{S})$ for $t \in [0, \epsilon]$ (in particular $\Upsilon^t(\hat{S})$ is convergent). Using the recursive definition of Υ^t (2.4) and the fact that Υ^t is a shuffle algebra homomorphism, we get from Eq. (2.2) that

$$\Upsilon^t(\hat{S}) = \Upsilon^t(a_0 + n_1\,\hat{S}\cdot a_1 + n_2\,\hat{S}\sqcup\!\sqcup\hat{S}\cdot a_2 + \ldots + \hat{S}^{\sqcup\!\sqcup n}\cdot a_n),$$

where we abbreviate $n_i = \begin{pmatrix} n \\ i \end{pmatrix}$, and $\hat{S}^{\sqcup\!\sqcup n}$ is defined recursively in a natural way, i.e., $\hat{S}^{\sqcup\!\sqcup 0} = 1$ and $\hat{S}^{\sqcup\!\sqcup n} = \hat{S}\sqcup\!\sqcup\hat{S}^{\sqcup\!\sqcup(n-1)}$. Now, the point is that we can forget, for a moment, about the homomorphism and consider only the algebraic equation

$$\hat{S} = a_0 + n_1\,\hat{S}\cdot a_1 + n_2\,\hat{S}\sqcup\!\sqcup\hat{S}\cdot a_2 + \ldots + \hat{S}^{\sqcup\!\sqcup n}\cdot a_n. \tag{2.5}$$

Proposition 2.1 *There exists the unique solution $\hat{S} \in \mathbb{R}\langle\langle A\rangle\rangle$ of the algebraic Eq. (2.5).*

Proof The equation under consideration must be satisfied for each homogeneous part, so we can split it into the following series of equations:

$$\hat{S}_0 = 0, \qquad\qquad \hat{S}_1 = a_0,$$
$$\hat{S}_2 = n_1\,\hat{S}_1\cdot a_1, \qquad \hat{S}_3 = n_1\,\hat{S}_2\cdot a_1 + n_2\,\hat{S}_1\sqcup\!\sqcup\hat{S}_1\cdot a_2,$$

and for arbitrary $k \in \mathbb{N}$

$$\hat{S}_{k+1} = \sum_{i=1}^{n} \begin{pmatrix} n \\ i \end{pmatrix} \sum_{\mathbf{l}\in M(i)} \hat{S}_{l_1}\sqcup\!\sqcup\cdots\sqcup\!\sqcup\hat{S}_{l_i}\cdot a_i, \tag{2.6}$$

where the second sum is taken over multi-indices $\mathbf{l} = (l_1, \ldots, l_i)$ in

$$M(i) = \{(l_1, \ldots, l_i) \in \mathbb{N}^i \mid l_1, \ldots, l_i \geq 1,\ l_1 + \cdots + l_i = k\}.$$

We see that the homogeneous parts of the series $\hat{S}$ are defined recursively; therefore, the series is defined uniquely. $\qquad\square$

Observe that from the recursive definition of Υ^t and a property $\Upsilon^t(v\sqcup\!\sqcup w) = \Upsilon^t(v)\,\Upsilon^t(w)$ we get

$$\hat{x}_{k+1}(t) = \sum_{i=1}^{n} \begin{pmatrix} n \\ i \end{pmatrix} \sum_{\mathbf{l}\in M(i)} \int_0^t \hat{x}_{l_1}(s)\cdots\hat{x}_{l_i}(s)\,u_i(s)\,ds, \tag{2.7}$$

where we use an abbreviation $\hat{x}_k(t) = \Upsilon^t(\hat{S}_k)$. By this definition, $\hat{x}(t) = \sum_{k=0}^{\infty}\hat{x}_k(t)$. Moreover, any permutation of $(l_1, \ldots, l_i)$ gives the same expression under the integral. For $\mathbf{l} \in M(i)$ denote by $R(\mathbf{l})$, the number of such permutations, i.e.,

$$R(\mathbf{l}) = \#\{\sigma \in \Sigma_i \mid l_1 = l_{\sigma(1)}, \ldots, l_i = l_{\sigma(i)}\}.$$

Then

$$\hat{x}_{k+1}(t) = \sum_{i=1}^{n} \binom{n}{i} \sum_{\mathbf{l} \in M_{\leq}(i)} R(\mathbf{l}) \int_0^t \hat{x}_{l_1}(s) \cdots \hat{x}_{l_i}(s) \, u_i(s) \, ds, \qquad (2.8)$$

where the second sum is taken over

$$M_{\leq}(i) = \{(l_1, \ldots, l_i) \in \mathbb{N}^i \,|\, 1 \leq l_1 \leq l_2 \leq \cdots \leq l_i, \, l_1 + \cdots + l_i = k\}.$$

Let us state it in the following theorem.

Theorem 1 *Let $\hat{x}_1(t) = \int_0^t u_0(s)\,ds$ and recursively for $k \geq 1$, $\hat{x}_{k+1}(t)$ is given by* Eq. (2.7) *or* (2.8). *Then $\hat{x}(t) = \sum_{k=1}^{\infty} \hat{x}_k(t)$ is a formal solution of the differential* Eq. (2.1).

Remark 2.2 It is worth noticing that for a fixed $k \geq 1$, the number of integrals in formula (2.8) is the cardinality of $\sum_{i=1}^{n} M_{\leq}(i)$. This is less than the cardinality of $\sum_{i=1}^{\infty} M_{\leq}(i)$, which is the number of partitions of k. The first ten of these numbers are 1, 2, 3, 5, 7, 11, 15, 22, 30, and 42. It means that the number of integrals that we have to perform to compute $\hat{x}_{k+1}$ grows quite slowly. In Section 5, we show that this growth is much slower than the growth of the number of nonzero components in the Chen-Fliess expansion.

