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Research Article

Asymptotic Behavior of Solutions to Some Homogeneous Second-Order Evolution Equations of Monotone Type

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We study the asymptotic behavior of solutions to the second-order evolution equation $p(t)u''(t)+r(t)u'(t) \in Au(t)$ a.e. $t \in (0,+\infty)$, $u(0)=u_0$, $\sup_{t\geq 0}|u(t)| < +\infty$, where A is a maximal monotone operator in a real Hilbert space H with $A^{-1}(0)$ nonempty, and $p(t)$ and $r(t)$ are real-valued functions with appropriate conditions that guarantee the existence of a solution. We prove a weak ergodic theorem when A is the subdifferential of a convex, proper, and lower semicontinuous function. We also establish some weak and strong convergence theorems for solutions to the above equation, under additional assumptions on the operator A or the function $r(t)$.

1. Introduction

Let H be a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $|\cdot|$. We denote weak convergence in H by $\rightharpoonup$ and strong convergence by $\to$. We will refer to a nonempty subset A of $H\times H$ as a (nonlinear) possibly multivalued operator in H. A is called monotone (resp., strongly monotone) if $(y_2-y_1,x_2-x_1)\geq 0$ (resp., $(y_2-y_1,x_2-x_1)\geq \beta|x_1-x_2|^2$ for some $\beta>0$) for all $[x_i,y_i]\in A$, $i=1,2$. A is called maximal monotone if A is monotone and $R(I+A)=H$, where I is the identity operator on H.

Existence, as well as asymptotic behavior of solutions to second-order evolution equations of the form

$$\begin{aligned} &p(t)u''(t)+r(t)u'(t)\in Au(t) \quad \text{a.e. on } \mathbb{R}^+,\\ &u(0)=u_0, \qquad \sup_{t\geq 0}|u(t)|<+\infty, \end{aligned} \tag{1.1}$$

in the special case $p(t) \equiv 1$ and $r(t) \equiv 0$, were studied by many authors, see, for example, Barbu [1], Moroşanu [2, 3], and the references therein, Mitidieri [4, 5], Poffald and Reich [6], and Véron [7].

Véron [8, 9] studied the existence and uniqueness of solutions to (1.1) with the following assumptions on $p(t)$ and $r(t)$:

$$\begin{gathered} p \in W^{2,\infty}(0,+\infty), \qquad r \in W^{1,\infty}(0,+\infty), \\ \exists \alpha > 0 \quad \text{such that } \forall t \geq 0,\ p(t) \geq \alpha, \end{gathered} \tag{1.2}$$

$$\int_0^{+\infty} e^{-\int_0^t (r(s)/p(s))ds} dt = +\infty. \tag{1.3}$$

The following theorem is proved in [9].

Theorem 1.1. *Assume that A is a maximal monotone, $0 \in A(0)$, and (1.2) and (1.3) are satisfied. Then for each $u_0 \in D(A)$, there exists a continuously differentiable function $u \in H^2((0,+\infty);H)$, satisfying*

$$\begin{gathered} p(t)u''(t) + r(t)u'(t) \in Au(t) \quad \text{a.e. on } \mathbb{R}^+, \\ u(0) = u_0, \quad u(t) \in D(A) \quad \text{a.e. on } \mathbb{R}^+. \end{gathered} \tag{1.4}$$

If u (resp., v) are solutions to (1.1) with initial conditions u_0 (resp., v_0), then for each $t \geq 0$,

$$|u(t) - v(t)| \leq |u_0 - v_0|. \tag{1.5}$$

In addition, $|u(t)|$ is nonincreasing.

Véron [8, 9] also proved another existence theorem by assuming A to be strongly monotone, instead of (1.3).

It is easy to show that without loss of generality, the condition $0 \in A(0)$ in Theorem 1.1 can be replaced by the more general assumption $A^{-1}(0) \neq \phi$.

In Section 2, we present our main results on the asymptotic behavior of solutions to (1.1).

2. Main results

In this section, we study the asymptotic behavior of solutions to the evolution equation (1.1) under appropriate assumptions on the operator A and the functions $p(t)$ and $r(t)$, similar to those assumed by Véron [8, 9], implying the existence of solutions to (1.1). Throughout the paper, we assume that (1.2) holds and $A^{-1}(0) \neq \phi$.

First we prove two lemmas.