There remains the problem under what assumption the solution for the algebraic equation is in the domain of the homomorphism $\Upsilon^t : \mathbb{R}\langle\langle A \rangle\rangle \supset D(\Upsilon^t) \to \mathbb{R}$, i.e., when $\sum_k \Upsilon^t(\hat{S}_k)$ is convergent. In order to solve it, we need to compute the number of words (with multiplicities) in each homogeneous part of $\hat{S}$. So for $S \in \mathbb{R}\langle A \rangle$, let us introduce the following definition:

$$\#S = \sum_{v \in A^*} |(S|v)|.$$

Proposition 2.3 *If $\hat{S}_k$ is the k-degree homogeneous part of the solution $\hat{S}$ of the algebraic* Eq. (2.5), *then for $k \geq 1$, $\#\hat{S}_k = ((n-1)(k-1)+1) \cdot ((n-1)(k-2)+1) \cdots n$ (in particular $\#\hat{S}_1 = 1$) and $\#\hat{S}_0 = 0$.*

In particular, for $n = 0$, $\#\hat{S}_1 = 1$ and $\#\hat{S}_k = 0$ otherwise; for $n = 1$, $\#\hat{S}_k = 1$; for $n = 2$, $\#\hat{S}_k = k!$; for $n = 3$, $\#\hat{S}_k = (2k-1)!!$; for $n = 4$, $\#\hat{S}_k = (3k-2)!!!$; and so on.

The proposition will be proved in Section 3.

Now, we state the theorem about convergence of the expansion.

Theorem 2 *Let $n \in \mathbb{N}$ and assume $u_0, \ldots, u_n : [0, T] \to \mathbb{R}$ are measurable functions such that $|u_i| \leq M$ for an $M > 0$. Let $\hat{S} \in \mathbb{R}\langle\langle A \rangle\rangle$ be the unique solution of the algebraic* Eq. (2.5). *Then the series $\sum_k \hat{x}_k(t) = \sum_k \Upsilon^t(\hat{S}_k)$ is absolutely convergent for $0 \leq t < \min\{T, 1/(M(n-1))\}$ if $n \geq 2$, $0 \leq t \leq T$ if $n = 0, 1$, and $\hat{x}(t) = \Upsilon^t(\hat{S})$ is the solution of the differential* Eq. (2.1) *on the same segment.*

Proof For $v \in A_k^*$, the iterated integral $\Upsilon^t(v)$ is in fact taken over a k-dimensional simplex of k-dimensional measure $t^k/k!$. Since u_i's are bounded by M, we have $|\Upsilon^t(v)| \leq (Mt)^k/k!$. Therefore,

$$|\Upsilon^t(\hat{S}_k)| \leq \#\hat{S}_k \frac{(Mt)^k}{k!} = \frac{((n-1)(k-1)+1) \cdot ((n-1)(k-2)+1) \cdots n}{k!} (Mt)^k.$$

Since $\#\hat{S}_{k+1}/\#\hat{S}_k = \frac{(n-1)k+1}{k+1} \xrightarrow[k\to\infty]{} n-1$, the series $\sum_k \Upsilon^t(\hat{S}_k)$ is convergent for $t < 1/(M(n-1))$ if $n \geq 2$, and $t < T$ if $n = 1$. For $n = 0$, the statement is obvious. $\qquad\square$

3 Counting Trees

In this section, we prove Proposition 2.3. In order to do this, we consider certain classes of trees. It occurs that the number of trees in these particular classes equals $\#\hat{S}_k$ on the one hand and $((n-1)(k-1)+1) \cdot ((n-1)(k-2)+1) \cdots n$ on the other hand.

For $k, n \in \mathbb{N}$ let $\mathcal{T}_k^n$ denote the set of planar, rooted, full n-ary and increasingly labeled trees on k vertices. Recall that a tree is rooted if there exists a distinguished vertex called the root; is full n-ary if each vertex has exactly none or n children; is on k vertices if the number of vertices with n children (parent vertices) is equal k; and is increasingly labeled if the parent vertices are labeled by natural numbers from 1 to k, and the labels increase along each branch starting at the root (in particular, the root is labeled by "1"). A leaf of a tree is a nonparent vertex, i.e., a vertex without children. It is important to note that the number of leafs in each tree in $\mathcal{T}_k^n$ is constant and equals $(n-1)k+1$. Indeed, using induction on k, we see that for $k = 0$, the only tree in $\mathcal{T}_0^n$ has 0 children, so the root is the only leaf; each tree $\mathcal{T}_k^n$ can be obtained from a tree $t \in \mathcal{T}_{k-1}^n$ by adding n leafs to a certain leaf of t, so the number of leafs increases by $(n-1)$.

Now, we count the cardinality of $\mathcal{T}_k^n$ in two different ways.

Lemma 3.1 *The cardinality of $\mathcal{T}_k^n$ equals $\#\mathcal{T}_k^n = ((n-1)(k-1)+1) \cdot ((n-1)(k-2)+1) \cdots n$ for $k \geq 1$ and $\mathcal{T}_0^n = 1$.*

Proof The case $n = 0$ is trivial. Fix $n \in \mathbb{N}$ s.t. $n \geq 1$. We proceed by induction on $k \in \mathbb{N}$. For $k = 0$, there is only one tree, so the statement is correct. Assume $\#\mathcal{T}_k^n = ((n-1)(k-1)+1) \cdot ((n-1)(k-2)+1) \cdots n$. Each tree in $\mathcal{T}_{k+1}^n$ comes from the unique tree t in $\mathcal{T}_k^n$ by adding label "$k+1$" and k vertices to a leaf of t. Since the number of leafs of t is equal $(n-1)k+1$, we obtain the result. $\qquad\square$

Lemma 3.2 *For $k \in \mathbb{N}$ the cardinality of $\mathcal{T}_{k+1}^n$ equals*

$$\#\mathcal{T}_{k+1}^n = \sum_{\mathbf{l} \in M_0(n)} \frac{k!}{l_1! \cdots l_n!} \#\mathcal{T}_{l_1}^n \cdots \#\mathcal{T}_{l_n}^n,$$

where the sum is taken over multi-indices $\mathbf{l} = (l_1, \ldots, l_n)$ in

$$M_0(n) = \{(l_1, \ldots, l_n) \in \mathbb{N}^n | l_1 + \cdots + l_n = k\}.$$

Proof First of all, observe that for $k \in \mathbb{N}$, each tree in $\mathcal{T}_{k+1}^n$ is uniquely given by n trees $t_1 \in \mathcal{T}_{l_1}^n, \ldots, t_n \in \mathcal{T}_{l_n}^n$ such that $l_1 + \cdots + l_n = k$, and a partition of the set $\{2, \ldots, k+1\}$ into n disjoint sets $I_1, \ldots, I_n$ of the cardinality $\#I_i = l_i$ for $i = 1, \ldots, n$ (we do not assume $I_i \neq \emptyset$), i.e.,

$$\mathcal{T}_{k+1}^n \sim \bigsqcup_{\mathbf{l} \in M_0(n)} \mathcal{T}_{l_1}^n \times \cdots \times \mathcal{T}_{l_n}^n \times I(\mathbf{l}),$$

where $I(\mathbf{l})$ is the set of all partitions of the set $\{2, \ldots, k+1\}$ into n disjoint sets $I_1, \ldots, I_n$ s.t. $\#I_i = l_i$ for $i = 1, \ldots, n$. Indeed, the root of a given tree $t \in \mathcal{T}_{k+1}^n$ has n child vertices $v_1, \ldots, v_n$. Each v_i is the root of a certain maximal subtree $\tilde{t}_i$ of t. We assume that $\tilde{t}_i$ has l_i

parent vertices, which are labeled by some numbers $2 \leq a_i^1 < \cdots < a_i^{l_i} \leq k+1$. Obviously, $l_1 + \cdots + l_n = k$. Changing the label "a_i^j" into a label "j," we obtain a tree $t_i \in \mathcal{T}_{l_i}^n$. Defining $I_i = \left\{ a_i^1, \ldots, a_i^{l_i} \right\}$ for $i = 1, \ldots, n$, we have a partition of $\{2, \ldots, k+1\}$ into n disjoint sets, i.e., $\{2, \ldots, k+1\} = I_1 \cup \cdots \cup I_n$. It is clear how to invert this procedure in order to get its uniqueness.

Using the above correspondence, it is easy to establish the formula in the lemma since there are $\frac{k!}{l_1! \cdots l_n!}$ possible partitions of the set $\{2, \ldots, k+1\}$ into n disjoint parts $I_1, \ldots, I_n$ such that $\#I_i = l_i \in \mathbb{N}$, i.e., $\#I(\mathbf{l}) = \frac{k!}{l_1! \cdots l_n!}$. $\qquad\square$

We are now ready to prove Proposition 2.3.

Proof of Proposition 2.3 First of all, observe that for $i \in \mathbb{N}$, $l_1, \ldots, l_i \in \mathbb{N}$, and words $v_1 \in A_{l_1}^*, \ldots, v_i \in A_{l_i}^*$, the number of words in the shuffle product $v_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup v_i$ equals

$$\#(v_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup v_i) = \frac{(l_1 + \cdots + l_i)!}{l_1! \cdots l_i!}.$$

Using this fact, homogeneity of polynomials $\hat{S}_l$, and the recursive formula (2.6), we get

$$\#\hat{S}_{k+1} = \sum_{i=1}^{n} \binom{n}{i} \sum_{\mathbf{l} \in M(i)} \# \left(\hat{S}_{l_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \hat{S}_{l_i} \right)$$
$$= \sum_{i=1}^{n} \binom{n}{i} \sum_{\mathbf{l} \in M(i)} \frac{(l_1 + \cdots + l_i)!}{l_1! \cdots l_i!} \cdot \#\hat{S}_{l_1} \cdots \#\hat{S}_{l_i} \qquad (3.1)$$

where $M(i)$ contains multi-indices $(l_1, \ldots, l_i) \in \mathbb{N}^i$ such that $l_1 + \cdots + l_i = k$ and, what is important, $l_1, \ldots, l_i \geq 1$. In order to get rid of the first sum, we introduce the following notation:

$$N_k = \begin{cases} 1 & \text{for } k = 0 \\ \#\hat{S}_k & \text{for } k \neq 0 \end{cases}$$

and allow $l_1, \ldots, l_i$ to be equal 0. Then, we rewrite Eq. (3.1) as

$$N_{k+1} = \sum_{\mathbf{l} \in M_0(n)} \frac{(l_1 + \cdots + l_n)!}{l_1! \cdots l_n!} \cdot N_{l_1} \cdots N_{l_n}, \qquad (3.2)$$

where $M_0(n) = \{ (l_1, \ldots, l_n) \in \mathbb{N}^n \mid l_1 + \cdots + l_n = k \}$. Indeed, if $l_1, \ldots, l_n \in \mathbb{N}$ and only i of them, say $\hat{l}_1, \ldots, \hat{l}_i$, are not equal 0, then

$$\frac{(l_1 + \cdots + l_n)!}{l_1! \cdots l_n!} \cdot N_{l_1} \cdots N_{l_n} = \frac{(\hat{l}_1 + \cdots + \hat{l}_i)!}{\hat{l}_1! \cdots \hat{l}_i!} \cdot N_{\hat{l}_1} \cdots N_{\hat{l}_i}.$$

Clearly, there are $\binom{n}{i}$ different multi-indices $(l_1, \ldots, l_n)$ satisfying this condition, and this is the reason for the Newton symbol to disappear in formula (3.2).