Lemma 2.1. *Assume that $u(t)$ is a solution to (1.1). Then for each $p \in A^{-1}(0)$, $|u(t) - p|$ is either nonincreasing, or eventually increasing.*

Proof. Let $p \in A^{-1}(0)$. By monotonicity of A and (1.1), we have

$$(p(t)u''(t) + r(t)u'(t), u(t) - p) \geq 0 \quad \text{a.e. on } (0,+\infty). \tag{2.1}$$

It follows that

$$p(t)\frac{d^2}{dt^2}\left|u(t)-p\right|^2+r(t)\frac{d}{dt}\left|u(t)-p\right|^2\geq 0. \tag{2.2}$$

Dividing both sides of the above inequality by $p(t)$ and multiplying by $e^{\int_0^t(r(s)/p(s))ds}$, we obtain

$$\frac{d}{dt}\left(e^{\int_0^t(r(s)/p(s))ds}\frac{d}{dt}\left|u(t)-p\right|^2\right)\geq 0. \tag{2.3}$$

We consider two cases.

If $(d/dt)|u(t)-p|^2\leq 0$ for each $t>0$, then $|u(t)-p|^2$ is nonincreasing. Otherwise, there exists $t_0>0$ such that $(d/dt)|u(t)-p|^2_{|t=t_0}>0$. Integrating (2.3), we get for each $t\geq t_0$ that

$$e^{\int_0^t(r(s)/p(s))ds}\frac{d}{dt}\left|u(t)-p\right|^2\geq 2e^{\int_0^{t_0}(r(s)/p(s))ds}\left(u'(t_0),u(t_0)-p\right)>0. \tag{2.4}$$

Hence, $(d/dt)|u(t)-p|^2>0$ for each $t>t_0$. This means that $|u(t)-p|$ is eventually increasing. □

Note that in the proof of Lemma 2.1, we did not use the boundedness of u.

LEMMA 2.2. *Suppose that $u(t)$ is a solution to (1.1). Then for each $p\in A^{-1}(0)$, $\lim_{t\to+\infty}|u(t)-p|^2$ exists and $\liminf_{t\to+\infty}(d/dt)|u(t)-p|^2\leq 0$. In addition, if either (1.3) is satisfied or A is strongly monotone, then $|u(t)-p|^2$ is nonincreasing.*

Proof. The existence of $\lim_{t\to+\infty}|u(t)-p|^2$ follows from Lemma 2.1.

By contradiction, assume that $\liminf_{t\to+\infty}(d/dt)|u(t)-p|^2>0$. Then there exist $t_0>0$ and $\lambda>0$, such that for each $t\geq t_0$,

$$\frac{d}{dt}\left|u(t)-p\right|^2\geq\lambda. \tag{2.5}$$

Integrating from $t=t_0$ to $t=T$, we get

$$\left|u(T)-p\right|^2-\left|u(t_0)-p\right|^2\geq\lambda T-\lambda t_0. \tag{2.6}$$

Letting $T\to+\infty$, we deduce that u is not bounded, a contradiction. If in addition (1.3) is satisfied, assume that $|u(t)-p|$ is eventually increasing. Then there exists $t_0>0$ such that $(u'(t_0),u(t_0)-p)>0$. Dividing both sides of (2.4) by $e^{\int_0^t(r(s)/p(s))ds}$ and integrating from $t=t_0$ to $t=T$, we get

$$\left|u(T)-p\right|^2-\left|u(t_0)-p\right|^2\geq 2e^{\int_0^{t_0}(r(s)/p(s))ds}\left(u'(t_0),u(t_0)-p\right)\int_{t_0}^{T}e^{-\int_0^t(r(s)/p(s))ds}dt. \tag{2.7}$$

Letting $T\to+\infty$, we obtain a contradiction to assumption (1.3). This implies that $|u(t)-p|$ is nonincreasing.