Finally, we see that by Lemma 3.2, the recursive formula (3.2) for the numbers N_k overlaps with the one for the cardinality of trees $\#\mathcal{T}_k^n$. Since the series coincide for $k = 0$, i.e., $N_0 = \#\mathcal{T}_0^n = 1$, we conclude using Lemma 3.1 that

$$\#\hat{S}_k = N_k = ((n-1)(k-1) + 1) \cdot ((n-1)(k-2) + 1) \cdots n$$

for $k \geq 1$. The fact that $\#\hat{S}_0 = 0$ is trivial.   □

Remark 3.3 The above proof can be simplified for $n = 0, 1$ when $\#\hat{S}_k \leq 1$, but also for $n = 2$. In this case, the recursive formula (2.6) gives

$$\#\hat{S}_{k+1} = 2 \#\hat{S}_k + \sum_{j=1}^{k-1} \binom{k}{j} \#\hat{S}_j \, \#\hat{S}_{k-j}.$$

Assuming the inductive hypothesis $\#\hat{S}_l = l!$ for $l \leq k$, we obtain

$$\#\hat{S}_{k+1} = 2\,k! + \sum_{j=1}^{k-1} \binom{k}{j} j!(k-j)! = 2\,k! + (k-1)\,k! = (k+1)!.$$

4 Examples

In this section, we discuss the three simplest cases when $n = 0, 1, 2$, and the case where only u_0 and u_n are not vanishing. We will need one additional intuitive notation, namely for $S \in \mathbb{R}\langle\langle A \rangle\rangle$ such that $(S|1) = 0$, we define the shuffle exponent

$$\exp_{\sqcup\!\sqcup}(S) = \sum_{k=0}^{\infty} \frac{S^{\sqcup\!\sqcup k}}{k!},$$

where we recall that $S^{\sqcup\!\sqcup 0} = 1$ and $S^{\sqcup\!\sqcup k} = S \sqcup\!\sqcup S^{\sqcup\!\sqcup (k-1)}$.

If $n = 0$, then the Eq. (2.1) is $\dot{x}(t) = u_0(t), x(0) = 0$, and obviously, a solution is $\hat{x}(t) = \Upsilon^t(\hat{S})$, where $\hat{S} = \hat{S}_1 = a_0$ is homogeneous of degree 1.

Let us pass to the case $n = 1$ when Eq. (2.1) is a linear equation $\dot{x}(t) = u_0(t) + u_1(t)x(t)$, $x(0) = 0$, which can be solved by a variation of parameter. Let us see how it can be done using the series $\hat{S}$. Using recursive formula (2.6),

$$\hat{S}_0 = 0, \qquad\qquad \hat{S}_1 = a_0, \qquad\qquad \hat{S}_{k+1} = \hat{S}_k \cdot a_1,$$

we get $\hat{S} = a_0 \cdot \left(1 + a_1 + a_1^2 + a_1^3 + \cdots \right)$. If we use formula $a_1^k = a_1^{\sqcup\!\sqcup k}/k!$, we get

$$\hat{S} = a_0 \cdot \exp_{\sqcup\!\sqcup}(a_1 1). \qquad\qquad (4.1)$$

This expression looks nice, but there is a problem since a_0 factor is on the left-hand side, and therefore the expression will not simplify if we apply Υ^t to it. In order to obtain the solution in a common form, we prove the following lemma.

Lemma 4.1 *For $a_0, a_1 \in A$ it follows that*

$$\sum_{k=0}^{\infty} a_0 a_1^k = \sum_{k,l=0}^{\infty} (-1)^l \, a_1^k \sqcup\!\sqcup \left(a_1^l a_0\right).$$

Proof Observe first that for $k, l \in \mathbb{N}$

$$a_1^k \sqcup\!\sqcup \left(a_1^l a_0\right) = a_1^l a_0 a_1^k + \binom{l+1}{1} a_1^{l+1} a_0 a_1^{k-1} + \cdots + \binom{l+k}{k} a_1^{l+k} a_0. \qquad (4.2)$$

Indeed, for $k = 0$, the formula is correct. Using the inductive hypothesis for each $m \le k$ and the defining formula for shuffle product (2.3), we get

$$a_1^{k+1} \, \sqcup\!\sqcup \, \left(a_1^l a_0\right) = \left(a_1^k \, \sqcup\!\sqcup \, \left(a_1^l a_0\right)\right) a_1 + \left(a_1^{k+1} \, \sqcup\!\sqcup \, a_1^l\right) a_0$$

$$= a_1^l a_0 a_1^{k+1} + \binom{l+1}{1} a_1^{l+1} a_0 a_1^k + \cdots + \binom{l+k}{k} a_1^{l+k} a_0 a_1 + \left(a_1^{k+1} \, \sqcup\!\sqcup \, a_1^l\right) a_0.$$

Since $a_1^{k+1} \, \sqcup\!\sqcup \, a_1^l = \binom{l+k+1}{k+1} a_1^{l+k+1}$, we obtain formula (4.2).