Finally, assume that A is strongly monotone, and let $p \in A^{-1}(0)$. Then we have

$$\left(p(t)u''(t)+r(t)u'(t),u(t)-p\right) \geq \beta\,|u(t)-p|^2. \tag{2.8}$$

This implies that

$$p(t)\frac{d^2}{dt^2}|u(t)-p|^2+r(t)\frac{d}{dt}|u(t)-p|^2 \geq 2\beta|u(t)-p|^2. \tag{2.9}$$

Suppose to the contrary that $|u(t)-p|$ is increasing for $t \geq T_0 > 0$. Let K (resp., M) be an upper bound for $p(t)$ (resp., $|r(t)|$). Integrating both sides of this inequality from $t = T_0$ to $t = T$, we get

$$\begin{aligned}
2\beta\int_{T_0}^{T}&|u(t)-p|^2dt\\
&\leq K\left(\frac{d}{dT}|u(T)-p|^2-2(u'(T_0),u(T_0)-p)+\int_{T_0}^{T}\frac{r(t)}{p(t)}\frac{d}{dt}|u(t)-p|^2dt\right)\\
&\leq K\left(\frac{d}{dT}|u(T)-p|^2-2(u'(T_0),u(T_0)-p)+\frac{M}{\alpha}|u(T)-p|^2-\frac{M}{\alpha}|u(T_0)-p|^2\right).
\end{aligned} \tag{2.10}$$

Since $|u(t)-p|$ is increasing for $t \geq T_0 > 0$, we have

$$\begin{aligned}
2\beta|u(T_0)-p|^2(T-T_0)&\\
\leq K\Big(&\frac{d}{dT}|u(T)-p|^2-2(u'(T_0),u(T_0)-p)\\
&+\frac{M}{\alpha}|u(T)-p|^2-\frac{M}{\alpha}|u(T_0)-p|^2\Big).
\end{aligned} \tag{2.11}$$

Taking lim inf as $T \to +\infty$ of both sides in the above inequality, by the first part of this lemma we deduce that $u(t)$ is unbounded, a contradiction. □

In the following, we prove a mean ergodic theorem when A is the subdifferential of a proper, convex, and lower semicontinuous function.

Theorem 2.3. *Suppose that $u(t)$ is a solution to (1.1) and $A = \partial\varphi$, where $\varphi : H \to]-\infty,+\infty]$ is a proper, convex, and lower semicontinuous function. If (1.3) is satisfied, then $\sigma_T := (1/T)\int_0^T u(t)dt \rightharpoonup p \in A^{-1}(0)$, as $T \to +\infty$.*

Proof. By the subdifferential inequality and (1.1), we get for each $p \in A^{-1}(0)$ that

$$\begin{aligned}
\varphi(u(t))-\varphi(p)&\leq\left(p(t)u''(t)+r(t)u'(t),u(t)-p\right)\\
&\leq\frac{p(t)}{2}\frac{d^2}{dt^2}|u(t)-p|^2+\frac{r(t)}{2}\frac{d}{dt}|u(t)-p|^2\\
&=\frac{p(t)}{2}e^{-\int_0^t(r(s)/p(s))ds}\frac{d}{dt}\left(e^{\int_0^t(r(s)/p(s))ds}\frac{d}{dt}|u(t)-p|^2\right).
\end{aligned} \tag{2.12}$$

Let K be an upper bound for $p(t)/2$. Integrating the above inequality from $t=0$ to $t=T$, and using integration by parts, we get

$$\begin{aligned}\int_0^T (\varphi(u(t))-\varphi(p))\,dt &\le K\Big(\frac{d}{dT}|u(T)-p|^2-2(u'(0),u(0)-p)+\int_0^T\frac{r(t)}{p(t)}\frac{d}{dt}|u(t)-p|^2dt\Big)\\ &\le K\Big(-2(u'(0),u(0)-p)+\int_0^T\frac{r(t)}{p(t)}\frac{d}{dt}|u(t)-p|^2dt\Big)\end{aligned} \tag{2.13}$$

(the second inequality holds by Lemma 2.2). Let R be an upper bound for $|r(t)|$, which exists by assumption (1.2). Since $|u(t)-p|$ is nonincreasing (by Lemma 2.2), we get from (2.13) that

$$\begin{aligned}\limsup_{T\to+\infty}\frac{1}{T}\int_0^T(\varphi(u(t))-\varphi(p))\,dt &\le \limsup_{T\to+\infty}\frac{K}{T}\int_0^T\frac{r(t)}{p(t)}\frac{d}{dt}|u(t)-p|^2dt\\ &\le \frac{-KR}{\alpha}\limsup_{T\to+\infty}\frac{1}{T}\Big[|u(T)-p|^2-|u(0)-p|^2\Big]=0.\end{aligned} \tag{2.14}$$