Using the above proved formula, we see that

$$\sum_{k,l=0}^{\infty} (-1)^l \, a_1^k \, \sqcup\!\sqcup \, \left(a_1^l a_0\right) = \sum_{k,l=0}^{\infty} \sum_{m=0}^{k} (-1)^l \binom{l+m}{m} a_1^{l+m} a_0 a_1^{k-m}$$

$$= \sum_{k',l'=0}^{\infty} \left[\sum_{m=0}^{l'} (-1)^{l'-m} \binom{l'}{m} \right] a_1^{l'} a_0 a_1^{k'},$$

where in the last line, we change a method of summation by putting $k' = k - m$ and $l' = l + m$. Since the expression in the squared brackets equals $0^{l'}$, the sum over l' reduces to the one summand with $l' = 0$, and therefore

$$\sum_{k,l=0}^{\infty} (-1)^l \, a_1^k \, \sqcup\!\sqcup \, \left(a_1^l a_0\right) = \sum_{k'=0}^{\infty} a_0 a_1^{k'}.$$

This ends the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

From the lemma, it follows that

$$\hat{S} = \sum_{k=0}^{\infty} a_0 a_1^k = \sum_{k,l=0}^{\infty} (-1)^l \, a_1^k \, \sqcup\!\sqcup \, \left(a_1^l a_0\right) = \sum_{k=0}^{\infty} \frac{a_1^{\sqcup\!\sqcup k}}{k!} \, \sqcup\!\sqcup \, \left(\sum_{k=0}^{\infty} \frac{(-a_1)^{\sqcup\!\sqcup l}}{l!} \cdot a_0 \right)$$

$$= \exp_{\sqcup\!\sqcup}(a_1) \, \sqcup\!\sqcup \, (\exp_{\sqcup\!\sqcup}(-a_1) \cdot a_0).$$

Since $\Upsilon^t : \mathbb{R}\langle\langle A \rangle\rangle \to \mathbb{R}$ is a shuffle-algebra homomorphism, it follows that $\Upsilon^t(\exp_{\sqcup\!\sqcup}(S)) = \exp(\Upsilon^t(S))$ for all series S such that $(S|1) = 0$, and therefore

$$\hat{x}(t) = \Upsilon^t(\hat{S}) = \exp\left(\int_0^t u_1(s)\,ds \right) \int_0^t \exp\left(-\int_0^s u_1(\tau)\,d\tau \right) u_0(s)\,ds,$$

which is the standard formula.

For $n = 2$, the equation under consideration is

$$\dot{x}(t) = u_0(t) + 2u_1(t)x(t) + u_2(t)x^2(t), \qquad\qquad x(0) = 0, \qquad (4.3)$$

that is a general Riccati equation. In this case, the series $\hat{S}$ is the unique solution of

$$\hat{S} = a_0 + 2\,\hat{S} \cdot a_1 + \hat{S} \, \sqcup\!\sqcup \, \hat{S} \cdot a_2, \qquad\qquad\qquad (4.4)$$

and therefore $\hat{S} = \sum_k \hat{S}_k$, where $\hat{S}_k$ are given by the recurrence

$$\hat{S}_0 = 0, \quad \hat{S}_1 = a_0, \quad \hat{S}_2 = 2\,\hat{S}_1 \cdot a_1, \quad \hat{S}_{k+1} = 2\,\hat{S}_k \cdot a_1 + \sum_{l=1}^{k-1} \hat{S}_l \, \sqcup\!\sqcup \, \hat{S}_{k-l} \cdot a_2. \quad (4.5)$$

Let us mention that the Riccati equation is a Lie-Scheffers system of the type $\mathfrak{a}_1$ (see [3, 7] and [17, 18]). More precisely, if we take vector fields

$$X_0(x) = \frac{\partial}{\partial x}, \qquad X_1(x) = 2x\frac{\partial}{\partial x}, \qquad X_2(x) = x^2\frac{\partial}{\partial x}$$

on $\mathbb{R}$, then they satisfy the following commutation relations:

$$[X_0, X_1] = 2X_0, \qquad [X_0, X_2] = X_1, \qquad [X_1, X_2] = 2X_2.$$

It means the vector fields span a simple Lie algebra of the type $\mathfrak{a}_1$ (isomorphic to $\mathfrak{sl}(2, \mathbb{R})$), and thus (4.3)—equivalent to $\dot{x}(t) = \sum u_i(t)X_i$—is a Lie-Scheffers system of this type. The solution in terms of iterated integrals of u_i's for such a system was given in [16]. Let us recall the main theorems of this article.

Theorem 3 (Theorem 1 in [16]) *Let X_a, X_b, $X_c \in \Gamma(\mathfrak{M})$ be smooth tangent vector fields on a manifold $\mathfrak{M}$ satisfying $[X_a, X_b] = 2X_a$, $[X_a, X_c] = -X_b$, $[X_b, X_c] = 2X_c$. Let $u_a, u_b, u_c : [0, T] \to \mathbb{R}$ be fixed measurable functions. Then (locally), the solution $x : [0, T] \to \mathfrak{M}$ of the differential equation*

$$\dot{x}(t) = u_c(t)X_c + u_b(t)X_b + u_a(t)X_a, \qquad x(0) = x_0 \in \mathfrak{M}$$

is of the form

$$x(t) = \exp\left(\Xi_c(t)X_c\right)\exp\left(\Xi_b(t)X_b\right)\exp\left(\Xi_a(t)X_a\right)(x_0). \qquad (4.6)$$

Here, $\Xi_a, \Xi_b, \Xi_c : [0, T] \to \mathbb{R}$ are given by $\Xi_d(t) := \Upsilon^t(S^d)$ (for $d = a, b, c$), where

$$S^a = a \cdot \exp_{\sqcup\sqcup}(2S_{\mathfrak{a}_1}), \qquad S^b = S_{\mathfrak{a}_1}, \qquad S^c = \exp_{\sqcup\sqcup}(2S_{\mathfrak{a}_1}) \cdot c, \qquad (4.7)$$

and $S_{\mathfrak{a}_1} \in \mathbb{R}\langle\langle A \rangle\rangle$ is the unique solution of the algebraic equation

$$S_{\mathfrak{a}_1} = b - a \cdot \exp_{\sqcup\sqcup}(2S_{\mathfrak{a}_1}) \cdot c.$$

In particular, we have

$$b - a \cdot S^c = S^b = b - S^a \cdot c.$$

Theorem 4 (Theorem 2 in [16]) *For fixed measurable functions $u_a, u_b, u_c : [0, T] \to \mathbb{R}$ the function $\Xi_a : [0, T] \to \mathbb{R}$, defined in Theorem 3 by $\Xi_a(t) = \Upsilon^t(a \cdot \exp_{\sqcup\sqcup}(2S_{\mathfrak{a}_1}))$, is (locally) the solution of the Riccati equation:*

$$\dot{\Xi}_a(t) = u_a(t) + 2u_b(t)\,\Xi_a(t) - u_c(t)\,\Xi_a^2(t), \qquad \Xi_a(0) = 0.$$

Observe that taking $X_a = X_0$, $X_b = X_1$, $X_c = -X_2$ (and therefore $c = -a_2$), and $u_a = u_0, u_b = u_1, u_c = u_2$, and $\Xi_a(t) = x(t)$, the system (4.3) can be put into the context of the above theorems in the following way. From Theorem 4, we conclude that the solution of (4.3) is $x(t) = \Upsilon^t(\hat{S})$, where

$$\hat{S} = a_0 \cdot \exp_{\sqcup\sqcup}(2S_{\mathfrak{a}_1}),$$

and $S_{\mathfrak{a}_1} \in \mathbb{R}\langle\langle A \rangle\rangle$ is, by Theorem 3, the unique solution of the algebraic equation

$$S_{\mathfrak{a}_1} = a_1 + a_0 \cdot \exp_{\sqcup\sqcup}(2S_{\mathfrak{a}_1}) \cdot a_2. \tag{4.8}$$

Additionally, from the last line in Theorem 3, we conclude that

$$S_{\mathfrak{a}_1} = a_1 + \hat{S} \cdot a_2. \tag{4.9}$$

In the discussed article, there was also given a recursive formula for $S_{\mathfrak{a}_1}$, but observe that in fact the algebraic Eq. (4.4) for $\hat{S}$ is simpler than the Eq. (4.8) for $S_{\mathfrak{a}_1}$. In consequence, it is reasonable to invert this statement saying that the series $S_{\mathfrak{a}_1}$ is given by (4.9), where $\hat{S}$ is the solution of Eq. (4.4). Now, using Theorem 2, we get the following corollary about the $\mathfrak{a}_1$-type Lie-Scheffers system considered in Theorem 3.

Proposition 4.2 *In the context of Theorem 3, if $|u_d| \leq M$ for an $M > 0$ ($d = a, b, c$), then the solution (4.6) exists for $0 \leq t < \min\{T, 1/M\}$.*

Proof By Theorem 1, the function $\Upsilon^t(\hat{S})$ is well defined for $0 \leq t < \min\{T, 1/M\}$. The above observations (in particular formula (4.9)) implies that $\Upsilon^t(S_{\mathfrak{a}_1})$ is also well defined in this interval. Finally, by formula (4.7), each function $\Xi_d(t) := \Upsilon^t(S^d)$ ($d = a, b, c$) is defined for $0 \leq t < \min\{T, 1/M\}$, too. □

Let us observe that in each of the discussed cases, the solution is of the form $\hat{S} = a_0 \cdot \exp_{\sqcup\sqcup}(L)$, where $L \in \mathbb{R}\langle\langle A \rangle\rangle$ such that $(L|1) = 0$, and the series L in case n reduces to L in case $n - 1$ if taking $u_n \equiv 0$. Indeed, $L = 0$ for $n = 0$, $L = a_1$ for $n = 1$, and $L = S_{\mathfrak{a}_1}$ for $n = 2$ reduces, by Eq. (4.8), to a_1 for $u_2 \equiv 0$. This observation suggests a question: if the same holds for all $n \in \mathbb{N}$? Since the Riccati equation is essentially the only differential equation on a real line which is connected with the action of a group (namely the special linear group $SL(2)$) [4, 5], one could anticipate that a generalization is impossible. Nevertheless, the problem is open.

Another example we are going to consider is the one where there are only two nonvanishing summands, i.e., $\dot{x}(t) = \binom{n}{m} u(t)x^m(t) + u_n(t)x^n(t)$, $x(0) = 0$, and $0 \leq m < n$ are fixed. The case $m \neq 0$ has the trivial solution $x(t) \equiv 0$, so in fact we consider

$$\dot{x}(t) = u_0(t) + u_n(t)x^n(t), \quad x(0) = 0 \tag{4.10}$$

with $n \geq 1$ fixed.

Proposition 4.3 *Let $u_0, u_n : [0, T] \to \mathbb{R}$ be measurable, bounded functions. Then the solution of (4.10) is $x(t) = \sum_{k=0}^{\infty} x_k(t)$, where $x_k(t)$ are recursively given by $x_0(t) = \int_0^t u_0(s)\,ds$, and*

$$x_k(t) = \sum_{\mathbf{l} \in N(k)} \frac{n!}{l_1! \cdots l_k!} \int_0^t (x_0(s))^{l_1} \cdots (x_{k-1}(s))^{l_k} u_n(s)\,ds,$$

where $N(k) = \{(l_1, \ldots, l_k) \in \mathbb{N}^k \mid n = l_1 + \cdots + l_k, \ k - 1 = l_2 + 2l_3 + \cdots + (k-1)l_k\}$.