Since $p\in A^{-1}(0)$ and $A=\partial\varphi$, p is a minimum point of φ. Convexity of φ implies that

$$0\le\varphi(\sigma_T)-\varphi(p)\le\frac{1}{T}\int_0^T\varphi(u(t))\,dt-\varphi(p). \tag{2.15}$$

Taking the lim sup as $T\to+\infty$ in the above inequality, we get by (2.14)

$$\limsup_{T\to+\infty}\varphi(\sigma_T)\le\varphi(p). \tag{2.16}$$

Assume that $\sigma_{T_n}\rightharpoonup q$ for some sequence $\{T_n\}$ converging to $+\infty$ as $n\to+\infty$. Since φ is lower semicontinuous, we have

$$\liminf_{n\to+\infty}\varphi(\sigma_{T_n})\ge\varphi(q). \tag{2.17}$$

Therefore,

$$\varphi(p)\ge\limsup_{T\to+\infty}\varphi(\sigma_T)\ge\liminf_{n\to+\infty}\varphi(\sigma_{T_n})\ge\varphi(q). \tag{2.18}$$

Hence, $q\in A^{-1}(0)$ and by Lemma 2.2 $\lim_{t\to+\infty}|u(t)-q|^2$ exists. Now if p is another weak cluster point of σ_T, then $\lim_{t\to+\infty}(|u(t)-p|^2-|u(t)-q|^2)$ exists. It follows that $\lim_{t\to+\infty}(u(t),p-q)$ exists, hence $\lim_{T\to+\infty}(\sigma_T,p-q)$ exists. This implies that $p=q$, and therefore $\sigma_T\rightharpoonup p\in A^{-1}(0)$, as $T\to+\infty$. $\square$

Theorem 2.4. *Let u be a solution to (1.1). If (1.3) is satisfied and there exist $t_0 > 0$ and a positive constant M, such that $r(t) \geq -Mt^{-2}$ for $t \geq t_0$, then*

$$\lim_{T\to+\infty} \left| u(T) - \frac{1}{T}\int_0^T u(t)dt \right| = 0. \tag{2.19}$$

Proof. From (2.1), we have

$$|u'(t)|^2 \leq \frac{1}{2}\frac{d^2}{dt^2}|u(t)-p|^2 + \frac{1}{2}\frac{r(t)}{p(t)}\frac{d}{dt}|u(t)-p|^2. \tag{2.20}$$

Multiplying both sides of the above inequality by t^2, integrating from $t = 0$ to $t = T$, and dividing by T, since $|u(t) - p|^2$ is nonincreasing, we get after integration by parts that

$$\frac{1}{T}\int_0^T t^2|u'(t)|^2dt \leq -|u(T)-p|^2 + \frac{1}{T}\int_0^T |u(t)-p|^2dt + \frac{1}{2T}\int_0^T \frac{t^2r(t)}{p(t)}\frac{d}{dt}|u(t)-p|^2dt. \tag{2.21}$$

Since $|u(t) - p|^2$ is nonincreasing (by Lemma 2.2), $r(t) \geq -Mt^{-2}$ for $t \geq t_0$, and $p(t)$ is bounded from below and by α, we get

$$\begin{aligned}\limsup_{T\to+\infty}\frac{1}{T}\int_0^T t^2|u'(t)|^2dt &\leq \limsup_{T\to+\infty}\frac{1}{2T}\int_0^T \frac{t^2r(t)}{p(t)}\frac{d}{dt}|u(t)-p|^2dt \\ &\leq \frac{-M}{2\alpha}\limsup_{T\to+\infty}\frac{1}{T}\left[|u(T)-p|^2 - |u(t_0)-p|^2\right] = 0.\end{aligned} \tag{2.22}$$

Integrating by parts and using the Cauchy-Schwartz inequality, we have

$$\begin{aligned}\left|u(t) - \frac{1}{t}\int_0^t u(s)ds\right|^2 &= \left|\frac{1}{t}\int_0^t su'(s)ds\right|^2 \leq \left(\frac{1}{t}\int_0^t s|u'(s)|ds\right)^2 \\ &\leq \frac{1}{t^2}\left(\int_0^t ds\right)\left(\int_0^t s^2|u'(s)|^2ds\right) = \frac{1}{t}\int_0^t s^2|u'(s)|^2ds.\end{aligned} \tag{2.23}$$

Thus by (2.22),

$$\limsup_{t\to+\infty}\left|u(t) - \frac{1}{t}\int_0^t u(s)ds\right|^2 \leq \limsup_{t\to+\infty}\frac{1}{t}\int_0^t s^2|u'(s)|^2ds = 0. \tag{2.24}$$

□

As a corollary to Theorem 2.4, we have the following weak convergence theorem.