Let us write the first few components of the expansion given in the above proposition.

$$x_0(t) = \Upsilon^t(a_0) = \int_0^t u_0(s)\,ds\,,$$

$$x_1(t) = \int_0^t (x_0(s))^n\, u_n(s)\,ds\,,$$

$$x_2(t) = n\int_0^t (x_0(s))^{n-1}x_1(s)\,u_n(s)\,ds\,,$$

$$x_3(t) = \binom{n}{2}\int_0^t (x_0(s))^{n-2}(x_1(s))^2\, u_n(s)\,ds + n\int_0^t (x_0(s))^{n-1}x_2(s)\,u_n(s)\,ds\,,$$

$$x_4(t) = \binom{n}{3}\int_0^t (x_0(s))^{n-3}(x_1(s))^3\, u_n(s)\,ds$$

$$+ n(n-1)\int_0^t (x_0(s))^{n-2}x_2(s)x_1(s)\,u_n(s)\,ds + n\int_0^t (x_0(s))^{n-1}x_3(s)\,u_n(s)\,ds\,.$$

Proof The algebraic Eq. (2.5) associated with the differential Eq. (4.10) is

$$\hat{S} = a_0 + \hat{S}^{\sqcup\!\sqcup n}\cdot a_n. \tag{4.11}$$

Let us first show that the only nonvanishing homogeneous parts of $\hat{S}$ are $\hat{S}_{kn+1}$, where $k \in \mathbb{N}$. We prove it by the induction on k. The k-th hypothesis is that $\hat{S}_{kn+l} = 0$ for all $l = 2,\ldots,n$. For $k = 0$, the hypothesis is clearly correct. Assume that it is correct for $k < K$, and let us prove that $\hat{S}_{Kn+2} = \cdots = \hat{S}_{Kn+n} = 0$. Using (4.11) and the induction hypothesis, we see that

$$\hat{S}_{Kn+l} = \sum_{(\mathbf{p},\mathbf{m})\in\tilde{N}} C(\mathbf{p},\mathbf{m})\cdot \hat{S}_1^{\sqcup\!\sqcup p_1}\sqcup\!\sqcup\cdots\sqcup\!\sqcup \hat{S}_{(K-1)n+1}^{\sqcup\!\sqcup p_K}\sqcup\!\sqcup \hat{S}_{Kn+2}^{\sqcup\!\sqcup m_2}\sqcup\!\sqcup\cdots\sqcup\!\sqcup \hat{S}_{Kn+l-1}^{\sqcup\!\sqcup m_{l-1}}\cdot a_n,$$

where the sum is taken over all $(\mathbf{p},\mathbf{m}) = (p_1,\ldots,p_K,m_2,\ldots,m_{l-1}) \in \tilde{N} \subset \mathbb{N}^{K+l-2}$,

$$\tilde{N} = \{n = p_1 + \cdots + p_K + m_2 + \cdots + m_{l-1},$$
$$Kn + l - 1 = p_1 + \cdots + p_K((K-1)n+1)$$
$$+ m_2(Kn+2) + \cdots + m_{l-1}(Kn+l-1)\},$$

and $C(\mathbf{p},\mathbf{m}) = \frac{n!}{\mathbf{p}!\,\mathbf{m}!}$, $\mathbf{p}! = p_1!\cdots p_K!$, $\mathbf{m}! = m_2!\cdots m_{l-1}!$. If $m_2 + \cdots + m_{l-1} \geq 1$, then from the second equality defining $\tilde{N}$, we have

$$l - 1 = p_1 + p_2(n+1) + \cdots + p_K((K-1)n+1)$$
$$+ 2m_2 + \cdots + (l-1)m_{l-1} + (m_2 + \cdots + m_{l-1} - 1)Kn,$$

which is not less than n (by the first equality defining $\tilde{N}$), a contradiction. If $m_2 + \cdots + m_{l-1} = 0$, then $p_1 + \cdots + p_K = n$ and therefore

$$l - 1 = n(1 + p_2 + 2p_3 + \cdots + (K-1)p_K - K).$$

But n does not divide $l - 1 \in \{1,\ldots,n-1\}$. This implies $\tilde{N} = \emptyset$ and $\hat{S}_{kn+l} = 0$ for $l = 2,\ldots,n$. We conclude that the solution of (4.11) is of the form $\hat{S} = \sum_{k=0}^{\infty}\hat{S}_{kn+1}$.

Now, similarly as above, we see from (4.11) that

$$\hat{S}_{Kn+1} = \sum_{\mathbf{l}\in N(k)} \frac{n!}{l_1!\cdots l_k!}\cdot \hat{S}_1^{\sqcup\!\sqcup l_1}\sqcup\!\sqcup \hat{S}_{n+1}^{\sqcup\!\sqcup l_2}\sqcup\!\sqcup\cdots\sqcup\!\sqcup \hat{S}_{(k-1)n+1}^{\sqcup\!\sqcup l_k}\cdot a_n,$$

where the sum is taken over

$$N(k) = \left\{ (l_1, \ldots, l_k) \in \mathbb{N}^k \mid n = l_1 + \cdots + l_k, \ kn = l_1 + l_2(n+1) + \cdots + l_k((k-1)n+1) \right\}.$$

Using the first equation defining $N(k)$, we simplify the second equation defining $N(k)$ as follows:

$$kn = l_1 + \cdots + l_k + n(l_2 + 2l_3 + \cdots + (k-1)l_k)$$
$$= n(1 + l_2 + 2l_3 + \cdots + (k-1)l_k).$$

Therefore,

$$N(k) = \left\{ (l_1, \ldots, l_k) \in \mathbb{N}^k \mid n = l_1 + \cdots + l_k, \ k-1 = l_2 + 2l_3 + \cdots + (k-1)l_k \right\}.$$

Denoting $x_k(t) = \Upsilon^t(\hat{S}_{kn+1})$ and using the homomorphic property of Υ^t, we obtain the hypothesis of the proposition. $\qquad\square$

5 Comparison with the Chen-Fliess Approach

In this section, we compare the number of nonzero iterated integrals in two approaches: the one given in this article and the Chen-Fliess one. Recall that in the latter approach [9], we assume that we have a differential equation:

$$\dot{x}(t) = u_0(t)\, X_0(x(t)) + u_1(t)\, X_1(x(t)) + \cdots + u_n(t)\, X_n(x(t)),$$
$$x(0) = 0,$$

where $X_i(x) = x^i \frac{\partial}{\partial x}$, for $i = 0, \ldots, n$, are differential vector fields on $\mathbb{R}$. The solution is given by

$$x(t) = \sum_{v \in A^*} \Upsilon^t(v)\, X_v(x)(0), \qquad (5.1)$$

where for $v = a_{i_1} \cdots a_{i_k} \in A^*$ we define $X_v(x)(0) := X_{i_1} \cdots X_{i_k}(x)(0)$ as a composition of vector fields acting on the function $h(x) = x$ and evaluated at the initial value $x_0 = 0$. Since $X_i(x)(0) \neq 0$ only for $i = 0$, the sum can be significantly reduced. Our aim is to eliminate all unnecessary summands. Since the second derivative $\frac{\partial^2}{\partial x^2} x = 0$, we need to compute $X_v(x)(0)$ modulo the second and higher derivatives.

Lemma 5.1 *For $k \geq 2$ and $v = a_{i_1} \cdots a_{i_k} \in A_k^*$, we have*

$$X_v = i_k(i_k + i_{k-1} - 1)(i_k + i_{k-1} + i_{k-2} - 2) \cdots (i_k + \cdots + i_2 - k + 2) x^{i_1 + \cdots + i_k - k + 1} \frac{\partial}{\partial x} \quad \mod \frac{\partial^2}{\partial x^2}.$$

Proof We use the induction on k. The case $k = 2$ is clear. Assume $w = a_{i_2} \cdots a_{i_{k+1}} \in A_k^*$ and $v = a_{i_1} w \in A_{k+1}^*$. By the induction hypothesis $X_w = I x^\alpha \frac{\partial}{\partial x} \mod \frac{\partial^2}{\partial x^2}$, where $I = i_{k+1}(i_{k+1} + i_k - 1) \cdots (i_{k+1} + \cdots + i_3 - k + 2)$, $\alpha = i_2 + \cdots + i_{k+1} - k + 1$, and thus $X_v = x^{i_1 + \alpha - 1} I \alpha \frac{\partial}{\partial x} \mod \frac{\partial^2}{\partial x^2}$. This ends the proof. $\qquad\square$

Using this lemma, we conclude that for $v = a_{i_1} \cdots a_{i_k} \in A_k^*$, $X_v(x)(0) \neq 0$ only if $i_1 + \cdots + i_k = k - 1$, and therefore (5.1) simplifies to

$$x(t) = \sum_{k=1}^{\infty} \sum_{\mathbf{i} \in M_0(k)} \Upsilon^t(a_{\mathbf{i}}) i_k(i_k + i_{k-1} - 1) \cdots (i_k + \cdots + i_2 - k + 2),$$

where $a_{\mathbf{i}} := a_{i_1} \cdots a_{i_k}$, and the second sum is taken over all multi-indices $\mathbf{i} = (i_1, \ldots, i_k)$ in the set $M_0(k) \subset (\{0, \ldots, n\})^k$ given by a one equality and $k - 1$ inequalities:

$$i_k + \cdots + i_1 - k + 1 = 0, \qquad i_k + \cdots + i_2 - k + 2 \geq 0,$$

$$\vdots$$

$$i_k + i_{k-1} - 1 \geq 0,$$

$$i_k \geq 0.$$

On the k-th step of approximation, there are $\#M_0(k)$ nontrivial integrals to compute. If we assume $n = \infty$, one can compute that these are Catalan numbers, i.e., $\#M_0(k) = \frac{1}{k+1}\binom{2k}{k}$. The first ten of these numbers are 1, 2, 5, 14, 42, 132, 429, 1,430, and 4,862. We see that this growth is much faster than the growth 1, 2, 3, 5, 7, 11, 15, 22, 30, and 42 of integrals needed to compute the k-th step of the expansion in our approach, as we mentioned in Remark 2.2.

6 Concluding Remarks

In the article, we formulated a scheme for expanding a solution of a general nonautonomous polynomial differential equation. The time-dependent homogeneous parts of the expansion were expressed in terms of iterated integrals. The formula for each of this part was given recursively by Eq. (2.6). The advantage of our approach is that it is made on algebraic level. We use the shuffle product which is an algebraic analog of multiplication of iterated integrals. Therefore, the algebraic formula can be easily transformed into the analytic one giving the expansion of the solution of the initial problem, as we stated in Theorem 1.

Finally, there is some work to be done. One way of development is to write an explicit formula for the algebraic series $\hat{S}$ preferably with the use of a shuffle product. It would also be important to find a deeper algebraic structure of this solution. Another way is to rewrite the scheme for systems of nonautonomous polynomial differential equations and estimate the radius of convergence in this case. This is important, for example, to integrate higher-order Lie-Sheffers systems.

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