Theorem 2.5. *Suppose that the assumptions in Theorems 2.3 and 2.4 are satisfied. Then $u(t) \rightharpoonup p \in A^{-1}(0)$ as $t \to +\infty$.*

In our next theorem, we prove the strong convergence of u by assuming A to be strongly monotone.

Theorem 2.6. *Assume that the operator A is strongly monotone, and let u be a solution to (1.1). Then $u(t)$ converges strongly to $p \in A^{-1}(0)$ as $t \to +\infty$.*

Proof. By the strong monotonicity of A, and for $p \in A^{-1}(0)$ (in this case $A^{-1}(0)$ is a singleton), we have

$$(p(t)u''(t)+r(t)u'(t),u(t)-p) \geq \beta|u(t)-p|^2. \tag{2.25}$$

Let K be an upper bound for $p(t)$. Integrating this inequality from $t=0$ to $t=T$ and using Lemma 2.2, we obtain

$$2\beta\int_0^T |u(t)-p|^2 dt \leq K\left(\frac{d}{dT}|u(T)-p|^2 - 2(u'(0),u(0)-p) + \int_0^T \frac{r(t)}{p(t)}\frac{d}{dt}|u(t)-p|^2 dt\right). \tag{2.26}$$

Let R be an upper bound for $|r(t)|$, which exists by assumption (1.2). Dividing both sides of this inequality by T and using Lemma 2.2, we get

$$\begin{aligned} 2\beta\lim_{T\to+\infty}|u(T)-p|^2 &= \limsup_{T\to+\infty}\frac{\beta}{T}\int_0^T |u(t)-p|^2 dt \\ &\leq \limsup_{T\to+\infty}\frac{K}{T}\int_0^T \frac{r(t)}{p(t)}\frac{d}{dt}|u(t)-p|^2 dt \\ &\leq \frac{-KR}{\alpha}\limsup_{T\to+\infty}\frac{1}{T}\left[|u(T)-p|^2 - |u(0)-p|^2\right] = 0. \end{aligned} \tag{2.27}$$

This completes the proof of the theorem. □

Now, we apply our results to an example presented by Véron [8] and Apreutesei [10].

Example 2.7. Let $H = L^2(\Omega)$ where $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with smooth boundary Γ. Let $j : \mathbb{R} \to (-\infty,+\infty]$ be proper, convex, and lower semicontinuous and $\beta = \partial j$. We assume for simplicity that $0 \in \beta(0)$. Define

$$Au = -\Delta u = -\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2} \tag{2.28}$$

with

$$D(A) = \left\{u \in H^2(\Omega),\ \frac{-\partial u}{\partial \eta}(x) \in \beta(u(x)) \text{ a.e. on } \Gamma\right\}, \tag{2.29}$$

where $((\partial u/\partial\eta)(x))$ is the outward normal derivative to Γ at $x \in \Gamma$. We know that $A = \partial\phi$, where $\phi : L^2(\Omega) \to (-\infty,+\infty]$ is the Brézis functional:

$$\phi(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla u|^2 dx + \int_\Gamma \beta(u(x))d\sigma & \text{if } u \in H^1(\Omega),\ \beta(u) \in L^1(\Gamma), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.30}$$

Consider the following equation:

$$p(t)\frac{\partial^2 u}{\partial t^2}(t,x)+r(t)\frac{\partial u}{\partial t}(t,x)+\sum_i \frac{\partial^2 u}{\partial x_i^2}(t,x)=0 \quad \text{a.e. on } \mathbb{R}^+\times\Omega,$$
$$-\frac{\partial u}{\partial \eta}(t,x)\in\beta u(t,x) \quad \text{a.e. on } \mathbb{R}^+\times\Gamma, \tag{2.31}$$
$$u(0,x)=u_0(x) \quad \text{a.e. on } \Omega.$$

Assume that $p(t)$ and $r(t)$ are real functions satisfying (1.2) and (1.3). Then Theorem 2.3 implies the weak mean ergodic convergence of $u(t,\cdot)$. In addition, if $r(t)\geq -Mt^{-2}$ eventually, Corollary 2.5 implies the weak convergence of the solution to the above equation.

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